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THE EXPONENT FOR THE MARKOFF-HURWITZ EQUATIONS

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In this paper, we study the Markoff-Hurwitz equations $x_0^2+\ldots+x_n^2 = ax_0\cdots x_n$. The variety V defined by this equation admits a group of automorphisms $\mathcal{A}\cong\mathbb{Z}/2*\cdots*\mathbb{Z}/2$ (an $n+1$ fold free product). For a solution P on this variety, we consider the number $N_P(t)$ of points Q in the $\mathcal{A}$-orbit of P with logarithmic height $h(Q)$ less than t. We show that if a is rational, and P is a non-trivial rational solution to this equation, then the limit

$$\lim_{t\to\infty}\frac{\log N_P(t)}{\log t}=\alpha(n)$$

exists and depends only on n. We give an effective algorithm for determining these exponents. For large n, this gives the asymptotic result

$$\frac{\log n}{\log 2}<\alpha(n)<\frac{\log n}{\log 2}+o(n^{-.58}).$$

Introduction.

Consider the Markoff-Hurwitz equations

$$x_0^2+\cdots+x_n^2-ax_0\cdots x_n=0 \tag{0.1}$$

first studied by Markoff (for $n = 2$, $a = 3$) [**M**], and Hurwitz [**H**]. Let V denote the affine variety defined by the zero-locus of Eq. (0.1) over the rationals. Note that Eq. (0.1) is a quadratic in each of its variables, so we can define the automorphism on V

$$\sigma_0 : (x_0, ..., x_n) \mapsto (ax_1\cdots x_n - x_0, x_1, ..., x_n)$$

which takes the root $T = x_0$ of the equation

$$T^2 - ax_1\cdots x_nT + x_1^2 + \cdots + x_n^2 = 0$$

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to the other root $T = x_0' = ax_1 \cdots x_n - x_0$. We can similarly define the automorphisms $\sigma_1, ..., \sigma_n$, and the group of automorphisms $\mathcal{A}$ generated by $\sigma_0, \sigma_1, ..., \sigma_n$.

We define a logarithmic height on V by setting

$$h(x_0, ..., x_n) = \max_i \{h(x_i)\},$$

where the height on the right hand side is the usual logarithmic height on $\mathbb{Q}$. We set

$$N_P(t) = \#\{Q \in \mathcal{A}(P) : h(Q) < t\}.$$

In this paper, we show:

Theorem 0.1. *If a is rational and P is a rational point on V not equal to $(0, ..., 0)$, then the limit*

$$\lim_{t\to\infty} \frac{\log N_P(t)}{\log t} = \alpha(n) \tag{0.2}$$

exists and depends only on the dimension n. Furthermore, $\alpha(2) = 2$,

$$\begin{aligned} 2.430 < \alpha(3) &< 2.477 \\ 2.730 < \alpha(4) &< 2.798 \\ 2.963 < \alpha(5) &< 3.048 \end{aligned}$$

and

$$\frac{\log n}{\log 2} < \alpha(n) < \frac{\log n}{\log 2} + o(n^{-.58}).$$

This improves on the results found in an earlier work [**Ba1**]: We have extended the result to orbits of rational points; the limit in Eq. (0.2) replaces a limit supremum; the bounds on $\alpha(n)$ for small n are an order of magnitude sharper; and the asymptotic bounds on $\alpha(n)$ improve on the previous result $\alpha(n) \gg\ll \log n$. We also give a (slowly) convergent algorithm for calculating $\alpha(n)$. Zagier [**Z**] credits Cohn with the result $\alpha(2) = 2$.

Cohn's idea is to compare the orbit $\mathcal{A}(P)$ with the Euclid tree. In [**Ba1**], for $n > 2$, we compared the orbits $\mathcal{A}(P)$ with the Euclid-like tree $\mathcal{E}_{\mathbf{r}}$ rooted at $\mathbf{r} = (r_1, ..., r_n)$, with $r_i > 0$ for all i, and generated by the branching operations

$$T_i : (a_1, ..., a_n) \mapsto (a_i, a_1 + a_i, ..., \widehat{a_i + a_i}, ..., a_n + a_i),$$

for $i = 1, ..., n$. The hat ˆ indicates that that component is omitted. We think of these trees as n-branch generalizations of the two branch Euclid tree, so named since moving down the tree (toward the root) is the Euclidean algorithm (see [**Z**]). We use the same notation h for a height function on $\mathcal{E}_{\mathbf{r}}$ defined by

$$h(a_1, ..., a_n) = a_1 + ... + a_n.$$

We set

$$\mathcal{E}_{\mathbf{r}}(t) = \{\mathbf{a} \in \mathcal{E}_{\mathbf{r}} : h(\mathbf{a}) < t\},$$

and define $n_{\mathbf{r}}(t)$ to be the cardinality of $\mathcal{E}_{\mathbf{r}}(t)$. We showed in [**Ba1**]:

Theorem 0.2 ([**Ba1**]). *Suppose a is an integer and P is a non-trivial integer solution to Eq. (0.1). Then there exist roots $\mathbf{r}$ and $\mathbf{s}$, with $r_i \geq s_i > 0$ for all i, which satisfy*

$$n_{\mathbf{r}}(t) \ll N_P(t) \ll n_{\mathbf{s}}(t).$$

This indicates that we should concentrate our efforts on the study of $\mathcal{E}_{\mathbf{r}}$, and do so in Sections 1 through 5.

We study the growth of $n_{\mathbf{r}}(t)$ by considering the function

$$f_{\mathbf{r}}(s) = \sum_{\mathbf{a} \in \mathcal{E}_{\mathbf{r}}} (h(\mathbf{a}))^{-s}.$$

There exists a real number $\alpha(\mathbf{r})$ such that $f_{\mathbf{r}}(s)$ converges for all s with $\mathrm{Re}(s) > \alpha(\mathbf{r})$, and diverges for all s with $\mathrm{Re}(s) < \alpha(\mathbf{r})$. This boundary of convergence is related to $n_{\mathbf{r}}(t)$ by the classical result

$$\limsup_{t\to\infty} \frac{\log n_{\mathbf{r}}(t)}{\log t} = \alpha(\mathbf{r}).$$

We therefore call $\alpha(\mathbf{r})$ the *exponent* of $n_{\mathbf{r}}(t)$. In [**Ba1**], we showed $\alpha(\mathbf{r}) = \alpha(n)$ depends only on n and found some bounds on $\alpha(n)$. Those bounds cannot be improved using the methods of [**Ba1**].

