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RIESZ COMPOSITION OPERATORS

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We give a sufficient condition for a univalently induced composition operator on the Hardy space H^2 to be a Riesz operator. We then establish that every Riesz composition operator has a Koenigs model and explore connections our work has with the model theory and spectral theory of composition operators.

1. Introduction.

If φ is a holomorphic function on the unit disc U with $\varphi(U) \subset U$ then the *composition operator* C_φ induced by φ is the linear map:

$$C_\varphi f = f \circ \varphi \qquad (f \text{ holomorphic on } U).$$

Although C_φ is initially defined on the full space $H(U)$ of functions holomorphic on U, a famous result of J.E. Littlewood shows that it takes the Hardy space H^2 into itself, and its restriction to H^2 is a bounded operator [9]. The connection between operator-theoretic properties of C_φ and function-theoretic properties of the inducing map φ has led to a rapidly expanding body of research, some major threads of which you can find developed in the recent books [6] and [16].

Much of the recent interest in composition operators arises from connections with function theory that emerge from the study of *compactness* of these operators (see for example [11], [15] and [18]). In this paper the motivating concept from operator theory is the closely related concept of "Rieszness".

A *Riesz operator* on a Hilbert space is a bounded operator whose essential spectrum is the singleton $\{0\}$. We will discuss this condition in more detail in the next section, but right now it is enough to know that the class of Riesz operators includes all operators that have a compact positive power, and that the Riesz operators are "spectrally indistinguishable" from the compact ones in that their nonzero spectral points occur as isolated, finite-multiplicity eigenvalues.

We show here that if C_φ is a Riesz operator on H^2 then the mapping φ has to arise from the geometric model associated with Koenigs's solution to Schroeder's functional equation (the eigenfunction equation for C_φ).

The corresponding result for compact composition operators has long been known, and forms the basis for the investigation begun in [18] connecting the compactness problem with the geometry of Schroeder-Koenigs models. Our work suggests that in this study it is the question of *Rieszness,* rather than of compactness that should be the main operator-theoretic issue.

Our interest in Riesz operators began with a question posed by Michael Neumann, who asked if the Hardy space H^2 supports a Riesz composition operator that is not power-compact. We answer Neumann's question affirmatively by deriving an easily checked sufficient condition for a univalently induced composition operator on H^2 to be Riesz. The connection with Schroeder-Koenigs models arises when we apply our condition to show that certain non-power-compact examples that motivated the work of [18] are in fact Riesz operators. These matters occupy Sections 3 and 4, with the necessary background material collected for the reader's convenience in Section 2.

In Section 5 we make the general connection between Riesz operators and Schroeder-Koenigs models by showing that if C_φ is Riesz, then φ must have an attractive fixed point a in U. Koenigs' work ([8]) on Schroeder's equation then shows that if, in addition, φ is univalent, then φ has the following model:

$$\varphi = \sigma^{-1} \circ (\varphi'(a)\sigma),$$

where σ is a univalent mapping on U, which is uniquely determined (up to a constant multiple). Our work suggests that the geometry of $\sigma(U)$ in this Koenigs model for φ determines Rieszness; more specifically it leads us to conjecture that the "no-twisted-sectors" property introduced by Shapiro, Smith, and Stegenga in [18] is equivalent to Rieszness. (After this work was completed Pietro Poggi-Corradini established the validity of this conjecture [12].)

In the final section of this paper we discuss the "no-twisted-sectors" conjecture in more detail, and point out a connection between our work and a recent result of Cowen and MacCluer [5, Corollary 19] concerning composition-operator spectra.

2. Background.

For completeness of exposition we summarize here some necessary prerequisites.

2.1. The Hardy space H^2. This is the space of functions that are analytic in the unit disc U and whose Taylor coefficients in the expansion about the origin are square summable. Thus H^2 is a Hilbert space isomorphic to ℓ^2, where the norm is defined by:

$$\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2,$$

with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f.

2.2. Riesz operators. For our purposes the class of Riesz operators is best defined to be those bounded operators T on Hilbert space for which the distance between T^n and the compacts tends to zero super-exponentially. To be more precise it helps to recall that the distance (in the operator norm) from an operator T to the space of compact operators is called the *essential norm* of T, denoted by $\|T\|_e$. To say that T is a *Riesz operator* means that

$$\lim_{n\to\infty} \|T^n\|_e^{1/n} = 0.$$

This definition makes it clear that every power-compact operator is Riesz.

