PACIFIC JOURNAL OF MATHEMATICS
Vol. 184, No. 1, 1998

CUNTZ-KRIEGER ALGEBRAS OF DIRECTED GRAPHS

ALEX KUMJIAN, DAVID PASK AND IAIN RAEBURN

We associate to each row-finite directed graph E a universal Cuntz-Krieger C^*-algebra $C^*(E)$, and study how the distribution of loops in E affects the structure of $C^*(E)$. We prove that $C^*(E)$ is AF if and only if E has no loops. We describe an exit condition (L) on loops in E which allows us to prove an analogue of the Cuntz-Krieger uniqueness theorem and give a characterisation of when $C^*(E)$ is purely infinite. If the graph E satisfies (L) and is cofinal, then we have a dichotomy: if E has no loops, then $C^*(E)$ is AF; if E has a loop, then $C^*(E)$ is purely infinite.

If A is an $n \times n$ $\{0,1\}$-matrix, a *Cuntz-Krieger A-family* consists of n partial isometries S_i on Hilbert space satisfying

$$S_i^* S_i = \sum_{j=1}^{n} A(i,j) S_j S_j^*. \tag{1}$$

Cuntz and Krieger proved that, provided A satisfies a fullness condition (I) and the partial isometries are all nonzero, two such families generate isomorphic C^*-algebras; thus the *Cuntz-Krieger algebra* $\mathcal{O}_A$ can be well-defined as the C^*-algebra generated by any such family $\{S_i\}$ [5]. Subsequently, for A satisfying a stronger condition (II), Cuntz analysed the ideal theory of $\mathcal{O}_A$ [4]. The relations (1) make sense for infinite matrices A, provided the rows of A contain only finitely many 1's; under a condition (K) analogous to (II), the Cuntz-Krieger uniqueness theorem and Cuntz's description of the ideals in $\mathcal{O}_A$ carry over [6].

In [6], A arose as the connectivity matrix of a directed graph E, and $\mathcal{O}_A$ was realised as the C^*-algebra of a locally compact groupoid $\mathcal{G}_E$ with unit space the infinite path space of the graph E. The condition (K) has a natural graph-theoretic interpretation, and the main theorem of [6] relates the loop structure in E to the ideal structure of $C^*(\mathcal{G}_E)$. From this point of view, it is natural to ask if there is a graph-theoretic analogue of the original condition (I) which allows one to extend the uniqueness theorem of [5] to infinite matrices and graphs.

Here we shall discuss such a condition (L): a graph E satisfies (L) if all loops in E have exits. It is important to realise that in an infinite graph E, there can be very few loops, and thus condition (L) may be trivially satisfied. We shall show that if E satisfies (L) and a cofinality hypothesis, then $C^*(E)$ is simple and there is a dichotomy: if E has no loops, $C^*(E)$ is AF, whereas if E has a loop, $C^*(E) = C^*(\mathcal{G}_E)$ is purely infinite.

We begin with our analysis of the case where E has no loops. To prove that $C^*(E)$ is AF requires approximating E by finite subgraphs; these subgraphs may have sinks (vertices which emit no edges), and hence do not belong to the class studied in [6]. We therefore introduce a slightly different notion of Cuntz-Krieger E-family, which involves projections parametrised by the vertices as well as partial isometries parametrised by the edges, and a C^*-algebra $C^*(E)$ which is universal for such families (Theorem 1.2). We then prove that $C^*(E)$ is AF if and only if E has no loops (Theorem 2.4).

When E has no sinks, the results of [6] show that $C^*(E) = C^*(\mathcal{G}_E)$, and we can therefore use groupoid techniques to analyse $C^*(E)$. Our main contribution here is the introduction of the condition (L), which we show is a good analogue for infinite graphs of the condition (I) of [5]. In particular, we prove a version of the Cuntz-Krieger uniqueness theorem for graphs E satisfying (L) (Theorem 3.7). We then prove that $C^*(E)$ is purely infinite if and only if E satisfies (L) and every vertex of E connects to a loop (Theorem 3.9); from this, our dichotomy follows easily.

