

FINITE GROUPS ACTING ON HOMOLOGY MANIFOLDS

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To the memory of Olga Taussky-Todd

1. Introduction.

In this paper we study homology manifolds T admitting the action of a finite group preserving the structure of a regular CW-complex on T . The CW-complex is parameterized by a poset and the topological properties of the manifold are translated into a combinatorial setting via the poset. We concentrate on n -manifolds which admit a fairly rigid group of automorphisms transitive on the n -cells of the complex. This allows us to make yet another translation from a combinatorial into a group theoretic setting. We close by using our machinery to construct representations on manifolds of the Monster, the largest sporadic group. Some of these manifolds are of dimension 24, and hence candidates for examples to Hirzebruch's Prize Question in [HBJ], but unfortunately closer inspection shows the \hat{A} -genus of these manifolds is 0 rather than 1, so none is a Hirzebruch manifold.

In order to state our results precisely we need to recall some definitions and introduce a few concepts. The geometric realization functor T associates to each poset X a topological space $T(X)$ and to each $x \in X$ a closed subspace $T(x)$ of $T(X)$. The cells $T(x)$, $x \in X$, supply a cell structure on $T(X)$ parametrized by X . In addition there is a simplicial complex $\mathcal{O}(X)$ associated to X called the *order complex* of X and a canonical triangulation $\varphi : \mathcal{O}(X) \rightarrow T(X)$ of $T(X)$ by $\mathcal{O}(X)$.

An *automorphism* of a space T with cell structure $(T(x) : x \in X)$ is a homeomorphism α of T permuting the cells and preserving the triangulation $\varphi : \mathcal{O}(X) \rightarrow T$. See Section 7 for a precise definition. In particular there is a natural isomorphism between the automorphisms of the space with cell structure and the automorphisms of X . The automorphism α is *lower admissible* if α induces the identity map on $T(x) \cap T(x)\alpha$ for each cell $T(x)$.

Assume T is a connected homology n -manifold with cell structure supplied by a poset X . A *reflection* through an $(n - 1)$ -cell $T(x)$ is a nontrivial automorphism of the space T with cell structure fixing each member of $T(x)$. There is at most one reflection through $T(x)$.

Let \mathcal{A} be a poset of height n with a least element a_0 , I the set of elements of \mathcal{A} of height 1, G a group, and $\rho : I \rightarrow G$ a map from I into the set of involutions of G such that $G = \langle \rho(I) \rangle$ is generated by the image of ρ . For $a \in \mathcal{A}$ let

$$\mathcal{A}(\leq a) = \{b \in \mathcal{A} : b \leq a\}, \quad I(a) = I \cap \mathcal{A}(\leq a), \quad Q_a = \langle \rho(I(a)) \rangle$$

and $P(G, \rho, \mathcal{A})$ the poset with vertex set $\coprod_{a \in \mathcal{A}} G/Q_a$ and $Q_a g \leq Q_b h$ if and only if $a \leq b$ and $Q_a g \subseteq Q_b h$. The triple (G, ρ, \mathcal{A}) is *thin* if $\mathcal{A}(> a_0)$ is thin and residually connected. In the literature thin residually connected posets are sometimes called *abstract polytopes*; cf. [MS]. We show there is a bijection between the class of abstract polytopes P admitting a lower admissible group of automorphisms transitive on maximal elements of P and the class of thin triples; cf. Remark 1 in Section 4.

The triple (G, ρ, \mathcal{A}) is a *manifold triple* of height n if $\mathcal{A}(> a_0)$ is a homology $(n-1)$ -manifold and sphere, and for each $a \in \mathcal{A}$ of height at least 2, $P(Q_a, \rho_a, \mathcal{A}(< a))$ is a homology sphere, where ρ_a is the restriction of ρ to $I(a)$. We prove the following two theorems which show that the notion of a connected homology manifold with cell structure admitting the rigid action of a group transitive on n -cells is equivalent to the notion of a manifold triple.

Theorem 1. *Let T be a connected homology n -manifold with cell structure supplied by a poset X , and G a lower admissible group of automorphisms of T transitive on n -cells of T . Let $T(x)$ be a n -cell of T . Then*

- (1) *There is a reflection $\rho(i)$ through each $(n-1)$ -cell i below x , G is generated by the set of all such reflections, and G is the subgroup generated by all reflections of the automorphism group of the space T with cell structure.*
- (2) *Let \mathcal{A} be the dual of $X(\leq x)$. Then (G, ρ, \mathcal{A}) is a manifold triple of rank n and X is isomorphic to the dual of $P(G, \rho, \mathcal{A})$.*

Theorem 2. *Let (G, ρ, \mathcal{A}) be a manifold triple of height n and X the dual of $P(G, \rho, \mathcal{A})$. Then the geometric realization $T = T(X)$ is a connected homology n -manifold with cell structure supplied by X and G is a lower admissible group of automorphisms of the space T transitive on the n -cells of T .*

If (G, ρ, \mathcal{A}) is a manifold triple then for each $a \in \mathcal{A}$, $(Q_a, \rho_a, \mathcal{A}(< a))$ is also a manifold triple and $P_a = P(Q_a, \rho_a, \mathcal{A}(< a))$ is a homology sphere, so it is important to investigate the case where $P = P(G, \rho, \mathcal{A})$ is a homology sphere. One very strong result in this direction uses an idea of Davis in [D]

and shows the isotropy groups Q_a , $a \in \mathcal{A}$, in our rigid action of G on $T(P)$ are all spherical Coxeter groups:

Theorem 3. *If (G, ρ, \mathcal{A}) is a manifold triple of height n such that $P(G, \rho, \mathcal{A})$ is a homology n -sphere then $(G, \rho(I))$ is a spherical Coxeter system.*

It remains to decide the possible structures for the big cell \mathcal{A} and the nature of the map $a \mapsto Q_a$ from \mathcal{A} into the poset of all subgroups of G , particularly in the case where $P(G, \rho, \mathcal{A})$ is a homology sphere. Perhaps the following two results are our strongest in this direction.

Theorem 4. *Let (G, ρ, \mathcal{A}) be a manifold triple of height n such that $P = P(G, \rho, \mathcal{A})$ is a homology sphere, ρ is injective, and $I(a) \neq I(b)$ for $a < b$ in \mathcal{A} . Then $(G, \rho(I))$ is a spherical Coxeter system of rank $n + 1$ or n and P is the Coxeter complex C of $(G, \rho(I))$ or its suspension, (i.e. the join $C * S^0$ of C with a 0-sphere) respectively.*

Theorem 5. *Let (G, ρ, \mathcal{A}) be a thin triple such that \mathcal{A} is of height n with $\mathcal{A}(> a_0)$ a homology $(n - 1)$ -sphere and manifold, and for each $a \in \mathcal{A}$, $(Q_a, \rho(I(a)))$ is a spherical Coxeter system of rank the height of a . Then (G, ρ, \mathcal{A}) is a manifold triple.*

Theorem 5 supplies one means for constructing representations of a group G on manifolds. We use it in Section 13 to construct representations of the Monster of dimension ranging from 7 to 14, and observe in Example 2 in Section 12 that it can be used to construct representations on 2-dimensional manifolds of most any group generated by involutions. Then taking products of such examples we obtain many 24 dimensional representations of the Monster. See Section 11 for a brief discussion of Hirzebruch's Prize Question.

In Section 4 of [D], Davis proves results similar to Theorems 1 through 3 under the assumption that the big cell $\mathcal{A}(> a_0)$ is a simplex and ρ is injective. Of course he does so in a different language. The preprint [A] contains results on manifolds parametrized by the cell complexes of [CCC], which are more general than regular CW-complexes.

2. Generalities on simplicial complexes and posets.

Let n be a nonnegative integer, $I = \{0, \dots, n\}$, and K an (abstract) n -dimensional simplicial complex. Write K for the vertex set of K and $\Sigma(K)$ for the set of simplices of K . Write $\Sigma^k(K)$ for the set of k -simplices of K . We call the n -simplices of K *chambers*. A *wall* of a chamber C is an $(n - 1)$ -simplex of C . The *chamber graph* of K is the graph on the chambers of K in which chambers C and D are adjacent if $C \cap D$ is a wall of C .

We say K is *residually connected* if K is connected, $\text{Link}_K(s)$ is connected for each k -simplex s with $k < n - 1$, and $\text{Link}_K(s) \neq \emptyset$ for each $(n - 1)$ -simplex s .

A complex K is a *typed complex* with type function τ if

$$\tau : K \rightarrow I'$$

is a function from the vertex set of K into some index set I' which is injective on simplices. For example if $I' = I$ is of order $n + 1$ this just says the graph of K is a Tits geometry with type function τ . The *type* of a simplex s is its image $\tau(s) \subseteq I'$.

A *morphism* $\alpha : K \rightarrow L$ of typed complexes over I' is a simplicial map preserving type. That is $\tau(v\alpha) = \tau(v)$ for each vertex v of K . We write $\text{Aut}(K, \tau)$ for the group of automorphisms of K in the category of typed complexes.

A *geometric complex* is a typed complex over I in which each simplex is contained in a chamber.

For example if P is a poset, the *order complex* $\mathcal{O}(P)$ of P is the simplicial complex with vertex set P and simplices the finite chains of P . Define P to be *residually connected* if $\mathcal{O}(P)$ is residually connected and define the *chamber graph* of P to be that of $\mathcal{O}(P)$. If the set $P(\leq x)$ of vertices y of P with $y \leq x$ is of finite height then $\mathcal{O}(P)$ is a typed complex with type function the height function. The *barycentric subdivision* of a complex K is the order complex of its poset of simplices, partially ordered by inclusion. Write $\text{sd}(K)$ for the barycentric subdivision of K ; thus $\text{sd}(K)$ is a typed complex with respect to the height function.

An automorphism g of K is said to be *admissible* if whenever g fixes a simplex s of K then g fixes each vertex of s . Notice if g is admissible then the set $\text{Fix}(g)$ of all simplices of K fixed by g is a subcomplex of K .

Similarly g is *strongly admissible* if g fixes each vertex of $s \cap sg$ for each simplex s of K . A group G of automorphisms is *admissible*, *strongly admissible* if each of its elements is admissible, strongly admissible, respectively. We say G is *topologically regular* if whenever $g_0, \dots, g_k \in G$ and $\{v_0, \dots, v_k\}$ and $\{v_0g_0, \dots, v_kg_k\}$ are k -simplices of K , then there exists $g \in G$ with $v_i g = v_i g_i$ for each i .

(2.1) *Let G be a group of automorphisms of K . Then*

- (1) *G is strongly admissible if and only if $vG \cap s = \{v\}$ for each simplex s of K and each $v \in s$.*
- (2) *If G is topologically regular then G is strongly admissible.*
- (3) *G is topologically regular if and only if G_s controls fusion in $\text{Link}_K(s)$ for each simple s of K . That is for each vertex v of K , the pointwise*

stabilizer G_s of the simplex s is transitive on $vG \cap \text{Link}_K(s)$.

Proof. Parts (1) and (2) are easy and appear for example in Section 1 of Chapter III of [Br]. Let $s = \{v_0, \dots, v_k\}$ be a k -simplex of K . Assume G is topologically regular and let $g \in G$ and $x, xg \in \text{Link}_K(s)$. Then $s \cup \{x\}$ and $s \cup \{xg\}$ are simplices of K , so as G is topologically regular there is $h \in G$ with $v_i h = v_i$ for each i and $xh = xg$. That is $h \in G_s$ and $xh = xg$, so G_s is transitive on $xG \cap \text{Link}_K(s)$. Conversely assume G_s controls fusion in $\text{Link}_K(s)$ for each s . Let $\{v_0 g_0, \dots, v_k g_k\}$ be a simplex; we must show there is $g \in G$ with $v_i g = v_i g_i$ for each i . We proceed by induction on k ; the remark is trivial when $k = 0$. Let $t = s - \{v_k\}$; by induction on k there is $h \in G$ with $v_i h = v_i g_i$ for $i \neq k$, so replacing t by th , we may assume $v_i = v_i g_i$ for $i \neq k$. Then as G_t controls fusion in $\text{Link}_K(t)$, there is $g \in G_t$ with $v_k g = v_k g_k$, completing the proof. \square

(2.2) *Let G be a group of automorphisms of K . Then*

- (1) *G is strongly admissible on $sd(K)$.*
- (2) *G is topologically regular on $sd^2(K)$.*
- (3) *If G is admissible on K then $\text{Fix}_{sd(K)}(G) = sd(\text{Fix}_K(G))$ and $\dim(K) = \dim(sd(K))$.*
- (4) *K and $sd(K)$ have the same homotopy type.*

Proof. Part (4) is well known. Parts (1) and (2) are easy and appear as Proposition 1.1 in Chapter III of [Br]. Part (3) is trivial. \square

(2.3) *Let G be a group of automorphisms of K and $C = \{v_0, \dots, v_n\}$ a chamber of K . Then*

- (1) *If there exists a G -invariant type function $\tau : K \rightarrow I'$ making K into a typed complex then G is strongly admissible.*
- (2) *If G is strongly admissible then G is admissible.*
- (3) *If G is strongly admissible, transitive on chambers, and each simplex of K is contained in a chamber, then there is a unique type function $\tau : K \rightarrow I$ preserved by G with $\tau(v_i) = i$ and making K into a geometric complex.*

Proof. If K is a typed complex with type function τ preserved by G , then for $g \in G$ and s a simplex of K , $t = s \cap sg$ is the simplex of s and of sg of type $\tau(t)$, so as g preserves type, g fixes each vertex of t . That is G is strongly admissible and (1) is established.