The main result of this paper is the development of a convergent algorithm to calculate $\alpha(n)$. In Section 1, we develop a crude algorithm which we show (in Theorem 2.3) converges. As a consequence, we find (in Theorem 3.2) that

$$\liminf_{t\to\infty} \frac{\log n_{\mathbf{r}}(t)}{\log t} = \alpha(n),$$

so the limit in Eq. (0.2) of Theorem 0.1 exists. In Section 4 we refine the algorithm and use it to find bounds on $\alpha(n)$ for small n (see Table 1). The rate of convergence of this algorithm is unfortunately prohibitively slow, so these bounds are good to only two significant digits. In Theorem 5.1, we derive the asymptotic bounds on $\alpha(n)$.

In the last section, we extend Theorem 0.2 to include rational solutions to Eq. (0.1) with $a \in \mathbb{Q}$. The purpose of this result is to demonstrate that the exponent $\alpha(n)$ is a characterization of the group $\mathcal{A}$ and not just of specific orbits. We remind the reader that the number of orbits of integer solutions is finite [**H**], so $\alpha(n)$ is also a characterization of the integer solutions to an equation of the form (0.1) with $a \in \mathbb{Z}$ and at least one non-trivial integer solution. The set of integer solutions though seems special. Silverman showed that the Markoff equation ($n = 2$) is birational to $\mathbb{P}^2$ minus six lines [**S**], so the number of rational points on the Markoff surface with height bounded by t is exponential in t, and hence the number of $\mathcal{A}$-orbits of rational solutions must be infinite. One can also use this bijection to show that if S contains the place at infinity and at least one local place, then the number of $\mathcal{A}$-orbits of S-integer solutions is infinite. Silverman also studied the Markoff surface over quadratic imaginary fields, and found the number of orbits of integer solutions there is often finite. However, over a real number field K not equal to $\mathbb{Q}$, the set of $\mathcal{A}$-orbits of non-trivial K-integer solutions is either empty or infinite. In this case, though, the full group of K-rational automorphisms is quite a bit larger than $\mathcal{A}$ [**Ba2**].

The techniques we use were inspired in part by Boyd's work on the Apollonian packing problem [**Bo1**, **Bo2**, **Bo3**]. I would also like to thank Cam Stewart for his helpful advice, and Mike Mossinghoff for his help implementing this algorithm.

1. The Algorithm.

Let us begin with a few more definitions: Let

$$f_{\mathbf{r}}(s,y) = \sum_{\mathbf{a}\in\mathcal{E}_{\mathbf{r}}(y)} (h(\mathbf{a}))^{-s};$$

let $\partial\mathcal{E}_{\mathbf{r}}(y)$ be the boundary of $\mathcal{E}_{\mathbf{r}}(y)$ — those nodes in $\mathcal{E}_{\mathbf{r}}$ that are not in $\mathcal{E}_{\mathbf{r}}(y)$ but are attached to $\mathcal{E}_{\mathbf{r}}(y)$ by one branch; for $a > -1$, let

$$\zeta(s,a) = \sum_{n=1}^{\infty} (n+a)^{-s}$$

be the Hurwitz zeta function; and for the specific choice $\mathbf{r} = (1, 1, 2, ..., 2)$, define

$$L(s,y) = (n-1) \sum_{\mathbf{a}\in\partial\mathcal{E}_{\mathbf{r}}(y)} \zeta\left(s, \frac{a_n}{a_1}\right) a_1^{-s},$$

$$U(s,y) = (n-1) \sum_{\mathbf{a}\in\partial\mathcal{E}_{\mathbf{r}}(y)} \zeta\left(s, \frac{a_2}{a_1} - 1\right) a_1^{-s}.$$

In this section, we prove:

Theorem 1.1. *For each $n \geq 2$, the exponent $\alpha = \alpha(n)$ satisfies*

$$s_1 \leq \alpha \leq s_2$$

where s_1 and s_2 depend on n and y and are the unique solutions in $(1, \infty)$ to $L(s_1, y) = 1$ and $U(s_2, y) = 1$.

The following lemma is an amalgam of related results which will be required at various points in this paper. It is perhaps a bit premature to include (iii), since I have not yet defined $f_{\mathbf{a}}(s, V)$. I include it here for completeness.

Lemma 1.2. *If $c > 0$, then*

(i) $$f_{c\mathbf{a}}(s) = c^{-s} f_{\mathbf{a}}(s),$$
(ii) $$f_{c\mathbf{a}}(s, y) = c^{-s} f_{\mathbf{a}}(s, y/c),$$
(iii) $$f_{c\mathbf{a}}(s, V) = c^{-s} f_{\mathbf{a}}(s, V),$$
(iv) $$n_{c\mathbf{a}}(x) = n_{\mathbf{a}}(x/c);$$

and if $a_i \geq b_i$ for all i, then

(i) $$f_{\mathbf{a}}(s) \leq f_{\mathbf{b}}(s),$$
(ii) $$f_{\mathbf{a}}(s, y) \leq f_{\mathbf{b}}(s, y),$$
(iii) $$f_{\mathbf{a}}(s, V) \leq f_{\mathbf{b}}(s, V),$$
(iv) $$n_{\mathbf{a}}(x) \leq n_{\mathbf{b}}(x).$$

To prove these, just make node by node comparisons for each tree, and check the number of nodes in each tree. Lemma 1.2 also demonstrates why we deal with the function f: When we dilate the root by a constant c, we can pull it out of f.

