PACIFIC JOURNAL OF MATHEMATICS
Vol. 182, No. 1, 1998

MEROMORPHIC CONTINUATION OF MINAKSHISUNDARAM-PLEIJEL SERIES FOR SEMISIMPLE LIE GROUPS

Floyd L. Williams

To the Memory of Marshall H. Stone

An explicit meromorphic continuation of the Minakshisundaram-Pleijel zeta function for a compact Riemann surface has been obtained by B. Randol, using the Selberg trace formula. Such zeta functions can be defined in the context of a general rank 1 symmetric space form $\Gamma\backslash G/K$. We find, similarly, their continuation to the full complex plane.

1. Introduction.

Let G be a connected non-compact semisimple Lie group with finite center. We assume the split rank of G is 1. Let $K \subset G$ be a maximal compact subgroup and let $\Gamma \subset G$ be a co-compact torsion free discrete subgroup. Then $X_\Gamma \overset{\text{def}}{=} \Gamma\backslash G/K$ is a compact smooth manifold, referred to as a space form of $X = G/K$. We fix a finite-dimensional unitary representation χ of Γ and we also denote by χ the corresponding character. There is a discrete decomposition of the unitary representation π_χ of G induced by χ:

$$\pi_\chi = \sum m_\pi(\Gamma)\pi, \quad \pi\epsilon\widehat{G} = \text{the unitary dual of } G \tag{1.1}$$

where each multiplicty $m_\pi(\Gamma) \geq 0$ is finite. Considering the set of class 1 representations $\{\pi_j\}_{j\geq 0}$ (with respect to K) in $\widehat{G}$ for which $n_j(\chi) \overset{\text{def}}{=} m_{\pi_j}(\Gamma) > 0$, and considering the eigenvalues $\{\lambda_j\}_{j\geq 0}$ of the Laplacian $-\Delta_\Gamma$ on smooth sections of the vector bundle over X_Γ induced by χ (see Equation (3.7) below), one has a Minakshisundaram-Pleijel spectral zeta function, with parameter $b > 0$,

$$D_\Gamma(s; b, \chi) = \sum_{j=0}^{\infty} \frac{n_j(\chi)}{(b + \lambda_j)^s} \tag{1.2}$$

of the complex variable s for $\operatorname{Re} s >> 0$. The purpose of this paper is to give a reasonably explicit meromorphic continuation of $D_\Gamma(\,.\,; b, \chi)$ to the full complex plane. For this we use the Selberg trace formula, following the method of B. Randol [14] (who considered the case $G = SL(2, R)$), and we use the explicit form of the Harish-Chandra Plancherel measure of X. The main result is Theorem 4.2 which has been announced (without proof) in [19], [21], [22] and applied in [20], [22], and which simplifies and generalizes the result obtained in a preliminary version [18] of the present work. Some remarks on applications of Theorem 4.2 follow its statement.

2. Notation and normalization of measures.

g_0, k_0 will denote the Lie algebras of G, K. If $p_0 = \{x \in g_0 \mid (x, k_0) = 0\}$, where $(\, ,)$ is the Killing form of g_0 then $g_0 = k_0 + p_0$ is a Cartan decomposition of g_0. Let θ denote the corresponding Cartan involution of g_0 and of G. Extend a maximal abelian subspace a_p of p_0 to a θ-stable Cartan subalgebra a of $g_0 : a \cap p_o = a_p$. We write $g, k, p, a^{\mathbb{C}}$ for the respective complexifications of g_0, k_0, p_o, a. For $\Phi = \Phi(g, a^{\mathbb{C}})$ the set of non-zero roots of $(g, a^{\mathbb{C}})$ we choose a system of positive roots Φ^+ in Φ which is a_p-compatible, say with respect to some lexicographic order on the dual space a_p^* induced by an ordered real basis of a_p: If $\alpha \in \Phi \ni \alpha|_{a_p} \not\equiv 0$ and $\alpha|_{a_p} > 0$, then $\alpha \in \Phi^+$. Accordingly we set

$$
\begin{aligned}
P^+ &= \{\alpha \in \Phi^+ \mid \alpha \not\equiv 0 \text{ on } a_p\} \\
P^- &= \{\alpha \in \Phi^+ \mid \alpha \equiv 0 \text{ on } a_p\} \\
2\rho &= \sum_{\alpha \in P^+} \alpha, \Sigma^+ = \{\alpha|_{a_p} | \alpha \in P^+\}.
\end{aligned} \tag{2.1}
$$

If $n_0 \overset{\text{def}}{=} g_0 \cap \sum_{\alpha \in P^+} g_\alpha$ where g_α is the root space of $\alpha \in \Phi$, one has Iwasawa decompositions $G = KA_pN$, $g_0 = k_0 + a_p + n_0$ of G, g_0 for $A_p = \exp a_p$, $N = \exp n_0$. As the split rank of G is 1 by assumption (i.e. $\dim a_p = 1$) the cardinality $|\sum^+|$ of $\sum^+$ is at most 2. More specifically

$$
\Sigma^+ = \{\beta\} \text{ or } \Sigma^+ = \{\beta, 2\beta\}. \tag{2.2}
$$

The choice of a basis vector H_0 of a_p is normalized by $\beta(H_0) = 1$. Let

$$
\begin{aligned}
p &= |\{\alpha \in P^+ \mid \alpha|_{a_p} = \beta\}| \\
q &= |\{\alpha \in P^+ \mid \alpha|_{a_p} \neq \beta\}|, \quad \rho_0 = \rho(H_0).
\end{aligned} \tag{2.3}
$$

In (2.2), $\sum^+ = \{\beta\} \Leftrightarrow q = 0$. Also $2\rho_0 = p + 2q$. The complexification of a_p^* is given by

$$
a_p^{*,\mathbb{C}} = \text{ space of R-linear maps } a_p \to \mathbb{C}. \tag{2.4}
$$

Each $x \in G$ has an Iwasawa decomposition $x = k(x) \exp H(x) n(x) \in KA_pN$ where $H : G \to a_p$ is a smooth map. Haar measures $dn, da, dx, d\nu$ on N, A_p, G, a_p^* are normalized by

$$\begin{aligned} &\int_N e^{-2\rho(H(\theta n))} dn = 1, \int_{A_p} h(a)da = \int_R h(\exp tH_0)dt, \\ &\int_G f(x)dx = \int_N \int_{A_p} \int_K f(kan)e^{2\rho(\log a)} dkdadn \\ &\int_{a_p^*} \psi(\nu)d\nu = \frac{1}{2\pi} \int_R \psi(t\beta)dt \end{aligned} \tag{2.5}$$

for continuous compactly supported functions h, f, ψ, where dt is Lebesgue measure on R and dk is normalized Haar measure on K. Given dx we have a unique G-invariant measure dm_Γ on $\Gamma\backslash G$ such that

$$\int_G f(x)dx = \int_{\Gamma\backslash G} \left[\sum_{\gamma \in \Gamma} f(\gamma x) \right] dm_\Gamma(\Gamma x) \tag{2.6}$$

for $f \in C_c(G)$. As usual one writes $\operatorname{vol}(\Gamma\backslash G) = \int_{\Gamma\backslash G} 1\, dm_\Gamma$. The Hilbert space of the induced representation π_χ in (1.1) is the space of functions f in $L^2(\Gamma\backslash G$, space of $\chi, dm_\Gamma) \ni f(\gamma x) = \chi(\gamma) f(x)$ for $(\gamma, x) \in \Gamma \times G$, on which G acts by right translation.

For $\nu \in a_p^{*,\mathbb{C}}$ in (2.4) the corresponding Harish-Chandra spherical function ϕ_ν on G is given by

$$\phi_\nu(x) = \int_K e^{(i\nu - \rho)(H(xk))} dk \tag{2.7}$$

for $x \in G$; see [1].

Up to local isomorphism we take $G = SU(n,1), Sp(n,1)$ with $n \geq 2$, $SO_1(2n,1)$, $SO_1(2n+1,1)$ with $n \geq 1$, or $F_{4(-20)}$.

