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RANK 2 VECTOR BUNDLES ON HIGHER DIMENSIONAL PROJECTIVE MANIFOLDS

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Fix a smooth complex projective n-fold X, $n \geq 5$, with $H^1(X, \mathcal{O}_X) = 0$. Then there is a smooth projective n-fold Y birational to X and $H \in \mathrm{Pic}(Y)$, H ample, with the following property. For any $d \in \mathbf{Z}$ let $M(Y, 2, \mathcal{O}, d, H)$ be the moduli scheme of H-stable rank 2 vector bundles, E, on Y with $\det(E) \cong \mathcal{O}_Y$ and $c_2(E) \cdot H^{n-2} = d$. Let $m(Y, 2, \mathcal{O}, d, H)$ be the number of its irreducible components. Then $\limsup_{d \to \infty} m(Y, 2, \mathcal{O}, d, H) = +\infty$.

Usually it is very hard to construct vector bundles of low rank on a projective manifold X of large dimension. Among the vector bundles, by far the more interesting ones are the stable ones. Here we will show that if we take a suitable birational model Y of X, then the construction is "easy" in the following sense.

Theorem 0.1. *Assume characteristic 0. Fix a smooth projective n-fold X, $n \geq 5$, with $H^1(X, \mathcal{O}_X) = 0$. Then there is a smooth projective n-fold Y birational to X and $H \in \mathrm{Pic}(Y)$, H ample, with the following property. For any $d \in \mathbf{Z}$ let $M(Y, 2, \mathcal{O}, d, H)$ be the moduli scheme of H-stable rank 2 vector bundles, E, on Y with $\det(E) \cong \mathcal{O}_Y$ and $c_2(E) \cdot H^{n-2} = d$. Let $m(Y, 2, \mathcal{O}, d, H)$ be the number of its irreducible components. Then $\limsup_{d \to \infty} m(Y, 2, \mathcal{O}, d, H) = +\infty$.*

Here, H-stability is in the sense of Mumford-Takemoto.

Remark 0.2. $M(Y, 2, \mathcal{O}, d, H)$ is an algebraic scheme and not only a locally algebraic scheme because the vector bundles in $M(Y, 2, \mathcal{O}, d, H)$ may have only finitely many Hilbert polynomials with respect to H. Hence $M(Y, 2, \mathcal{O}, d, H)$ is a the union of finitely many Maruyama's moduli schemes. A priori this number may grow to infinity as d grows.

Theorem 0.1 will be proved in the unique section of this note. For similar statements on the number of components for $Y = \mathbf{P}^3$ and rank 2 vector bundles, see [**E**], and for $Y = \mathbf{P}^5$ and rank 3 vector bundles, see [**AO**]. Here we will use in an essential way the rank 2 vector bundles on $\mathbf{P}^3$ constructed in [**E**] ("generalized null correlation bundles"). Our manifold Y will have a morphism $\pi : Y \to \mathbf{P}^3$ and our bundles will be the pull-back bundles $\pi^*(E)$

with E generalized null correlation bundle. The main points are to find a polarization H on Y (the same for all integers d) such that these bundles $\pi^*(E)$ are H-stable and to prove that these families are Zariski dense in their irreducible component of the moduli space $M(Y, 2, \mathcal{O}, d, H)$. A related construction was used in [**BM**].

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1. Proof of Theorem 0.1.

Fix a very ample line bundle A on X such that $h^i(X, A^{\otimes t}) = 0$ for all (i, t) with either $i < n$, $t < 0$ or $i > 0$ and $t > 0$. Let M be the complete intersection of 4 general divisors in $|A|$ and let Y be the blowing-up of X along M. Any such Y will be a birational model of X satisfying the hypothesis of Theorem 0.1. Note that $h^1(Y, \mathcal{O}_Y) = h^1(X, \mathcal{O}_X) = 0$. The 4 general divisors in $|A|$ cutting M define a morphism $\pi : Y \to \mathbf{P}^3$. The morphism π is flat because $\mathbf{P}^3$ and Y smooth and each fiber of π has dimension $n-3$ ([**H**], Ex. III.10.9). Our bundles on Y will be the pull-backs by π of the generalized null correlation bundles with trivial determinant introduced and studied in [**E**].

