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SUFFICIENT CONDITIONS FOR MULTIPLY CONSTRAINED EXTREMA

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Assume that I and J_i, $i = 1, \ldots, n$ are smooth functions defined on a Hilbert space $\mathcal{H}$. We derive sufficient conditions for I to have a strong local minimum at a point y subject to the n constraints $J_i = 0$.

1. Introduction.

Suppose that I and J_i, $i = 1, \ldots, n$ are functions with at least two Fréchet derivatives defined on a Hilbert space $\mathcal{H}$. Suppose that $y \in \mathcal{H}$ satisfies the n constraints $J_i(y) = 0$. The first order necessary condition for I to have a strong local minimum at $y \in H$ subject to these constraints is that there exist Lagrange multipliers λ_i so that $I' + \sum \lambda_i J_i'$ is identically zero ([1]). Here I' and J_i' are Fréchet derivatives evaluated at y. We will assume that this condition is satisfied. A second order necessary condition is that $I'' + \sum \lambda_i J_i''$ be non-negative definite on the tangent space to the constraint manifold at y ([3]). In order to prove that y is a strong local constrained minimum (i.e., that $I(y + h) > I(y)$ for all non-zero h of sufficiently small magnitude satisfying $J_i(y + h) = 0$, $i = 1, \cdots, n$), a first step is often to show that $I'' + \sum \lambda_i J_i''$ is positive definite on the tangent space. Even in the unconstrained case, however, this is not by itself sufficient ([1]).

Maddocks ([3], see also [4]) derived sufficient conditions for $I'' + \sum \lambda_i J_i''$ to be positive definite on the space tangent to the constraint manifold. These conditions involve the spectrum of the bounded, self-adjoint operator A defined by $(I'' + \sum \lambda_i J_i'')(h, h) = \langle h, Ah \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product of $\mathcal{H}$. In this paper, I will start with Maddocks' criteria to obtain sufficient conditions for constrained strong local extrema. This paper generalizes [6] in two ways. The first is obvious: [6] dealt with single constraints. In addition, in [6] the assumption was made that A was invertible. We will weaken this assumption in two different ways. In Theorem 2.1, A might fail to be invertible if one of the non-positive eigenvalues happens to equal zero. On the other hand, in Theorem 3.1 the kernel of A is treated separately. The result of this is that when A is not invertible, the two theorems yield slightly different criteria. For a (one constraint) example in which A is singular, see [5].

Throughout this paper, the kernel of A will be finite dimensional. This is certainly necessary for A to be positive definite on the tangent space, which is of finite co-dimension. However, I won't state this as a separate assumption, since it will be implied by the hypotheses of the two theorems. In both of the theorems of this paper, 0 is an isolated point of the spectrum of A. Because of this, the range of A (which I shall call $\mathcal{R}(A)$) is closed, so that $\mathcal{H} = \mathcal{R}(A) \oplus \ker(A)$ ([2]). We shall be using this fact throughout the paper.

2. Non-positive eigenvalues.

As in [6], we define $\nabla J_i \in \mathcal{H}$ to be the vector such that $\langle \nabla J_i, h\rangle = J_i'(h)$.

Lemma 2.1. *For each ∇J_i there are η_i, β_i, with $\beta_i \in \ker(A)$, $\eta_i \perp \ker(A)$, so that $\nabla J_i = A(\eta_i) + \beta_i$. (We do not exclude the possibility that $\ker(A)$ is trivial. In that case $\beta_i = 0$.)*

Proof. Certainly $\nabla J_i = \alpha_i + \beta_i$, where $\alpha_i \in \mathcal{R}(A)$ and $\beta_i \in \ker(A)$. Take $\hat{\eta}_i$ so that $A(\hat{\eta}_i) = \alpha_i$. Defining η_i to be the orthogonal projection of $\hat{\eta}_i$ onto the range of A, one finds that $A(\eta_i) = \alpha_i$, concluding the proof.