Theorem 1.1 embodies two results — an upper bound and a lower bound. Though the two results are derived in much the same way, the upper bound involves a few more technicalities, so let us begin with the lower bound. The philosophy that guides our argument is this: If $f_{\mathbf{r}}(s)$ converges (i.e. $s > \alpha(n)$,) then $f_{\mathbf{r}}(s, y)$ is a good approximation to $f_{\mathbf{r}}(s)$ provided y is large enough. So we write

$$f_{\mathbf{r}}(s) = f_{\mathbf{r}}(s, y) + \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y)} f_{\mathbf{a}}(s).$$

We wish to use Lemma 1.2 to estimate $f_{\mathbf{a}}(s)$. First, we note that Lemma 1.2 has the following corollary:

Corollary 1.3. *If $c_1 r_i \leq a_i \leq c_2 r_i$ for all i and some $0 < c_1 < c_2$, then*

$$c_2^{-s} f_{\mathbf{r}}(s) \leq f_{\mathbf{a}}(s) \leq c_1^{-s} f_{\mathbf{r}}(s).$$

Though this will always give us an estimate on $f_{\mathbf{a}}(s)$, the estimate may not be too good, since the constants c_1 and c_2 might not be close to each other. However, we note that

$$T_1^k(\mathbf{a}) = (a_1, ka_1 + a_2, ..., ka_1 + a_n),$$

from which the jth branch is

$$T_jT_1^k(\mathbf{a}) = (ka_1 + a_j, (k+1)a_1 + a_j, 2ka_1 + a_2 + a_j, ..., 2ka_1 + a_n + a_j) \tag{1.1}$$

for $j = 2, 3, ..., n$. For large k, this branch looks a lot like $ka_1(1, 1, 2, ..., 2)$, which suggests we set (and do so from now on) $\mathbf{r} = (1, 1, 2, ..., 2)$. Then we get:

Lemma 1.4. *Suppose* $1 \leq a_1 \leq ... \leq a_n$*, and* $s > \alpha(n)$*. Then*

$$\begin{aligned} f_{\mathbf{a}}(s) &\geq \sum_{k=0}^{\infty}(n-1)((k+1)a_1 + a_n)^{-s} f_{\mathbf{r}}(s) \\ &\geq (n-1)\zeta\left(s, a_n/a_1\right) a_1^{-s} f_{\mathbf{r}}(s). \end{aligned}$$

Thus

$$\begin{aligned} f_{\mathbf{r}}(s) &\geq f_{\mathbf{r}}(s, y) + L(s, y) f_{\mathbf{r}}(s) \\ f_{\mathbf{r}}(s)(1 - L(s, y)) &\geq f_{\mathbf{r}}(s, y), \end{aligned}$$

so in particular, $L(s, y) \leq 1$. Since $L(s, y)$ is a decreasing function in s for $s > 1$, we get

$$\alpha \geq s_1.$$

To get an upper bound, we exploit the divergence of $f_{\mathbf{r}}(s)$ for $s < \alpha(n)$, which we express as

$$\lim_{y \to \infty} f_{\mathbf{r}}(s, y) = \infty.$$

So we choose y_1 much larger than y_2, write

$$f_{\mathbf{r}}(s, y_1) = f_{\mathbf{r}}(s, y_2) + \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y_2)} f_{\mathbf{a}}(s, y_1)$$

and use Lemma 1.2 to conclude:

Lemma 1.5. *If* $1 \leq a_1 \leq ... \leq a_n$*, then*

$$\begin{aligned} f_{\mathbf{a}}(s, y) &\leq \sum_{k=0}^{\infty} \frac{n-1}{(ka_1 + a_2)^s} f_{\mathbf{r}}\left(s, \frac{y}{ka_1 + a_2}\right) + \sum_{k=0}^{\infty}(k(n-1)a_1 + h(\mathbf{a}))^{-s} \\ &\leq (n-1)\zeta\left(s, a_2/a_1 - 1\right) a_1^{-s} f_{\mathbf{r}}(s, y) + O(1). \end{aligned}$$

So we conclude, for a fixed y_2,

$$f_{\mathbf{r}}(s, y_1) \leq f_{\mathbf{r}}(s, y_2) + U(s, y_2) f_{\mathbf{r}}(s, y_1) + O(1)$$

$$f_{\mathbf{r}}(s, y_1)(1 - U(s, y_2)) \leq f_{\mathbf{r}}(s, y_2) + O(1), \tag{1.2}$$

and if $s < \alpha(n)$, then we can choose y_1 so that $f_{\mathbf{r}}(s, y_1)$ is arbitrarily large. Thus, we must have $U(s, y_2) > 1$, so

$$\alpha \leq s_2.$$

2. Convergence.

To show $s_2 - s_1$ converges to zero as y goes to infinity, we show that $U(s, y)$ is bounded above by a constant multiple of $L(s, y)$, and that the slope of $L(s, y)$ (with respect to s) gets steeper as y grows.

Lemma 2.1. *If* $\mathbf{a} \in \mathcal{E}_{\mathbf{r}}$, *then*

$$a_n \leq 2a_2.$$

Proof. We use induction to show

$$a_1 + \cdots + a_n - (n-1)a_j \geq 0$$

for $j = 1, ..., n$. Then

$$a_1 + \cdots + a_n - (n-1)a_n + \frac{1}{2} \sum_{j=3}^{n-1} (a_1 + \cdots + a_n - (n-1)a_j) \geq 0,$$

so

$$(a_1 + a_2) \left(\frac{n-1}{2} \right) + (a_3 + \cdots + a_{n-1})(0) - a_n \left(\frac{n-1}{2} \right) \geq 0.$$

□

Also, for $a \geq 1$, we have

$$\zeta(s, 2a) \geq \zeta(s, 3a-1) = \sum_{k=0}^{\infty} 3^{-s} \left(\frac{k}{3} + a \right)^{-s} \geq 3^{-s} \zeta(s, a-1).$$

In [**Ba1**] we found the crude bounds on $\alpha(n)$ of 2 and n, so let us assume that $s \in [2, n]$. Then, using Lemma 2.1 and the above, we have

$$\zeta\left(s, \frac{a_n}{a_1}\right) \geq 3^{-n} \zeta\left(s, \frac{a_2}{a_1} - 1\right),$$

so

$$U(s,y) \leq 3^n L(s,y). \tag{2.1}$$

Lemma 2.2.
$$\frac{\partial}{\partial s} L(s,y) < -\log\left(\frac{y}{2n}\right) L(s,y).$$

Proof.

$$\begin{aligned}\frac{\partial}{\partial s} L(s,y) &= \sum_{\mathbf{a}\in\partial\mathcal{E}_{\mathbf{r}}(y)} \frac{\partial}{\partial s}\left(\zeta\left(s, \frac{a_2}{a_1} - 1\right) a_1^{-s}\right) \\ &< \sum_{\mathbf{a}\in\partial\mathcal{E}_{\mathbf{r}}(y)} \left(-\log\left(\frac{a_2}{a_1}\right) - \log(a_1)\right) \zeta\left(s, \frac{a_2}{a_1} - 1\right) a_1^{-s} \\ &< -\log\left(\frac{y}{2n}\right) L(s,y).\end{aligned}$$