Lemma 1.1.

(a) *We have* $\pi_*(\mathcal{O}_{\mathbf{P}^3}) \cong \mathcal{O}_{\mathbf{P}^3}$.

(b) *If* $n \geq 5$, *we have* $R^1\pi_*(\mathcal{O}_Y) = 0$. *the same is true if* $X = \mathbf{P}^4$ *and* $\deg(A) = 1$ (*i.e.* Y *is a* $\mathbf{P}^1$-*bundle over* $\mathbf{P}^3$).

Proof. Since the general fibers of π are reduced and connected and π is flat, part (a) is obvious. Now we will check part (b). Assume $n \geq 5$. We take $n-2$ general divisors A_i, $1 \leq i \leq n-2$, in $|A|$ and we obtain $n-2$ exact sequences

$$0 \to \mathcal{O}_B(-(x+1)A) \to \mathcal{O}_B(-xA) \to \mathcal{O}_{B\cap A'}(-xA) \to 0 \tag{1}$$

with B and $B \cap A'$ complete intersection of divisors in $|A|$. Using Kodaira vanishing and the exact sequences (1) for all integers $x \leq 0$ we obtain the well-known fact that any complete intersection, U, of at most $n-2$ divisors of $|A|$ have $h^0(U, \mathcal{O}_T) = 1$, $h^i(U, \mathcal{O}_T(-yA)) = 0$ if $y < 0$ and $i < \dim(U)$, and that $h^1(U, \mathcal{O}_U) = 0$. In particular for every fiber T of π we have $h^1(T, \mathcal{O}_T) = 0$. Since $\mathbf{P}^3$ is reduced by a base — change theorem for cohomology ([**OSS**, p. 11]) we obtain $R^1\pi_*(\mathcal{O}_Y) = 0$. Now assume $X = \mathbf{P}^4$ and $\deg(A) = 1$. Hence $Y \cong \mathbf{P}(\mathcal{O}_{\mathbf{P}^3} \oplus \mathcal{O}_{\mathbf{P}^3}(-1))$ and π is the projection as $\mathbf{P}^1$-bundle. Again $h^1(T, \mathcal{O}_T(1)) = 0$ for every fiber of π and we conclude in the same way. □

Corollary 1.2. *Let U be vector bundle U on $\mathbf{P}^3$ and set $U' := \pi^*(U)$. We have $h^1(Y, U') = h^1(\mathbf{P}^3, U)$ and $h^0(Y, U') = h^0(\mathbf{P}^3, U)$.*

Proof. Use 1.1, the projection formula and the Leray spectral sequence of π. □

Corollary 1.3. *For every vector bundle E on $\mathbf{P}^3$ the tangent space to the deformation functor of E on $\mathbf{P}^3$ has the same dimension as the tangent space of the deformation functor of $\pi^*(E)$ on Y. For any two stable vector bundles E, E' with the same rank on $\mathbf{P}^3$ the bundle $\pi^*(E)$ is isomorphic to $\pi^*(E')$ if and only if E is isomorphic to E'.*

Proof. For the first part apply the first equality of 1.2 to the bundle $U := \mathrm{End}(E)$. For the second part apply the second equality of 1.2 to the bundle $U := \mathrm{Hom}(E, E')$. □

In [**E**] L. Ein constructed enough families of rank 2 vector bundles (called "generalized null correlation bundles") to prove the thesis of Theorem 0.1 taking $\mathbf{P}^3$ manifold Y. Each of these families were Zariski dense in a generically reduced component of the moduli scheme $M(\mathbf{P}^3, \mathcal{O}(1), \mathcal{O}, d)$ ([**E**, Th. 2.2]). Thus by Corollary 1.3 to prove Theorem 0.1 on Y it is sufficient to find a polarization H on Y such that all the bundles $\pi^*(E)$, E generalized null correlation bundle are H-stable. We will do this in two steps.