2.3. The essential spectrum. Suppose H is a Hilbert space. The quotient $\mathcal{B}/\mathcal{K}$ of the algebra $\mathcal{B}$ of bounded operators on H by the closed ideal $\mathcal{K}$ of compact operators is called the *Calkin Algebra* of H. The essential norm of an operator $T \in \mathcal{B}$ is thus revealed as the Calkin Algebra norm of the coset $T + \mathcal{K}$. The *spectrum* of this coset is called the *essential spectrum* of T. The spectral radius formula, applied to the Calkin Algebra, reveals a spectral interpretation of our definition of Riesz operator:

An operator is Riesz if and only if its essential spectrum is the singleton $\{0\}$.

This interpretation, along with a little operator theory, leads to the assertion made in the Introduction that Riesz operators are spectrally indistinguishable from compacts (see [7, Chapter 3] or [17]).

2.4. Essential norms and angular derivatives. We rely heavily on the formula obtained in [15] for the essential norm of a composition operator. We will discuss this in more detail in Section 3, but for now it is enough to know that if φ is univalent then this formula reduces to:

$$\|C_\varphi\|_e = \liminf_{|z|\to 1-} \left[\frac{1-|\varphi(z)|}{1-|z|}\right]^{-\frac{1}{2}}.$$

This equation relates the study of composition operators with classical work on boundary properties of holomorphic self-maps of the disc. In these studies φ is said to have an *angular derivative* at a point $\zeta \in \partial U$ if two things happen:

(a) $\varphi'(z)$ has a finite angular (nontangential) limit as $z \to \zeta$, and

(b) The angular limit of φ (which necessarily exists at ζ because of condition (a)) has modulus 1.

We denote the angular limit in (a) by $\varphi'(\zeta)$. More generally, whenever f is a complex-valued function defined on U that has an angular limit at $\zeta \in \partial U$, then we denote this limit by $f(\zeta)$. The *Julia-Carathéodory theorem* asserts that φ has an angular derivative at $\zeta \in \partial U$ if and only if

$$\liminf_{z\to\zeta} \frac{1-|\varphi(z)|}{1-|z|} < \infty. \tag{1}$$

The theorem also asserts that when (1) holds, then:

- The left-hand side of (1) is precisely $|\varphi'(\zeta)|$, and
- The angular limit of the difference quotient in (1) exists, and equals $|\varphi'(\zeta)|$

(see e.g. [2, §298-300], [6, page 51] or [16, page 57] for the full story).

The Julia-Carathéodory theorem allows us to think of $|\varphi'(\zeta)|$ as a function on ∂U with values in the extended interval $[0, \infty]$. Just think of $|\varphi'(\zeta)|$ as defined by the left-hand side of (1), and make the connection with angular limits of the derivative by interpreting "$|\varphi'(\zeta)| = \infty$" to mean "the angular limit of φ' either doesn't exist at ζ, or it exists and the corresponding angular limit of φ has modulus < 1." Note that that $|\varphi'|$ is bounded away from zero on ∂U: for example, if $\varphi(0) = 0$ then by the Schwarz Lemma and (1), $|\varphi'| \geq 1$ on ∂U.

2.5. Lemma. *The function* $|\varphi'| : \partial U \to (0, \infty]$ *is lower semicontinuous.*

Proof. We have to show that for each $M > 0$ the set $\{|\varphi'| > M\}$ is open, or equivalently,

$$E_M = \{\zeta \in \partial U : |\varphi'(\zeta)| \leq M\}$$

is closed.

To this end suppose $\{\zeta_n\}$ is a sequence of points in E_M that converges to ζ. We have to show $\zeta \in E_M$. Given $\epsilon > 0$, the Julia-Carathéodory Theorem guarantees that for each n there exists a point $z_n \in U$ at distance less than $1/n$ from ζ_n such that

$$\frac{1-|\varphi(z_n)|}{1-|z_n|} < M + \epsilon.$$

Since $z_n \to \zeta$, another application of the Julia-Carathéodory theorem shows that the angular derivative of φ exists at ζ, and shows that $|\varphi'(\zeta)| \leq M + \epsilon$. Since ϵ is an arbitrary positive number, $|\varphi'(\zeta)| \leq M$, hence $\zeta \in E_M$. □

Angular derivatives of iterates of φ may be computed via the usual chain rule.