Part (2) is trivial. Assume G is chamber transitive and each simplex is contained in a chamber. Then each vertex is in $v_i G$ for some i . Therefore if G is strongly admissible, we can define $\tau : K \rightarrow I$ by $\tau(v_i g) = i$, and τ is

well defined, since if $v_i g = v_j$ then $v_j \in C \cap Cg$, so by strong admissibility, $v_i = v_i g = v_j$. Visibly τ is the unique G -invariant function from K to I with $\tau(v_i) = i$. Finally if s is a simplex of K then $sg \subseteq C$ for some $g \in G$, so as τ is injective on C and G -invariant, τ is injective on s . Thus K is a geometric complex with type function τ , completing the proof of (3).

(2.4) *Let G be a group of automorphisms of K and $C = \{v_0, \dots, v_n\}$ a chamber of K . Assume G is admissible and chamber transitive and each simplex of K is contained in a chamber. Then the following are equivalent:*

- (1) G is strongly admissible.
- (2) G_{v_i} is chamber transitive on $\text{Link}_K(v_i)$ for each $i \in I$.
- (3) G_s is chamber transitive on $\text{Link}_K(s)$ for each simplex s of K .
- (4) G is topologically regular on K .

Proof. Assume G_{v_i} is chamber transitive on $\text{Link}_K(v_i)$ for each $i \in I$. Suppose $v \in s \cap sg$ for some $g \in G$, some vertex v , and some simplex s . As G is chamber transitive and s is contained in a chamber, we may take $v = v_i$ and $s = C$. Then as G_v is chamber transitive on $\text{Link}_K(v)$, there is $h \in G_v$ with $Cgh = C$. Now as G is admissible, gh fixes each vertex of C , so $v = vgh$ and hence $vg = vh^{-1} = v$. That is G is strongly admissible. Thus (2) implies (1). On the other hand if G is strongly admissible, $s \subseteq C$, and d is a chamber in $\text{Link}_K(s)$ then $D = d \cup s$ is a chamber of K , so $D = Cg$ for some $g \in G$. Then $s \subseteq C \cap Cg$, so as G is strongly admissible, $g \in G_s$. That is (1) implies (3). Trivially, (3) implies (2). As G is chamber transitive and each simplex is contained in a chamber, (3) is equivalent to G_s controlling fusion in $\text{Link}_K(s)$, which is in turn equivalent to (4) by 2.1.3.

3. Upper admissible representations.

In this section P is a poset of finite height n . A k -cell of P is an element of P of height k .

A group G of automorphisms of P is said to be *lower admissible* if each $g \in G$ fixes $P(\leq x) \cap P(\leq xg)$ pointwise for each $x \in P$. We define upper admissibility dually.

Examples. (1) If G is a strongly admissible group of automorphisms of the simplicial complex K then G is a lower admissible group of automorphisms of the poset $sd(K)$.

(2) Let \mathcal{A} be a poset and $\mathcal{F} = (Q_a : a \in \mathcal{A})$ a family of subgroups of a group G . Assume \mathcal{F} is *compatible*; that is $Q_a \leq Q_b$ for all $a, b \in \mathcal{A}$ with $a \leq b$. Define $P(\mathcal{F})$ to be the poset consisting of the disjoint union of the coset spaces G/Q_a with $Q_a g \leq Q_b h$ if and only if $a \leq b$ and $Q_a g \subseteq Q_b h$. Then visibly

(3.1) G is represented as a upper admissible group of automorphisms of $P(\mathcal{F})$ by right multiplication.

We have a partial converse to Lemma 3.1:

(3.2) Let G be represented as a upper admissible group of automorphisms of P transitive on 0-cells, let y be a 0-cell of P , $\mathcal{A} = P(\geq y)$, and for $a \in \mathcal{A}$ let $Q_a = G_a$ and $\mathcal{F} = (Q_a : a \in \mathcal{A})$. Then \mathcal{A} is a set of representatives for the orbits of G on P and the map $ag \mapsto Q_ag$ is a G -equivariant isomorphism of P with $P(\mathcal{F})$.

Proof. As P is of finite height, each cell is above a 0-cell, so as G is transitive on 0-cells, each cell is conjugate to a member of \mathcal{A} . Then as G is upper admissible, \mathcal{A} is a set of representatives for the orbits of G on P .

Let $\varphi : P \rightarrow P(\mathcal{F})$ be the map $\varphi : ag \mapsto Q_ag$. As \mathcal{A} is a set of representatives for the orbits of G on P and $Q_a = G_a$, the map φ is a well defined bijection. Next $ag \leq bh$ iff $a \leq bhg^{-1}$. So if $ag \leq bh$ then $y \leq b, bhg^{-1}$, so $b = bhg^{-1}$ and hence $bh = bg$ so that $Q_bh = Q_bg$. Also $ag \leq bh = bg$ so $a \leq b$ and hence $\varphi(ag) = Q_ag \leq Q_bg = Q_bh = \varphi(bh)$. Similarly we can reverse the direction of the argument, and so conclude φ is an isomorphism of posets. \square

For $a \in \mathcal{A}$, let $\mathcal{F}_a = (Q_b : b < a)$ be the compatible family of subgroups of Q_a indexed by the poset $\mathcal{A}(< a)$.

(3.3) Let $\mathcal{F} = (Q_a : a \in \mathcal{A})$ be a compatible family of subgroups of a group G indexed by a poset \mathcal{A} , let $P = P(\mathcal{F})$, $a \in \mathcal{A}$, and $x = Q_a \in P$. Then

- (1) $P(> x) = \{Q_b : a < b\}$ is isomorphic to $\mathcal{A}(> a)$.
- (2) $P(< x) \cong P(\mathcal{F}_a)$.
- (3) P is connected iff \mathcal{A} is connected and $G = \langle \mathcal{F} \rangle$.
- (4) P is residually connected iff \mathcal{A} is residually connected, $G = \langle \mathcal{F} \rangle$, and $Q_a = \langle \mathcal{F}_a \rangle$ for each $a \in \mathcal{A}$ of height at least 2.

Proof. First $x < Q_bg$ iff $a \leq b$ and $Q_a \subseteq Q_bg$ iff $Q_bg = Q_b$ and $a \leq b$, so (1) holds. Similarly $Q_bg < x$ iff $b < a$ and $Q_bg \subseteq Q_a$ iff $b < a$ and $g \in Q_a$, so (2) holds.

Let $\mathcal{A}_i, i \in I$, be the connected components of \mathcal{A} , $H_i = \langle Q_a : a \in \mathcal{A}_i \rangle$, and

$$\Delta_i = \bigcup_{a \in \mathcal{A}_i} Q_a H_i.$$

Now for $a \in \mathcal{A}_i$, we just saw that Q_a is adjacent to Q_b for $a \leq b$ and to Q_cg for $c < a$ and $g \in Q_a$, so Δ_i contains all cells adjacent to Q_a . Further Q_a acts on the connected component $\Gamma(a)$ of P containing Q_a and $\{Q_a : a \in \mathcal{A}_i\} \cong \mathcal{A}_i$

is connected, so $\Gamma(a) = \Gamma(b) = \Gamma(i)$ for $a, b \in \mathcal{A}_i$, so H_i acts on $\Gamma(i)$. Thus as Δ_i is the union of H_i orbits of cells in \mathcal{A}_i , $\Delta_i \subseteq \Gamma(i)$ and then as all cells adjacent to Q_a are contained in Δ_i , we conclude $\Delta_i = \Gamma(i)$. Therefore P is connected iff \mathcal{A} is connected and $G = Q_a G = Q_a H_i = H_i$. That is (3) holds. Finally parts (1)-(3) imply part (4). \square

4. Thin complexes and posets.

In this section K is an n -dimensional simplicial complex. We say K is *thin* if each simplex of K is contained in a chamber and for each $(n-1)$ -simplex s of K , $\text{Link}_K(s)$ is of order 2. If $\text{Link}(s)$ is of order 2, a *reflection* through s is an automorphism of K fixing each vertex of s and interchanging the two vertices of $\text{Link}(s)$.

Let P be a poset of finite height n . The poset is *thin* if its order complex is thin.

Remark. (1) Thin residually connected posets of height n are called *abstract n -polytopes*; see [MS] for example. An abstract polytope is *regular* if it possesses a chamber transitive group of automorphisms. Regular abstract polytopes have been studied extensively; eg. [MS]. We will study abstract polytopes with a weaker symmetry condition: a lower admissible group of automorphisms transitive on n -cells. For if G is transitive on chambers of a poset P then by 2.3.1 and Example 1 in Section 3, G is lower admissible and transitive on n -cells of $sd(\mathcal{O}(P))$.

If P is thin and x is an $(n-1)$ -cell, then there are 2 n -cells above x , and a *reflection* through x is an automorphism of P fixing $P(\leq x)$ pointwise and interchanging the two n -cells above x . Similarly we define reflections through 1-cells dually.

(4.1) *Let K be thin with a connected chamber graph and G a group of automorphisms of K . Then*

- (1) $G_C = 1$ for each chamber C of K .
- (2) For each simplex s of K , G_s acts faithfully on $\text{Link}_K(s)$.
- (3) If s is an $(n-1)$ -simplex of K then there is at most one reflection through s and either $G_s = 1$ or $G_s = \langle r_s \rangle$, where r_s is the reflection through s .

Proof. As K is thin with a connected chamber graph, a standard argument establishes (1). Then (1) implies (2). If s is an $n-1$ -simplex, then as K is thin, $|\text{Link}_K(s)| = 2$. Then as G_s is faithful on $\text{Link}_K(s)$ by (2), (3) holds. \square

(4.2) *Let P be thin poset of height n with a connected chamber graph, G*

a group of automorphisms of P , z an n -cell of P , \mathcal{W} the set of $(n-1)$ -cells below z , and $O(P)$ the group generated by all reflections through $(n-1)$ -cells of P . Then

- (1) If $G_{P(\leq w)} \neq 1$ for each $w \in \mathcal{W}$ then $O(P) = \langle r_w : w \in \mathcal{W} \rangle \leq G$ and $O(P)$ is transitive on the n -cells of P .
- (2) If G is lower admissible and transitive on the n -cells of P then $G_{P(\leq w)} \neq 1$ for each $w \in \mathcal{W}$ and $G = O(P)$.

Proof. Let $w \in \mathcal{W}$. As P is thin there is exactly one more n -cell y above w . If G is transitive on n -cells there is $g \in G$ with $zg = y$. Therefore

$$P(\leq w) \subseteq P(\leq z) \cap P(\leq zg),$$

so if G is lower admissible then $g \in G_{P(\leq w)}$, so $G_{P(\leq w)} \neq 1$, proving half of (2).

Conversely assume $G_{P(\leq w)} \neq 1$ for each $w \in \mathcal{W}$. Then by 4.1.3 applied to $K = O(P)$, $G_{P(\leq w)} = \langle r_w \rangle \cong \mathbf{Z}_2$, where r_w is the unique reflection through w and $zr_w = y$. Let $H = \langle r_w : w \in \mathcal{W} \rangle$. To complete the proof of (2) and to establish (1), it remains to show $H = O(P)$ is transitive on n -cells and if G is lower admissible then $H = G$. If H is transitive on n -cells and G is lower admissible then $G = HG_z = HG_{P(\leq z)} = H$, since by 4.1.1, $G_{P(\leq z)} = 1$. Further as H contains each reflection through a member of \mathcal{W} , and as H is transitive on n -cells, H contains all reflections, so $H = O(P)$. Therefore it remains to show H is transitive on n -cells.