Step 1. Here we take as Y the blowing-up $\sigma : Y \to X := \mathbf{P}^n$ (any $n \geq 4$) of $\mathbf{P}^n$ along a linear subspace M of dimension $n-4$, i.e. we take $X = \mathbf{P}^n$ and $\deg(A) = 1$. The 4 hyperplanes defining M induce a morphism $\pi : Y \to \mathbf{P}^3$. Let D be the exceptional divisor of σ. Since the normal bundle of M in X is the direct sum of 4 line bundles of degree 1 we have $D \cong \mathbf{P}^3 \times \mathbf{P}^{n-4}$ and $\pi|D$ (resp. $\sigma|D$) are the projection on the first (resp. second factor). If $n = 4$, then $D \cong \mathbf{P}^3$ and the normal bundle of D in Y has degree -1. We leave the case $n = 4$ to the reader. From now on, we assume $n \geq 5$. We have $\mathrm{Pic}(D) \cong \mathbf{Z}^{\oplus 2}$ and as generators of $\mathrm{Pic}(D)$ we take $\mathbf{O}(1,0) := (\pi|D)^*(\mathcal{O}_{\mathbf{P}^3}(1))$ and $\mathbf{O}(0,1) := (\sigma|D)^*(\mathcal{O}_{\mathbf{P}^{n-4}}(1))$. We have $\mathrm{Pic}(Y) \cong \mathbf{Z}^{\oplus 2}$ and the restriction map $\mathrm{Pic}(Y) \to \mathrm{Pic}(D)$ is bijective. Set $O(1,0) := \pi^*(\mathcal{O}_{\mathbf{P}^3}(1))$ and $O(0,1) := \sigma^*(\mathcal{O}_X(1))$. Hence for all integers a, b we have $O(a,b)|D \cong \mathbf{O}(a,b)$. Fix a generalized null correlation bundle E on $\mathbf{P}^3$ and set $F := \pi^*(E)$. Let $M := O(x,y)$ be a saturated line bundle contained in F. Restricting $M|D$ and E to a general fiber of π we obtain $x \leq 0$. Since E is stable, restricting $M|D$ and $F|D$ to a general fiber of $\sigma|D$ we obtain $y \leq 0$. Fix an integer b with $0 \leq b \leq n-1$. Since D is a product of projective spaces, the product of b copies of $\mathbf{O}(1,0)$ and $n-1-b$ copies of $\mathbf{O}(0,1)$ in the Chow ring $A(D)$ of D is 0 if $b \neq 3$ and 1 if $b = 3$. Fix $H \in \mathrm{Pic}(Y)$, $H = O(u,v)$ with $u > 0$

and $v > 0$, and ample. Since both $O(1,0)$ and $O(0,1)$ are effective, we have $H^{n-1} \cdot M < 0$. Hence F is H-stable in the sense of Mumford-Takemoto.

Step 2. Now we consider the general case (here $n \geq 5$). We take a general n-dimensional projective subspace $|V| \subseteq |A|$ containing the $n-4$ divisors defining M. We may assume $|V|$ base point free and hence defining a morphism $f : X \to \mathbf{P}^n$ such that the image of M is a linear subspace M' of dimension $n-4$. Call Y' the blowing-up of $\mathbf{P}^n$ along M' and apply Step 1 to Y'. By construction we have a finite map $g : Y \to Y'$ and we will take $H := g^*(H')$ as polarization on Y with H' any polarization on Y' used in Step 1. We have to check that if F is a rank 2 H'-stable bundle on Y', then $g^*(F)$ is H-stable. Let $u : C \to B$ be a finite map of smooth projective curves and G a rank 2 stable vector bundle on B. Assume that u does not factor through an étale cover of degree > 1. Then, assuming characteristic 0, $u^*(G)$ is stable ([**L**, Remark 2.3]). Taking as B a general complete intersection curve of divisors in $\mathbf{P}(H^0(Y', H'^{,\otimes t}))$ with t large and as u the restriction of g, we conclude if this restriction map does not factor through an étale cover of degree > 1. Taking A sufficiently ample and $|V|$ general we may assume that $g : Y \to Y'$ has irreducible ramification locus R and that for a general $x \in R$ we have $\text{card}\,(g^{-1}(x)) = \deg(g) - 1$. In this way we reduce to a case in which the monodromy group of $u : C \to B$ is transitive and generated by simple transpositions, i.e. it is the full symmetric group $S_{\deg(u)}$. Hence u does not factor non trivially and we conclude the proof of Theorem 0.1.

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