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OLGA TAUSSKY–TODD'S WORK IN CLASS FIELD THEORY

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To the memory of Olga Taussky-Todd,
a friend, a collegue and an inspiration

Olga Taussky's interest in class field theory began when she was a student with Ph. Furtwängler at the University of Vienna (she received her doctorate under his supervision in 1930). At this time, E. Artin, using his general reciprocity law, reformulated the "Principal Ideal Theorem" as a purely group-theoretic question regarding the triviality of the "Verlagerung" homomorphism from a finite group to its commutator subgroup. Since Furtwängler had been developing group theoretic methods in his own number theoretic investigations, Artin communicated this new idea to him. This was considered to be one of the central problems in the subject at the time, and so there was a good deal of excitement when Furtwängler eventually succeeded in proving the result using this new approach.

Olga Taussky had been looking for a thesis topic and Furtwängler suggested several questions related to the finer structure of the Principal Ideal Theorem. The optimistic atmosphere, and the prospect of contributing at the frontiers of class field theory, sparked in Olga a deep interest in this problem which lasted her entire career. She returned to the questions of "capitulation", a term coined by one of her co-authors Arnold Scholz, several times in her life, always with the sense that these were questions of deep arithmetic significance. I believe that it was the experience of her early contact with class field theory which dominated her research interests in number theory and which provided her a fertile source of inspiration for most of her career. I will describe some of her work in this area below. I would like to take this opportunity to thank the referee for many valuable suggestions.

For a number field F, (i.e., an extension of finite degree over the rational field $\mathbb{Q}$) its *Hilbert class field* $H = H(F)$ is the maximal Galois extension of F which is everywhere unramified and whose Galois group $\mathrm{Gal}(H/F)$ is *abelian*. It is a consequence of Artin's reciprocity law that $\mathrm{Gal}(H/F)$ is isomorphic to the ideal class group $C(F)$ of F. The Principal Ideal Theorem is the statement that every ideal of F becomes a principal ideal when considered as an ideal in $H(F)$. If $H_2 = H_2(F) = H(H(F))$ is the "second" Hilbert class field, then H_2/F is a Galois extension, and the Galois group $\mathrm{Gal}(H_2/H)$ is the commutator subgroup of $\mathrm{Gal}(H_2/F)$. Since the ideal class group of H is isomorphic to $\mathrm{Gal}(H_2/H)$, Artin re-interpreted the extension map on ideal class groups

$$e : C(F) \longrightarrow C(H)$$

as the (group-theoretic) transfer map or the Verlagerung map

$$\mathrm{Ver} : \mathrm{Gal}(H/F) = \mathrm{Gal}(H_2/F)/\mathrm{Gal}(H_2/H) \longrightarrow \mathrm{Gal}(H_2/H).$$

The Principal Ideal Theorem follows by proving that, for a group G whose commutator subgroup G' is abelian, the Verlagerung map from G to G' is the trivial map.

Furtwängler suggested to Olga that she consider the question of determining which subgroups of the ideal class group of F become principal in the various subfields of H. Hilbert had proved that such subgroups could not be trivial (Theorem 94 in his *Zahlbericht*). Since an ideal class $c \in C(F)$ of order n would still have order n in any extension of degree prime to n, it was natural that one restricted attention to the p–primary subgroups $C_p = C_p(F)$ of the ideal class group $C(F)$ and to extensions of p-power degree. In this context, Furtwängler himself had shown that for a field F with an elementary 2–group as ideal class group, $C_2(F) \simeq (\mathbb{Z}/2\mathbb{Z})^n$, there is a basis $\{c_1, \dots, c_n\}$ of C_2 such that each c_i becomes principal in an unramified quadratic extension of F.

He asked Olga to consider the generalization of this result for odd primes $p \geq 3$. Using group-theoretic methods of Schreier, she showed [**OTT1**] that the pattern of capitulation in cyclic unramified extensions of degree p behaved in a rather chaotic manner and depended both on p and $p - 2$ (see also [**OTT3**]). She reproved Furtwängler's theorem in the process.