2.6. Lemma (The Chain Rule). *Suppose φ is a holomorphic self-map of U and $\zeta \in U$. Suppose further that for some positive integer n the iterate φ_n has an angular derivative at ζ. Then:*

(a) *For each $1 \leq j \leq n$ the iterate φ_j has an angular derivative at ζ.*

(b) *φ has an angular derivative at $\varphi_j(\zeta)$ for $1 \leq j < n$.*

(c) *Letting $\zeta_j = \varphi_j(\zeta)$, we have $\varphi_n'(\zeta) = \prod_{j=0}^{n-1} \varphi'(\zeta_j)$.*

We omit the proof, which is a routine exercise in applying the Julia-Carathéodory Theorem. Note that the "$j = n$" part of statement (a) is just the hypothesis on φ_n, and that existence of the angular limits $\zeta_j = \varphi_j(\zeta)$ used in (c) is insured by (a), as is the fact that these limits lie on the unit circle.

2.7. The Denjoy-Wolff Theorem. If a holomorphic self-map φ of U fixes a point $a \in U$ then the Schwarz Lemma guarantees that $|\varphi'(a)| \leq 1$, with equality if and only if φ is an elliptic automorphism (i.e. conformally equivalent to a rotation). In case $|\varphi'(a)| < 1$ the fixed point is *attractive* in the sense that the sequence $(\varphi_n : n \geq 1)$ converges to the constant function a uniformly on compact subsets of U.

The *Denjoy-Wolff Theorem* asserts that something similar happens even if φ fixes *no* point of U. In this case there is a unique point $\alpha \in \partial U$, called the *Denjoy-Wolff point* of φ such that $\varphi_n \to \alpha$ uniformly on compact subsets of U. In addition, φ has remarkable regularity at α. For example, $\varphi(\alpha) = \alpha$ in the sense of angular limits, so α serves as a "boundary fixed point" for φ. Moreover, φ has some degree of smoothness at α in the sense that the angular derivative exists there. Although not required for the sequel, it is worth noting the angular derivative of φ at the Denjoy-Wolff point is a strictly positive number that is ≤ 1. For complete details see e.g. [16, Chapter 5].

3. Sufficient condition for Rieszness.

As usual, the symbol φ denotes a holomorphic self-map of U, and as in the proof of Lemma 2.5, $E_M = \{\zeta \in \partial U : |\varphi'(\zeta)| \leq M\}$. We also continue with the convention of extending φ to ∂U by defining $\varphi(\zeta)$ to be the angular limit of φ at $\zeta \in \partial U$, whenever this limit exists. Recall that by Fatou's theorem, this limit exists at (Lebesgue) almost every point of ∂U, and (as pointed out in the preceding section) at every point where the map φ has an angular derivative.

3.1. Theorem. *Suppose φ is univalent, and that for every $M > 0$ there exists a positive integer $N = N(M)$ such that*

$$n \geq N \quad \Rightarrow \quad \varphi_n(E_M) \cap E_M = \emptyset.$$

Then C_φ is a Riesz operator (i.e., its essential spectrum is $\{0\}$).

Proof. We know from [15] that the essential norm of C_φ on H^2 is given by the formula

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} \tag{2}$$

where N_φ is the Nevanlinna counting function of φ:

$$N_\varphi(w) = \sum_{z\in\varphi^{-1}\{w\}} \log \frac{1}{|z|}$$

the sum being interpreted as 0 when $w \notin \varphi(U)$. Now if φ is univalent, then the sum has just one term; hence, using the notation $w = \varphi(z)$ (and interpreting $\log(1/|z|)$ to be 0 whenever $w \notin \varphi(U)$), we have, as advertised in Section 2.4,

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1-} \frac{\log \frac{1}{|z|}}{\log \frac{1}{|w|}} = \limsup_{|z|\to 1-} \frac{1-|z|}{1-|\varphi(z)|} = \left[\liminf_{|z|\to 1-} \frac{1-|\varphi(z)|}{1-|z|}\right]^{-1}.$$

Upon applying the Julia-Carathéodory Theorem to the term on the right, and noting that by lower semicontinuity $|\varphi'|$ attains its infimum on ∂U, we obtain

$$\|C_\varphi\|_e^2 = \left[\min_{\zeta\in\partial U} |\varphi'(\zeta)|\right]^{-1}. \tag{3}$$

Our goal is to show that the essential spectral radius of C_φ is zero; i.e.,

$$\lim_{n\to\infty} \|C_\varphi^n\|_e^{1/n} = \{0\},$$

which in view of (3) above, means

$$\lim_{n\to\infty} \min_{\zeta\in\partial U} |\varphi_n'(\zeta)|^{1/n} = \infty. \tag{4}$$

To prove (4) for our mapping φ, fix $M > 1$ and recall that by hypothesis there exists a positive integer N such that $\varphi_n(E_M)$ is disjoint from E_M whenever $n \geq N$.