In the last step, we used

$$y < a_1 + ... + a_n < na_n < n2a_2.$$

□

Theorem 2.3. *If $y > 2n$, then*

$$s_2 - s_1 < \frac{1 - 3^{-n}}{\log y - \log(2n)}.$$

Proof. From Eq. (2.1),

$$U(s_2, y) = 1 < 3^n L(s_2, y),$$

and by the mean value theorem, there is an s_0 in (s_1, s_2) so that

$$\frac{L(s_2,y) - L(s_1,y)}{s_2 - s_1} = \frac{\partial}{\partial s} L(s_0, y).$$

Thus
$$\frac{3^{-n} - 1}{s_2 - s_1} < \frac{\partial}{\partial s} L(s_0, y) < -\log\left(\frac{y}{2n}\right) L(s_1, y).$$

□

Hence, our algorithm converges. Note that the rate of convergence guaranteed by this argument is only inversely proportional to $\log y$. Unfortunately, experiments indicate the actual rate of convergence is as poor. This is why the approximations found for $\alpha(n)$ are good only to two significant digits.

As a corollary of Theorems 1.1 and 2.3, we get the following result which we use in the next section:

Corollary 2.4. *Recall s_1 depends on y and n. For fixed n,*

$$\lim_{y\to\infty} s_1 = \alpha(n).$$

3. A lower bound.

In this section we show

$$\liminf_{t\to\infty} \frac{\log n_{\mathbf{r}}(t)}{\log t} \geq \alpha(n).$$

Thus, $\alpha(n)$ is in fact a limit.

Let us abstract our setting a bit. Let A be some arbitrary infinite set, and let

$$h : A \to \mathbb{Z}^+$$

be a function on A such that the subsets

$$A(t) = \{a \in A : h(a) < t\}$$

are finite for any t. Let $N(t)$ be the cardinality of $A(t)$. Define the Dirichlet series

$$f(s) = \sum_{a\in A} (h(a))^{-s}$$

and suppose $f(s)$ converges for some s. Then there exists an $\alpha \in \mathbb{R}$ so that $f(s)$ converges for all $s > \alpha$ and diverges for all $s < \alpha$.

As mentioned in the introduction, the exponent α of A is related to the growth of $N(t)$ by the classical result:

$$\limsup_{t\to\infty} \frac{\log(N(t))}{\log t} = \alpha.$$

This is the type of setting encountered by Boyd and others in the study of the Apollonian packing [**Bo1**, **Bo2**, **Bo3**]. In that case, though, Boyd [**Bo3**] pointed out that the supremum condition could be dropped. His argument is marvelously simple and applicable to our situation.

Theorem 3.1 ([**Bo3**]). *Suppose $N(t) \geq 1$ for all $t > t_0$, and suppose we can write*

$$N(t) \geq \sum_{m=1}^{\infty} N(t/d_m)$$

for a set of ordered integers $2 \leq d_1 \leq d_2 \leq \ldots$. Suppose also that

$$g(s) = \sum_{m=1}^{\infty} d_m^{-s}$$

converges in the interval (r, ∞) and $g(s_1) = 1$ for some $s_1 \in (r, \infty)$. Then for any $\beta \in (r, s_1)$, there exists a constant $C > 0$ such that

$$N(t) \geq Ct^{\beta}$$

for all $t \geq t_0$.

Proof. Note that $g(s)$ is a decreasing function on (r, ∞), so $g(\beta) > 1$. Hence there exists an M so that

$$\sum_{m=1}^{M} d_m^{-\beta} > 1.$$

Set $t_1 = d_M t_0$, and

$$C = \inf_{t_0 \leq t \leq t_1} N(t) t^{-\beta}.$$

Then by construction, $N(t) \geq Ct^{\beta}$ for all $t \in [t_0, t_1]$.

Let $t_m = d_1^{m-1} t_1$ for $m = 2, 3, \ldots$, and suppose we have shown $N(t) \geq Ct^{\beta}$ for all $t \in [t_0, t_{m-1}]$. Then for $k = 1, \ldots, M$ and $t \in [t_{m-1}, t_m]$, we know

$$t_0 = t_1/d_M \leq t/d_k \leq t_m/d_1 = t_{m-1},$$

so

$$N(t/d_k) \geq Ct^{\beta}/d_k^{\beta}.$$

Thus

$$N(t) \geq \sum_{m=1}^{M} C(t/d_m)^{\beta} \geq Ct^{\beta}$$

for all $t \in [t_{m-1}, t_m]$. Hence, by induction, $N(t) \geq Ct^{\beta}$ for all $t \geq t_0$. □

To apply this result to our situation, we note that for $t > y$,

$$n_{\mathbf{r}}(t) = n_{\mathbf{r}}(y) + \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y)} n_{\mathbf{a}}(t),$$

and estimate $n_{\mathbf{a}}(t)$ by using Eq. (1.1) and (iv) of Lemma 1.2 to get

$$n_{\mathbf{r}}(t) \geq \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y)} \sum_{k=0}^{\infty} (n-1) n_{\mathbf{r}} \left(\frac{t}{(k+1)a_1 + a_n} \right).$$

We note that $(n+1)a_1 + a_n \in \mathbb{Z}$ and is ≥ 2, so we order these for all $\mathbf{a} \in \partial\mathcal{E}_{\mathbf{r}}(y)$, $k = 0, 1, ...$, and each with multiplicity $n-1$. We relabel the set $d_1, d_2, ...$ so that

$$n_{\mathbf{r}}(t) \geq \sum_{m=1}^{\infty} n_{\mathbf{r}}(t/d_m),$$

as is desired in Theorem 3.1. The function $g(s)$ then depends on y and is precisely $L(s, y)$. We are now ready to prove:

Theorem 3.2.