Olga recounted that E. Artin had expressed interest in this work, but that when she saw him several years later, he asked whether she was still working "on these hopeless questions." Furtwängler also came to the belief that these problems were very difficult and (in Olga's words) withdrew from them turning to the geometry of numbers.

Olga Taussky continued working on this question for the case of imaginary quadratic fields F, with p-class groups $C_p(F) \simeq (\mathbb{Z}/p\mathbb{Z})^2$. In an important paper with Arnold Scholz [**OTT-Sc**], she examined in detail, for $p = 3$, the fields $\mathbb{Q}(\sqrt{-4027})$ and $\mathbb{Q}(\sqrt{-3299})$ (which have 3-class groups $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ respectively). By studying the capitulation pattern in each of the four cyclic unramified extensions of degree 3 of F, they were able to show that, in these cases, the 3-class towers of the respective fields had length exactly two.

Furtwängler had earlier posed the question of the finiteness of the class tower of a number field (i.e., whether the sequence $F \subseteq H_1 \subseteq H_2 \subseteq H_3 \cdots$ is in fact finite, where $H_n = H(H_{n-1})$). The importance of this problem lies in the following fact:
F has a finite class tower if and only if F can be embedded in a number field with class number one. A slight refinement of this statement is that F has a finite p-class tower if and only if F can be embedded in a number field with class number relatively prime to p.

It is fairly easy to see that if the p-class group of a field is cyclic then the p-class tower is finite and has length one. The paper of Scholz–Taussky provided the first non-trivial examples of finite 3-class towers. Previously, Olga had also contributed to this question by giving a simple proof of a result of Furtwängler, that a field with 2-class group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ has a 2-class tower which terminates at the second step.

E. Artin remarked that any "significant" improvement of the Minkowski bound would lead to a positive solution of the "class tower problem" (i.e., that the class tower is finite). It was felt at the time, that one might somehow be able to exploit the special nature of the lattices coming from the rings of integers of number fields to achieve such an improvement.

This work led Olga to formulate the "Group Tower Problem", a group theoretic interpretation of Furtwängler's question. Specifically she asked whether there exist infinite (pro-)p-groups G such that all terms of the derived series $G^{(i)}/G^{(i+1)}$ are finite, where $G^{(i+1)}$ is the closed commutator subgroup of $G^{(i)}$. Magnus produced examples of such group towers of arbitrary (finite) length. The first examples of infinite towers for 2–groups were found in the thesis of her student Charles Hobby [**Ho**], and then Serre [**Se**] constructed examples of infinite p-group towers.

In 1962 Shafarevich ([**Sh**]) announced a negative solution to the class tower problem, and in 1964 together with Golod ([**G-Sh**]) demonstrated the existence of (infinitely many) fields with infinite p-class towers. Later, Koch and Venkov ([**Ko-V**]) showed that imaginary quadratic fields with ideal class groups containing a subgroup isomorphic to $(\mathbb{Z}/p\mathbb{Z})^3$ for p odd, or $(\mathbb{Z}/4\mathbb{Z})^3$ for $p = 2$ have an infinite p–class tower. Farshid Hajir ([**Ha**]) has simplified and strengthened this result for $p = 2$, and Benjamin ([**Be**]) has produced examples of imaginary quadratic fields with 2-class group isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ which have infinite 2-class towers.

In this context, the examples of Scholz and Taussky take on a renewed interest since they provide non-trivial examples of fields with 2-generator 3-class groups and with *finite* 3-class towers.

The question (in the arithmetic setting) was recently taken up again from the point of view of p-adic analytic groups by Nigel Boston [**Bo**] and Farshid

Hajir ([**Ha2**]). Boston gives examples of number fields K for which the maximal unramified p-extension with p-adic analytic Galois group is finite. Hajir and Boston (independently) show that if a field F has a class group with sufficiently large p-rank, then the p-ranks of the class groups of the fields in their p–class tower tends to infinity. These results support the conjecture of Fontaine-Mazur ([**F-Ma**]), a consequence of which asserts that if L/F is an infinite everywhere unramified Galois pro-p-extension, then the Galois group $\mathrm{Gal}(L/F)$ cannot be embedded in a linear group $\mathbf{GL}_n(\mathbb{Z}_p)$ for any integer $n \geq 1$.