Now fix $n \geq 2N$. We are going to show that the quantity on the left-hand side of (4) is $> \sqrt{M}$, which will establish (4), and prove that C_φ is a Riesz operator.

To this end, we need only consider those $\zeta \in \partial U$ for which $|\varphi_n'(\zeta)| < \infty$. The idea is to use the Chain Rule (part (c) of Lemma 2.6) to compute

$|\varphi_n'(\zeta)|$. At this point it is convenient to assume that $\varphi(0) = 0$, so that (as we observed just before stating Lemma 2.5) $|\varphi'| \geq 1$ on ∂U. We will show in Theorem 5.3 that no loss of generality results from the assumption $\varphi(0) = 0$.

To estimate the size of the product on the right-hand side of the chain rule formula, let

$$J(\zeta) = \{0 \leq j < n : \varphi_j(\zeta) \notin E_M\}.$$

That is, $J(\zeta)$ is the set of indices in $[0, n-1]$ for which $|\varphi_j'(\zeta)| > M$. Observe that our choice of N guarantees that the number of indices in $J(\zeta)$ is $\geq n - N$ for every $\zeta \in \partial U$. Thus by the Chain Rule (using the notation $\zeta_j = \varphi_j(\zeta)$, and writing $J = J(\zeta)$),

$$|\varphi_n'(\zeta)| = \prod_{j=0}^{n-1} |\varphi'(\zeta_j)| \geq \prod_{j \in J} |\varphi'(\zeta_j)| \geq M^{n-N}$$

from which follows

$$|\varphi_n'(\zeta)|^{1/n} \geq M^{1-\frac{N}{n}}$$

so $|\varphi_n'(\zeta)|^{1/n} \geq \sqrt{M}$ if $n \geq 2N$, as promised. □

Here is an easy-to-use special case of Theorem 3.1 that we will apply in the next section to construct a Riesz composition operator that is not power-compact.

3.2. Corollary. *Suppose φ is a holomorphic self-map of U that is univalent on U and extends continuously to $\overline{U}$ (the closed unit disc). Suppose further that*

(a) *At each fixed point on the unit circle, φ does not have an angular derivative, and*

(b) *If $\zeta \in \partial U$ is not a fixed point of φ then $\varphi_n(\zeta) \in U$ for some positive integer n.*

Then C_φ is a Riesz operator on H^2.

Proof. Let FP denote the (possibly empty) set of fixed points of φ on ∂U. Fix $M > 0$ and note that by hypothesis (a) we have $E_M \subset \partial U \backslash FP$. Note that for each nonnegative integer n, the set

$$G_n = \{\zeta \in \partial U : \varphi_n(\zeta) \in U\} = \varphi_n^{-1}(U) \cap \partial U$$

is open (continuity of φ), and that (G_n) is an increasing sequence whose union contains $\partial U \backslash FP$ (by hypothesis (b)). Thus this union contains the compact set E_M and there is a positive integer N such that $G_n \supset E_M$ whenever $n \geq N$. In particular:

$$n \geq N \quad \Rightarrow \quad \varphi_n(E_M) \subset U \quad \Rightarrow \quad \varphi_n(E_M) \cap E_M = \emptyset,$$

so C_φ is Riesz by Theorem 3.1. □

4. A non-power-compact Riesz composition operator.

4.1. A strip with a bulge. Our example is a holomorphic self-map of U that played a key role in the work of [18]. Let G be the union of the open unit disc and the open horizontal strip of unit width whose lower boundary is the real axis. So G is a strip with a bulge that contains the origin, and $\frac{1}{2}G \subset G$. Let σ be the univalent holomorphic mapping of U onto G that fixes the origin and has derivative > 0 there (Riemann Mapping Theorem). Define φ on U by

$$\varphi(z) = \sigma^{-1}\left(\frac{1}{2}\sigma(z)\right) \qquad (z \in U). \tag{5}$$

We think of φ in the following way: its action on U is conformally similar, via σ, to the action of the mapping $M_{1/2}$ of "multiplication by $1/2$" acting on G. We think of $M_{1/2} : G \to G$ as a "model" for $\varphi : U \to U$, and deduce the properties of φ from properties of the model. In particular, by going back and forth between U and G it is easy to check that φ extends continuously to $\overline{U}$ and has two fixed points on ∂U, which correspond to "the points at plus and minus infinity on the boundary of G".