Now define the matrix B by

$$B = \begin{pmatrix} \langle \eta_1, A\eta_1\rangle & \cdots & \langle \eta_1, A\eta_n\rangle \\ \vdots & \ddots & \vdots \\ \langle \eta_n, A\eta_1\rangle & \cdots & \langle \eta_n, A\eta_n\rangle \end{pmatrix}.$$

□

Lemma 2.2. *Let $P = (p_{ij})$ be the orthogonal matrix which diagonalizes B. (Since A is self-adjoint, B is symmetric.) Define vectors ζ_k to be $\sum_j p_{jk}\eta_j$. Then upon diagonalizing B, $P^tBP = (\langle \zeta_i, A\zeta_j\rangle)$ (so that $\langle \zeta_i, A\zeta_j\rangle = 0$ for $i \neq j$, i.e., the ζ_i's form an A-orthogonal set, see [3]). Moreover, we may renumber the functionals J_i so that $\langle \zeta_1, A\zeta_1\rangle \leq \langle \zeta_2, A\zeta_2\rangle \leq \cdots \leq \langle \zeta_n, A\zeta_n\rangle$ (where these are the eigenvalues of B).*

Proof. This is essentially Lemma 1 of [3], obtained by writing out the matrix multiplication. Note that ζ_k is orthogonal to $\ker(A)$. □

Lemma 2.3. *Suppose that Y and Z are subspaces of a Hilbert space $\mathcal{H}$ such that Y is closed, Z is finite dimensional, and $Y \cap Z$ is trivial. Then there exists a $\delta > 0$ so that no element of Z is within δ of an element of the unit ball of Y.*

Proof. If the lemma is false, then there exist sequences $y_i \in Y$ and $z_i \in Z$ so that $\|y_i\| = 1$ and $\|y_i - z_i\| \to 0$. By taking a subsequence, we may assume that z_i converges (strongly) to $z \in Z$. Therefore y_i converges to z as well. Since Y is closed, $z \in Y \cap Z = 0$, so y_i converges to the zero vector. This contradicts the assumption that $\|y_i\| = 1$. □

Theorem 2.1. *Suppose that $I' + \sum \lambda_i J_i'$ is zero at a point $y \in \mathcal{H}$ which satisfies the constraints $J_i(y) = 0$, $i = 1, \ldots, n$. Suppose that for some $\epsilon > 0$, $\sigma(A) \cap (-\infty, \epsilon)$ consists of k non-positive eigenvalues (counting multiplicity) with corresponding eigenvectors $\phi_0, \ldots, \phi_{k-1}$. (Here $\sigma(A)$ is the spectrum of A.) Suppose that the matrix B, defined above, has at least k negative eigenvalues. Then I has a strong constrained local minimum at y.*

Proof. Let $\mathcal{C}$ be the span of the vectors $\{\nabla J_1, \ldots, \nabla J_n\}$, and $\mathcal{C}^\perp$ its orthogonal complement. $I'' + \sum \lambda_j J_j''$ is said to be *strongly positive* on the space $\mathcal{C}^\perp$ if for some $\epsilon > 0$ there holds $\langle x, Ax \rangle \geq \epsilon \|x\|^2$ for all $x \in \mathcal{C}^\perp$. This condition is sufficient to imply that y is a strong local constrained minimum (see [1, Chapter 3]), so this is what we need to prove.

Let x be an arbitrary element of $\mathcal{C}^\perp$. I will first show that there exist unique scalars $c_1, \ldots, c_k$ so that $v = x - c_1\zeta_1 - \cdots - c_k\zeta_k$ is perpendicular to $\phi_0, \ldots, \phi_{k-1}$. We seek $c_1, \ldots, c_k$ to solve

$$\begin{aligned} \langle x, \phi_0 \rangle &= c_1 \langle \zeta_1, \phi_0 \rangle + \cdots + c_k \langle \zeta_k, \phi_0 \rangle \\ \vdots &= \vdots \\ \langle x, \phi_{k-1} \rangle &= c_1 \langle \zeta_1, \phi_{k-1} \rangle + \cdots + c_k \langle \zeta_k, \phi_{k-1} \rangle \end{aligned}$$

so that it certainly suffices to show that the matrix

$$\begin{pmatrix} \langle \zeta_1, \phi_0 \rangle & \cdots & \langle \zeta_k, \phi_0 \rangle \\ \vdots & \ddots & \vdots \\ \langle \zeta_1, \phi_{k-1} \rangle & \cdots & \langle \zeta_k, \phi_{k-1} \rangle \end{pmatrix}$$