3. The trace formula.

We will not need the general form of the Selberg trace formula, but only the formula that results from its application to a certain specific K-biinvariant Harish-Chandra Schwartz function $h_t, t > 0$, on G. Up to scaling, h_t is the fundamental solution of the heat equation on $X = G/K$. Given the normalization of measures in (2.5) one has by spherical Fourier inversion

$$h_t(1) = \frac{e^{-\rho_0^2 t}}{4\pi} \int_R e^{-r^2 t} |c(r)|^{-2} dr \tag{3.1}$$

where $c(\cdot)$ is Harish-Chandra's c-function: $c(r) \overset{\text{def}}{=} c(r\beta)$ for $r \in \mathbb{C}$ where for $\nu \in a_p^{*,\mathbb{C}}$

$$c(\nu)^{-1} = \frac{\Gamma(\frac{p+q}{2})\Gamma(i\nu(H_0)+\frac{p}{2})\Gamma(\frac{i\nu(H_0)}{2}+\frac{p}{4}+\frac{q}{2})}{\Gamma(p+q)\Gamma(i\nu(H_0))\Gamma(\frac{i\nu(H_0)}{2}+\frac{p}{4})}; \tag{3.2}$$

see (2.3). For $\gamma \in \Gamma - \{1\}$ define $t_\gamma > 0$ by

$$\begin{aligned} &e^{t_\gamma} = \max\{|c| \,|\, c = \text{ an eigenvalue of } \operatorname{Ad}(\gamma) : g \to g\} \\ &\text{if } \Sigma^+ = \{\beta\} \text{ in (2.2), with } |c| \text{ replaced by } |c|^{\frac{1}{2}} \text{ if } \Sigma^+ = \{\beta, 2\beta\}. \end{aligned} \tag{3.3}$$

Then $\exists\, m_\gamma \in M$, the centralizer of A_p in K, unique up to conjugation in M such that γ is G-conjugate to $m_\gamma \exp t_\gamma H_0$; see [**16**]. The following function C on $\Gamma - \{1\}$ is therefore well-defined:

$$C(\gamma)^{-1} = e^{t_\gamma \rho_0} \left|\det{}_{n_0}(\operatorname{Ad}(m_\gamma \exp t_\gamma H_0)^{-1} - 1)\right|. \tag{3.4}$$

Also, as Γ is torsion free, each $\gamma \in \Gamma - \{1\}$ can be represented uniquely as some power of a primitive element $\delta : \gamma = \delta^{j(\gamma)}$ where $j(\gamma) \geq 1$ is an integer and δ cannot be written as γ_1^j for $\gamma_1 \in \Gamma, j > 1$ an integer. The form of the trace formula which we need, as developed by Gangolli in [**3**], [**4**] is

$$\begin{aligned} \sum_{j=0}^{\infty} n_j(\chi) e^{-(\rho_0^2 + \nu_j(\chi)(H_0)^2)t} &= \chi(1) \operatorname{vol}(\Gamma\backslash G) h_t(1) \\ &+ \frac{1}{(4\pi t)^{1/2}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) e^{-(\rho_0^2 t + t_\gamma^2/4t)} \end{aligned} \tag{3.5}$$

where C_Γ is a complete set of representatives in Γ for its conjugacy classes, $n_j(\chi) = m_{\pi_j}(\Gamma)$ as defined in Section 1, with $\phi_{\nu_j(\chi)}$ the positive definite spherical function of π_j ; see (2.7), (3.1), (3.3), (3.4). $\phi_{\nu_j(\chi)}(x) = \langle v_j, \pi_j(x) v_j \rangle$ where v_j is a $\pi_j(K)$-fixed unit vector in the Hilbert space of π_j. As ϕ_ν in (2.7) is determined up to the action of the Weyl group of (g_0, a_p) on ν we normalize the choice of the $\nu_j(\chi)$ by

$$\begin{aligned} \nu_j(\chi)(H_0) \geq 0 &\text{ if } \nu_j(\chi)(H_0) \in R \\ i\nu_j(\chi)(H_0) < 0 &\text{ if } \nu_j(\chi)(H_0) \in iR - \{0\}. \end{aligned} \tag{3.6}$$

The eigenvalues $\lambda_j = \lambda_j(\chi)$ of Section 1 are given by

$$\lambda_j = \nu_j(\chi)(H_0)^2 + \rho_0^2. \tag{3.7}$$

We label the π_j so that $\pi_0 = 1$, the trivial representation: $\nu_0(\chi) = i\rho$; $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots; \lim_{j\to\infty} \lambda_j = \infty$. The multiplicity $n_j(\chi)$ coincides with the multiplicity of the eigenvalue λ_j. For the continuation of $D_\Gamma(.\,; b, \chi)$ which we seek in the next section the explicit form of the spherical Plancherel density $|c(r)|^{-2}$ of (3.1) is required. With our normalization of measures Miatello's computation [9], [11] of this density takes the form

(3.8)

$$|c(r)|^{-2} = \begin{cases} C_G \pi r P(r) \tanh \pi r & \text{for } G = SO_1(2n,1) \\ C_G \pi r P(r) \begin{bmatrix} \tanh \frac{\pi}{2} r \\ \text{or} \\ \coth \frac{\pi}{2} r \end{bmatrix} & \text{for } G = SU(n,1) \text{ with the cotangent choice for } n \text{ even} \\ C_G \pi r P(r) \tanh \frac{\pi}{2} r & \text{for } G = Sp(n,1) \text{ or } F_{4(-20)} \\ C_G \pi P(r) & \text{for } G = SO_1(2n+1,1) \end{cases}$$

where $P(r)$ is an even polynomial of degree $d-2$ for $G \neq SO_1(2n+1,1)$, and of degree $d-1 = 2n$ for $G = SO_1(2n+1,1)$ for $d \overset{\text{def}}{=} \dim X$ (see Table 1 below) and where

$$C_G = \frac{1}{2^{4n-4}\Gamma(n)^2}$$

(3.9)

for $G = SO_1(2n,1)(n \geq 1)$

$$P(r) = \prod_{j=2}^{n} \left[r^2 + \left(n - j + \frac{1}{2} \right)^2 \right]$$

$$C_G = \frac{1}{2^{2n-1}\Gamma(n)^2}$$

for $G = SU(n,1)(n \geq 2)$

$$P(r) = \prod_{j=1}^{n-1} \left[\frac{r^2}{4} + \frac{(n-2j)^2}{4} \right]$$

$$C_G = \frac{1}{2^{4n+1}\Gamma(2n)^2}$$

for $G = Sp(n,1)(n \geq 2)$

$$P(r) = \left[\frac{r^2}{4} + \frac{1}{4} \right] \prod_{j=3}^{n+1} \left[\frac{r^2}{4} + \left(n - j + \frac{3}{2} \right)^2 \right] \left[\frac{r^2}{4} + \left(n - j + \frac{5}{2} \right)^2 \right]$$

$$C_G = \frac{1}{2^{21}\Gamma(8)^2}$$

for $G = F_{4(-20)}$

$$P(r) = \left[\frac{r^2}{4} + \frac{1}{4}\right]^2 \left[\frac{r^2}{4} + \left(\frac{3}{2}\right)^2\right]^2 \left[\frac{r^2}{4} + \left(\frac{5}{2}\right)^2\right] \left[\frac{r^2}{4} + \left(\frac{7}{2}\right)^2\right] \left[\frac{r^2}{4} + \left(\frac{9}{2}\right)^2\right]$$

$$C_G = \frac{1}{2^{4n-2}\Gamma(n+\frac{1}{2})^2}$$

for $G = SO_1(2n+1,1)(n \geq 1)$

$$P(r) = \prod_{j=1}^{n} \left[r^2 + (n-j)^2\right].$$

Note that Haar measures in [11] are subject to normalizations which differ from those given in (2.5). Given $z \in \mathbb{C}$ we define the following polynomials $\Pi(r\,;\,z)$ in r, whose coefficients will play a key role:

$$\Pi(r;\,z) = \prod_{j=2}^{n} \left[r + \left(n - j + \frac{1}{2}\right)^2 - z\right]$$

for $G = SO_1(2n,1)(n \geq 1)$

$$\Pi(r;\,z) = \prod_{j=1}^{n-1} \left[\frac{r + (n-2j)^2 - z}{4}\right]$$

(3.10)

for $G = SU(n,1)(n \geq 2)$

$$\Pi(r;\,z) = \left[\frac{r+1-z}{4}\right].$$

$$\prod_{j=3}^{n+1} \left[\frac{r}{4} + \left(n - j + \frac{3}{2}\right)^2 - \frac{z}{4}\right] \left[\frac{r}{4} + \left(n - j - \frac{5}{2}\right)^2 - \frac{z}{4}\right]$$

for $G = Sp(n,1)(n \geq 2)$

$$\Pi(r;\,z) = \left[\frac{r}{4} + \left(\frac{1}{2}\right)^2 - \frac{z}{4}\right]^2 \left[\frac{r}{4} + \left(\frac{3}{2}\right)^2 - \frac{z}{4}\right]^2 \left[\frac{r}{4} + \left(\frac{5}{2}\right)^2 - \frac{z}{4}\right] \cdot \left[\frac{r}{4} + \left(\frac{7}{2}\right)^2 - \frac{z}{4}\right] \left[\frac{r}{4} + \left(\frac{9}{2}\right)^2 - \frac{z}{4}\right]$$

for $G = F_{4(-20)}$

$$\Pi(r;\,z) = \prod_{j=1}^{n} \left[r + (n-j)^2 - z\right]$$

for $G = SO_1(2n+1,1)(n \geq 1)$.