Let Γ be the graph on the n -cells of P with a adjacent to b if there is a common $(n-1)$ -cell below a and b . As the chamber graph of P is connected, so is Γ . Further if $w < y, z$ is an $(n-1)$ -cell then $zr_w = y$. That is if y is adjacent to z then $y \in zH$. Now if H is not transitive on n -cells, there exists y at minimal distance d from z subject to $y \notin zH$. By minimality of d there is x adjacent to y with $x \in zH$. Then as $u \in zH$ for each u adjacent to z and $x \in zH$, $y \in xH = zH$, contrary to the choice of y . This completes the proof of the lemma. \square

(4.3) Let \mathcal{A} be a poset of height n with a unique 0-cell a_0 and $\mathcal{F} = (Q_a : a \in \mathcal{A})$ a compatible family of subgroups of a group G . Then $P(\mathcal{F})$ is an abstract polytope if and only if the following conditions hold:

- (1) $Q_{a_0} = 1$.
- (2) For each 1-cell a of \mathcal{A} , Q_a is of order 2.
- (3) $\mathcal{A}(> a_0)$ is an abstract polytope.
- (4) $Q_b = \langle Q_a : a \leq b \text{ and } h(a) = 1 \rangle$ for each $b \in \mathcal{A}$.
- (5) $G = \langle \mathcal{F} \rangle$.

Proof. Let $P = P(\mathcal{F})$ and $Q_0 = Q_{a_0}$. By 3.3, P is residually connected iff \mathcal{A} is residually connected, $G = \langle \mathcal{F} \rangle$, and $Q_b = \langle \mathcal{F}_b \rangle$ for each $b \in \mathcal{A}$ of height at least 2. Notice the last condition is equivalent to (4) when $Q_0 = 1$.

Next $\mathcal{A} \cong P(\geq Q_0)$ by 3.3, so if P is an abstract polytope then (3) holds. By 4.1.1, $Q_0 = 1$. Further if a is a 1-cell in \mathcal{A} then by 3.3, $P(< Q_a) \cong Q_a/Q_0 \cong Q_a$ is of order 2, so (2) holds. Thus (1)-(5) are necessary conditions.

Conversely assume (1)-(5); then we have seen that P is residually connected. Next if $Q_{ag} \leq Q_{bh}$ with $h(Q_{bh}) = h(Q_{ag}) + 2$ then $a < b$ with $h(a) = h(b) + 2$ and by 3.3, the intervals $(Q_{ag}, Q_{bh})_P \cong (a, b)_{\mathcal{A}}$ are of order 2 by (3). If Q_{bh} is a 1-cell then b is a 1-cell and $P(< Q_{bh}) \cong Q_b/Q_0$ is of order 2 by (1) and (2). Finally if Q_{bg} is an $n-1$ -cell then by 3.3, $P(> Q_{bg}) \cong \mathcal{A}(> b)$ is of order 2 by (3). So P is thin and (1)-(5) are sufficient. \square

Let I be a finite index set and $\rho : I \rightarrow G$ a map such that $\rho(i)$ is an involution for each $i \in I$. For $J \subseteq I$ define

$$P_J = \langle \rho(j) : j \in J \rangle.$$

The groups P_J , $J \subseteq I$, are the *parabolics* of the pair (G, ρ) .

Define the *Coxeter matrix* of (G, ρ) to be the matrix $M = (m_{ij})$ defined by $m_{ij} = |\rho(i)\rho(j)|$ and the *Coxeter diagram* Δ of (G, ρ) to be the Coxeter diagram of its Coxeter matrix. That is Δ is the diagram whose nodes are the elements of I and with an edge of weight $m_{i,j} - 2$ between distinct nodes i and j .

We have a graph induced on I by the Coxeter diagram Δ via i adjacent to j if there is an edge between i and j of weight at least 1; i.e. if $m_{i,j} > 2$. We say Δ is *connected* if its graph is connected, and more generally define the *connected components* of Δ to be the connected components of its graph.

Define a *thin triple* of height n to be a triple (G, ρ, \mathcal{A}) such that

(T1) G is a group.

(T2) \mathcal{A} is a poset of height n with a unique element a_0 of height 0, $\mathcal{A}(> a_0)$ is thin, and $\mathcal{A}(> a_0)$ is residually connected if $n > 1$.

(T3) Let I be the set of 1-cells of \mathcal{A} . Then $\rho : I \rightarrow G$ is a function with $\rho(i)$ an involution for each $i \in I$ and $G = \langle \rho(I) \rangle$.

Let $\mathcal{F} = \mathcal{F}(G, \rho, \mathcal{A})$ be the family of subgroups $\mathcal{F} = (Q_a : a \in \mathcal{A})$ with $Q_a = P_{I(a)}$ the parabolic determined by $I(a) = \{i : i \leq a\}$. The family \mathcal{F} is the *family of parabolics* of (G, ρ, \mathcal{A}) and the poset $P(\mathcal{F})$ is the *parabolic poset* of (G, ρ, \mathcal{A}) and is denoted by $P(G, \rho, \mathcal{A})$.

(4.4) *Let (G, ρ, \mathcal{A}) be a thin triple, $\mathcal{F} = (Q_a : a \in \mathcal{A})$ the parabolic family and $P = P(\mathcal{F})$ the parabolic poset of (G, ρ, \mathcal{A}) . Then*

(1) *P is an abstract polytope and G is an upper admissible group of auto-*

morphisms of P transitive on 0-cells of P .

- (2) \mathcal{F} is a set of representatives for the orbits of G on P and for $a \in \mathcal{A}$, $h(Q_a) = h(a)$ and Q_a is the stabilizer in G of the cell a .
- (3) For each $i \in I$, $\rho(i)$ is the reflection through the 1-cell Q_i of P .

Proof. By definition, $Q_0 = Q_{a_0} = P_{I(a_0)} = P_\emptyset = 1$ and for $i \in I$, $Q_i = P_i = \langle \rho(i) \rangle$ is of order 2. By hypothesis, $\mathcal{A}(\geq a_0)$ is an abstract polytope and $G = \langle \mathcal{F} \rangle$. By construction $Q_a = P_{I(a)} = \langle Q_i : i \leq a \text{ and } h(i) = 1 \rangle$. Therefore by 4.3, P is an abstract polytope. By 3.1, G is an upper admissible group of automorphisms of P and by construction (2) holds so as a_0 is the unique 0-cell in \mathcal{A} , G is transitive on 0-cells. Finally part (3) holds by the dual of 4.1.3. \square

(4.5) Let P be an abstract polytope, a_0 a 0-cell of P , and G an upper admissible group of automorphisms of P transitive on 0-cells. Let $\mathcal{A} = P(\geq a_0)$, I the set of 1-cells in \mathcal{A} , and $\rho(i)$ the reflection through $i \in I$. Then

- (1) $G = O(P)$ is the group generated by all reflections through 1-cells of P .
- (2) (G, ρ, \mathcal{A}) is a thin triple and $P \cong P(G, \rho, \mathcal{A})$ via the G -equivariant map $ag \mapsto Q_a g$.
- (3) $Q_a = O(P(< a))$ for each $a \in \mathcal{A}$.

Proof. Part (2) follows from 3.2 and part (1) from the dual of 4.2.2. For $a \in \mathcal{A}$, $P(< a)$ is an abstract polytope and $Q_a = G_a$ is transitive on 0-cells of $P(< a)$, so (1) implies (3). \square

Remark. (1) Theorems 4.4 and 4.5 show the notion of a thin triple (G, ρ, \mathcal{A}) and the notion of an abstract polytope admitting G as an upper admissible group of automorphisms transitive on 0-cells are equivalent. Namely the map $(G, \rho, \mathcal{A}) \mapsto P(G, \rho, \mathcal{A})$ induces a bijection between the collection of isomorphism classes of thin triples and the collection of isomorphism classes of abstract polytopes with an upper admissible group of automorphisms transitive on 0-cells. The inverse of this bijection is induced by $P \mapsto (O(P), \rho, \mathcal{A})$, where a_0 is some fixed 0-cell of P , $\mathcal{A} = P(\geq a_0)$ and $\rho(i)$ is the reflection through i .

5. Generalities on triangulations.

Let \mathbf{R}^n be n -dimensional Euclidean space. An *affine subspace* of \mathbf{R}^n is a coset $U + x$ of a linear subspace U of \mathbf{R}^n . The *dimension* of the affine subspace $U + x$ is $\dim(U)$, with the empty set of dimension -1 .

A subset C of \mathbf{R}^n is *convex* if for each $x, y \in C$ and each real number t with $0 \leq t \leq 1$, $tx + (1-t)y \in C$. The intersection of any family of convex sets is convex, so for each subset S of \mathbf{R}^n there is a smallest convex subset $[S]$ of \mathbf{R}^n containing S . We call $[S]$ the *convex closure* of S . Recall that if $S = \{v_0, \dots, v_k\}$ then

$$[S] = \left\{ \sum_{i=0}^k a_i v_i : 0 \leq a_i \in \mathbf{R}, \text{ and } \sum_i a_i = 1 \right\}$$

with this expression being unique if S is *affine independent*; that is if the affine subspace generated by S is of affine dimension k .

Let K be a finite dimensional simplicial complex with vertex set K and simplices Σ . A *triangulation* of a topological space T by K is a map φ of Σ into the set of closed subspaces of T together with homeomorphisms

$$\varphi_s : \varphi(s) \rightarrow \hat{\varphi}(s) = [u(s, v) : v \in s] \subset \mathbf{R}^k$$

for each k -simplex s of K such that:

(T1) For $s, t \in \Sigma$, $\varphi(s) \cap \varphi(t) = \varphi(s \cap t)$, where $\varphi(\emptyset) = \emptyset$.

(T2) $T = \bigcup_{s \in \Sigma} \varphi(s)$ and $C \subseteq T$ is closed in T if and only if $C \cap \varphi(s)$ is closed in $\varphi(s)$ for all $s \in \Sigma$.

(T3) For each k -simplex s of K and $t \subseteq s$, $\hat{\varphi}(s) = [u(s, v) : v \in s]$ is of affine dimension k and $\varphi_{t,s} = \varphi_s \circ \varphi_t^{-1}$ acts on $\hat{\varphi}(t) = [u(t, v) : v \in t]$ via $\varphi_{t,s} : \sum_{v \in t} a_v u(t, v) \mapsto \sum_{v \in t} a_v u(s, v)$.

A *morphism of topological spaces* $\varphi^i : K^i \rightarrow T^i$, $i = 1, 2$, with triangulation is a pair (α, β) where $\alpha : K^1 \rightarrow K^2$ is a simplicial map, $\beta : T^1 \rightarrow T^2$ is continuous, and for each $s \in \Sigma^1$,

(T₁) $\beta(\varphi^1(s)) \subseteq \varphi^2(\alpha(s))$, and

(T₂) $\alpha_s \circ \varphi_s^1 = \varphi_{\alpha(s)}^2 \circ \beta$, where $\alpha_s : \hat{\varphi}^1(s) \rightarrow \hat{\varphi}^2(\alpha(s))$ is defined by

$$\alpha_s : \sum_{v \in s} a_v u(s, v) \mapsto \sum_{v \in \alpha(s)} a_v u(\alpha(s), \alpha(v)).$$

Any pair $\varphi_i : K \rightarrow T_i$, $i = 1, 2$, of triangulations of topological spaces T_i by K are isomorphic and there exists a canonical triangulation $\varphi : K \rightarrow T = T(K)$ called the *geometric realization* of K . Moreover T extends to a functor from the category of simplicial complexes to the category of triangulated topological spaces. Namely if $\alpha : K \rightarrow \bar{K}$ is a simplicial map, then α extends to a map $T(\alpha) : \varphi \rightarrow \bar{\varphi}$. In particular this shows that $\text{Aut}(K) \cong \text{Aut}(\varphi)$.

Let β be a homeomorphism of $T = T(K)$ permuting the cells $\varphi(s)$, $s \in \Sigma$. Then β induces a permutation $\alpha : \Sigma \rightarrow \Sigma$ of Σ via $\beta(\varphi(s)) = \varphi(\alpha(s))$. As $t \subseteq s$ if and only if $\varphi(t) \subseteq \varphi(s)$, the map α is an automorphism $\alpha : K \rightarrow K$.

We say that β *preserves the cell structure of T* if $\alpha_s \circ \varphi_s = \varphi_{\alpha(s)} \circ \beta$, where α_s is the map defined in Axiom (T_2) for morphisms of triangulations. In other words the set G of all homeomorphisms β preserving the cell structure on T forms a group and the maps

$$\alpha \mapsto (\alpha, \beta) \mapsto \beta$$

are isomorphisms $\text{Aut}(K) \cong \text{Aut}(\varphi) \cong G$. Thus we can regard the automorphism group of the triangulated space T to be the group of all homeomorphisms preserving the cell structure on T , and that group is naturally isomorphic to $\text{Aut}(K)$.