4.2. Theorem. *For the mapping φ just described, C_φ is a Riesz operator on H^2 that is not power-compact.*

Proof. The fact that C_φ is not power-compact was noted in [18]. For completeness we review the argument. Let L denote the half-line $(2, \infty)$ on the real axis, and note that $M_{1/2}(L) \subset \partial G$. Let $I = \sigma^{-1}(L)$, an open arc of ∂U. The behavior of the model translates into the fact that $\varphi(I) \subset \partial U$, in particular φ has radial limits of modulus one on the arc I. It is a standard result about composition operators that whenever the inducing map has radial limits of modulus one on a subset of ∂U having positive measure, the induced composition operator is not compact (see e.g. [18, §2.5, page 32]). Thus our operator C_φ is not compact.

Now if n is a positive integer, then the n-th iterate φ_n of φ is obtained by changing the multiplier $1/2$ in the definition (5) of φ to $1/2^n$. The same analysis shows that $C_\varphi^n = C_{\varphi_n}$ is not compact. Thus C_φ is not power-compact.

To show that C_φ is a Riesz operator we check that φ satisfies the hypotheses of Corollary 3.2. We have already noted that φ is univalent on U and extends continuously to $\overline{U}$. Since every (finite) point of ∂G is eventually taken into G by successive multiplications by $1/2$, we see that each non-fixed point of ∂U is taken into U by some iterate of φ, so hypothesis (b) of Corollary 3.2 is satisfied.

It remains to show that hypothesis (a) is satisfied, i.e. that φ does not have an angular derivative at either of its boundary fixed points. For this we require a crucial result of [18], which states that whenever a self-map φ of U defined by a formula like (5) has an angular derivative at a boundary fixed point, then the "model domain" G contains a "twisted sector"

$$S_\epsilon(\Gamma) = \bigcup_{w\in\Gamma} \{z : |z-w| < \epsilon|w|\},$$

for some $\epsilon > 0$ and some simple arc Γ that connects the origin to ∞ ([18, Proposition 3.3], see also [16, §9.6]). We will say more about twisted sectors in the final section, but for now it is enough to note that the strip-like region G in our present example doesn't contain one. Thus φ does not have an angular derivative at either of its boundary fixed points.

We have shown that C_φ satisfies all the hypotheses of Corollary 3.2, so it is therefore a Riesz operator. □

4.3. Remarks. The construction of φ can be refined to create non-power-compact Riesz composition operators whose inducing functions have infinitely many fixed points on the unit circle. For this, take a sequence of rays emanating from the origin, and making angles with the real axis that decrease to zero. For definiteness, suppose also that these angles are $< \pi/2$ (you can also symmetrize the situation with respect to the real axis to increase its aesthetic appeal). Make each ray the top edge of a little open strip, taking care to make the strips thin enough so that they don't intersect outside the unit disc. Let G be the union of this countable collection of strips and the unit disc, and proceed just as before.

A bit more work, using Theorem 3.1 directly, shows that the examples of non-power-compact composition operators constructed in [18] from "jellyfish models" also induce Riesz operators.

5. Riesz operators and Koenigs models.

5.1. Koenigs Models. The method we used in §4 to construct our non-power-compact Riesz composition operator is actually part of a much more general scheme. If λ is any complex number of modulus < 1 and G a simply connected plane domain with $0 \in G$ and $\lambda G \subset G$, then the Riemann Mapping Theorem provides a univalent map σ of U onto G with $\sigma(0) = 0$, and just as in the last section we can use σ to define a holomorphic self-map φ of U. This time the formula is

$$\varphi(z) = \sigma^{-1}(\lambda\sigma(z)) \qquad (z \in U). \tag{6}$$

Once again σ establishes a conformal similarity between the action of φ on U and that of the mapping M_λ of "multiplication-by-λ" on G. We call $M_\lambda : G \to G$ the *Koenigs Model* for φ, and observe that the subtleties of how φ acts on U are now coded into the geometry of G.