Thus by definition $\Pi(r\,;\,z)$ is of degree $\frac{d-2}{2}$ for $G \neq SO_1(2n+1,1)$, and of degree $\frac{d-1}{2} = n$ for $G = SO_1(2n+1,1)$. Moreover $\Pi(r^2+z\,;\,z)$ is independent of z and in fact

$$P(r) = \Pi(r^2 + z\,;\, z)\,. \tag{3.11}$$

Of particular interest is the case $z = b + \rho_0^2$ where $b \in \mathbb{C}$ is fixed. We define coefficients $m_j(b)$ by

$$\Pi(r\,;\, b + \rho_0^2) = \sum_{j=0}^{m} m_j(b) r^j \tag{3.12}$$

where $m = \frac{d-2}{2}$ for $G \neq SO_1(2n+1,1)$ and $m = \frac{d-1}{2} = n$ for $G = SO_1(2n+1,1)$.

Table 1.

G up to local isomorphism	ρ_0	d	p	q	$a(G)$	$m_{\frac{d}{2}-1}(b)$
$SO_1(2n,1), n \geq 1$	$n - \frac{1}{2}$	$2n$	$2n-1$	0	π	1
$SO_1(2n+1,1), n \geq 1$	n	$2n+1$	$2n$	0		$m_n(b) = 1$
$SU(n,1), n \geq 2$	n	$2n$	$2n-2$	1	$\frac{\pi}{2}$	$\frac{1}{4^{n-1}}$
$Sp(n,1), n \geq 2$	$2n+1$	$4n$	$4n-4$	3	$\frac{\pi}{2}$	$\frac{1}{4^{2(n-1)}}$
$F_{4(-20)}$	11	16	8	7	$\frac{\pi}{2}$	$\frac{1}{4^7}$

4. The Minakshisundaram-Pleijel Series.

For $b > 0$ we assign to the class 1 spectral data $\{\pi_j, n_j(\chi)\}_{j \geq 0}$ the Minakshisundaram-Pleijel series

$$D_\Gamma(s;\, b, \chi) = \sum_{j=0}^{\infty} \frac{n_j(\chi)}{(b + \lambda_j)^s} \tag{4.1}$$

which converges absolutely for $\operatorname{Re} s > \frac{d}{2}$, where λ_j is given by (3.7); cf. [6], [12], [17]. Note that $D_\Gamma(s;\, b, \chi)$ is a Dirichlet series with exponents $\{\beta_j \overset{\text{def}}{=} \log(b + \lambda_j)\}_{j \geq 0}$:

$$D_\Gamma(s;\, b, \chi) = \sum_{j=0}^{\infty} n_j(\chi) e^{-\beta_j s}. \tag{4.2}$$

Thus $s \to D_\Gamma(s;\, b, \chi)$ is holomorphic in the domain $\operatorname{Re} s > \frac{d}{2}$, and we seek an explicit meromorphic continuation of this function to the full complex plane.

The starting point is to represent our Dirichlet series in integral form in a standard way:

$$\Gamma(s)D_\Gamma(s;\, b,\chi) = \int_0^\infty w_\Gamma(t)t^{s-1}dt, \quad \operatorname{Re} s > \frac{d}{2} \tag{4.3}$$

where for $t > 0$

$$w_\Gamma(t) = \sum_{j=0}^\infty n_j(\chi)e^{-(b+\lambda_j)t}. \tag{4.4}$$

On the other hand by the trace formula (3.5), Equation (3.1), and definition (3.7)

$$\begin{aligned} w_\Gamma(t) = {} & \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi}\int_R e^{-(r^2+b+\rho_0^2)t}|c(r)|^{-2}dr \\ & + \frac{1}{\sqrt{4\pi t}}\sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)e^{-(bt+\rho_0^2 t+t_\gamma^2/4t)} \end{aligned} \tag{4.5}$$

so that by (4.3) (for $\operatorname{Re} s > \frac{d}{2}$)

$$\begin{aligned} \Gamma(s)D_\Gamma(s;\, b,\chi) = {} & \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi}\int_0^\infty\int_R e^{-(r^2+b+\rho_0^2)t}|c(r)|^{-2}t^{s-1}drdt \\ & + \int_0^\infty \frac{t^{s-1}}{\sqrt{4\pi t}}\sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)e^{-(bt+\rho_0^2 t+t_\gamma^2/4t)}dt. \end{aligned} \tag{4.6}$$

Now

$$\begin{aligned} & \int_R\int_0^\infty e^{-(r^2+b+\rho_0^2)t}|c(r)|^{-2}t^{\operatorname{Re} s-1}dtdr \\ & = \int_R |c(r)|^{-2}\left[\int_0^\infty e^{-(r^2+b+\rho_0^2)t}t^{\operatorname{Re} s-1}dt\right]dr \\ & = \Gamma(\operatorname{Re} s)\int_R \frac{|c(r)|^{-2}dr}{(b+\rho_0^2+r^2)^{\operatorname{Re} s}}, \quad \text{where} \\ & I_s \overset{\text{def}}{=} \int_R \frac{|c(r)|^{-2}dr}{(b+\rho_0^2+r^2)^{\operatorname{Re} s}} < \infty \text{ for } \operatorname{Re} s > \frac{d}{2}. \end{aligned}$$

Indeed $r \to \frac{r^n}{(b_0+r^2)^s} \in L^1(R)$ for $\operatorname{Re} s > \frac{n+1}{2}, b_0 > 0, n = 0,1,2,3,\dots,$ and in (3.8) $rP(r)$ is a polynomial of degree $d-1$ for $G \neq SO_1(2n+1,1)$ with $P(r)$ of degree $d-1$ in case $G = SO_1(2n+1,1)$. Thus as $|\tanh r| < 1$ we do have $I_s < \infty$ for $\operatorname{Re} s > \frac{d-1+1}{2} = \frac{d}{2}$, apart from the cotangent case. In the cotangent case (i.e. $G = SU(n,1)$ with n even) we deduce the finiteness of I_s

for $\operatorname{Re} s > \frac{d}{2}$ by writing, for example, $\coth \frac{\pi}{2}r = \tanh \frac{\pi}{2}r + (\operatorname{csch}\frac{\pi}{2}r)(\operatorname{sech}\frac{\pi}{2}r)$ and using the boundedness of $r \operatorname{csch} \frac{\pi}{2}r$. Of course $\operatorname{sech} \frac{\pi}{2}r$ is bounded by 1 and one in fact has an estimate $r \operatorname{csch} ar \leq Me^{-a|r|/2}$ for $a > 0$, using the continuity of $r \operatorname{csch} ar$ at 0 (whereas $\operatorname{csch} ar$ is not continuous at 0). The application of Fubini's theorem is therefore justified and we write the 1st term on the right hand side of (4.6) as

$$\begin{aligned} \text{(4.7)} \qquad & \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi} \int_R \int_0^\infty e^{-(r^2+b+\rho_0^2)t} |c(r)|^{-2} t^{s-1} dt dr \\ &= \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)\Gamma(s)}{4\pi} \int_R \frac{|c(r)|^{-2} dr}{(b+\rho_0^2+r^2)^s}. \end{aligned}$$