Finally define an automorphism β of the triangulated space T to be *admissible* if whenever β acts on a cell $\varphi(s)$ then β induces the identity map on $\varphi(s)$ and define β to be *strongly admissible* if β induces the identity on $\varphi(s) \cap \varphi(s)\beta$ for each cell $\varphi(s)$. Notice that β is admissible if and only if the automorphism α of K corresponding to β is an admissible automorphism of K and β is strongly admissible if and only if α is strongly admissible.

6. Manifolds.

Let n be a nonnegative integer. A *topological n -manifold* is a nonempty Hausdorff space T which is locally isomorphic to n -dimensional Euclidean space; that is each point of T has a neighborhood homeomorphic to a neighborhood of \mathbf{R}^n . Let $R = \mathbf{Z}$ or a field. A *R -homology n -manifold* is a nonempty Hausdorff space T such that for each point $t \in T$,

$$H_i(T, T - \{t\}, R) \cong \begin{cases} 0 & \text{if } i \neq n \\ R & \text{if } i = n \end{cases}.$$

We say T is a *homology n -manifold* if $R = \mathbf{Z}$ and T is a *p -homology n -manifold* if R is a field of prime characteristic p .

(6.1) *Each topological n -manifold T is a homology n -manifold.*

Proof. See Lemma 35.1 and Example 1 in Section 35 of [M]. □

Let K be a simplicial complex. We say K is a *R -homology n -sphere* if $\dim(K) = n$ and K has the R -homology of an n -sphere. That is $\tilde{H}_i(K, R) = 0$ for $i \neq n$ and $\tilde{H}_n(K, R) \cong R$. A *R -homology (-1) -sphere* is an empty simplicial complex. We define K to be a *R -homology n -manifold* if $\text{Link}_K(s)$ is a R -homology $(n - k - 1)$ -sphere for each k -simplex s and each $0 \leq k < n$. For s a simplex of K and $\varphi : K \rightarrow T(K)$ the geometric realization of K , let

$$I(s) = \varphi(s) - \bigcup_{t \subset s} \varphi(t)$$

be the interior of $\varphi(s)$.

(6.2) *Let K be a finite dimensional simplicial complex. Then*

- (1) *K is a R -homology n -manifold if and only if the geometric realization T of K is a R -homology n -manifold.*
- (2) *Assume K is a connected R -homology n -manifold. Then K is thin and residually connected. Further $\text{Link}_K(s)$ is finite for each simplex s of K .*

Proof. If the geometric realization T of K is a R -homology n -manifold then by Theorem 63.2 in [M], K is a R -homology n -manifold. Conversely assume K is a R -homology n -manifold and let $x \in T$. Then $x \in I(s)$ for a unique simplex s of K . Let $k = \dim(s)$. If $k = n$ then

$$I(s) = T - \bigcup_{s \neq t \in \Sigma(K)} \varphi(t)$$

so $I(s)$ is open in T and as $\varphi(s)$ is homeomorphic to a closed n -ball, $I(s) \cong \mathbf{R}^n$. So $I(s)$ is a neighborhood of T homeomorphic to \mathbf{R}^n , and hence by Theorem 35.1 and Example 1 in Section 35.1 of [M], $H_i(T, T - \{x\}, R)$ is isomorphic to R or 0 for $i = n$, $i \neq n$, respectively.

So assume $k < n$. Then by Lemma 63.1 in [M],

$$H_i(T, T - \{x\}, R) \cong \tilde{H}_{i-k-1}(\text{Link}_K(s), R)$$

so that $H_i(T, T - \{x\}, R)$ is 0 unless $i = n$, where it is R .

So (1) is established. Thus we may assume K is a connected R -homology n -manifold. Then $H_0(\text{Link}_K(s), R) \cong R$ so $\text{Link}_K(s)$ is connected for each k -simplex s such that $0 \leq k < n - 1$, while $\text{Link}_K(s)$ is of order 2 if $k = n - 1$. Therefore K is residually connected and thin of dimension n . Finally by Exercise 4 on page 377 of [M], $\text{Link}_K(s)$ is finite for each simplex s , completing the proof of (2). \square

Suppose K is finite of dimension n . We say K is *R -orientable* if there exists an orientation of the n -simplices of K such that

$$\sum_{s \in \Sigma^n(K)} s$$

is a cycle in $C_n(K, R)$.

(6.3) *If K is a connected R -homology n -manifold with $H_n(K, R) \neq 0$ then K is finite and orientable and $H_n(K, R) \cong R$.*

Proof. If $H_n(K, R) \neq 0$ then by Exercise 4 on page 377 of [M], K is finite and then Corollary 65.3 in [M] completes the proof. \square

(6.4) *Let K be a finite dimensional simplicial complex which is a p -homology n -sphere for some prime p and G a p -group acting strongly admissibly on K . Then*

- (1) $\text{Fix}_K(G)$ is a p -homology r -sphere for some $-1 \leq r \leq n$.
- (2) If K is a homology p -manifold then $r = \dim(\text{Fix}_K(G))$ and either $r = -1$ and $\text{Fix}_K(G) = \emptyset$, or $r = 0$ and $\text{Fix}_K(G)$ is disconnected of order 2, or $r > 0$ and $\text{Fix}_K(G)$ is a connected p -homology r -manifold.

Proof. Part (1) is a special case of Smith's Theorem. If G is topologically regular on K it appears as Theorem 5.1 in [Br], so it remains to reduce to the case G topologically regular. But by 2.2, K and $sd^2(K)$ have the same homotopy type, G is topologically regular on $sd^2(K)$, and $\text{Fix}_{sd^2(K)}(G) = sd^2(\text{Fix}_K(G))$. By the first remark, $sd^2(K)$ is a p -homology n -sphere, so by the second remark and Smith's Theorem, $\text{Fix}_{sd^2(K)}(G)$ is a p -homology r -sphere, and then by the first and third remarks, $\text{Fix}_K(G) \simeq sd^2(\text{Fix}_K(G)) = \text{Fix}_{sd^2(K)}(G)$ is too.

Similarly (2) is well known and can be retrieved using Theorem 2.2 in Chapter 5 of [Bo]. Alternatively [A] contains a sketch of a proof of this fact in our special situation, following Bredon in Theorem 2.1 in Chapter IV of [Br]. \square

(6.5) *Let K be a homology n -manifold. Then:*

- (1) K is a p -homology manifold for each prime p .
- (2) If K is a homology n -sphere then K is a p -homology n -sphere for each prime p .

Proof. Notice (2) implies (1), so assume K is a homology sphere. Let \tilde{C}_* be the augmented chain complex for K . Thus $\tilde{H}_k(K)$ is 0 if $k \neq n$ and \mathbf{Z} if $k = n$. Let $x \in \tilde{C}_k$ with $\partial(x) = py$ for some $y \in \tilde{C}_{k-1}$. Then $0 = \partial^2(x) = \partial(py) = p\partial(y)$, so $y \in \tilde{Z}_{k-1}(K)$ and hence there is $z \in \tilde{C}_k$ with $\partial(z) = y$. Then $0 = \partial(x - pz)$, so if $k \neq n$ there is $u \in \tilde{C}_{k+1}$ with $\partial(u) = x - pz \equiv x \pmod{p\tilde{C}_k}$. Hence $\tilde{H}_k(K, \mathbf{F}_p) = 0$ for $k \neq n$, so it remains to show the top homology is 1-dimensional.

Now relax the assumption that K is a homology manifold and sphere, and assume instead that K is thin and residually connected. Let

$$x = \sum_{s \in \Sigma^n(K)} a_s s \in \tilde{C}_n.$$

Then

$$\partial(x) = \sum_{s,i} (-1)^i a_s s^i = \sum_{t \in \Sigma^{n-1}(K)} (\epsilon_{t,s} a_s + \epsilon_{t,s'} a_{s'}) t$$

where s, s' are the n -simplices above the $(n-1)$ -simplex t and $\epsilon_{t,s} = (-1)^i$, where $t = s^i$. Thus $\partial(x) \in p\tilde{C}_{n-1}$ if and only if

$$a_{s'} \equiv -\epsilon_{t,s}\epsilon_{t,s'}a_s \pmod{p}$$

for all $t \in \Sigma^{n-1}(K)$. Hence as the graph on $\Sigma^n(K) \cup \Sigma^{n-1}(K)$ with s adjacent to t if $t \subset s$, is connected, we conclude that if K is a homology n -sphere then $\partial(x) \in p\tilde{C}_{n-1}$ if and only if $x \in p\mathbf{Z} + \pi$, where $\pi = \sum_s c_s s$ is the fundamental cycle with $c_s = \pm 1$ for all s and $\epsilon_{t,s}c_s + \epsilon_{t,s'}c_{s'} = 0$ for all $t \in \Sigma^{n-1}(K)$. This completes the proof of (2). Further if K is only thin and residually connected, we see that $\pi = \sum_s s$ is a cycle over \mathbf{F}_2 , so we have shown:

(6.6) *If K is thin and residually connected then K is \mathbf{F}_2 -orientable.*

7. Combinatorial cell complexes.

The notion of a combinatorial cell complex (X, f) is defined in [CCC]. We will consider only the case where (X, f) is the simplicial cell complex of the poset X . That is X is a poset in which each element is of finite height and for $x \in X$, $f(x) = X(\leq x)$. The posets $f(x)$, $x \in X$, are the *cells* of (X, f) and the *boundary* of the cell $f(x)$ is $\dot{f}(x) = f(x) - \{x\}$. The *faces* of the cell are its subposets $f(x)(\leq v)$, $v \in f(x)$.

From Section 5, there is a functor T from the category of posets to the category of topological spaces which associates to X its *geometric realization* $T(X) = T(\mathcal{O}(X))$. Further $T(X)$ is the union of the closed subspaces $T(x) = T(f(x))$, $x \in X$, which supply $T(X)$ with a cell structure parametrized by X . A *topological space with cell structure over X* is just a space homeomorphic to $T(X)$. By construction

(7.1) *There is a triangulation $\varphi : \mathcal{O}(X) \rightarrow T(X)$ of the geometric realization of X by $\mathcal{O}(X)$.*

Because of 7.1, the homology, cohomology, and fundamental group of the geometric realization $T(X)$ of X are the same as that of $\mathcal{O}(X)$ (defined combinatorially), and we define $H_*(X) = H_*(\mathcal{O}(X))$, $H^*(X) = H^*(\mathcal{O}(X))$, and $\pi_1(X) = \pi_1(\mathcal{O}(X))$.

Recall a space with cell structure over X is just a copy of $T(X)$, so we can regard any such space as $T(X)$. An *automorphism of a space with cell structure* is an automorphism α of the triangulated space $\varphi : \mathcal{O}(X) \rightarrow T(X)$ (as defined in Section 5) permuting the cells $T(x)$, $x \in X$. In particular from Section 5, α is induced from a unique automorphism of $\mathcal{O}(X)$ and then as α permutes the cells $T(x)$, $x \in X$, α preserves height, so α is induced by an automorphism of X . That is the group of automorphisms of a space with cell structure over X is isomorphic to $\text{Aut}(X)$.

We next discuss cellular homology. Define the n -skeleton of X to be

$$X^n = \{x \in X : h(x) \leq n\}.$$

Let $\mathcal{O}^n = \mathcal{O}(X^n)$ be the order complex of the n -skeleton of X . If s is a k -simplex of $K = \mathcal{O}(X)$ then order $s = \{v_0, \dots, v_k\}$ so that $v_i \leq v_{i+1}$ for each i . Use this ordering of s to define the oriented simplex

$$s = v_0 \cdots v_k = v_0 \wedge \cdots \wedge v_k.$$

Then these oriented simplices are generators for $C_k(K)$ the k th member of the simplicial chain group $C_*(K) = (C_n(K) : n)$ with boundary map ∂ defined by

$$\partial(s) = \sum_{i=0}^k (-1)^i s^i$$

where $s^i = v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k$. If $h(x) = n$, a typical member of $C_n(f(x))$ is of the form $\sum_s a_s s x$, where the sum is over all $(n-1)$ -simplices s of $\dot{f}(x)$ and $a_s \in \mathbf{Z}$. Notice $\tilde{H}_{n-1}(\dot{f}(x)) = \tilde{Z}_{n-1}(\dot{f}(x))$. Let

$$D(x) = \left\{ \sum_s a_s s x : \sum_s a_s s \in \tilde{Z}_{n-1}(\dot{f}(x)) \right\} \leq C_n(f(x))$$

so that $D(x) \cong \tilde{H}_{n-1}(\dot{f}(x))$, and define

$$D_n(X) = \bigoplus_{h(x)=n} D(x) \cong \bigoplus_{h(x)=n} \tilde{H}_{n-1}(\dot{f}(x)).$$

Finally define the boundary map on the chain complex $D_*(X) = (D_n(X) : 0 \leq n \in \mathbf{Z})$ by

$$\begin{aligned} \partial_n : D_n(X) &\rightarrow D_{n-1}(X) \\ \sum_{x,s} a_{x,s} s x &\mapsto \sum_{x,s} a_{x,s} s. \end{aligned}$$

We call $D_*(X)$ the *cellular chain complex* of X and ∂ the *cellular boundary map*. The *cellular homology* of X is $H_*^c(X) = H_*(D_*(X))$.