1. The universal C^*-algebra of a graph.

A *directed graph* E consists of countable sets E^0 of vertices and E^1 of edges, and maps $r, s : E^1 \to E^0$ describing the range and source of edges. The graph E is *row-finite* if for every $v \in E^0$, the set $s^{-1}(v) \subseteq E^1$ is finite; if in addition $r^{-1}(v)$ is finite for all $v \in E^0$, then E is *locally finite*. For $n \geq 2$, we define

$$E^n := \{\alpha = (\alpha_1, \dots, \alpha_n) : \alpha_i \in E^1 \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for } 1 \leq i \leq n-1\},$$

and $E^* = \cup_{n \geq 0} E^n$. For $\alpha \in E^n$, we write $|\alpha| := n$. The maps r, s extend naturally to E^*; for $v \in E^0$, we define $r(v) = s(v) = v$. The infinite path space is

$$E^\infty = \{(\alpha_i)_{i=1}^\infty : \alpha_i \in E^1 \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for } i \geq 1\}.$$

A vertex $v \in E^0$ which emits no edges is called a *sink*.

If E is a row finite directed graph, a *Cuntz-Krieger E-family* consists of a set $\{P_v : v \in E^0\}$ of mutually orthogonal projections and a set $\{S_e : e \in E^1\}$ of partial isometries satisfying

$$(2) \qquad S_e^* S_e = P_{r(e)} \text{ for } e \in E^1, \quad \text{and} \quad P_v = \sum_{\{e : s(e) = v\}} S_e S_e^* \text{ for } v \in s(E^1).$$

The *edge matrix* of E is the $E^1 \times E^1$ matrix defined by

$$A_E(e,f) := \begin{cases} 1 & \text{if } r(e) = s(f) \\ 0 & \text{otherwise.} \end{cases}$$

A Cuntz-Krieger E-family $\{P_v, S_e\}$ satisfies

$$S_e^* S_e = \sum_{\{f : s(f) = r(e)\}} S_f S_f^* = \sum_{f \in E} A_E(e,f) S_f S_f^*$$

for every e such that $A_E(e,\cdot)$ has nonzero entries. Thus if E has no sinks, $\{S_e : e \in E^1\}$ is a Cuntz-Krieger A_E-family in the sense of (1). (We warn that the projections $\{P_v\}$ are the *initial* projections of the partial isometries S_e with $r(e) = v$, and not the range projections as in [5].) The point of our new definition is that the projection P_v can be nonzero even if there are no edges coming out of v.

Not every $\{0,1\}$-matrix is the edge matrix of a directed graph, but for any $V \times V$ matrix B with entries in $\{0,1\}$, we can construct a graph E with vertex set $E^0 = V$ by joining v to w iff $B(v,w) = 1$, and then there is a natural bijection between Cuntz-Krieger B-families associated to B and those associated to the corresponding edge matrix A_E [8, Proposition 4.1].

If E is a directed graph and $\{P_v, S_e\}$ is a Cuntz-Krieger E-family, then $S_e S_f \neq 0$ only if $r(e) = s(f)$; if each S_e is non-zero, so is $S_e S_f$. More generally, if $\alpha \in E^n$, then $S_\alpha = S_{\alpha_1} \dots S_{\alpha_{|\alpha|}}$ is a nonzero partial isometry with $S_\alpha^* S_\alpha = P_{r(\alpha)}$ and $S_\alpha S_\alpha^* \leq P_{s(\alpha)}$. ($S_v := P_v$ for $v \in E^0$).

Lemma 1.1. *Let $\{S_e, P_v\}$ be a Cuntz-Krieger E-family, and $\beta, \gamma \in E^*$. Then*

$$S_\beta^* S_\gamma = \begin{cases} S_{\gamma'} & \text{if } \gamma = \beta\gamma',\ \gamma' \notin E^0 \\ P_{r(\gamma)} & \text{if } \gamma = \beta \\ S_{\beta'}^* & \text{if } \beta = \gamma\beta',\ \beta' \notin E^0 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Moreover, every non-zero word in S_e, P_v and S_f^* is a partial isometry of the form $S_\alpha S_\beta^*$ for some $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$.

Proof. If β or $\gamma \in E^0$, then (3) is an easy calculation. Now for $e, f \in E^1$ we have $S_e^* S_f = 0$ unless $e = f$, so

$$S_e^* S_\gamma = \delta_{e,\gamma_1} S_{\gamma_1}^* S_{\gamma_1} S_{\gamma_2} \dots S_{\gamma_{|\gamma|}};$$

because $r(\gamma_1) = s(\gamma_2)$, we have $S_{\gamma_1}^* S_{\gamma_1} \geq S_{\gamma_2} S_{\gamma_2}^*$, and

$$S_e^* S_\gamma = \delta_{e,\gamma_1} S_{\gamma_2} S_{\gamma_2}^* S_{\gamma_2} \dots S_{\gamma_{|\gamma|}} = \delta_{e,\gamma_1} S_{\gamma_2} \dots S_{\gamma_{|\gamma|}}. \tag{4}$$

Repeated calculations of the form (4) show that $S_\beta^* S_\gamma = 0$ unless either γ extends β or β extends γ; suppose for the sake of argument that $\gamma = \beta\gamma'$ extends β. Then

$$S_\beta^* S_\beta S_{\gamma'} = S_{\beta_{|\beta|}}^* S_{\beta_{|\beta|}} S_{\gamma'} = P_{r(\beta)} S_{\gamma'} = S_{\gamma'}$$

as required.