Olga Taussky–Todd returned ([**OTT2**, **OTT3**]) to the capitulation problem in the late 1960's and early 1970's. She again considered the case of imaginary quadratic fields F, with p-class groups $C_p(F) \simeq (\mathbb{Z}/p\mathbb{Z})^2$ but for arbitrary primes p. In [**OTT2**], she introduced a partition of the question into two subcases: If K/F is a cyclic unramified extension of degree p, then by class field theory, K corresponds to a subgroup $H \subseteq C_p$ of index p. She says K/F satisfies condition "A" if a nontrivial element of H capitulates in K, and K/F satisfies condition "B" otherwise. Re-examining her earlier work with Scholz in this light revealed that they had proved that for the case $p = 3$ not all the extensions K/F defined above could satisfy condition A. (A gap in their list of capitulation patterns was filled by Brink and Gold [**Br-Go**].) For $p \geq 5$ she proved that there was no group theoretic obstacle to having all the cyclic unramified extensions satisfy condition A. In [**Ki**], the conditions A and B were interpreted cohomologically and this was used in [**Ki2**] to give a complete description of the capitulation in the 2-class tower of an inaginary quadratic field with class group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

Many authors have taken up this question and there has been substantial progress made toward understanding the capitulation process in unramified extension fields. However, the overall picture remains elusive and as yet, no general theory governing the capitulation pattern has emerged. There was a major advance by Hiroshi Suzuki ([**Su**]) who proved that in an abelian everywhere unramified extension of number fields K/F, the order of the subgroup of the ideal class group of F which capitulates in K is at least equal to the degree $[K : F]$.

Resonances of the question of capitulation have also appeared in other contexts. For example, the vanishing of the Iwasawa invariants of "$\mathbb{Z}_p$–extensions" can be related to the capitulation in such extensions (see Greenberg [**Gr**]), and one can think of Greenberg's conjecture as analoguous to the Principal Ideal Theorem. Also Gras ([**Gra2**]) has recently turned the question in a new and interesting direction by asking to characterize the abelian extensions of a number field F which trivialize the class group $C(F)$ of F.

Other areas of her number theoretic research were influenced by her parallel interest in matrix theory. When working on a problem involving the factorization of circulant matrices, Olga Taussky–Todd was led to consider real, totally positive, units in cyclotomic fields. When she proved ([**OTT4**]) that the factorization condition turned out to be equivalent to the relevant unit being a norm from the full cyclotomic field to its maximal real subfield, she posed the question: For which real subfields of prime cyclotomic fields is it true that a totally positive unit is a norm? Her student, D. Davis, wrote his doctoral thesis on this subject, and following some suggestions of E. Dade, was able to give a criterion for this property to hold for prime cyclotomic fields. This work was related to questions of the parity of class numbers of number fields, and stimulated the interest of many number theorists (e.g., Garbanati [**Ga**], Gras [**Gra**] and others) .

Another example of this sort of cross fertilization occurred when she was studying the discriminant matrix. This led Olga to consider the trace form and in particular she asked her student D. Maurer to determine the Sylvester inertial invariant of the associated quadratic form, which he did in his Ph. D. thesis ([**M**]). It turned out that Serre had been interested in the Witt invariant of this form, and had independently calculated this in [**Se2**].

It is evident from her choice of research topics, that Olga Taussky–Todd's mathematical tastes and interests reflected a deep arithmetic intuition. She initiated several questions and topics which have proven to be a rich source of research problems for subsequent generations of mathematicians. Her insight, her enthusiasm and her vision were certainly inspiring for those of us who knew her, as is her mathematical legacy for all mathematicians.

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