Classical work of Gabriel Koenigs [8, 1884] shows that:

Every univalent self-map of U with an attractive fixed point in U has a Koenigs model, which is uniquely determined (*up to a constant multiple of the model domain G*)*.*

More generally Koenigs showed that if φ is any holomorphic self-map of U with a fixed point in U, say $\varphi(0) = 0$ without loss of generality, and if $0 < |\varphi'(0)| < 1$, then there is a holomorphic function σ on U such that

$$\sigma(\varphi(z)) = \varphi'(0)\sigma(z) \qquad (z \in U). \tag{7}$$

Koenigs showed further that σ is unique up to multiplication by a constant, and is univalent whenever φ is univalent (see e.g., [16, §6.1] for an exposition of these matters). Thus if φ is univalent then (7), which is usually called *Schroeder's equation*, can be rewritten in the form (6), with $\lambda = \varphi'(0)$. Note that the Schwarz Lemma guarantees that $|\varphi'(0)| \leq 1$ for any holomorphic self-map of U that fixes the origin — by insisting on strict inequality we simply prevent φ from being a rotation.

5.2. Koenigs eigenfunctions. We continue to assume that φ is a holomorphic self-map of U that fixes the origin, and that $0 < |\varphi'(0)| < 1$. The map σ discussed above is often called the *Koenigs eigenfunction* of φ. This terminology comes from observing that equation (7) can be rewritten

$$C_\varphi \sigma = \varphi'(0)\sigma,$$

hence σ is an eigenfunction of the operator $C_\varphi : H(U) \to H(U)$. The essential uniqueness of σ says that the eigenvalue $\varphi'(0)$ has multiplicity one. Upon raising both sides of (7) to the n-th power we see that σ^n is an eigenfunction and $\varphi'(0)^n$ the corresponding eigenvalue ($n = 0, 1, 2, \dots$). Koenigs showed that *all* these eigenvalues have multiplicity one.

Caughran and Schwartz [3] (proof of their Theorem 3) showed that the point $\varphi'(0)^n$ always belongs to the *spectrum* of $C_\varphi : H^2 \to H^2$ (see also [4, Theorem 4.1] and [1]). The subtlety here is that in general σ^n may not belong to H^2, so $\varphi'(0)^n$ need not always be an eigenvalue of $C_\varphi : H^2 \to H^2$. However if C_φ is a *Riesz operator* on H^2, then $\varphi'(0)^n$, being a non-zero spectral point, *must* be an eigenvalue — of multiplicity one by the work of Koenigs. This forces the corresponding eigenfunction σ^n to lie in H^2. So if C_φ is Riesz, then $\sigma^n \in H^2$ for every positive integer n. Equivalently:

$$C_\varphi \text{ Riesz on } H^2 \Rightarrow \sigma \in \bigcap_{p<\infty} H^p. \tag{8}$$

For power-compact operators this result is implicit in the work of Caughran and Schwartz [3, Theorem 3].

In the context of Riesz operators no restriction is imposed by assuming, as we have done above, that φ has an attractive fixed point in U. We show now that this condition is, in fact, *necessary* for C_φ to be Riesz. Consequently:

> *Every univalently induced Riesz composition operator has a Koenigs model.*

5.3. Theorem. *Suppose φ is a holomorphic self-map of U and C_φ is a Riesz operator on H^2; then φ fixes a point $a \in U$, and $|\varphi'(a)| < 1$.*

Proof. Suppose first that φ does not fix any point of U. Will show that the essential norm of C_φ^n does not go to zero super-exponentially, and so C_φ is not a Riesz operator on H^2 (Section 2.2).

For this we employ the boundary fixed point α guaranteed for φ by the Denjoy-Wolff Theorem (see Section 2.7). Recall that φ has an angular derivative at α, and so by the Julia-Carathéodory Theorem (Section 2.4)

$$\lim_{r\to 1-} \frac{\log \frac{1}{r}}{\log \frac{1}{|\varphi(r\alpha)|}} = \lim_{r\to 1-} \frac{1-r}{1-|\varphi(r\alpha)|} = \frac{1}{|\varphi'(\alpha)|}. \tag{9}$$

For $0 \le r < 1$ write $w(r) = \varphi(r\alpha)$, so that

$$N_\varphi(w(r)) \ge \log \frac{1}{r}. \tag{10}$$

The essential-norm formula (2) combined with (9) and (10) yields:

$$\|C_\varphi\|_e^2 = \limsup_{|w|\to 1-} \frac{N_\varphi(w)}{\log \frac{1}{|w|}} \ge \lim_{r\to 1-} \frac{\log \frac{1}{r}}{\log \frac{1}{|w(r)|}} = \frac{1}{|\varphi'(\alpha)|}. \tag{11}$$