$$\liminf_{t\to\infty} \frac{\log n_{\mathbf{r}}(t)}{\log t} \geq \alpha(n).$$

Proof. We first note that $\zeta(s, a)$ converges in $(1, \infty)$ for all $a > 0$, so $L(s, y)$ converges in $(1, \infty)$ for any y. Suppose $\beta \in (1, \alpha(n))$. Then by Corollary 2.4, there exists a y so that $s_1(y) > \beta$. Hence, by Theorem 3.1 there exists a constant $C = C(\beta)$ so that

$$n_{\mathbf{r}}(t) \geq Ct^{\beta}$$

for all $t > 2n$ (since $t_0 = 2n$). Taking logarithms, dividing by $\log t$, and letting t go to infinity, we get

$$\liminf_{t\to\infty} \frac{\log n_{\mathbf{r}}(t)}{\log t} \geq \beta,$$

for all $\beta \in (1, \alpha(n))$, which gives our result. □

4. Refinements.

We know the difference of the bounds on α converges to zero like a constant over $\log y$. Though we cannot as yet improve the $(\log y)^{-1}$, it is possible to significantly improve the constant. We do this by using Boyd's observation that $f_{\mathbf{a}}(s)$ is convex with respect to $\mathbf{a}$. Unfortunately, the finite sum $f_{\mathbf{a}}(s, y)$ might not be convex, but we can define a finite sum that is. Let W be the monoid generated by T_1, $T_2, ..., T_n$. Let V be any subset of W and define

$$f_{\mathbf{a}}(s, V) = \sum_{w\in V} (h(w(\mathbf{a})))^{-s}.$$

So $f_{\mathbf{a}}(s) = f_{\mathbf{a}}(s, W)$. This function is computationally difficult to deal with, since it does not converge very fast, but we only use it as a theoretical tool. More precisely, we will substitute $f_{\mathbf{r}}(s, y_1)$ with $f_{\mathbf{r}}(s, V)$ at Eq. (1.2), so all we need to observe is that we can pick V finite but still have $f_{\mathbf{r}}(s,V)$ as large as we want for fixed $s<\alpha(n)$.

Theorem 4.1 (Boyd [**Bo2**]). *Suppose* $0<u<1$*, and for* V *infinite, let* $s>\alpha(n)$*, but in any case, let* $s>0$*. Then*

$$f_{u\mathbf{a}+(1-u)\mathbf{b}}(s,V)\leq uf_{\mathbf{a}}(s,V)+(1-u)f_{\mathbf{b}}(s,V).$$

Proof. If $s>0$, then the function x^{-s} is convex in $(0,\infty)$. That is to say, if $x,\ y>0$, then

$$(ux+(1-u)y)^{-s}\leq ux^{-s}+(1-u)y^{-s}.$$

Note that

$$h(w(\mathbf{x}))=\mathbf{1}^T w\mathbf{x},$$

where the $\mathbf{1}^T$ indicates the transpose of $\mathbf{1}=(1,...,1)$. Hence

$$\begin{aligned} h(w(u\mathbf{a}+(1-u)\mathbf{b})) &= \mathbf{1}^T w(u\mathbf{a}+(1-u)\mathbf{b}) \\ &= uh(w(\mathbf{a}))+(1-u)h(w(\mathbf{b})), \end{aligned}$$

and

$$\begin{aligned} f_{u\mathbf{a}+(1-u)\mathbf{b}}(s,V) &= \sum_{w\in V}(uh(w(\mathbf{a}))+(1-u)h(w(\mathbf{b})))^{-s} \\ &\leq uf_{\mathbf{a}}(s,V)+(1-u)f_{\mathbf{b}}(s,V). \end{aligned}$$

In the last step, we used the fact that $h(w(\mathbf{x}))$ is always positive for every non-zero $\mathbf{x}$ with non-negative entries. □

Note that this observation is true for any tree whose branching operations are linear and whose nodes never have negative entries.

This theorem is useful because $f_{\mathbf{a}}(s)$ is invariant under permutations of the components of $\mathbf{a}$. This can be proved using induction on the number of generators in an element w in W, and depends very much on the character of W. The function $f_{\mathbf{a}}(s,V)$ is not always invariant under these permutations, but is for certain classes of subsets.

Let the set of permutations S_n on $\{1,...,n\}$ act on W by

$$\tau(T_{i_1}\cdots T_{i_n})=T_{\tau i_1}\cdots T_{\tau i_n}.$$

This is a well defined action. We say a subset V of W is symmetric if for any $w\in V$, $\tau w\in V$ for all $\tau\in S_n$. If V is symmetric, then the function $f_{\mathbf{a}}(s,V)$ is invariant under permutations of the variables in $\mathbf{a}$.

The following lemmas use this observation to improve the upper and lower bounds.

Lemma 4.2. *If $\mathbf{a} = (a_1, ..., a_n) \in \mathcal{E}_{\mathbf{r}}$ and V is symmetric, then*

$$f_{\mathbf{a}}(s, V) \leq a_1^{-s} f_{\mathbf{r}}(s, V).$$

Proof. We set

$$\mathbf{b} = \sum_{j=2}^{n} \left(\frac{a_1 + ... + a_n}{n-1} - a_j \right) \mathbf{r}_j$$

where $\mathbf{r}_j = (1, 2, ..., 2, 1, 2, ..., 2)$ is the vector with every component equal to two except for the first and jth, which are equal to one. Thus $b_1 = a_1$ and for $j = 2, ..., n$,

$$b_j = 2a_1 - \frac{a_1 + ... + a_n}{n-1} + a_j.$$

It is clear, using induction, that

$$1 \leq b_j \leq a_j$$

for all j, so from Lemma 1.2,

$$f_{\mathbf{a}}(s, V) \leq f_{\mathbf{b}}(s, V).$$

Since V is symmetric, $f_{\mathbf{r}_j}(s, V) = f_{\mathbf{r}}(s, V)$, so using Theorem 4.1 we conclude

$$\begin{aligned} f_{\mathbf{a}}(s, V) &\leq \left(\sum_{j=2}^{n} \left(\frac{a_1 + ... + a_n}{n-1} - a_j \right) \right)^{-s} f_{\mathbf{r}}(s, V) \\ &\leq a_1^{-s} f_{\mathbf{r}}(s, V). \end{aligned}$$

∎

Lemma 4.3. *If $\mathbf{a} \in \mathcal{E}_{\mathbf{r}}$, then*

$$f_{\mathbf{a}}(s) \geq \left(\frac{a_1 + a_2}{2} \right)^{-s} f_{\mathbf{r}}(s).$$