Consider the case $|c(r)|^{-2} = C_G \pi r P(r) \tanh ar$ in (3.8) with $a = \pi$ or $\frac{\pi}{2}$. Take $z = b_0 \overset{\text{def}}{=} b + \rho_0^2$ in (3.11) and use (3.12):

$$P(r) = \Pi(r^2 + b_0;\, b_0) = \sum_{j=0}^m m_j(b)(r^2+b_0)^j \Longrightarrow$$

$$\text{(4.8)} \qquad \frac{|c(r)|^{-2}}{(b_0+r^2)^s} = C_G \pi r(\tanh ar) \sum_{j=0}^m \frac{m_j(b)}{(b_0+r^2)^{s-j}}$$

which implies

$$\begin{aligned} \text{(4.9)} \qquad & \int_R \frac{|c(r)|^{-2}}{(b+\rho_0^2+r^2)^s} = C_G \pi \sum_{j=0}^m m_j(b) \int_R \frac{r \tanh ar dr}{(b_0+r^2)^{s-j}} \\ &= \frac{C_G a\pi}{2} \sum_{j=0}^m \frac{m_j(b)}{(s-j-1)} \int_R \frac{\operatorname{sech}^2 ar dr}{(b_0+r^2)^{s-j-1}}, \end{aligned}$$

integrating by parts, where

$$\text{(4.10)} \qquad K_0(s;\, b, a) \overset{\text{def}}{=} \int_R \frac{\operatorname{sech}^2 ar dr}{(b+\rho_0^2+r^2)^s}$$

is an entire function of s, as $\operatorname{sech}^2 ar < Me^{-a|r|}$ for some $M > 0$ (for $a > 0$). That is, the integral in (4.10) converges uniformly on compact subsets of the plane. By (4.7), (4.9), (4.10) the 1st term on the right hand side of (4.6) is (for $\operatorname{Re} s > \frac{d}{2}$)

$$\text{(4.11)} \quad \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)\Gamma(s)}{8} C_G a(G) \sum_{j=0}^m \frac{m_j(b)}{s-j-1} K_0(s-j-1;\, b, a(G))$$

where $a(G) = \pi$ or $\frac{\pi}{2}$ is given by Table 1 and $m = \frac{d-2}{2}$. Here G is locally isomorphic to $SO_1(2n,1), Sp(n,1), F_{4(-20)}$, or $SU(\ell,1)$ with ℓ odd. In the cotangent case write (as before) coth $ar = \tanh ar + (\operatorname{csch} ar)(\operatorname{sech} ar)$ so that

$$\begin{aligned}|c(r)|^{-2} &= C_G \pi r P(r) \coth ar \\ &= C_G \pi r P(r) \tanh ar + C_G \pi P(r)(r \operatorname{csch} ar)(\operatorname{sech} ar).\end{aligned}$$

In place of (4.9) we have

(4.12)

$$\begin{aligned}\int_R \frac{|c(r)|^{-2}}{(b+\rho_0^2+r^2)^s} &= \frac{C_G a\pi}{2} \sum_{j=0}^{m} \frac{m_j(b)}{(s-j-1)} \int_R \frac{\operatorname{sech}^2 ar dr}{(b_0+r^2)^{s-j-1}} \\ &+ C_G \pi \sum_{j=0}^{m} m_j(b) \int_R \frac{(r \operatorname{csch} ar)\operatorname{sech} ar dr}{(b_0+r^2)^{s-j}}\end{aligned}$$

where (recalling the estimate $r \operatorname{csch} ar \leq M e^{-a|r|/2}$)

$$L_0(s;\, b, a) \overset{\text{def}}{=} \int_R \frac{(r \operatorname{csch} ar)\operatorname{sech} ar dr}{(b+\rho_0^2+r^2)^s} \tag{4.13}$$

is an entire function of s. In place of (4.11) we have for G locally isomorphic to $SU(\ell, 1)$ with ℓ even, the 1st term on the right hand side of (4.6) written as

(4.14)

$$\begin{aligned}&\frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)\Gamma(s)}{8} C_G \frac{\pi}{2} \sum_{j=0}^{m} \frac{m_j(b)}{s-j-1} K_0\left(s-j-1;\, b, \frac{\pi}{2}\right) \\ &+ \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)\Gamma(s)}{4\pi} C_G \pi \sum_{j=0}^{m} m_j(b) L_0\left(s-j;\, b, \frac{\pi}{2}\right)\end{aligned}$$

for $\operatorname{Re} s > \frac{d}{2} = \ell$, where $m = \frac{d-2}{2} = \ell - 1$.

The final case to consider is $G = SO_1(2n+1, 1)$ (up to local isomorphism):

$$|c(r)|^{-2} = C_G \pi P(r) = C_G \pi \sum_{j=0}^{n} m_j(b)(r^2+b_0)^j \text{ in (3.8)} \Longrightarrow$$

$$\begin{aligned}\int_R \frac{|c(r)|^{-2} dr}{(b+\rho_0^2+r^2)^s} &= C_G \pi \sum_{j=0}^{n} m_j(b) \int_R \frac{dr}{(r^2+b_0)^{s-j}} \\ &= C_G \pi \sum_{j=0}^{n} m_j(b) b_0^{1/2-s+j} \pi^{1/2} \Gamma(s-j-1/2)/\Gamma(s-j) \qquad (4.15)\end{aligned}$$

so that (by (4.7)) the 1st term on the right hand side of (4.6) is

(4.16)

$$\frac{\pi^{1/2}\chi(1)\operatorname{vol}(\Gamma\backslash G)\Gamma(s)}{4} C_G \sum_{j=0}^{n} m_j(b) b_0^{1/2-s+j} \Gamma(s-j-1/2)/\Gamma(s-j)$$

with $b_0 \overset{\text{def}}{=} b + \rho_0^2 = b + n^2$. In summary:

Theorem 4.1. *The* 1^{st} *term on the right hand side of* (4.6) *is given by* (4.11) *for* $G = SO_1(2n,1)$, $Sp(n,1)$, $F_{4(-20)}$, *or* $SU(\ell,1)$ *with* ℓ *odd (up to local isomorphism), by* (4.14) *for* $G = SU(\ell,1)$ *with* ℓ *even, and by* (4.16) *for* $G = SO_1(2n+1,1)$*; here* $\operatorname{Re} s > \frac{d}{2}$.

We consider now the 2^{nd} term in (4.6) which can be written

$$T_\Gamma(s\,;\,b,\chi) \overset{\text{def}}{=} \int_0^\infty \theta_\Gamma(t) t^{s-1} dt \qquad \left(\text{for } \operatorname{Re} s > \frac{d}{2}\right) \tag{4.17}$$

where for $t > 0$

$$\theta_\Gamma(t) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) e^{-(bt+\rho_0^2 t + t_\gamma^2/4t)}\,. \tag{4.18}$$

We will show that $T_\Gamma(s\,;\,b,\chi)$ is an entire function of s. Let

$$\theta_1(t) = \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi} \int_R e^{-(r^2+b+\rho_0^2)t} |c(r)|^{-2} dr \tag{4.19}$$

be the 1^{st} term in (4.5); $t > 0$. For $t \geq 1$, $e^{-(r^2+b)t} \leq e^{-(r^2+b)} \implies \theta_1(t) \leq \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi} \cdot \; e^{-b} e^{-\rho_0^2 t} \int_R e^{-r^2} |c(r)|^{-2} dr = \chi(1)\operatorname{vol}(\Gamma\backslash G) e^{-b+\rho_0^2} h_1(1) e^{-\rho_0^2 t}$ by (3.1). That is, for $A(b) = 4\pi e^{-b+\rho_0^2} h_1(1)$, $\theta_1(t) \overset{(i)}{\leq} \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi} A(b) e^{-\rho_0^2 t}$ for $t \geq 1$. By (4.4), (4.5), (4.18), (4.19)

$$\theta_1(t) + \theta_\Gamma(t) = w_\Gamma(t) \overset{\text{(ii)}}{=} n_0(\chi) e^{-bt} + \textstyle\sum_{j=1}^\infty n_j(\chi) e^{-(b+\lambda_j)t}$$
$$(0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots).$$

Lemma 4.1. *$\exists\, B(b) > 0$ such that* $\sum_{j=1}^\infty n_j(\chi) e^{-(b+\lambda_j)t} \leq B(b) e^{-\lambda_1 t}$ *for* $t \geq 1$.