Now for each positive integer n we have $C_\varphi^n = C_{\varphi_n}$. Since the Denjoy-Wolff point of φ is the limit of the sequence of successive iterates of φ, we see that φ_n also does not fix any points of U, and that α must be *its* Denjoy-Wolff point. Thus φ_n has an angular derivative at α, and our Chain Rule (Lemma 2.6) shows that $\varphi_n'(\alpha) = \varphi'(\alpha)^n$. This allows us to apply the essential-norm estimate (11) to φ_n, obtaining

$$\|C_\varphi^n\|_e^2 = \|C_{\varphi_n}\|_e^2 \ge \frac{1}{|\varphi'(\alpha)|^n},$$

from which follows

$$\lim_n \|C_\varphi^n\|_e^{1/n} \ge \frac{1}{|\varphi'(\alpha)|^{1/2}} > 0.$$

Thus, by our definition (Section 2.2), C_φ is not a Riesz operator.

Summary: *If φ has no fixed point in U then C_φ is not a Riesz operator on H^2.*

Suppose now that C_φ *is* a Riesz operator. Then φ fixes a point of U, which by a standard similarity argument we may assume to be the origin: $\varphi(0) = 0$. It remains to show that $|\varphi'(0)| < 1$.

The Schwarz Lemma insures that $|\varphi'(0)| \leq 1$, and in case of equality that φ has to be a rotation of U. Thus if we do not have the desired strict inequality on $\varphi'(0)$, then C_φ must be a unitary operator on H^2. We claim this contradicts the assumption that C_φ is Riesz.

More generally:

Every linear isometry of Hilbert space into itself has essential spectral radius $= 1$.

This is well-known, but to keep our exposition complete, here is a proof. Suppose T is a Hilbert-space isometry. We begin by showing that the essential norm of T is 1. Since the essential norm is just the distance to the subspace of compact operators, it is clear that $\|T\|_e \leq \|T\| \leq 1$. To prove the opposite inequality, fix a compact operator K, and an orthonormal sequence (e_n). Since T is an isometry, $\|Te_n\| = 1$ for each n, so it follows that for each positive integer n:

$$\|T - K\| \geq \|(T - K)e_n\| \geq \|Te_n\| - \|Ke_n\| = 1 - \|Ke_n\|. \tag{12}$$

Now every orthonormal sequence converges weakly to zero, so because K is compact, $\|Ke_n\| \to 0$ as $n \to \infty$. Thus upon letting $n \to \infty$ in the right-hand side of (12), we see that $\|T - K\| \geq 1$, hence $\|T\|_e \geq 1$, as desired.

Since every Hilbert-space isometry T has essential norm one, and since every power of T is again an isometry, it follows from the last paragraph that $\|T^n\|_e = 1$ for every positive integer n; hence, T cannot be a Riesz operator. In particular, no rotation of the disc can induce a Riesz composition operator. Thus, if φ fixes the origin and induces a Riesz composition operator, then $|\varphi'(0)| < 1$. □

5.4. Remarks. (a) The argument given in the first part of Theorem 5.3 shows that if φ fixes no point of U, the essential spectral radius of $C_\varphi : H^2 \to H^2$ is bounded below by $|\varphi'(\alpha)|^{-1/2}$, where $\alpha \in \partial U$ is the Denjoy-Wolff point of φ. For this we could also have referenced [1, Lemma 5.3], where we showed that for this situation $r_e(C_\varphi)$ is *precisely* $|\varphi'(\alpha)|^{-1/2}$.

However the argument given above, in addition to offering a more self-contained exposition, can be easily modified to show that for *any* holomorphic self-map φ of U:

$$r_e(C_\varphi) \geq \max_{\zeta \in FP} |\varphi'(\zeta)|^{-1/2} \ ,$$

where on the right, FP denotes the collection of boundary fixed points of φ. For this we only need to use the general version of the Chain Rule given in [16, Chapter 4, Exercise 10, page 74], to show that if φ has an angular derivative at a boundary fixed point ζ then so does every iterate φ_n, and $\varphi_n'(\zeta) = \varphi'(\zeta)^n$.

Thus if C_φ is a Riesz operator on H^2 then φ cannot have an angular derivative at any boundary fixed point. Since every positive power of a Riesz operator is also Riesz, the same holds for each iterate of φ. Now boundary fixed points of iterates of φ are, in an obvious way, "boundary periodic points" of φ, so we have:

If $C_\varphi : H^2 \to H^2$ is a Riesz operator, then φ does not have an angular derivative at any boundary periodic point.