Proof. Using Theorem 4.1 we know

$$\begin{aligned} f_{\mathbf{a}}(s) &= \frac{1}{2} f_{\mathbf{a}}(s) + \frac{1}{2} f_{(a_2, a_1, a_3, ..., a_n)}(s) \\ &\geq f_{\left(\frac{a_1+a_2}{2}, \frac{a_1+a_2}{2}, a_3, ..., a_n\right)}(s) \\ &\geq \left(\frac{a_1 + a_2}{2} \right)^{-s} f_{\mathbf{r}}(s), \end{aligned}$$

where in the last step, we used

$$a_1 + a_2 - a_j \geq 0,$$

which is easy enough to prove using induction. □

The proof of Lemma 4.2, though straight forward, appears contrived. The result and proof were inspired by looking at the dual tree (see [**Ba1**] or Section 6) where the choice

$$\mathbf{B} = \sum_{j=1}^{k-1} A_j \mathbf{R}_j$$

is more obvious. Here, $\mathbf{R}_j$ is the vector with every component equal to zero except for the jth and kth, which are equal to one. We chose to keep the focus of this paper on just one tree, rather than switch between the two. If we had chosen to instead focus on the dual tree, then the proof of the lower bound would have looked contrived.

We now use Lemmas 4.2 and 4.3 to get the better bounds:

$$U(s,y) = \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y)} \left(a_1^{-s} \sum_{j=2}^{n} \zeta\left(s, \frac{a_j}{a_1} - 1\right)\right),$$

$$L(s,y) = \sum_{\mathbf{a} \in \partial \mathcal{E}_{\mathbf{r}}(y)} \left(a_1^{-s} \sum_{j=2}^{n} \zeta\left(s, \frac{a_j}{a_1} - \frac{1}{2}\right)\right).$$

Using these functions, we present in Table 1 the solutions s_1 and s_2 for various values of n and y.

n	s_1	s_1^*	s_2^*	s_2	y	n	s_1^*	s_2^*	y
3	2.425	2.430	2.477	2.494	400	7	3.317	3.435	400
4	2.724	2.730	2.798	2.813	400	8	3.453	3.595	300
5	2.957	2.963	3.048	3.063	400	9	3.578	3.734	300
6	3.148	3.155	3.256	3.270	400	10	3.692	3.858	300

Table 1.

The bounds s_1^* and s_2^* are roots of L^* and U^*, which are obtained by making one more refinement: For some values of $\mathbf{b} \in \mathcal{E}_{\mathbf{r}}(y)$, the bound

$$f_{\mathbf{b}}(s) > (b_1)^{-s} \sum_{j=2}^{n} \zeta\left(s, \frac{b_j}{b_1} - \frac{1}{2}\right) f_{\mathbf{r}}(s)$$

is sharper than the bound

$$f_{\mathbf{b}}(s) > \sum_{\mathbf{a}\in\mathcal{E}_{\mathbf{b}}(y)} \left((a_1)^{-s} \sum_{j=1}^{n} \zeta\left(\frac{a_j}{a_1} - \frac{1}{2}\right)\right) f_{\mathbf{r}}(s).$$

We construct L^* (and similarly U^*) by optimizing these choices.

This table raises an intriguing question: Does $\alpha(5) = 3$?

5. The asymptotic growth of the exponent.

In this section, we show:

Theorem 5.1.
$$\frac{\log n}{\log 2} < \alpha(n) < \frac{\log n}{\log 2} + o(n^{-.58}).$$

Proof. For the lower bound, let us set $\mathbf{r} = (1, 2, ..., 2)$. Then, for any $s > \alpha(n)$, we get

$$\begin{aligned} f_{\mathbf{r}}(s) &> f_{(1,3,\ldots,3)}(s) + (n-1)f_{(2,3,4,\ldots,4)}(s) \\ &> n2^{-s} f_{\mathbf{r}}(s). \end{aligned}$$

Thus,

$$\begin{aligned} 1 &> n2^{-s} \\ 0 &> \log n - s\log 2. \end{aligned}$$

Since this is true for any $s > \alpha(n)$, we get our lower bound for $\alpha(n)$.

For the upper bound, we again set $\mathbf{r} = (1, 1, 2, ..., 2)$, and note

$$f_{\mathbf{r}}(s, V) \leq 2f_{(1,2,3,\ldots,3)}(s, V) + (n-2)f_{(2,3,3,4,\ldots,4)}(s, V),$$

so using Lemma 4.2 and Eq. (1.1), we get

$$\begin{aligned} &f_{\mathbf{r}}(s, V) \\ &\leq 2(\zeta(s, 1)f_{\mathbf{r}}(s, V) + (n-2)\zeta(s, 2)f_{\mathbf{r}}(s, V)) + (n-2)2^{-s}f_{\mathbf{r}}(s, V) + O(1) \\ &\leq 2f_{\mathbf{r}}(s, V)\left(2^{-s} + (n-1)3^{-s} + (n-1)\int_3^\infty \frac{dx}{x^s} + (n-2)2^{-s-1}\right) + O(1). \end{aligned}$$

So, if $s < \alpha(n)$, then we can choose V so that $f_{\mathbf{r}}(s, V)$ is arbitrarily large, and divide through by it to get

$$\begin{aligned} 1 &\leq n2^{-s}\left(1 + 2\left(\frac{2}{3}\right)^s + \frac{2\cdot 3}{s-1}\left(\frac{2}{3}\right)^s\right) \\ 0 &\leq \log n - s\log 2 + O\left((2/3)^s\right). \end{aligned}$$

Since $\alpha(n) > \log n/\log 2$, we can find an $s \in (\log n/\log 2, \alpha(n))$, and for such an s, we get

$$s \leq \frac{\log n}{\log 2} + O\left(n^{\frac{\log 2 - \log 3}{\log 2}}\right).$$

Since this is true for all s in $(\log n/\log 2, \alpha(n))$, this gives our upper bound for $\alpha(n)$. □

6. Orbits of rational points.

In this section, we prove a generalization of Theorem 0.2 which includes non-trivial rational solutions to Eq. (0.1):

Theorem 6.1. *Suppose $a \in \mathbb{Q}$ and P is a non-trivial rational solution to the Hurwitz equation Eq. (0.1). Then there exist roots $\mathbf{r}$ and $\mathbf{s}$, with $r_i > 0$ and $s_i > 0$ for all i, such that*

$$n_{\mathbf{r}}(t) \ll N_P(t) \ll n_{\mathbf{s}}(t).$$

For integer solutions to Eq. (0.1) with a an integer, the height of a point is determined entirely by the valuation at infinity. For a rational point, other valuations contribute. However, if S is a set of valuations including the valuation at infinity, and a is an S-integer, then the set of S-integer solutions to Eq. (0.1) is closed under the action of $\mathcal{A}$. Thus, for a particular solution P, we can choose a finite set S such that P is an S-integer solution, a is an S-integer, and S contains the place at infinity. Then, the $\mathcal{A}$-orbit of P is a set of S-integer solutions, and the heights of points in $\mathcal{A}(P)$ are determined by the finite set of valuations in S.