is invertible. If not, then some non-trivial linear combination of the columns is the zero vector, implying that there is a non-trivial linear combination ζ of the vectors $\zeta_1, \ldots, \zeta_k$ which is perpendicular to $\phi_0, \ldots, \phi_{k-1}$. This leads to a contradiction as follows. Since ζ is a non-trivial vector which is orthogonal to all of the non-positive eigenvectors of A, $\langle \zeta, A\zeta \rangle > 0$. On the other hand, a non-trivial linear combination $d_1\zeta_1 + \cdots + d_k\zeta_k$ satisfies

$$\langle d_1\zeta_1 + \cdots + d_k\zeta_k, d_1 A\zeta_1 + \cdots + d_k A\zeta_k \rangle = \sum_{i=1}^{k} d_i^2 \langle \zeta_i, A\zeta_i \rangle < 0$$

since $\langle \zeta_i, A\zeta_i \rangle$ is the i^{th} eigenvalue of B.

Since x is orthogonal to all of the ∇J_i's, $\langle x, A\eta_i + \beta_i\rangle = 0$ for $i = 1, \dots, n$. (Of course if $0 \notin \sigma(A)$, $\beta_i = 0$ for all i.) It follows that

$$\left\langle x, A\zeta_i + \sum_j p_{ij}\beta_j \right\rangle = 0 \tag{2.1}$$

for all i. Let $\gamma_i = \sum p_{ij}\beta_j \in \ker(A)$. Then

$$0 = \langle x - v + v, A\zeta_i + \gamma_i\rangle = \left\langle \sum_{j=1}^{k} c_j\zeta_j, A\zeta_i \right\rangle + \langle v, A\zeta_i + \gamma_i\rangle \tag{2.2}$$

holds for all i.

Now, v is certainly perpendicular to $\ker(A)$, since v is perpendicular to all eigenspaces corresponding to non-positive eigenvalues. Using this fact and the A-orthogonality of the ζ_i's, Equation 2.2 becomes

$$0 = c_i \langle \zeta_i, A\zeta_i\rangle + \langle v, A\zeta_i\rangle .$$

Substituting this into

$$\begin{aligned}\langle x, Ax\rangle &= \left\langle v + \sum_{i=1}^{k} c_i\zeta_i, Av + \sum c_i A\zeta_i \right\rangle \\ &= \langle v, Av\rangle + 2\left\langle v, \sum_{i=1}^{k} c_i A\zeta_i \right\rangle + \sum_{i=1}^{k} c_i^2 \langle \zeta_i, A\zeta_i\rangle\end{aligned}$$

we obtain

$$\langle x, Ax\rangle = \langle v, Av\rangle - \sum_{i=1}^{k} c_i^2 \langle \zeta_i, A\zeta_i\rangle \geq \langle v, Av\rangle ,$$

with the last inequality due to the fact that the first k eigenvalues of B are negative. Since v is orthogonal to $\phi_0, \dots, \phi_{k-1}$, it follows that

$$\langle x, Ax\rangle \geq \langle v, Av\rangle \geq \epsilon \|v\|^2 \tag{2.3}$$

and we therefore must bound $\|v\|$ from below in terms of $\|x\|$. We will do this by applying Lemma 2.3, where Y will be $\mathcal{C}^\perp$ and Z is the span of $\{\zeta_1, \dots, \zeta_k\}$. To apply this lemma, it is necessary to show that the intersection of $\mathcal{C}^\perp$ and the span of $\{\zeta_1, \cdots, \zeta_k\}$ is trivial.

Suppose that $\zeta = d_1\zeta_1 + \cdots + d_k\zeta_k \in \mathcal{C}^\perp$, i.e., that $\langle \zeta, \nabla J_j\rangle = 0$ for $j = 1, \dots, n$. By equation 2.1, $\langle \zeta, A\zeta_i + \gamma_i\rangle = 0$ for all i. Since the ζ_i's are chosen to be orthogonal to $\ker(A)$, we have $\langle \zeta, A\zeta_i\rangle = 0$ holding for all i. By the

A-orthogonality of the ζ_i's and the assumption that the first k eigenvalues of B are negative, it follows that $d_1 = \cdots = d_k = 0$. Thus the intersection of $\mathcal{C}^\perp$ and the span of $\{\zeta_1, \cdots, \zeta_k\}$ is the zero vector.