Poggi-Corradini [13] has recently proved the converse of this result for *univalent maps* φ. In general the converse is *false*, as shown by the fact that there exist inner functions with no angular derivative at any point of ∂U (see [16, Section 10.2] for example) — the last part of the proof of Theorem 5.3 shows that no inner function can induce a Riesz composition operator. (Every inner function with a fixed point in U is conjugate, via a conformal disc automorphism, to an inner function that vanishes at the origin, and by [10, Section 1] such inner functions induce isometries on H^2. Thus every composition operator induced by an inner function is similar to an isometry, and therefore is not Riesz.)

(b) There is a parallel between the results discussed here and the corresponding ones for compactness. Theorem 5.3 was first obtained in the context of power-compact operators by Caughran and Schwartz [3, 1975]. It has long been known that non-existence of the angular derivative at every point of ∂U is necessary for C_φ to be compact on H^2, and that this condition is sufficient if φ is univalent (see [16, Chapter 2] for details and historical notes). Our results, along with those of Poggi-Corradini show that the same is true for *Rieszness* if non-existence of the angular derivative is required not everywhere on the boundary, but only at the *boundary periodic points* of the inducing map.

(c) Shapiro and Taylor [19, 1973] proved that for $0 < p < \infty$, a composition operator is compact on H^p if and only if it is compact on H^2. The same is true for Riesz operators. The notions of essential norm and essential spectral radius, make sense for any Banach space, and more generally for any p-Banach space where $0 < p \le 1$. Thus Riesz operators can be defined in these more general settings, and it turns out that all the spectral theory that makes Riesz-operator spectra identical with compact-operator spectra still holds (see [17]).

Now in [1] we have shown that for any holomorphic self-map φ of U, and any $0 < p < \infty$,

$$r_e(C_\varphi : H^p \to H^p) = \left[r_e(C_\varphi : H^2 \to H^2)\right]^{2/p}.$$

It follows from this identity that for $0 < p < \infty$:

C_φ is a Riesz operator on H^p if and only if it is a Riesz operator on H^2.

6. Connections and conjectures.

In [18] Shapiro, Smith and Stegenga study the relationship between the geometry of Koenigs models and compactness of composition operators on H^2. They establish, for example, that if the image $\sigma(U)$ of a (univalent) Koenigs function σ contains a twisted sector, then σ cannot belong to H^p for all $p < \infty$. Thus in view of the work of the previous section:

If C_φ is Riesz, then the image $\sigma(U)$ of its Koenigs eigenfunction cannot contain a twisted sector.

On the other hand, in [11] Pietro Poggi-Corradini shows that if $\sigma(U)$ contains no twisted sectors, then $\sigma \in H^p$ for all $p < \infty$, and raises the problem of describing an operator-theoretic property of C_φ that is equivalent to $\sigma(U)$'s containing no twisted sectors. As we have seen, Rieszness of C_φ is sufficient to ensure no twisted sectors lie inside $\sigma(U)$. Analysis of models presented in [18] provides some evidence that the Riesz property is also necessary: Each model having the *no-twisted-sector property* (including the "bulging-strip" model we discussed in detail in Section 4) can be shown by application of Theorem 3.1 to yield a Riesz composition operator on H^2. This observation and further results in papers [1], [18], and [11] lead us to conjecture the following:

Suppose φ is univalent and σ is the Koenigs eigenfunction of φ; then the composition operator C_φ on H^2 is Riesz if and only if $\sigma(U)$ does not contain a twisted sector.

More generally, we conjecture that for an arbitrary holomorphic self-map $\varphi(U)$ (satisfying $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$), the implication (8) goes both ways:

$C_\varphi : H^2 \to H^2$ is Riesz if and only if $\sigma \in H^p$ for all $p < \infty$.

As we mentioned in the Introduction, Poggi-Corradini [12] has recently established the first of these conjectures. We have just learned that he has proved the second as well [14].

We conclude by noting a connection between our work and the study of composition-operator spectra. In [5], Cowen and MacCluer characterize the spectrum of C_φ given that φ is univalent but not an automorphism, and $\varphi(0) = 0$; in particular they show that the spectrum of C_φ must contain the disc $\{z : |z| \leq r_e(C_\varphi)\}$, where $r_e(C_\varphi)$ denotes the essential spectral radius of C_φ on H^2. Our Theorem 3.1 provides a sufficient condition for this disc to be degenerate (i.e. $= \{0\}$), while Theorem 4.2 shows that this can happen non-trivially.

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