Theorem 7.2. *If $\mathcal{O}(\dot{f}(x))$ is homology spherical for each $x \in X$ then the ordinary homology $H_*(X)$ of X is isomorphic to the cellular homology $H_*^c(X)$.*

Proof. This is Theorem 12.16 in [CCC]. □

Define \hat{X} to be the dual poset of X . That is X and \hat{X} are the same as sets with the ordering \lesssim on \hat{X} defined by $x \lesssim y$ if and only if $y \leq x$. Then

for $x \in \hat{X}$, define $\hat{f}(x) = X(\geq x)$ and $\check{f}(x) = \hat{f}(x) - \{x\}$, giving rise to the dual complex (\hat{X}, \hat{f}) of (X, f) . Then of course

(7.3) $\mathcal{O}(X) = \mathcal{O}(\hat{X})$, so X and \hat{X} have the same homology and fundamental group.

Theorem 7.4. *Let X be a poset of height n and R be \mathbf{Z} or a field. Then the following are equivalent.*

- (1) *The geometric realization of X is a R -homology n -manifold.*
- (2) *For each $x \in X$ of height h , $\check{f}(x)$ is a R -homology $(n - h - 1)$ -sphere and $\dot{f}(x)$ is a R -homology $(h - 1)$ -sphere.*
- (3) *For each $x \in X$ of height 0 and each $y \in X$ of height n , $\check{f}(x)$ and $\dot{f}(y)$ are both R -homology $(n - 1)$ -spheres and R -homology $(n - 1)$ -manifolds.*

Proof. First we recall that the geometric realizations of X and $K = \mathcal{O}(X)$ are the same, and by 6.2.1, that space is a R -homology n -sphere if and only if $\text{Link}_K(s)$ is a homology $(n - k - 1)$ -sphere for each k -simplex s of K . So let $s = \{v_0, \dots, v_k\}$ be a k -simplex of K with $v_i \leq v_j$ for $i \leq j$. Notice

(a) If v_k is of height k then $\text{Link}_K(s) \cong \check{f}(v_k)$.

The dual of (a) is:

(b) If v_0 is of height $n - k$ then $\text{Link}_K(s) \cong \dot{f}(v_0)$.

Observe that (a) and (b) together with paragraph one of this proof show that (1) implies (2). Similarly if K is a R -homology manifold, so are all its links, so (a) and (b) show that (1) implies (3). Conversely

$$\text{Link}_K(s) = \dot{f}(v_0) * S_1 * \dots * S_k * \check{f}(v_k)$$

is the join of $\dot{f}(v_0)$, $\check{f}(v_k)$, and the intervals

$$S_i = \check{f}(v_{i-1}) \cap \dot{f}(v_i)$$

so as the join of a R -homology p -sphere with a R -homology q -sphere is a R -homology $p + q + 1$ -sphere, to show (2) implies (1), it suffices to show that (2) implies that S_i is a homology $\dim(S_i)$ -sphere, since $\dim(\text{Link}_K(s)) = n - k - 1$. This follows from (2) using the argument in the next paragraph.

Finally assume (3). For $x \in X$ of height h , $x \leq y$ for some y of height n . If $x = y$ then by hypothesis $\dot{f}(x)$ is a R -homology $(n - 1)$ -sphere. If not then as $\dot{f}(y)$ is a R -homology manifold, $\dot{f}(y)(< x) = \dot{f}(x)$ is a R -homology $(h - 1)$ -sphere. By duality, $\check{f}(x)$ is a R -homology $(n - h - 1)$ -sphere, so (3) implies (2). \square

Finally we find in Theorem 15.15 of [CCC] that:

(7.5) *There is an equivalence of categories between the category of regular CW-complexes and posets X in which $\mathcal{O}(\dot{f}(x))$ is an $(n-1)$ -sphere for each $x \in X$ of height n and each $1 \leq n \leq h(X)$.*

8. Poincaré duality.

In this section $R = \mathbf{Z}$ or R is a field and X is a poset of height n such that the geometric realization $T(X)$ of X is a R -homology n -manifold. Let $K = \mathcal{O}(X)$ be the order complex and \hat{X} the dual poset. We say X is a *R -homology n -manifold* if its geometric realization is an R -homology n -manifold.

From Section 7, we have the cellular chain complex $D_* = D_*(X)$ and the homology of X is isomorphic to the homology of D_* . Similarly let $\hat{D}_* = D_*(\hat{X})$ be the cellular chain complex of \hat{X} . By 7.3, \hat{D}_* has the same homology as D_* .

Recall from Section 7 that

$$D_k = \bigoplus_{h(x)=k} D(x)$$

where $D(x) \cong \tilde{H}_{k-1}(\dot{f}(x))$. As X is a R -homology n -manifold, $\tilde{H}_{k-1}(\dot{f}(x), R) \cong R$ by 7.4. Thus $D(x) = R\sigma_x$, where $\sigma_x = d_x x$ and d_x is a generator for $\tilde{H}_{k-1}(\dot{f}(x), R)$. Similarly $\hat{D}(x) = R\hat{\sigma}_x$ with $\sigma_x = \hat{d}_x x$.

Let D^* be the cochain complex for D_* . Thus

$$D^p(X, R) = \text{Hom}(D_p(X), R)$$

with the coboundary operator $\delta : D^p(X, R) \rightarrow D^{p+1}(X, R)$ the dual of

$$\partial_p : D_{p+1}(X, R) \rightarrow D_p(X, R).$$

Following [M], write $\langle a, b \rangle$ for the image of $b \in D_p$ under $a \in D^p = \text{Hom}(D_p, R)$. We have the natural basis $(\sigma_x : h(x) = p)$ for D_p and as X is finite this basis is finite. Let $(\sigma_x^* : h(x) = p)$ be the dual basis for D^p . Recall the coboundary operator δ is defined by

$$\langle \delta(a), b \rangle = \langle a, \partial(b) \rangle \quad \text{for each } a \in D^p \text{ and } b \in D_{p+1}.$$

The standard proof of Poincaré duality such as in Section 65 of [M] establishes the following result, which gives an explicit isomorphism between the cochain complex and the dual chain complex. Details can be found in [A].

Theorem 8.1. *Assume X is an orientable R -homology n -manifold. Then we can choose orientations for $\hat{\sigma}_x$, $x \in X$, so that the map $\phi : \sigma_x^* \mapsto \hat{\sigma}_x$*

defines an isomorphism $\phi : D^{n-p} \rightarrow \hat{D}_p$, $0 \leq p \leq n$, of the cellular cochain complex for X with the dual chain complex for X with $\phi \circ \delta = \partial \circ \phi$.

Corollary 8.2 (Poincaré Duality). *If X is an orientable R -homology n -manifold then $H^p(X, R) \cong H_{n-p}(X, R)$ for each $0 \leq p \leq n$.*

Proof. By 8.1, $H^p(X, R) \cong H_{n-p}(\hat{X}, R)$, while as remarked earlier, 7.3 says $H_{n-p}(\hat{X}, R) \cong H_{n-p}(X, R)$. \square

(8.3) *Let X be an orientable R -homology n -manifold. Then*

- (1) *If R is a field then $H_p(X, R) \cong H_{n-p}(X, R)$.*
- (2) *If $R = \mathbf{Z}$ then*

$$\mathrm{Tor}(H_{p-1}(X)) \cong \mathrm{Tor}(H_{n-p}(X)) \text{ and } r(H_p(X)) = r(H_{n-p}(X)),$$

where $r(G)$ is the rank of $G/\mathrm{Tor}(G)$ as a \mathbf{Z} -module for a finitely generated abelian group G .

Proof. This follows from 8.2 and the Universal Coefficient Theorem; cf. 53.1 and 53.5 in [M]. \square

9. Posets as manifolds.

In this section $R = \mathbf{Z}$ or R is a field. Further X is a poset of finite height n .

A subset Y of X is *lower closed* if $X(\leq y) \subseteq Y$ for each $y \in Y$. Define *upper closed* subsets dually.

(9.1) *Assume X is an R -homology n -manifold, $0 < m < n$, Y is an upper closed subset of X whose minimal elements are of height m in X , and Z is a lower closed subset of X whose maximal elements are of height m in X . Assume Y and Z are abstract polytopes and R -orientable, and $Y \cap Z = \{v\}$ is of order 1. Then $H_m(X, R) \neq 0$, so X is not a R -homology sphere.*

Proof. For $x \in X$, let $\sigma_x, \hat{\sigma}_x$ be the fundamental cycles for $D(x)$ and $\hat{D}(x)$ supplied in Section 8. As Z is an abstract polytope, R -orientable, and lower closed, $\gamma = \sum_x c_x \sigma_x \in C_m(Z, R)$ is a cycle, where the sum is over those $x \in Z$ of height m and $c_x = \pm 1$. As Z is lower closed, $C_m(Z, R) \subseteq D_m(X, R)$ and the boundary map on $C_m(Z, R)$ is the restriction of the boundary map on $D_m(X, R)$, so γ is a cycle in $D_m(X, R)$. We show γ is not a boundary in $D_m(X, R)$ to complete the proof. Assume otherwise; then there is $\alpha \in D_{m+1}(X, R)$ with $\partial(\alpha) = \gamma$.

Similarly \hat{Y} is thin, residually connected, R -orientable, and lower closed in \hat{X} , so \hat{Y} has a fundamental cycle β and β is a cycle in $D_{n-m}(\hat{X}, R)$. Now Theorem 8.1 supplies us with an isomorphism

$$\phi : D^m(X, R) \rightarrow D_{n-m}(\hat{X}, R)$$

with $\phi \circ \delta = \partial \circ \phi$ and $\phi(\sigma_x^*) = \hat{\sigma}_x$ for $x \in X$ of height m . Let $\theta = \phi^{-1}(\beta)$. Then

$$\phi(\delta(\theta)) = \partial(\phi(\theta)) = \partial(\beta) = 0$$

so as ϕ is injective, $\delta(\theta) = 0$. Hence

$$0 = \langle 0, \alpha \rangle = \langle \delta(\theta), \alpha \rangle = \langle \theta, \partial(\alpha) \rangle = \langle \theta, \gamma \rangle.$$

Now $\beta = \sum_x b_x \hat{\sigma}_x$, where the sum is over those $x \in Y$ of height m and $b_x = \pm 1$, so

$$\theta = \phi^{-1}(\beta) = \sum_x b_x \sigma_x^*.$$

Similarly $\gamma = \sum_x c_x \sigma_x$ with $c_x = \pm 1$ and thus

$$0 = \langle \theta, \gamma \rangle = \sum_x b_x c_x.$$

However $Y \cap Z = \{v\}$ is of order 1 and $h(y) \geq m$ and $h(z) \leq m$ for $y \in Y$ and $z \in Z$, so $h(v) = m$. Thus $0 = b_v c_v = \pm 1$, a contradiction. This completes the proof of the lemma. \square

Recall the notion of a thin triple from Section 4. An *R-homology manifold triple* of height n is a thin triple (G, ρ, \mathcal{A}) of height n such that:

(RM1) $\mathcal{A}(> a_0)$ is a *R-homology manifold* and *R-homology sphere*, and;

(RM2) For each $a \in \mathcal{A}$ of height at least 2, $P(Q_a, \rho_a, \mathcal{A}(< a))$ is an *R-homology sphere*, where ρ_a is the restriction of ρ to $I(a)$.