Let us begin with a few definitions and lemmas.

For any $\varphi \in \mathcal{A}$, let us always write

$$\varphi = \sigma_{i_r} \cdots \sigma_{i_1}$$

with r minimal. That is, $i_k \neq i_{k-1}$ for all $k = 2, ..., r$.

Let

$$\mathcal{A}_k = \{\varphi \in \mathcal{A} : \varphi = \sigma_{i_r} \cdots \sigma_{i_1}, i_1 \neq k, r = 0, 1, 2, ...\}.$$

Lemma 6.2. *Suppose $P = (p_0, ..., p_n)$ is a rational solution to Eq. (0.1) and let v be a valuation in S. If $v = v_\infty$ is the place at infinity, then there exists a k so that $v_\infty(p_k) \geq v_\infty(p_i)$ for all $i \neq k$. If v is a local valuation, then let us also suppose there exists a k so that $v(p_k) > v(p_i)$ for all $i \neq k$. Let*

$$p_i' = (ap_0 \cdots p_n/p_i) - p_i.$$

Then

$$v(p_k) = v(p_0) + ... + v(p_n) - v(p_k) + O(1)$$
$$v(p_i') = v(p_0) + ... + v(p_n) - v(p_i) + O(1) \tag{6.1}$$

and

$$v(p_i') > v(p_k).$$

Proof. For v a local valuation,

$$v(p_0^2 + ... + p_n^2) = v(p_k^2),$$

since $v(p_k) > v(p_i)$ for all $i \neq k$. Thus,

$$2v(p_k) = v(p_0) + ... + v(p_n) + v(a).$$

Also, since $v(p_k) > v(p_i)$ for all $i \neq k$,

$$v(p_i') = v(ap_0 \cdots p_n/p_i) = v(p_0) + ... + v(p_n) - v(p_i) + v(a).$$

Subtracting $v(p_k)$ from $v(p_i')$, we get

$$v(p_i') - v(p_k) = v(p_k) - v(p_i) > 0.$$

For the place at infinity, we note

$$p_k^2 \leq ap_0 \cdots p_n \leq (n+1)p_k^2$$

and

$$p_k^2 \leq p_0^2 + ... + p_n^2 = p_i p_i' \leq (n+1)p_k^2,$$

from which Eqs. (6.1) follow, and the last inequality trivially follows. □

Thus, if there exists a k such that $v(p_i) < v(p_k)$ for all $i \neq k$, then $v(p_i') > v(p_i)$ for all $i \neq k$, and we can sensibly define the *forward tree* from P with respect to v to be

$$\mathcal{A}_v^+ = \{\varphi P : \varphi \in \mathcal{A}_k\}.$$

Also, as in [**Ba1**], we can compare $\mathcal{A}_v^+(P)$ with the Euclid-like tree $\mathcal{E}^*$ defined by the branching operations

$$E_i : (A_1, ..., A_n) \mapsto (A_1, ..., \widehat{A}_i, ..., A_n, A_1 + ... + A_n).$$

We send this to the dual tree $\mathcal{E}$ defined in the introduction via the linear map Θ where

$$\Theta_i(\mathbf{A}) = A_1 + ... + A_n - A_i.$$

To bound $N_P(t)$ from above, we use the following:

Lemma 6.3. *Suppose $v(p_k) \geq v(p_i)$ for all $i \neq k$ with strict inequality if v is a local valuation. Let $u > 0$ bound the function implied by the $O(1)$ in Eqs. (6.1). Let $\mathbf{a} = (v(p_0), ..., \widehat{v(p_k)}, ..., v(p_n))$, and suppose $a_i > u$ for all i. Let $\mathbf{r} = \Theta(\mathbf{a} - (u, ..., u))$, so $r_i > 0$ for all i. Then*

$$N^+_{P,v}(t) = \#\{Q \in \mathcal{A}^+_v(P) : h(Q) < t\} \leq n_{\mathbf{r}}(t).$$

Proof. We note that

$$h(Q) \geq \max_i \{v(q_i)\},$$

so

$$N^+_{P,v}(t) \leq \#\{Q \in \mathcal{A}^+_v(P) : \max_i \{v(q_i)\} < t\}.$$

Thus we need only consider the valuation v. We also note that

$$n_{\mathbf{r}}(t) = \#\{\mathbf{s} \in \mathcal{E}^*_{\Theta^{-1}\mathbf{r}} : s_1 + ... + s_n < t\},$$

and refer the reader to [**Ba1**] for a proof. Thus, we can consider $\mathcal{E}^*$ instead of $\mathcal{E}$. We simultaneously define inductively a correspondence and comparison between $\mathcal{A}^+_v(P)$ and $\mathcal{E}^*_{\Theta^{-1}\mathbf{r}}$ in the following way: We let P correspond to $\mathbf{a} - (u, ..., u) = \Theta^{-1}\mathbf{r}$. We let an arbitrary Q correspond to $\mathbf{s}$ and let τ be a reordering of $\{0, ..., n\}$ such that

$$v(q_{\tau(0)}) \leq \cdots \leq v(q_{\tau(n-1)}) < v(q_{\tau(n)}),$$

and

$$v(q_{\tau(i-1)}) \geq s_i + u.$$

This last is true for P and $\mathbf{a} - (u, ..., u)$ by construction. Now, we let $\sigma_{\tau(i-1)}Q$ correspond to $E_i\mathbf{s}$ for $i = 1, ..., n$. Then, we need only check that

$$v(q_{\tau(n)}) \geq s_1 + ... + s_n + u.$$

But, from Lemma 6.2,

$$\begin{aligned} v(q_{\tau(n)}) &\geq v(q_{\tau(0)}) + ... + v(q_{\tau(n-1)}) - u \\ &\geq (s_1 + u) + ... + (s_n + u) - u \\ &\geq (s_1 + ... + s_n) + u, \end{aligned}$$

as desired.