Proof. Let $M(t) = e^{\lambda_1 t} \sum_{j=2}^\infty n_j(\chi) e^{-(b+\lambda_j)t}$ for $t > 0$. For $j \geq 2, \lambda_j > \lambda_1 \implies b + \lambda_j - \lambda_1 > 0 \implies (b + \lambda_j - \lambda_1)t \geq b + \lambda_j - \lambda_1$ for $t \geq 1$. That is, $e^{\lambda_1 t} e^{-(b+\lambda_j)t} \leq e^{\lambda_1}\; e^{-(b+\lambda_j)} \implies M(t) \leq e^{\lambda_1} \sum_{j=2}^\infty n_j(\chi) e^{-(b+\lambda_j)} = M(1)$ for $t \geq 1$. Define $B(b) = n_1(\chi) + M(1) > 0$ to obtain $\sum_{j=1}^\infty n_j(\chi) e^{-(b+\lambda_j)t} = n_1(\chi) e^{-(b+\lambda_1)t} + e^{-\lambda_1 t} M(t) = e^{-\lambda_1 t}[n_1(\chi) e^{-bt} + M(t)] \leq e^{-\lambda_1 t} B(b)$ for $t \geq 1$, as desired. □

By (i), (ii), and Lemma 4.1,

$$\theta_\Gamma(t) - n_0(\chi) e^{-bt} = -\theta_1(t) + \sum_{j=1}^\infty n_j(\chi) e^{-(b+\lambda_j)t}$$

$$\begin{aligned}&\Longrightarrow |\theta_\Gamma(t) - n_0(\chi)e^{-bt}|\\ &\leq A_1(b)e^{-\rho_0^2 t} + B(b)e^{-\lambda_1 t} \leq C(b)e^{-\alpha t}\end{aligned}$$

for $t \geq 1$, where $A_1(b) = \frac{\chi(1)\,\mathrm{vol}(\Gamma\backslash G)}{4\pi}A(b), C(b) = A_1(b) + B(b)$, $\alpha = \min(\lambda_1, \rho_0^2) > 0$. That is:

Lemma 4.2. *$\exists\, C(b), \alpha > 0$ such that $|\theta_\Gamma(t) - n_0(\chi)e^{-bt}| \leq C(b)e^{-\alpha t}$ for $t \geq 1$. We can take $\alpha = \min(\lambda_1, \rho_0^2)$.*

Lemma 4.2, which compares with Lemma 3 of Randol [**14**], gives us a sense of the behavior of $\theta_\Gamma(t)$ as $t \to \infty$. We need also its behavior as $t \to 0^+$, which is the content of Lemma 4 of [**14**] in case $G = SL(2, R)$. In the general situation which we are considering this matter is much more delicate. In principle the result we need follows by Theorem 5.1 of [**9**]. That theorem and its proof contains several generalities which we need not be concerned with, since we deal only with the class 1 spectrum. The result is:

Lemma 4.3. *$\exists\, \epsilon > 0$ such that $\lim_{t\to 0^+}\theta_\Gamma(t)e^{\epsilon/t} = 0$. We can choose $\epsilon = \epsilon_0^2/8$ where $\epsilon_0 > 0$ is a lower bound for each $t_\gamma \in C_\Gamma - \{1\}$.*

For the sake of completeness and self-containedness we give a direct simpler proof of Lemma 4.3, following [**9**] up to a point. For $x \geq 0$ let $E(x) = |\{\gamma \in C_\Gamma - \{1\} | t_\gamma \leq x\}|, \widetilde{E}(x) = |\{\gamma \in C_\Gamma - \{1\} | x \leq t_\gamma < x + 1\}|$. The asymptotic behavior of $E(x)$ as $x \to \infty$ is given in [**2**]; also cf. [**4**]: For some $a > 0, \lim_{x\to\infty} axe^{-ax}E(x) \overset{\text{(iii)}}{=} 1$; one can in fact take $a = 2\rho_0$. Using (iii) we first prove the following, from whence Lemma 4.3 can be readily deduced.

Scholium. *Let b be a positive real number. Then for $t > 0$ sufficiently small $\sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma e^{-t_\gamma^2/bt}$ converges. This sum in fact is dominated by terms $j_0E(j_0)e^{-\epsilon_0^2/bt} + \epsilon(t)$ where $\epsilon_0 > 0$ and $\epsilon(t)/t \to 0$ as $t \to 0^+$.*

Proof. By (iii) we can find an integer j_0 sufficiently large and a positive constant $\delta \ni \widetilde{E}(x) \overset{\text{(iv)}}{\leq} \frac{\delta}{x}e^{ax}$ for $x \geq j_0$. We have

$$\begin{aligned}&\sum_{t_\gamma\in[x,x+1)} t_\gamma e^{-t_\gamma^2/bt} \leq (x+1)e^{-x^2/bt}\widetilde{E}(x) \quad \text{(by definition of } \widetilde{E}(x))\\ &\Longrightarrow \sum_{j_0<t_\gamma} t_\gamma e^{-t_\gamma^2/bt} \leq \sum_{j_0<t_\gamma<j_0+1} + \sum_{j_0+1\leq t_\gamma<j_0+2} + \sum_{j_0+2\leq t_\gamma<j_0+3} + \cdots\\ &\leq (j_0+1)e^{-j_0^2/bt}\widetilde{E}(j_0) + (j_0+2)e^{-(j_0+1)^2/bt}\widetilde{E}(j_0+1)\\ &\quad + (j_0+3)e^{-(j_0+2)^2/bt}\widetilde{E}(j_0+2) + \cdots\\ &\leq \delta\sum_{n=1}^{\infty}\frac{j_0+n}{j_0+n-1}e^{-(j_0+n-1)^2/bt}e^{a(j_0+n-1)},\end{aligned}$$

by (iv). Now $\frac{j_0+n}{j_0+n-1} \le 1+\frac{1}{j_0}$ and $e^{-(j_0+n-1)^2/bt} \le e^{-j_0^2/bt}e^{-2j_0(n-1)/bt} \Longrightarrow$ the preceding sum is dominated by $\frac{\delta(j_0+1)}{j_0}e^{-j_0^2/bt}e^{2j_0/bt}e^{a(j_0-1)}\sum_{n=1}^{\infty} e^{-[2j_0/bt-a]n}$, which is a convergent geometric series for $2j_0/bt - a > 0$. That is

$$\begin{aligned}\sum_{j_0<t_\gamma} t_\gamma e^{-t_\gamma^2/bt} &\le \delta\frac{(j_0+1)}{j_0}e^{-j_0^2/bt}e^{2j_0/bt}e^{a(j_0-1)}\frac{e^{-(2j_0/bt-a)}}{1-e^{-(2j_0/bt-a)}}\\ &= \delta\frac{(j_0+1)}{j_0}\frac{e^{-j_0^2/bt}e^{aj_0}}{1-e^{-(2j_0/bt-a)}}\end{aligned}$$

for $0 < t < \frac{2j_0}{ab}$. On the other hand the numbers $\{t_\gamma | \gamma \in C_\Gamma - \{1\}\}$ are bounded away from zero [4]: $\exists\, \epsilon_0 > 0 \ni t_\gamma \ge \epsilon_0 \,\forall\, \gamma \in C_\Gamma - \{1\}$. Therefore

$$\begin{aligned}\sum_{t_\gamma \le j_0} t_\gamma e^{-t_\gamma^2/bt} &\le \sum_{t_\gamma \le j_0} t_\gamma e^{-\epsilon_0^2/bt}\\ &= j_0E(j_0)e^{-\epsilon_0^2/bt} \Longrightarrow \sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma e^{-t_\gamma^2/bt}\\ &\le j_0E(j_0)e^{-\epsilon_0^2/bt} + \epsilon(t)\end{aligned}$$

for $0 < t < \frac{2j_0}{ab}$, where $\epsilon(t) = \frac{\delta(j_0+1)e^{-j_0^2/bt}e^{aj_0}}{j_0(1-e^{-(2j_0/bt-a)})}$ satisfies $\lim_{t\to 0^+}\frac{\epsilon(t)}{t} = 0$, as desired. ☐

Proof of Lemma 4.3. $\forall\, \gamma \in \Gamma - \{1\}, j(\gamma) \ge 1, \frac{t_\gamma^2}{4t} \ge \frac{\epsilon_0^2}{8t} + \frac{t_\gamma^2}{8t}$, and $C(\gamma) \le M$ (for some M as in [9]). For $0 < t < 1, \frac{1}{\sqrt{t}} \le \frac{1}{t} \Longrightarrow \theta_\Gamma(t) \le \frac{M}{\sqrt{4\pi}}e^{-\epsilon_0^2/8t}\frac{1}{t}\sum_{\gamma\in C_\Gamma-\{1\}} t_\gamma e^{-t_\gamma^2/8t} \le \frac{M}{\sqrt{4\pi}}e^{-\epsilon_0^2/8t}\theta_0(t)$ by the Scholium, for t sufficiently small, where $\theta_0(t) = j_0E(j_0)\frac{e^{-\epsilon_0^2/8t}}{t} + \frac{\epsilon(t)}{t} \to 0$ as $t \to 0^+$. This completes the proof as we can choose $\epsilon = \frac{\epsilon_0^2}{8}$. ☐