(9.2) *If (G, ρ, \mathcal{A}) is a R-homology manifold triple of height n then its parabolic poset P is an R-homology n -manifold and G is an upper admissible group of automorphisms of P transitive on 0-cells of P .*

Proof. By 4.4 the last two remarks hold and by 3.3.1, $\mathcal{A} \cong P(\geq a_0)$. Then by 7.4, it remains to show $P(< x)$ and $P(> x)$ are *R-homology* $(n - m - 1)$ and m -spheres, respectively, for each $x \in P$ of height m . But by 4.4 we may take $x = Q_a$ for some $a \in \mathcal{A}$, so $P(> x) \cong \mathcal{A}(> a)$ is a *R-homology sphere* by (RM1) and 3.3.1. Similarly $P(< a) \cong P(Q_a, \rho_a, \mathcal{A}(< a))$ is a *R-homology sphere* by (RM2) and 3.3.2. \square

(9.3) *Assume X is a connected R-homology n -manifold and G is an upper admissible group of automorphisms of X transitive on 0-cells. Let a_0 be a 0-cell, $\mathcal{A} = X(\geq a_0)$, I the 1-cells in \mathcal{A} , and $\rho(i)$ the reflection through $i \in I$. Then (G, ρ, \mathcal{A}) is a R-homology manifold triple and $X \cong P(G, \rho, \mathcal{A})$.*

Proof. By 6.2.2, X is an abstract polytope. By 4.5, (G, ρ, \mathcal{A}) is a thin triple and $X \cong P(G, \rho, \mathcal{A})$. By 7.4, $\mathcal{A}(> a_0)$ is a *R-homology manifold* and

sphere and $X(< a)$ is a R -homology sphere for each $a \in \mathcal{A}$. Then by 3.3.2, $P(Q_a, \rho_a, \mathcal{A}(< a)) \cong X(< a)$, completing the proof. \square

Remark. (1) Lemmas 9.2 and 9.3 show that the notion of an R -homology manifold triple and the notion of a poset with an upper admissible group transitive on 0-cells, which is an R -homology manifold, are equivalent. Moreover 9.2 implies Theorem 2 and 4.2 and 9.3 imply Theorem 1.

Example. (1) Let (G, S) be a spherical Coxeter system of rank m and \mathcal{A} the poset of all proper subsets of S . Then $a_0 = \emptyset$ is the unique member of \mathcal{A} of height 0, $\mathcal{A}(> a_0)$ is an abstract $(m-2)$ -polytope, and S is the set of 1-cells of \mathcal{A} . Thus if we define $\rho : S \rightarrow G$ by $\rho(s) = s$, then (G, ρ, \mathcal{A}) is a thin triple of height $m-1$. The poset $P(G, \rho, \mathcal{A})$ is the *Coxeter complex* of the Coxeter system (G, S) and it is well known that the geometric realization of the Coxeter complex of a spherical Coxeter system of rank m is homeomorphic to the $(m-1)$ -sphere. See for example Theorem 66.28 in [CR].

Further for $a \in \mathcal{A}$ of height k , (Q_a, S_a) is well known to be a Coxeter system of rank k , where $S_a = \{s \in S : s \in a\}$, and the Coxeter complex is residually connected; cf. 29.13 in [FGT]. Thus (RM2) is also satisfied, so the Coxeter complex is a manifold triple of height $m-1$ and a sphere.

Remark. (2) We are now in a position to prove Theorem 5. Assume (G, ρ, \mathcal{A}) satisfies the hypotheses of Theorem 5. Then (RM1) is satisfied by hypothesis, so it remains to verify (RM2). But this follows from Example 1.

Example. (2) Let G be a group generated by a set S of m involutions. We construct a faithful representation of G on a 2-dimensional manifold. Let \mathcal{A} be the poset with least element a_0 and $\mathcal{A}(> a_0)$ a $2m$ -gon. Let I be the set of 1-cells of \mathcal{A} . Pick any bijection $\rho : I \rightarrow S$. Then (G, ρ, \mathcal{A}) is a manifold triple of height 2. This follows from Theorem 5. Namely as $2m$ -gons are 1-spheres, (RM1) is satisfied. Further if $a \in \mathcal{A}$ is of height 2, then as $\mathcal{A}(> a_0)$ is a $2m$ -gon, $(a_0, a)_{\mathcal{A}} = \{b, c\}$ is of order 2, so $\langle \rho(b), \rho(c) \rangle = Q_a$ is a dihedral group and $(Q_a, \{\rho(b), \rho(c)\})$ is a Coxeter system of rank 2. Thus (RM2) holds by Example 1.

(9.4) *Let S be a set of involutions generating a group G such that for each $r \in S$, there exists a subset P_r of G such that*

- (1) $1 \in P_r$.
- (2) $P_r \cap P_r r = \emptyset$.
- (3) For $r, s \in S$ and $g \in P_r$, if $sg \notin P_r$ then $s^g = r$.

Then (G, S) is a Coxeter system.

Proof. This is Lemma 4.2.1 in [D] and comes from page 18 of [Bb]. \square

(9.5) Let (G, ρ, \mathcal{A}) be a thin triple, $P = P(G, \rho, \mathcal{A})$ the parabolic poset of the triple, $U = P^1$ the 1-skeleton of P , I the set of 1-cells of \mathcal{A} , and $S = \rho(I)$.

Then either:

- (1) (G, S) is a Coxeter system, or;
- (2) There exists $i \in I$ such that 1 and $\rho(i)$ are in the same connected component of $U - \text{Fix}_U(\rho(i))$.

Proof. This is essentially Lemma 4.2 in [D]; we include a proof for completeness. Assume (2) is false and for $r \in S$ define P_r to consist of those $g \in G$ such that 1 and g are in the same connected component of $Y_r = U - \text{Fix}_U(r)$. We show the subsets P_r satisfy (1), (2), and (3) of lemma 9.4, so that by that lemma, (G, S) is a Coxeter system.

Recall that by definition of P , the 0-cells of P are the members of G and G acts by right multiplication on P , so the stabilizer of each 0-cell is trivial. In particular $G \subseteq Y_r$, so our definition of P_r makes sense. For $g, h \in G$, write \sim_r for the equivalence relation on G defined by $g \sim_r h$ if g and h are in the same connected component of Y_r . Thus P_r is the equivalence class of \sim_r containing 1 and in particular $1 \in P_r$.

Next r permutes the connected components of Y_r and as (2) fails, $r \notin P_r$, so as $r = 1 \cdot r \in P_r r$, $P_r r \neq P_r$, so $P_r \cap P_r r = \emptyset$.

Finally let $r, s \in S$ and $g \in P_r$ with $sg \notin P_r$. Then $g \sim_r 1 \not\sim_r sg$, so $g \not\sim_r sg$ and hence 1 and s are in distinct connected components of $Y_r g^{-1} = U - \text{Fix}_U(r^{g^{-1}})$. Now $s = \rho(i)$ for some $i \in I$ and if $Q_i \notin \text{Fix}(r^{g^{-1}})$ then $1 < Q_i > s$ is a path in $U - \text{Fix}(r^{g^{-1}})$, a contradiction. Thus $Q_i \in \text{Fix}(r^{g^{-1}})$, so as $\langle s \rangle$ is the stabilizer in G of Q_i , $r^{g^{-1}} = s$. That is $s^g = r$, completing the proof. \square

(9.6) Let (G, ρ, \mathcal{A}) be a 2-homology manifold triple of height n with parabolic poset X a 2-homology n -sphere. Let I be the set of 1-cells of \mathcal{A} and $S = \rho(I)$. Then (G, S) is a spherical Coxeter system.

Proof. The proof comes from Proposition 4.4 in [D]. If (G, S) is a Coxeter system then by 6.3, X is finite, so G is finite and hence (G, S) is spherical. Thus we may assume (G, S) is not a Coxeter system, so by 9.5 there exists $i \in I$ such that 1 and $r = \rho(i)$ are in the same connected component of $U - \text{Fix}_U(r)$, where $U = X^1$ is the 1-skeleton of X . Let x_1, \dots, x_k be a path in $U - \text{Fix}_U(r)$ from 1 to r with no repeats. Then $x_0, x_1, \dots, x_k, x_0$ is a cycle, where $x_0 = Q_i$. Hence $Z = \{x_0, \dots, x_k\}$ is a 1-submanifold of X . Notice Z is lower closed since if $h(x_i) = 1$ then $X(< x_i) = \{x_{i-1}, x_{i+1}\} \subset Z$. By construction the maximal members of Z are 1-cells.

Next let $Y = \text{Fix}_X(r)$. As the stabilizer in G of each 0-cell is trivial, the minimal members of Y are the 1-cells fixed by r , so $Y \cap Z = \{x_0\}$. As G

is upper admissible, Y is upper closed. By Smith's Theorem, 6.4, Y is a 2-homology $(n-1)$ -manifold. By 6.6, Y and Z are orientable. But now 9.1 says $H_1(X, \mathbf{F}_2) \neq 0$, contradicting X a 2-homology n -sphere. \square

Remark. (3) Notice 9.6 implies Theorem 3.

(9.7) *If (G, ρ, \mathcal{A}) is a 2-homology manifold triple and $a \in \mathcal{A}$, then $(Q_a, \rho(I(a)))$ is a spherical Coxeter system.*

Proof. Observe $(Q_a, \rho_a, \mathcal{A}(\leq a))$ is a 2-homology manifold triple whose parabolic poset is a 2-homology sphere, so by 9.6, $(Q_a, \rho(I(a)))$ is a spherical Coxeter system. \square

10. Regular 2-homology manifolds which are spheres.

In this section X is a poset of finite height n and G is an upper admissible group of automorphisms of X transitive on 0-cells. Assume X is a 2-homology n -manifold and a 2-homology n -sphere. Let a_0 be a 0-cell of X , $\mathcal{A} = X(\geq a_0)$, I the set of 1-cells in \mathcal{A} , $\rho(i)$ the reflection through $i \in I$, and $R = \{\rho(i) : i \in I\}$. By 9.3, (G, ρ, \mathcal{A}) is a 2-homology manifold triple and X is isomorphic to the parabolic poset of (G, ρ, \mathcal{A}) .

Let $Pow(I)$ be the power set of I and $\phi : \mathcal{A} \rightarrow Pow(I)$ be the map $\phi(a) = I(a)$ for $a \in \mathcal{A}$.

Let G_a be the stabilizer in G of $a \in \mathcal{A}$. By 4.4, $G_a = P_{I(a)}$ is the parabolic generated by $R_a = \rho(I(a))$, the set of reflections through the set $I(a)$ of 1-cells between a_0 and a . Throughout this section we assume:

Hypothesis 10.1. *Distinct 0-cells of X are below at most one 1-cell of X .*

(10.2) $\rho : I \rightarrow R$ is a bijection.

Proof. If $\rho(i) = \rho(j)$ then $a_0, a_0\rho(i) \leq i, j$, so as distinct 0-cells are below at most one 1-cell, $i = j$. Indeed the injectivity of ρ is equivalent to Hypothesis 10.1. \square

(10.3) (G, R) is a spherical Coxeter system.

Proof. This is 9.6. \square

(10.4) *Let $x \in \mathcal{A}$ be of height $m < n$, $L = I(x)$,*

$$M = \bigcup_{y>x} I(y) - L$$

$H = P_M$ the parabolic of G determined by M , and Y the subset of X which is the image of $\mathcal{B} = \mathcal{A}(\geq x)$ under H . Assume

$$(*) \quad |I(y)| = |I(x)| + 1 \text{ for each } (m+1)\text{-cell } y \text{ above } x.$$

Then Y is upper closed, the minimal members of Y are m -cells, and Y is an abstract polytope.

Proof. As \mathcal{B} is upper closed, so is Y and by construction the minimal members of Y are the m -cells xH of X .

Next H is an upper admissible group of automorphisms of the poset Y transitive on the set xH of 0-cells of Y with $\mathcal{B} = Y(\geq x)$, so setting $\mathcal{H} = (H_b : b \in \mathcal{B})$, 3.2 says Y is isomorphic to $P(\mathcal{H})$ and \mathcal{H} is a compatible family of subgroups of H . We verify the hypotheses of 4.3 for H, \mathcal{H} , and conclude from that lemma that $Y \cong P(\mathcal{H})$ is an abstract polytope. First $\mathcal{B}(> x)$ is a 2-homology $(n - m - 1)$ -manifold and sphere, and hence an abstract polytope as $\mathcal{A}(> a_0)$ is a 2-homology $(n - 1)$ -manifold and sphere. Second $H_x = P_M \cap P_L = P_{M \cap L}$ as (G, R) is a Coxeter system. But $M \cap L = \emptyset$, so $P_{L \cap M} = 1$.