Since $x = v + c_1\zeta_1 + \cdots + c_k\zeta_k$, Lemma 2.3 yields a positive δ so that $\|v\| > \delta$ if $\|x\| = 1$. By normalizing we obtain $\|v\| > \delta\|x\|$ for general $x \in \mathcal{C}$, where δ does not depend x. Substituting this into Equation 2.3 we obtain that

$$\langle x, Ax\rangle > \epsilon\delta^2\|x\|^2$$

for $x \in C^\perp$. This strong positivity on $\mathcal{C}^\perp$ implies that y is a strong constrained local minimum, as mentioned above. □

3. Negative eigenvalues.

The following lemma will be needed in the proof of Theorem 3.1, at a point where we will apply Lemma 2.3.

Lemma 3.1. *Suppose that C and D are closed subspaces of $\mathcal{H}$ of finite codimension. Let P_D be orthogonal projection onto D. Then $P_D(C)$ is a closed set.*

Proof. Let $\{\gamma_1, \ldots, \gamma_n\}$, $\{\delta_1, \ldots \delta_m\}$ be vectors such that $c \in C$ if and only if $\langle c, \gamma_i\rangle = 0$, $i = 1, \ldots, n$, and $d \in D$ if and only if $\langle d, \delta_j\rangle = 0$, $j = 1, \ldots, m$. Then a vector $d \in D$ is in $P_D(C)$ if and only if there are constants a_j so that $d + \sum a_j\delta_j \in C$. For this we need

$$\begin{array}{ccccccc} a_1\langle\delta_1,\gamma_1\rangle & + \cdots + & a_m\langle\delta_m,\gamma_1\rangle & = & - & \langle d,\gamma_1\rangle \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1\langle\delta_1,\gamma_n\rangle & + \cdots + & a_m\langle\delta_m,\gamma_n\rangle & = & - & \langle d,\gamma_n\rangle \end{array}$$

so that d is in $P_D(C)$ if and only if the vector

$$\begin{pmatrix} \langle d,\gamma_1\rangle \\ \vdots \\ \langle d,\gamma_n\rangle \end{pmatrix}$$

is in the span of the vectors

$$\begin{pmatrix} \langle\delta_1,\gamma_1\rangle \\ \vdots \\ \langle\delta_1,\gamma_n\rangle \end{pmatrix}, \cdots, \begin{pmatrix} \langle\delta_m,\gamma_1\rangle \\ \vdots \\ \langle\delta_m,\gamma_n\rangle \end{pmatrix}.$$

This characterization of the image of C under the projection P_D and the fact that inner product is continuous is enough to yield the result. $\square$

We now develop a criterion slightly different from that of the previous section. We suppose that $\sigma(A) \cap (-\infty, 0)$ consists of k negative eigenvalues (counting multiplicity), that $\ker(A)$ is finite dimensional (this will be implied by the requirement that $\ker(A) \cap \mathcal{C}^\perp$ is trivial), and $\sigma(A) \cap (0, \infty) \subseteq [\epsilon, \infty)$ for some $\epsilon > 0$. Suppose that $\mathcal{R}(A) \cap \mathcal{C}$ is spanned by $\{A\xi_1, \ldots, A\xi_l\}$. We may assume that $\xi_1, \ldots, \xi_l$ are orthogonal to $\ker(A)$ by projecting onto $\mathcal{R}(A)$. Define E to be

$$E = \begin{pmatrix} \langle \xi_1, A\xi_1 \rangle & \cdots & \langle \xi_1, A\xi_l \rangle \\ \vdots & \ddots & \vdots \\ \langle \xi_l, A\xi_1 \rangle & \cdots & \langle \xi_l, A\xi_l \rangle \end{pmatrix}.$$

Theorem 3.1. *Suppose that $I' + \sum \lambda_i J_i'$ is zero at a point $y \in \mathcal{H}$ which satisfies the constraints $J_i(y) = 0$, $i = 1, \ldots, n$. Suppose that $\sigma(A)$ is as described above. Suppose that E has at least k negative eigenvalues, and that $\ker(A) \cap \mathcal{C}^\perp$ is trivial. Then I has a strong constrained local minimum at y.*