In (4.17) write $T_\Gamma(s;\, b, \chi) \overset{(v)}{=} \int_0^1 \theta_\Gamma(t)t^{s-1}dt + \int_1^\infty \theta_\Gamma(t)t^{s-1}dt$, $\int_1^\infty \theta_\Gamma(t)t^{s-1}dt = \int_1^\infty[\theta_\Gamma(t) - n_0(\chi)e^{-bt}]\ t^{s-1}dt + n_0(\chi)\int_1^\infty e^{-bt}t^{s-1}dt$ where $\int_1^\infty e^{-bt}t^{s-1}dt$ is an entire function of s and where $\int_1^\infty[\theta_\Gamma(t) - n_0(\chi)e^{-bt}]t^{s-1}dt$ is an entire function of s by Lemma 4.2, which implies that the latter integral converges uniformly on compact subsets of $\mathbb{C}$. Similarly $\int_0^1 \theta_\Gamma(t)t^{s-1}dt = \int_1^\infty \theta_\Gamma(\frac{1}{t})t^{-s-1}dt$ is an entire function of s by Lemma 4.3. By Equation (v) we see therefore that $s \to T_\Gamma(s\,;\, b, \chi)$ extends to an entire function. We also see that the application of Fubini's theorem is valid: For $\sigma = \operatorname{Re} s$

(4.20)
$$\int_0^\infty \sum_{\gamma \in C_\Gamma - \{1\}} \left| \frac{t^{s-1}}{\sqrt{4\pi t}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) e^{-\left(bt + \rho_0^2 t + \frac{t_\gamma^2}{4t}\right)} \right| dt \\ \leq \chi(1) \int_0^\infty t^{\sigma-1} \theta_{\Gamma,1}(t) dt$$

where $\theta_{\Gamma,1}$ is θ_Γ in (4.18) with $\chi = 1$ and where $\int_0^\infty t^{\sigma-1}\theta_{\Gamma,1}(t)dt = \int_0^1 t^{\sigma-1}\theta_{\Gamma,1}(t)dt + \int_1^\infty t^{\sigma-1}[\theta_{\Gamma,1}(t) - n_0(1)e^{-bt}]dt + n_0(1)\int_1^\infty e^{-bt}t^{\sigma-1}dt < \infty$ for all s by Lemmas 4.2, 4.3; here $n_0(1) = 1$. Therefore we also have

(4.21)
$$T_\Gamma(s;\, b, \chi) \\ = \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) \int_0^\infty \frac{e^{-\left(bt + \rho_0^2 t + \frac{t_\gamma^2}{4t}\right)}}{\sqrt{4\pi t}} t^{s-1} dt \\ = \frac{\pi^{-1/2}}{[2\sqrt{b+\rho_0^2}]^{s-1/2}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) j(\gamma)^{-1} C(\gamma) t_\gamma^{s+1/2} K_{-s+1/2}\left(t_\gamma \sqrt{b+\rho_0^2}\right)$$

where $K_\nu, \nu \in \mathbb{C}$, is the K-Bessel function

$$K_\nu(r) \overset{\text{def}}{=} \frac{1}{2}\int_0^\infty e^{-r/2(t+\frac{1}{t})} t^{\nu-1} dt. \tag{4.22}$$

Namely, observe first that for $x, r > 0$ $\frac{xr}{2}(t + \frac{1}{t}) = \frac{x}{2}(rt + \frac{r}{t}) \Longrightarrow K_\nu(xr) = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(rt+\frac{r}{t})} t^{\nu-1} dt = \frac{1}{2}\int_0^\infty e^{-\frac{x}{2}(u+\frac{r^2}{u})} (\frac{u}{r})^{\nu-1}\frac{du}{r} = \frac{r^{-\nu}}{2}\int_0^\infty e^{-\frac{x}{2}(t+\frac{r^2}{t})} t^{\nu-1} dt$, or (by the functional equation $K_\nu(xr) = K_{-\nu}(xr)) = \frac{r^\nu}{2}\int_0^\infty e^{-\frac{x}{2}(t+\frac{r^2}{t})} t^{-\nu-1} dt$. Now choose $x = 2(b+\rho_0^2), r = \frac{t_\gamma}{2\sqrt{b+\rho_0^2}}$: $xr = t_\gamma\sqrt{b+\rho_0^2} \Longrightarrow K_{-s+1/2}(t_\gamma\sqrt{b+\rho_0^2}) = \frac{1}{2}\left[\frac{t_\gamma}{2\sqrt{b+\rho_0^2}}\right]^{-s+1/2} \int_0^\infty e^{-(bt+\rho_0^2 t + t_\gamma^2/4t)} t^{s-1/2-1} dt$, or $\int_0^\infty e^{-(bt+\rho_0^2 t+t_\gamma^2/4t)} \frac{t^{s-1}}{\sqrt{4\pi t}} dt = \frac{\pi^{-1/2} t_\gamma^{s-1/2}}{[2\sqrt{b+\rho_0^2}]^{s-1/2}} K_{-s+1/2}(t_\gamma\sqrt{b+\rho_0^2})$, which proves the 2nd statement in (4.21).

Divide by $\Gamma(s)$ in Equation (4.6). By Theorem 4.1 and the preceding observations we obtain the following main result.

Theorem 4.2. *For $b > 0$ define $T_\Gamma(s; b, \chi) = \int_0^\infty \theta_\Gamma(t;\, \chi) t^{s-1} dt$ where for $t > 0$ $\theta_\Gamma(t;\, \chi) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) t_\gamma j(\gamma)^{-1} C(\gamma) e^{-(bt+\rho_0^2 t+t_\gamma^2/4t)}$. Then*

$T_\Gamma(s;\, b, \chi)$ is an entire function of s and is also given in terms of the K-Bessel function by (4.21). For $a > 0$ define $K_0(s;\, b, a), L_0(s;\, b, a)$, which are also entire functions of s, by (4.10), (4.13) respectively. Define

(4.23)

$$\begin{aligned}\Phi_\Gamma(s;\, b, \chi) &= \frac{T_\Gamma(s;\, b, \chi)}{\Gamma(s)} + \frac{a(G)}{8} C_G \chi(1)\operatorname{vol}(\Gamma\backslash G)\\ &\quad \cdot \sum_{j=0}^{(d-2)/2} m_j(b) \frac{K_0(s-j-1;\, b, a(G))}{s-j-1},\\ \Phi_\Gamma^{(n)}(s;\, b, \chi) &= \frac{T_\Gamma(s;\, b, \chi)}{\Gamma(s)} + \frac{\pi^{1/2}}{4} C_G \chi(1)\operatorname{vol}(\Gamma\backslash G)\\ &\quad \cdot (b+n^2)^{-s+1/2} \sum_{j=0}^{n} m_j(b)(b+n^2)^j \frac{\Gamma(s-j-1/2)}{\Gamma(s-j)};\end{aligned}$$

see (3.9), (3.10), (3.12), Table 1. Let $D_\Gamma(s;\, b, \chi)(\operatorname{Re} s > \frac{d}{2})$ be the spectral series of (4.1). Then for $G = SO_1(2n, 1), Sp(n, 1),\ F_{4(-20)}$, or $SU(\ell, 1)$ with ℓ odd *(up to local isomorphism), $D_\Gamma(s;\, b, \chi) = \Phi_\Gamma(s;\, b, \chi)$ for $\operatorname{Re} s > \frac{d}{2}$. For $G = SO_1(2n+1, 1)$ (up to local isomorphism) $D_\Gamma(s;\, b, \chi) = \Phi_\Gamma^{(n)}(s;\, b, \chi)$ for $\operatorname{Re} s > \frac{d}{2} = n + \frac{1}{2}$. For $G = SU(\ell, 1)$ (up to local isomorphism) with ℓ* even*, $D_\Gamma(s\,;\, b, \chi) = \Phi_\Gamma(s\,;\, b, \chi) + \frac{1}{4}\chi(1)\operatorname{vol}(\Gamma\backslash G) C_G \sum_{j=0}^{\frac{d-2}{2}=\ell-1} m_j(b) L_0(s-j; b, a(G) = \frac{\pi}{2})$ for $\operatorname{Re} s > \frac{d}{2} = \ell$.*

The meromorphic continuation of $D_\Gamma(.\,;\, b,\, \chi)$ to $\mathbb{C}$ is provided therefore by $\Phi_\Gamma(.\,;\, b,\, \chi), \Phi_\Gamma^{(n)}(.\,;\, b,\, \chi)$, $L_0(\,;\, b, a(G))$. It is clear that $\Phi_\Gamma(.\,;\, b, \chi)$ is holomorphic except for possibly a simple pole at $s = j, j = 1, 2, \ldots, \frac{d}{2}$, with residue $\frac{1}{4} C_G \chi(1)\operatorname{vol}(\Gamma\backslash G) m_{j-1}(b)$ (using that $K_0(0;\, b, a) = \frac{2}{a}$). Similarly $\Phi_\Gamma^{(n)}(.\,;\, b,\, \chi)$ is holomorphic except for possibly a simple pole at $s = j + \frac{1}{2} - \ell$ where $j, \ell \geq 0$ are integers, $j \leq n$. In particular $\Phi_\Gamma(.\,;\, b,\, \chi), \Phi_\Gamma^{(n)}(.\,;\, b,\, \chi)$ are holomorphic at $s = 0$, and $s = \frac{d}{2}$ is a simple pole; cf. [**12**]. Special values of these functions at any negative integer are easily computed. Similarly we can immediately compute $L_0(s;\, b, a)$ at $s=$ any negative integer.