Similarly for $a \in \mathcal{B}$ of \mathcal{B} -height 1, $G_a = P_{I(a)}$ with $I(a) = L \cup \{i(a)\}$ for some $i(a) \in M$ by hypothesis (*). Now

$$H_a = P_M \cap P_{I(a)} = P_{M \cap I(a)} = P_{i(a)} = \langle \rho(i(a)) \rangle$$

so H_a is of order 2. Similarly for $b \in \mathcal{B}$,

$$\begin{aligned} H_b &= P_M \cap P_{I(b)} = P_{M \cap I(b)} = \langle P_{i(a)} : a \leq b \text{ and } h_{\mathcal{B}}(a) = 1 \rangle \\ &= \langle H_a : a \leq b \text{ and } h_{\mathcal{B}}(a) = 1 \rangle. \end{aligned}$$

Finally

$$H = P_M = \langle P_{i(a)} : h_{\mathcal{B}}(a) = 1 \rangle = \langle \mathcal{H} \rangle.$$

Thus we have shown that Y is an abstract polytope, completing the proof. \square

Define a k -subset J of I to be \mathcal{A} -simplicial if

$$\phi : \phi^{-1}(\text{Pow}(J)) \cap \mathcal{A}^k \rightarrow \text{Pow}(J)$$

is an isomorphism. Define $\lambda : \text{Pow}(I) \rightarrow \mathbf{Z}$ by

$$\lambda(J) = \max\{m \leq k : L \text{ is } \mathcal{A}\text{-simplicial for all } L \in \text{Pow}(J)^m\}.$$

(10.5) *Let J be an $(m+1)$ -subset of I with $\lambda(J) \geq m-1$ and assume each m -subset L of J is of the form $I(c_L)$ for some m -cell c_L with $\phi : \mathcal{A}(\leq c_L) \rightarrow \text{Pow}(L)$ an isomorphism. Let*

$$\mathcal{D} = \phi^{-1}(\text{Pow}(J)^{m-1}) \cup \{c_L : L \text{ is an } m\text{-subset of } J\},$$

$D = P_J$, and Z the subset of X consisting of all D -conjugates of members of \mathcal{D} . Then Z is a lower closed subset of X^m which is a 2-homology m -manifold.

Proof. The proof is much the same as that of the previous lemma. As \mathcal{D} is lower closed with $\mathcal{D} \subseteq X^m$, so is Z . Our hypotheses say $\phi : \mathcal{D} \rightarrow \text{Pow}(J)^m$ is an isomorphism. In particular $\mathcal{D}(> a_0)$ is a 2-homology $(m-1)$ -sphere and for $c \in \mathcal{D}$, $G_c = P_{I(c)} = D_c$. Now complete the proof as in the previous lemma. \square

(10.6) *Let $0 \leq h \leq n$ and assume:*

() For all $x \in \mathcal{A}^h$ and $y \in X$ with $x < y$ that $I(x) \neq I(y)$.*

Then either

- (1) *$\phi : \mathcal{A}^h \rightarrow \text{Pow}(I)^h$ is an isomorphism. Further if $h < n$ then $\phi : \mathcal{A}^{h+1} \rightarrow \text{Pow}(I)^{h+1}$ is a surjection, while if $h = n$ then $|I| = n+1$ and X is isomorphic to the Coxeter complex for (G, R) , or*
- (2) *$n = h$, X has exactly two n -cells u and v , $|I| = n$, $X = \{u, v\} * X^{n-1}$, $\phi : \mathcal{A}^{n-1} \rightarrow \text{Pow}(I)^{n-1}$ is an isomorphism, and X^{n-1} is isomorphic to the Coxeter complex for (G, R) .*

Proof. We prove the lemma by induction on h and n . For $h = 0$, (1) holds as a_0 is the unique 0-cell of \mathcal{A} and the empty set is the unique 0-subset of I . So a minimal counter example satisfies $n \geq h > 0$.

If ϕ is an isomorphism of \mathcal{A}^h with $\text{Pow}(I)^h$ and $h \neq n$, let $k = h+1$, while if ϕ is not an isomorphism, let $k = h$. If ϕ is an isomorphism and $h = n$, then as $\mathcal{A}(> a_0)$ is a 2-homology $(n-1)$ -sphere, $|I| = n+1$. But now X is isomorphic to the Coxeter complex for (G, R) and (1), holds, so if $h = n$ we may assume ϕ is not an isomorphism.

Let J be a k -subset of I and $m = k-1$. If $m = h$ then by hypothesis, $\phi : \mathcal{A}^m \rightarrow \text{Pow}(I)^m$ is an isomorphism, while if $m < h$ this holds by minimality of h .

Suppose first $J \neq I(a)$ for some k -cell a of \mathcal{A} . Define \mathcal{D} , D , and Z as in 10.5. By the previous paragraph, the map $c \mapsto I(c)$ is an isomorphism between \mathcal{D}^m and $\text{Pow}(J)^m$, so by 10.5, Z is a lower closed 2-homology m -manifold with maximal elements the images cD under D of the m -cells c in \mathcal{D} .

Notice that if y is an k -cell then $(G_y, \rho_y, \mathcal{A}(< y))$ satisfies our hypotheses and is of height $m < n$, so by minimality of n , $\mathcal{A}(< y) \cong \text{Pow}(I(y))^m$ and $I(y)$ is of order k .

Pick $j_0 \in J$ and let $L = J - \{j_0\}$. We have seen there is an m -cell x with $I(x) = L$. By the previous paragraph, if y is an k -cell then $|I(y)| = k = m + 1 = |I(x)| + 1$. Thus we may apply the construction of 10.4 to this x and obtain a subgroup H of G and a thin residually closed subcomplex Y which is upper closed with minimal elements xH . By 6.6, Z and Y are orientable.

Now $x \in Y \cap Z$ and as $Z \subseteq X^m$ and the minimal members of Y are in xH , if $y \in Y \cap Z$ then $y = xg$ for some $g \in H$. But also $y = cd$ for some $d \in D$ and some m -cell c of Z , so $c = x$ and $g \in G_x d = P_L d \subseteq D = P_J$, as $L \subset J$. Thus

$$g \in H \cap D = P_M \cap P_J = P_{M \cap J}.$$

But $J = L \cup \{j_0\}$ with $L \cap M = \emptyset$ and $j_0 \notin M$ as $J \neq I(u)$ for any u . Thus $g \in H \cap D = 1$, so $y = x$. That is $Y \cap Z = \{x\}$. But now 9.1 supplies a contradiction.

Thus we have established the surjectivity of our map. In particular if $k = h + 1$ then (1) holds, so from now on we may take $k = h$.

Next suppose $I(u) = J = I(v)$ for distinct h -cells u and v . We saw above that $\phi : \mathcal{A}(\leq w) \rightarrow \text{Pow}(J)^m$ is an isomorphism for $w = u, v$, so $X(< u) = X(< v) = Z$. Thus $Z_0 = \{u, v\} \cup Z = \{u, v\} * Z$ is a lower closed 2-homology h -manifold. Now if $h = n$ then as $X(> x)$ is of order 2 for each m -cell $x \in Z$, $X(> x) = \{u, v\}$. But now if we let Γ consist of n and $n - 1$ cells of X and $\Sigma = \Gamma \cap Z_0$, then $\Gamma(y) \subseteq \Sigma$ for each $y \in \Sigma$, so as Γ is connected, $\Gamma = \Sigma$ and hence $X = Z_0$, so that (2) holds. Thus we may assume $h < n$.

Let y be an $(h + 1)$ -cell. Then $(G_y, \rho_y, \mathcal{A}(< y))$ satisfies our hypotheses and is of height h , so by a previous case either $I(y)$ is of order $h + 1$ and

$$\phi : \mathcal{A}(< y) \rightarrow \text{Pow}(I(y))^h$$

is an isomorphism, or $\mathcal{A}(< y)$ has just 2 h -cells a and b and $I(y) = I(a) = I(b)$. But the latter case contradicts hypothesis (*), so the former holds.

Given this observation, we can repeat an argument above on u in place of x to obtain a thin residually connected upper closed subcomplex Y_0 with $Y_0 \cap Z_0 = \{u\}$. Hence once again 9.1 supplies a contradiction, completing the proof. \square

Remark. Notice that Theorem 10.6 implies Theorem 4.

11. Hirzebruch's Prize Question.

In [FLM], Frenkel, Lepowsky, and Meurman construct a vertex operator algebra whose automorphism group is the largest sporadic group the Monster. Vertex operator algebras are the algebraic version of physicist's conformal field theory. There are also geometric versions of conformal field theory, although as I understand it, no geometric formalism yet exists for conformal field theory which is satisfactory to mathematicians. However some kind of metaprinciple says there should exist a geometric object corresponding to the Monster vertex operator algebra.

In that direction Hirzebruch has posed the following problem in [HBJ]:

Prize Question: Does there exist a 24-dimensional compact, orientable, differentiable manifold X (admitting the action of the Monster) with $p_1(X) = 0$, $w_2(X) = 0$, $\hat{A}(X) = 1$, and $\hat{A}(X, T_{\mathbf{C}}) = 0$?

Here $p_1(X)$ is the first Pontrjagin class of X , $w_2(X)$ is the second Stiefel-Whitney class, $\hat{A}(X)$ is the \hat{A} -genus of X , and $\hat{A}(X, T_{\mathbf{C}})$ is the twisted \hat{A} -genus of the complex tangent bundle $T_{\mathbf{C}} = TX \otimes \mathbf{C}$.

Theorem 5 applies to groups G which are close to being spherical Coxeter groups, in the sense that G is generated by a set S of involutions such that for many subsets J of S , $(\langle J \rangle, J)$ is a spherical Coxeter system. Examples of such groups include certain subgroups of the Bimonster, including the Monster. Thus Theorem 5 can be used to construct representations of the Monster on manifolds.

The *Bimonster* is the wreath product of the Monster by a group of order 2. Thus the Bimonster B has a normal subgroup $M_1 \times M_2$ of index 2 and B is $M_1 M_2$ extended by an involution t interchanging M_1 and M_2 . In particular B is transitive on the involutions in $B - M_1 M_2$ and $C_{M_1 M_2}(t)$ is a diagonal subgroup of $M_1 \times M_2$ isomorphic to the Monster.

In [CNS], Conway, Norton, and Soicher show that there exists a set $\Gamma \subseteq B - M_1 M_2$ of 26 involutions generating B and admitting the structure of a projective plane of order 3 in such a way that if $i, j \in \Gamma$ are distinct then $|ij| = 3$ if i and j are incident in Γ and $|ij| = 2$ otherwise. Further the full group A of automorphisms of the projective plane is induced on Γ in B . Call Γ a set of *Conway-Norton-Soicher involutions*.

We have the usual Coxeter diagram Δ defined on Γ . Define a subset J of Γ to be *spherical* if the restriction Δ_J of Δ to J is spherical and define J to be *B-spherical* if $(\langle J \rangle, J)$ is a spherical Coxeter system. Thus if J is *B-spherical* then J is spherical. We seek manifold triples (G, ρ, \mathcal{A}) with $G \leq B$ and $\rho(I) \subseteq \Gamma$ satisfying the hypotheses of Theorem 5. Thus if $a \in \mathcal{A}$ then $\mathcal{A}(\leq a)$ is simplicial and $\rho(I(a))$ is a *B-spherical* subset of Γ .

In Section 12, we record various results about the action of A on spherical

and B -spherical subsets of Γ . Then in Section 13 we use Theorem 5 and these results to construct manifolds admitting the action of subgroups G of the Bimonster with dimensions ranging from 5 to 14. Notice that the height of our triple (G, ρ, \mathcal{A}) is $|I(a)|$ for a a maximal cell of \mathcal{A} and hence bounded by 16 by Lemma 12.1. The actual bound when G is the Monster or Bimonster is smaller; 14 may be best possible. Namely by 12.1, the only class of spherical sets of rank 16 has diagram $4D_4$. Up to conjugation it consists of all points distinct from a fixed point p plus the four lines through p . But it is not B -spherical by 12.3, so the bound of 16 cannot be attained.

The geometric realization $T(L)$ of any link L in any poset $P = P(G, \rho, \mathcal{A})$ we construct is smoothly isomorphic to a sphere, so $T(P)$ is a topological manifold admitting a canonical smooth differential structure. As G has a subgroup G_* of index 2 with $\rho(I) \subseteq G - G_*$, $T(P)$ is orientable. When $G = B$, the centralizer in G of each $t \in \Gamma$ is a copy of the Monster, and Smith Theory says the fixed points T of t on $T(P)$ is a 2- homology manifold of dimension $h(P) - 1$, which then admits the action of $C_{M_1 M_2}(t)$, a group isomorphic to the Monster. Presumably this space is actually a manifold. Then for example in the case when $T(P)$ is 9-dimensional, we could take the product of three copies of T to obtain a 24-dimensional manifold admitting the Monster.