Thus, if $h(Q) < t$, then

$$t \geq \max_i \{v(q_i)\} = v(q_{\tau(n)}) \geq s_1 + ... + s_n,$$

so

$$N^+_{P,v}(t) \leq n_{\mathbf{r}}(t).$$

□

For the lower bound, we use:

Lemma 6.4. *For a fixed v and P, suppose that $\mathcal{A}^+_v(P)$ is defined. For any $Q \in A^+_v(P)$, choose k so that $v(q_k) > v(q_i)$ for all $i \neq k$. Suppose also that*

$$h(q'_i) = h(q_0) + ... + h(q_n) - h(q_i) + O(1)$$

for all $Q \in A^+_v(P)$ and $i \neq k$. Let $\mathbf{a} = (h(q_0), ..., \widehat{h(q_k)}, ..., h(q_n))$ and $\mathbf{r} = \Theta(\mathbf{a} + (u, ..., u))$. Then

$$N_P(t) \geq \#\{Q \in A^+_v(P) : h(Q) < t\} \geq n_{\mathbf{r}}(t).$$

The proof is similar to the proof of Lemma 6.3.

We are now prepared to prove Theorem 6.1:

Proof of Theorem 6.1. Let us fix P and choose S appropriately. For a valuation $v \in S$, choose j so that $v(p_j) \geq v(p_i)$ for all $i \neq j$. For any $Q \in A(P)$, choose k so that $v(q_k) \geq v(q_i)$ for all $i \neq k$. Note that if $v(q_k) > v(p_j)$, then in fact $v(q_k) > v(q_i)$ for all $i \neq k$ (this follows from Lemma 6.2). Let us set

$$m = \max_{v \in S}\{(n-1)v(p_j), u\}$$

where u bounds all the functions implied by the $O(1)$ in Eqs. (6.1) for all $v \in S$. Let

$$U_v = \{Q \in \mathcal{A}(P) : v(q_0) + ... + v(q_n) - v(q_i) \leq m \text{ for some } i\},$$
$$U = \bigcap_{v \in S} U_v.$$

Since $P \in U_v$ for all v, $U \neq \emptyset$. Also, if $Q \in U$, then

$$v(q_0) + ... + v(q_n) - v(q_i) \leq m$$

for some i. The quantity on the right is smallest when $i = k$. If we have $v(q_i) = v(q_k)$ for some i, then $v(q_k) \leq v(p_j) < m$. Otherwise, from Eq. (6.1) we get

$$v(q_i) \leq v(q_k) \leq m + u \leq 2m.$$

Since this is true for all $v \in S$, and S is finite, we conclude that the height of Q is bounded. There are only a finite number of such Q, so U is finite.

Now, suppose $Q \notin U$. Then $Q \notin U_v$ for some v. If there is an i such that $v(q_k) = v(q_i)$, then $v(q_k) \leq v(p_j)$, and

$$v(q_0) + ... + v(q_n) - v(q_i) \leq (n-1)v(p_j) \leq m,$$

which means $Q \in U_v$, a contradiction. Therefore, $v(q_k) > v(q_i)$ for all $i \neq k$. Hence, by Lemma 6.3, there exists an $\mathbf{r}$ so that

$$N^+_{Q,v}(t) < n_{\mathbf{r}}(t).$$

Thus,

$$\begin{aligned} N_P(t) &= O(1) + \sum_{Q\in\partial U} N^+_{Q,v}(t) \\ &\leq \sum_{Q\in\partial U} n_{\mathbf{r}}(t). \end{aligned}$$

Here, the v and $\mathbf{r}$ depend on Q. Since this is a finite sum, we can define $\mathbf{s}$ so that $0 < s_i = \min_{Q\in\partial U}\{r_i\}$. Then, we get

$$N_P(t) \ll n_{\mathbf{s}}(t),$$

as desired.

For the lower bound, we argue as follows: For the place at infinity, P non-trivial and $i \neq j$,

$$v_\infty(p'_i) > v_\infty(p_j).$$

Thus, $\mathcal{A}(P)$ is infinite. Hence, there exists a v so that $U_v \neq \mathcal{A}(P)$. Write $v_0 = v$, and choose $Q_0 \notin U_{v_0}$. Label the rest of the elements of S as $S = \{v_0, ..., v_s\}$. Define Q_i and S_0 inductively for $i = 1, ..., s$ in the following fashion: If $U_{v_i} \supset \mathcal{A}^+_{v_{i-1}}(Q_{i-1})$, we set $Q_i = Q_{i-1}$ and place $v_i \in S_0$. Otherwise, we choose Q_i so that $Q_i \in \mathcal{A}^+_{v_{i-1}}(Q_{i-1})$, but $Q_i \notin U_{v_i}$. Suppose now that $Q \in \mathcal{A}^+_{v_s}(Q_s)$, and $v_s(q_k) > v_s(q_i)$ for all $i \neq k$. Then, for all $v \in S \setminus S_0$, and for the same k, $v(q_k) > v(q_i)$ for all $i \neq k$. Thus,

$$h(Q) = \sum_{v\in S\setminus S_0} v(q_k) + O(1),$$

where the $O(1)$ includes the contributions from all valuations in S_0. In particular, we have an analogue of Eq. (6.1): For all $Q \in \mathcal{A}^+_{v_s}(Q_s)$ and all $j \neq i$,

$$h(q'_i) = h(q_0) + ... + h(q_n) - h(q_i) + O(1). \tag{6.2}$$

We now appeal to Lemma 6.4 to conclude

$$N_P(t) \geq N^+_{Q_s,v_s}(t) \geq n_{\mathbf{r}}(t),$$

as desired. □

Remark. We have really proven a version of Theorem 6.1 for totally real fields. For an arbitrary field, one must deal with complex valuations. Over the reals, the singularity $(0, ..., 0)$ is isolated, but this is no longer the case over $\mathbb{C}$, which complicates the problem.

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