Since $K_{\frac{1}{2}}(r) = e^{-r}\sqrt{\frac{\pi}{2r}}$ for $r > 0$ formula (4.21) implies

$$T_\Gamma(0;\, b, \chi) = \sum_{\gamma \in C_\Gamma - \{1\}} \chi(\gamma) j(\gamma)^{-1} C(\gamma) e^{-t_\gamma \sqrt{b+\rho_0^2}}. \tag{4.24}$$

Hence

(4.25)

$$T_\Gamma(0;\, s(s-2\rho_0), \chi) = \sum_{\gamma \in C_{\Gamma - \{1\}}} \chi(\gamma) j(\gamma)^{-1} C(\gamma) e^{-t_\gamma (s-\rho_0)}$$

for $s \in R$ with $s > 2\rho_0$. By Proposition 2.1 of [**5**] we see that Equation (4.25) provides a link between our Φ_Γ (or $\Phi_\Gamma^{(n)}$) and Gangolli's Selberg zeta function $Z_\Gamma(\,\cdot\,,\chi)$ of X_Γ; cf. [**19**] for precise details. This link is exploited in [**20**] where it is shown to lead to a factorization of $Z_\Gamma(\,\cdot\,,\chi)$ in case $\chi = 1$ and $G \neq SU(n,1)$ with n *even*, the cotangent case in (3.8). However given the full generality of Theorem 4.2 we can now clearly carry out this factorization for arbitrary χ and for $G = SU(n,1)$ with n even - i.e. for all split rank 1 simple groups. In fact by working with the coefficients $m_j(b)$ of $\Pi(r\,;\,b+\rho_0^2)$ in (3.12) rather than the Miatello coefficients of $P(r)$ in (3.9) we obtain simpler formulas than those of [**18**].

Theorem 4.2 can also be used to prove a Weyl asymptotic law (Gelfand's conjecture) for the class 1 spectrum $\{\pi_j, n_j(\chi)\}_{j\geq 0}$: $\sum_{\lambda_j\leq t} n_j(\chi) \sim C(G)\chi(1)\,\mathrm{vol}(\Gamma\backslash G)t^{d/2}$ as $t \to \infty$ where $C(G)$ is a constant which depends only on G; see [**22**]. In principle, Theorem 4.2 contains more information on the class 1 spectrum than that provided in the asymptotic law. For this one needs, for example, suitable bounds like $|\Phi_\Gamma(s;\,1,\chi)| = O((\mathrm{Im}\,s)^P)$ on some vertical strip. We reserve such matters for future consideration.

5. Meromorphic continuation of $D_\Gamma(s;\,\chi)$.

A companion series to $D_\Gamma(s;\,b,\chi)$ in (4.1) is

$$D_\Gamma(s;\,\chi) \overset{\text{def}}{=} \sum_{j=1}^{\infty} \frac{n_j(\chi)}{\lambda_j^s}. \tag{5.1}$$

Note that for $j \geq 1$, $\left[\frac{1+\lambda_j}{\lambda_j}\right]^\sigma = \left[1+\frac{1}{\lambda_j}\right]^\sigma \to 1$ as $j \to \infty \Longrightarrow \left|\frac{(1+\lambda_j)^\sigma}{\lambda_j^\sigma} - 1\right| < 1$ for j sufficiently large, or $\frac{n_j(\chi)}{\lambda_j^\sigma} \leq \frac{2n_j(\chi)}{(1+\lambda_j)^\sigma}$ for j sufficiently large. Thus $D_\Gamma(s;\,\chi)$ converges absolutely for $\mathrm{Re}\,s > \frac{d}{2}$ since $D_\Gamma(s;\,1,\chi)$ does. Following much of the argument of Section 4 we can obtain the meromorphic continuation of $s \to D_\Gamma(s;\,\chi)$ to $\mathbb{C}$.

Since

$$\begin{aligned}
&\sum_{j=1}^{\infty}\int_0^\infty |n_j(\chi)e^{-\lambda_j t}t^{s-1}|dt \\
&= \sum_{j=1}^{\infty} n_j(\chi)\int_0^\infty e^{-\lambda_j t}t^{\mathrm{Re}\,s-1}dt \\
&= \left[\sum_{j=1}^{\infty}\frac{n_j(\chi)}{\lambda_j^{\mathrm{Re}\,s}}\right]\Gamma(\mathrm{Re}\,s) \quad (\text{as } \lambda_j > 0 \text{ for } j \geq 1) < \infty,
\end{aligned}$$

Fubini's theorem gives

$$
\begin{aligned}
(5.2) \qquad I(s;\,\chi) &\overset{\text{def}}{=} \int_0^\infty \left[\sum_{j=1}^\infty n_j(\chi)e^{-\lambda_j t}\right] t^{s-1}dt \\
&= \Gamma(s)D_\Gamma(s;\,\chi) \text{ for } \operatorname{Re} s > \frac{d}{2}.
\end{aligned}
$$

On the other hand, by (3.1), (3.5), (3.7), (4.18) where we take $b = 0$ in (4.17), (4.18)

$$
\begin{aligned}
(5.3) \qquad I(s;\,\chi) = \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)}{4\pi} &\int_0^\infty \int_R e^{-(\rho_0^2+r^2)t}|c(r)|^{-2}t^{s-1}drdt \\
&+ \int_0^\infty [\theta_\Gamma(t) - n_0(\chi)]\, t^{s-1}dt.
\end{aligned}
$$

The Fubini argument following (4.6) is valid for $b = 0$ and one sees therefore that Theorem 4.1 holds for $b = 0$ where the phrase "the right hand side of (4.6)" is replaced by the phrase "the right hand side of (5.3)". As in Equation (ii) preceding Lemma 4.1 we have (with $b = 0)\theta_1(t) + \theta_\Gamma(t) \overset{(\text{ii})'}{=} n_0(\chi) + \sum_{j=1}^\infty n_j(\chi)e^{-\lambda_j t}$, with $\theta_1(t) \overset{(\text{i})'}{\leq} \frac{\chi(1)\operatorname{vol}(\Gamma\backslash G)A(0)e^{-\rho_0^2 t}}{4\pi}$, as in (i). The proof of Lemma 4.1 works for $b = 0$ and thus given (i)′, (ii)′, Lemma 4.2 also holds for $b = 0$. It follows that in (5.3) $\int_1^\infty [\theta_\Gamma(t) - n_0(\chi)]t^{s-1}dt$ converges uniformly on compact subsets of the plane and is thus an entire function of s. Now $\int_0^1 [\theta_\Gamma(t) - n_0(\chi)]t^{s-1}dt = \int_0^1 \theta_\Gamma(t)t^{s-1}dt - \frac{n_0(\chi)}{s}$ for $\operatorname{Re} s > 0$ and if $\theta_\Gamma(t\,;\,1)$ is $\theta_\Gamma(t)$ in (4.18) with $b = 1$ there, then $\theta_\Gamma(t\,;\,1) = e^{-t}\theta_\Gamma(t)$, as we have written $\theta_\Gamma(t)$ for $\theta_\Gamma(t)$ in (4.18) with $b = 0$. By Lemma 4.3, $\lim_{t\to 0^+} e^{-t}\theta_\Gamma(t)e^{\epsilon/t} = 0 \implies \lim_{t\to 0^+}\theta_\Gamma(t)e^{\epsilon/t} = 0 \implies \int_0^1 \theta_\Gamma(t)t^{s-1}dt = \int_1^\infty \theta_\Gamma(\frac{1}{t})t^{-s-1}dt$ is also an entire function of s. We have proved (for the record):

Theorem 5.1. *Let $D_\Gamma(s;\,\chi)$ be the Dirichlet series of (5.1), which converges absolutely for $\operatorname{Re} s > \frac{d}{2}$. For $\operatorname{Re} s > \frac{d}{2}$ let $T_\Gamma(s;\,\chi) = \int_0^\infty [\theta_\Gamma(t) - n_0(\chi)]t^{s-1}dt$, where*

$$
(5.4) \qquad \theta_\Gamma(t) \overset{\text{def}}{=} \frac{1}{\sqrt{4\pi t}} \sum_{\gamma\in C_\Gamma - \{1\}} \chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)e^{-(\rho_0^2 t + t_\gamma^2/4t)}.
$$