However the manifolds we construct are unfortunately not candidates for the Hirzebruch manifold. This is because our reflections are inducing orientation reversing diffeomorphisms of these manifolds, so the Pontrjagin numbers of the manifolds are 0, and hence the \hat{A} -genus is 0 rather than 1.

12. The poset of spherical subsets of the plane of order 3.

Let V be a 3-dimensional vector space over the field F of order 3 and π the projective plane over the field of order 3. Thus π consists of the points (1-dimensional subspaces of V) and lines (2-dimensional subspaces of V) of V together with the incidence relation of inclusion between points and lines.

Associate a Coxeter matrix $M = (m_{i,j})$ of size 26 to π by letting $m_{i,i} = 1$ and for $i \neq j$ let $m_{i,j} = 3$ if i is incident to j in π and $m_{i,j} = 2$ otherwise. Then each subset J of π inherits a Coxeter diagram Δ_J from the Coxeter diagram defined on π by the Coxeter matrix M . A subset J is *spherical* if its Coxeter diagram is spherical. The *type* of J is the type of its Coxeter diagram Δ_J . Write J^\perp for the set of vertices $i \in \pi$ orthogonal to each member of J ; that is $m_{i,j} = 2$ for each $j \in J$.

Let P be the poset of all spherical subsets of π partially ordered by inclusion. We call the members of P of order k , k -cells. We regard the empty set as spherical, so ϕ is the unique 0-cell. The k -cells are the members of P of

height k .

Let A be the group of all automorphisms of π . Thus $A = L\langle\tau\rangle$ where $L = PSL(V) \cong L_3(3)$ is the projective special linear group on V and τ is a polarity of π . Thus τ induces an outer automorphism on L and $A = \text{Aut}(L) \cong \text{Aut}(L_3(3))$.

We record the following two results without proof. They can be retrieved from tables in Section 11 of [CNS], although those tables are supplied without proof too. Alternatively, proofs of 12.1 and 12.2 appear in [A].

(12.1) *The maximal cells of P have the following types and A is transitive on cells of each type:*

- (1) $13A_1$ of rank 13.
- (2) $4D_4$ of rank 16.
- (3) $3A_5$ of rank 15.
- (4) $A_{11} \oplus A_4$ of rank 15.
- (5) $D_5 \oplus 3A_3$ of rank 14.
- (6) $D_6 \oplus 2A_3 \oplus 2A_1$ of rank 14.
- (7) $D_8 \oplus 2A_3$ or rank 14.
- (8) $E_6 \oplus 2A_2 \oplus A_3$ of rank 13.
- (9) $E_7 \oplus A_1 \oplus A_2 \oplus A_3$ of rank 13.
- (10) $E_8 \oplus A_2 \oplus A_3$ of rank 13.

(12.2) *A is transitive on connected spherical cells of each type.*

(12.3) *A spherical subset J of I is not B -spherical if and only if $Y \subseteq J \subseteq X$, where X is maximal spherical and Y is one of the following sums of connected components of X : $2D_4$, $D_6 \oplus 2A_1$, $E_7 \oplus A_1$, D_8 , or E_8 . Further if $\sigma : W \rightarrow \langle\rho(Y)\rangle$ is the corresponding Coxeter cover then $\ker(\sigma) = \langle z_1 \cdots z_r \rangle$ where z_i generates the center of the i th direct factor of W .*

Proof. A stronger assertion seems to be made on page 44 of [CNS], but the assertion given here is the best that could be possible and is presumably what the authors of [CNS] intended.

Let J be spherical but not B -spherical. Then $J \subseteq X$ maximal spherical. Then $J \subseteq X_1 \cup \cdots \cup X_r$ with X_i a connected component of X and $J_i = X_i \cap J \neq \emptyset$. Let $H_i = \langle\rho(J_i)\rangle$ and $\sigma : W \rightarrow H = H_1 \cdots H_r$ the Coxeter covering of H and $W = W_1 \times \cdots \times W_r$, $\sigma(W_i) = H_i$.

From 12.2, A is transitive on connected spherical subsets Z of I of each type and from [CNS], particularly Table 3, Z is B -spherical except when Z is of type D_8 or E_8 , where the kernel of the Coxeter covering is the center of order 2. For example each Z is a subset of a maximal connected subset of type A_{11} , D_8 or E_8 and all subsets of A_{11} and all proper subsets of D_8 or E_8

are B -spherical. Thus unless $X_i = J_i$ is of type D_8 or E_8 , J_i is B -spherical, and we may assume the latter.

Then $[\ker(\sigma), W_i] \leq \ker(\sigma) \cap W_i = 1$, so $\ker(\sigma) \leq Z(W)$. Hence the projection z_i of $z \in \ker(\sigma)$ on W_i is contained in $Z(W_i)$ of order at most 2. Further if $z_i \neq 1$ then $\sigma(z_i) = \sigma(zz_i^{-1}) \in C_H(\langle \rho(X_i) \rangle)$ so $J_i = X_i$. Hence if J is chosen to be minimal non B -spherical then $\ker(\sigma) = z_1 \dots z_r$ with z_i of order 2 generating $Z(W_i)$ and $J_i = X_i$. It follows that J_i is of type A_1 , D_4 , D_6 , D_8 , E_7 , or E_8 . If J_i is of type D_8 or E_8 then $J = J_i$ by minimality of J and the lemma holds. If J_i of type D_4 then all components of X are of type D_4 and from page 44 of [CNS], $2D_4$ is not B -spherical, so the lemma holds. If J_1 is of type D_6 then $J_1^\perp = 3A_3 \oplus 2A_1$, so as $Z(W_i) \neq 1$, J_i must be of type A_1 for $i > 1$ and then from page 44 of [CNS], J is minimal non B -spherical if and only if J is of type $D_6 \oplus 2A_1$. A similar argument works for J_1 of type E_7 . This leaves the case J of type kA_1 . From 12.1 and the cases eliminated, X is of type $13A_1$, which is B -spherical from the discussion at the top of page 45 of [CNS]. \square

13. Manifolds for subgroups of the Bimonster.

Continue the hypotheses and notation of Section 12. In particular V is a 3-dimensional vector space over the field F of order 3, π is the projective plane of V , and P is the poset of all spherical subsets of π ordered by inclusion.

We construct manifold triples (G, ρ, \mathcal{A}) with G a subgroup of the Bimonster B , and \mathcal{A} a subposet of P , and $I \subseteq \pi$. The map $\rho : I \rightarrow B$ is induced by the Conway-Norton-Soicher embedding. As \mathcal{A} is a subposet of P , the map $\phi : \mathcal{A} \rightarrow \text{Pow}(I)$ is an injection whose image is contained in P and for $a \in \mathcal{A}$, $\mathcal{A}(\leq a) \cong \text{Pow}(I(a))$ via ϕ . We choose \mathcal{A} so that its members are B -spherical and hence $P(G_a, \rho_a, \mathcal{A}(\leq a))$ is the Coxeter complex of the spherical Coxeter system $(G_a, \rho(I(a)))$, and thus is a sphere.

Remark 1. Let $I \subseteq \pi$, $G = \langle \rho(I) \rangle$, $I = I_1 \cup \dots \cup I_m$ a partition of I with $|I_i| > 1$ for each i , and K_i a simplicial complex with vertex set I_i such that K_i is a homology sphere and manifold. Let $\mathcal{B}_i = \text{sd}(K_i)$ be the barycentric subdivision of K_i regarded as a poset, and let $\mathcal{A}_i = \mathcal{B}_i \cup \{\emptyset\}$ be the extended poset with unique 0-cell \emptyset and $\mathcal{B}_i = \mathcal{A}_i(> \emptyset)$. Finally let $\mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_m$ be the product poset consisting of all subsets $J = J_1 \cup \dots \cup J_m$ with $J_i \in \mathcal{A}_i$. Then

$$h(\mathcal{A}) = \sum_{i=1}^m h(\mathcal{A}_i)$$

and $\mathcal{A}(> \emptyset)$ is the join $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$ of homology spheres and manifolds, so $\mathcal{A}(> \emptyset)$ is a homology sphere and manifold. Thus if each cell J in \mathcal{A} is

B -spherical, then by remarks above and Theorem 5, (G, ρ, \mathcal{A}) is a manifold triple of height $h(\mathcal{A})$.

Notice that to check that J is B -spherical amounts by 12.3 to checking that the diagram of J is spherical and contains none of the excluded subdiagrams $2D_4$, $D_6 \oplus 2A_1$, $E_7 \oplus A_1$, D_8 , or E_8 . In particular if Δ_J is spherical and contains no $2D_4$ or D_6 subdiagrams then J is B -spherical.

Remark 2. In all but the last example in this section, \mathcal{A}_i is the poset of all proper subsets of I_i and hence $\mathcal{A}_i(> \emptyset)$ is a sphere and manifold and $h(\mathcal{A}_i) = |I_i| - 1$. Hence

$$h(\mathcal{A}) = \sum_{i=1}^m h(\mathcal{A}_i) = \sum_{i=1}^m (|I_i| - 1) = |I| - m.$$

In each case if $J \in \mathcal{A}$ then Δ_J contains no $2D_4$ or D_6 subdiagram, so by Remark 1, J is B -spherical and (G, ρ, \mathcal{A}) is a manifold triple of height $|I| - m$.

Example 1. Let I have the Y_{555} diagram of [CNS]. That is $|I| = 16$ and the diagram of I is a “Y” with a central node and three legs with A_5 diagrams connected to the central node at one end. From [CNS], $\langle \rho(I) \rangle = B$ is the Bimonster. Let I_1 be any subset of I of order at least 2 containing the central node and contained in the unique subset of I whose diagram is D_4 . Thus the diagram of I_1 is A_2 , A_3 , or D_4 . Finally let $I = I_1 \cup \dots \cup I_m$ be a partition of I containing I_1 with $|I_i| > 1$ for each i .

Then the diagram of each $J \in \mathcal{A}$ is the sum of diagrams of type A , so it is spherical and contains no subdiagram of type $2D_4$ or D_6 . Hence by Remark 1, we have a manifold triple (G, ρ, \mathcal{A}) of height $|I| - m = 16 - m$. As m can range between 2 and 8, we get manifolds of dimension between 8 and 14 by varying our partition.

Let $X = P(B, \rho, \mathcal{A})$. For $t \in \rho(I)$, $M = C_{M_1 M_2}(t)$ is the Monster and acts faithfully on the 2-homology manifold $\text{Fix}_X(t)$ of dimension $h(X) - 1$, ranging between 7 and 13.

Example 2. Let I have the Y_{553} diagram. From [CNS]

$$G = \langle \rho(I) \rangle = C_B(t) \cong \mathbf{Z}_2 \times M$$

where M is the Monster and $t \in \Gamma$. We proceed as in Example 1, obtaining partitions of I into m parts, $2 \leq m \leq 7$, and manifolds of dimension $|I| - m = 14 - m$ between 7 and 12.

Example 3. Let I have the Y_{443} , Y_{432} , or Y_{332} diagram. Then from [CNS], $G = \langle \rho(I) \rangle$ is isomorphic to $M(24)/\mathbf{Z}_2$, $M(23) \times \mathbf{Z}_2$, or $M(22)/E_4$, respectively, so G is essentially a Fischer group. Proceeding as in Examples 1 and

2, we obtain manifolds for these groups with dimensions ranging between 5 and 10.

Example 4. This example is a little different. Choose I to have the diagram in Figure 1.b on page 28 of [CNS]. Then $G = \langle \rho(I) \rangle$ is again isomorphic to $\mathbf{Z}_2 \times M$. There is a subset I_1 of I of order 6 with diagram of type \tilde{A}_5 and I is obtained by adding three legs with diagrams of type A_3 . Consider any partition $I = I_1 \cup \dots \cup I_m$ containing I_1 with $|I_i| > 1$. For $i \neq 1$ let \mathcal{A}_i be the poset of all proper subsets of I_i , but form \mathcal{A}_1 as in Remark 1 with respect to the 1-dimensional complex K_1 whose 1-simplices are the edges in the \tilde{A}_5 diagram on I_1 . Thus K_1 is a 1-sphere, so by Remark 1 we get a manifold triple (G, ρ, \mathcal{A}) of height

$$2 + |I| - 6 - (m - 1) = 12 - m$$

producing manifolds for $\mathbf{Z}_2 \times M$ of dimension between 7 and 10.

The preprint [A] contains other examples.

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This work was partially supported by NSF DMS-9101237 and NSF DMS-9622843.

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