Proof. Again we need to show that A is strongly positive on $\mathcal{C}^\perp$. By orthogonalizing the matrix E as in Lemma 2.2, we may assume that $\langle \xi_i, A\xi_j \rangle = 0$ if $i \neq j$ and that $\langle \xi_1, A\xi_1 \rangle \leq \cdots \leq \langle \xi_k, A\xi_k \rangle < 0$. Take an $x \in \mathcal{C}^\perp$. Then $x = y + z$, where $z \in \ker(A)$ and $y \in \mathcal{R}(A)$. One easily sees that $\langle x, Ax \rangle = \langle y, Ay \rangle$.

By arguing as in Theorem 2.1, one can show that there are constants c_1, $\ldots$, c_k so that $y = v + \sum_{i=1}^k c_i \xi_i$, where v is orthogonal to $\phi_0, \ldots, \phi_{k-1}$. In addition, v is orthogonal to $\ker(A)$, since y and the ξ_i's are. Substituting,

$$\langle y, Ay \rangle = \langle v, Av \rangle + 2 \sum_{i=1}^k c_i \langle v, A\xi_i \rangle + \sum_{i=1}^k c_i^2 \langle \xi_i, A\xi_i \rangle .$$

Since $\langle x, A\xi_j \rangle = 0$, we have $\langle y + z, A\xi_j \rangle = \langle y, A\xi_j \rangle = 0$, so that $\langle v, A\xi_j \rangle = -c_j \langle \xi_j, A\xi_j \rangle$ holds for all j. Therefore

$$\langle y, Ay \rangle = \langle v, Av \rangle - \sum_{i=1}^k c_i^2 \langle \xi_i, A\xi_i \rangle \geq \langle v, Av \rangle \geq \epsilon \|v\|^2$$

with the last inequality following from the fact that v is orthogonal to all non-positive eigenvectors of A. We must now relate the length of v to that of x. As before, we use Lemma 2.3. Since $\ker(A) \cap \mathcal{C}^\perp$ is trivial, there is a $\delta_1 > 0$ so that $\|y\| \geq \delta_1 \|x\|$, where δ_1 is independent of x.

For a similar inequality relating $\|y\|$ and $\|v\|$, we must first show that the orthogonal projection of $\mathcal{C}^\perp$ onto $\mathcal{R}(A)$ has only the trivial intersection with the span of $\{\xi_1, \dots, \xi_k\}$. Suppose that some $\hat{y} = \sum_{i=1}^k d_i \xi_i$ is also in the orthogonal projection of $\mathcal{C}^\perp$ onto $\mathcal{R}(A)$. There is a vector $\hat{z} \in \ker(A)$ so that $\hat{y} + \hat{z} \in \mathcal{C}^\perp$. Then for each j from 1 to k,

$$0 = \left\langle \sum_{i=1}^k d_i \xi_i + \hat{z}, A\xi_j \right\rangle = \left\langle \sum_{i=1}^k d_i \xi_i, A\xi_j \right\rangle = d_j \langle \xi_j, A\xi_j \rangle$$

so that $\hat{y}$ is the zero vector. To apply Lemma 2.3 we must also show that the image of $\mathcal{C}^\perp$ under orthogonal projection onto $\mathcal{R}(A)$ is closed. This follows from Lemma 2.3. Therefore, there is some $\delta_2 > 0$ (not depending on x or y) so that $\|v\| \geq \delta_2 \|y\|$. We now obtain that

$$\langle x, Ax \rangle = \langle y, Ay \rangle \geq \epsilon \|v\|^2 \geq \epsilon \delta_1^2 \delta_2^2 \|x\|^2$$

for all $x \in \mathcal{C}^\perp$, so that A is strongly positive on $\mathcal{C}^\perp$. As above, this is sufficient for the result to follow. □

Note. In the case that A has no negative eigenvalues, Theorem 3.1 reduces to the condition that $\ker(A) \cap \mathcal{C}^\perp$ be trivial, since there is no condition on E. This is the case which appears in [5].

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