Then $T_\Gamma(s;\,\chi) = \int_0^1 \theta_\Gamma(t)t^{s-1}dt - \frac{n_0(\chi)}{s} + \int_1^\infty [\theta_\Gamma(t) - n_0(\chi)]t^{s-1}dt$ where the 1^{st} and 3^{rd} integrals are entire functions of s. Therefore $\frac{T_\Gamma(s;\,\chi)}{\Gamma(s)}$ is an entire *function of* s (*where* $\frac{1}{\Gamma(s)s}$ *is defined to be* 1 *at* $s=0$). *Let*

$$\begin{aligned}\Phi_\Gamma(s;\,\chi) &= \frac{T_\Gamma(s;\,\chi)}{\Gamma(s)} + \frac{a(G)}{8}C_G\chi(1)\operatorname{vol}(\Gamma\backslash G)\\ &\quad\cdot\sum_{j=0}^{\frac{d-2}{2}}\frac{m_j(0)}{s-j-1}K_0(s-j-1;\,0,a(G))\\ \Phi_\Gamma^{(n)}(s;\,\chi) &= \frac{T_\Gamma(s;\chi)}{\Gamma(s)} + \frac{\pi^{1/2}}{4}C_G\chi(1)\operatorname{vol}(\Gamma\backslash G)\\ &\quad\cdot(n^2)^{-s+1/2}\sum_{j=0}^{n}m_j(0)(n^2)^j\frac{\Gamma(s-j-1/2)}{\Gamma(s-j)};\end{aligned}\tag{5.5}$$

see (3.9), (3.10), (3.12), *Table* 1, *and* (4.10), *where* K_0 *is also well-defined for* $b=0$. *Similarly we allow for* $b=0$ *in definition* (4.13) *of* L_0. *Then* (*as above*) $s\to K_0(s;\,0,a), L_0(s;0,a)$ *are entire functions of* s, *and for* $\operatorname{Re}s>\frac{d}{2}, D_\Gamma(s;\,\chi)=\Phi_\Gamma(s;\,\chi)$ *if* $G=SO_1(2n,1), Sp(n,1), F_{4(-20)}$, *or* $SU(\ell,1)$ *with* ℓ odd (*up to local isomorphism*). *If* $G=SU(\ell,1)$ (*up to local isomorphism*) *with* ℓ *even then* $D_\Gamma(s;\,\chi)=\Phi_\Gamma(s;\,\chi)+\frac{1}{4}\chi(1)\operatorname{vol}(\Gamma\backslash G)C_G\sum_{j=0}^{\ell-1}m_j(0)L_0(s-j,0,\frac{\pi}{2})$ *for* $\operatorname{Re}s>\frac{d}{2}=\ell$. *If* $G=SO_1(2n+1,1)$ (*up to local isomorphism*) *then for* $\operatorname{Re}s>\frac{d}{2}=n+\frac{1}{2}, D_\Gamma(s;\,\chi)=\Phi_\Gamma^{(n)}(s;\,\chi)$.

Consider the case $\operatorname{Re}s<0$: $\int_1^\infty t^{s-1}dt=-\frac{1}{s}\implies\int_1^\infty\theta_\Gamma(t)t^{s-1}dt=\int_1^\infty[\theta_\Gamma(t)-n_0(\chi)]t^{s-1}dt-\frac{n_0(\chi)}{s}=T_\Gamma(s;\,\chi)-\int_0^1\theta_\Gamma(t)t^{s-1}dt$; i.e.

$$T_\Gamma(s;\,\chi)=\int_0^\infty\theta_\Gamma(t)t^{s-1}dt\quad\text{for }\operatorname{Re}s<0.\tag{5.6}$$

The Fubini argument of (4.20) applies for $\operatorname{Re}s<0$ and we obtain

(5.7)

$$\begin{aligned}T_\Gamma(s;\,\chi) &= \sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)t_\gamma j(\gamma)^{-1}C(\gamma)\int_0^\infty\frac{e^{-(\rho_0^2t+t_\gamma^2/4t)}}{\sqrt{4\pi t}}t^{s-1}dt\\ &= \frac{\pi^{-1/2}}{[2\rho_0]^{s-1/2}}\sum_{\gamma\in C_\Gamma-\{1\}}\chi(\gamma)j(\gamma)^{-1}C(\gamma)t_\gamma^{s+1/2}K_{-s+1/2}(t_\gamma\rho_0)\end{aligned}$$

by the argument following (4.22):

Theorem 5.2. *In Theorem* 5.1, $T_\Gamma(s;\,\chi)$ *is given in terms of the* K*-Bessel function by Equation* (5.7) *in case* $\operatorname{Re}s<0$; *see* (4.22).

Randol's main result [**14**] (where $G=SL(2,R)$, with $\rho_0=\frac{1}{2}$) is contained in particular in Theorem 5.2.

References

[1] Harish-Chandra, *Representations of semisimple Lie groups* II, TAMS, **76** (1954), 26-65.

[2] D. DeGeorge, *Length spectrum for compact locally symmetric spaces of strictly negative curvature*, Ann. École Norm. Sup., **10** (1977), 133-152.

[3] R. Gangolli, *Asymptotic behavior of spectra of compact quotients of certain symmetric spaces*, Acta Math., **121** (1968), 151-192.

[4] ———, *On the length spectrum of certain compact manifolds of negative curvature*, J. Diff. Geom., **12** (1977), 403-424.

[5] ———, *Zeta functions of Selberg's type for compact space forms of symmetric spaces of rank one*, Ill. J. Math., **21** (1977), 1-42.

[6] R. Gangolli and G. Warner, *On Selberg's trace formula*, J. Math. Soc. Japan, **27** (1975), 328-343.

[7] S. Helgason, *Differential geometry and symmetric spaces*, Pure and Applied Math. Series, Vol. 12, Academic Press, 1962.

[8] J. Jorgenson and S. Lang, *Basic analysis of regularized series and products*, Springer-Verlag Lecture Notes in Math., **1564** (1993).

[9] R. Miatello, *The Minakshisundaram-Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature*, Thesis, Appendix 2, Rutgers Univ., 1976.

[10] ———, *The Minakshisundaram-Pleijel coefficients for the vector-valued heat kernel on compact locally symmetric spaces of negative curvature*, TAMS, **260** (1980), 1-33.

[11] ———, *On the Plancherel measure for linear Lie groups of rank one*, Manuscripta Math., **29** (1979), 249-276.

[12] S. Minakshisundaram and A. Pleijel, *Some properties of the eigenfunctions of the Laplace operator on Riemannian manifolds*, Canadian J. Math., **1** (1949), 242-256.

[13] G. Mostow, *Strong rigidity for locally symmetric spaces*, Annals of Math., Studies no. 78, Princeton Univ. Press, 1973.

[14] B. Randol, *On the analytic continuation of the Minakshisundaram-Pleijel zeta function for compact Riemann surfaces*, TAMS, **201** (1975), 241-246.

[15] A. Voros, *Spectral functions and the Selberg zeta function*, Comm. Math. Physics, **110** (1987), 439-465.

[16] N. Wallach, *On the Selberg trace formula in the case of compact quotient*, Bull. AMS, **2** (1976), 171-195.

[17] ———, *On an asymptotic formula of Gelfand and Gangolli for the spectrum of* $\Gamma\backslash G$, J. Diff. Geom., **11** (1976), 91-101.

[18] F. Williams, *Formula for the class* 1 *spectrum and the meromorphic continuation of Minakshisundaram-Pleijel series*, preprint, 1989.

[19] ———, *Some zeta functions attached to* $\Gamma\backslash G/K$, in 'New Developments in Lie Theory and Their Applications', Birkhäuser, Progress in Math., Series 105 (1992) 163-177 (J. Tirao, N. Wallach eds).

[20] ———, *A factorization of the Selberg zeta function attached to a rank* 1 *space form*, Manuscripta Math., **77** (1992) 17-39.

[21] ———, *Lectures on the spectrum of* $L^2(\Gamma\backslash G)$, Pitman Research Notes in Math., **242**, Longman House Pub., 1990.

[22] ———, *Spectral zeta series of rank* 1 *space forms*, in 'Representation Theory and Harmonic Anlaysis', A Conference in Honor of Ray A. Kunze, Contemporary Mathematics, **191** (1995), 245-254.

Received October 25, 1995 and revised April 22, 1997. This research was supported by NSF Grant No. DMS-8802597.

University of Massachusetts
Amherst, MA 01003-4515
E-mail address: williams@math.umass.edu