Since $S_\alpha^* S_\alpha = P_{r(\alpha)}$, S_α is a partial isometry with initial projection $P_{r(\alpha)}$. Thus S_β^* is a partial isometry with range space $P_{r(\beta)}$, and we can deduce from the orthogonality of the P_v that $S_\alpha S_\beta^*$ is a nonzero partial isometry only if $r(\alpha) = r(\beta)$. Repeated applications of (3) in various cases then gives us the desired result. □

Theorem 1.2. *Let E be a directed graph. Then there is a C^*-algebra B generated by a Cuntz-Krieger E-family $\{s_e, p_v\}$ of non-zero elements such that, for every Cuntz-Krieger E-family $\{S_e, P_v\}$ of partial isometries on $\mathcal{H}$, there is a representation π of B on $\mathcal{H}$ such that $\pi(s_e) = S_e$ and $\pi(p_v) = P_v$ for all $e \in E^1, v \in E^0$.*

Proof. We only give an outline here, as the argument closely follows that of [3, Theorem 2.1]. Let $S_E = \{(\alpha, \beta) : \alpha, \beta \in E^* \text{ and } r(\alpha) = r(\beta)\}$, and let k_E be the space of functions of finite support on S_E. The set of point masses $\{e_\lambda : \lambda \in S_E\}$ forms a basis for k_E. By thinking of $e_{(\alpha,\beta)}$ as $S_\alpha S_\beta^*$ and using the formulas in (3), we can define an associative multiplication and involution on k_E such that k_E is a *-algebra.

As a *-algebra, k_E is generated by $q_v := e_{(v,v)}$ and $t_e := e_{(e,r(e))}$: indeed, $e_{(\alpha,\beta)} = t_\alpha t_\beta^*$. The elements q_v are orthogonal projections such that $q_v \geq \sum_{\{e:s(e)=v\}} t_e t_e^*$. If we mod out the ideal J generated by the elements $q_v - \sum_{\{e:s(e)=v\}} t_e t_e^*$ for $v \in E^0$, then the images r_v of q_v and u_e of t_e in k_E/J form a Cuntz-Krieger E-family which generates k_E/J. The triple $(k_E/J, q_v, u_e)$ then has the required universal property, though k_E/J is not a C^*-algebra. However, a standard argument shows that

$$\|a\|_0 := \sup\{\|\pi(a)\| : \pi \text{ is a non-degenerate } ^*\text{-representation of } k_E/J\}$$

is a well-defined, bounded seminorm on k_E/J. The completion B of

$$(k_E/J)/\{b \in k_E/J : \| b \|_0 = 0\}$$

is a C^*-algebra with the same representation theory as k_E/J. Thus if p_v and s_e are the images of r_v and u_e in B, then (B, p_v, s_e) has all the required properties.

There is a Cuntz-Krieger E-family in which each P_v and S_e is non-zero: for each vertex take an infinite-dimensional Hilbert space $\mathcal{H}_v$, decompose it into orthogonal infinite-dimensional subspaces $\mathcal{H}_e$ corresponding to the edges e with source v, choose isometries S_e of each $\mathcal{H}_{r(e)}$ onto the subspaces $\mathcal{H}_e$, and set $\mathcal{H} = \oplus\mathcal{H}_v$. Thus each p_v and each s_e in $C^*(E)$ must be nonzero. □

Remark 1.3. The triple (B, p_v, s_e) is unique up to isomorphism, and hence we write $C^*(E)$ for B. If E has no sinks, then the projections p_v are redundant, and the Cuntz-Krieger E-families are the Cuntz-Krieger families for the edge matrix A_E. Thus [6, Theorem 4.2] implies that $C^*(E)$ has a groupoid model: $(C^*(E), s_e) = (C^*(\mathcal{G}_E), 1_{Z(e,r(e))})$.

Proposition 1.4. *The C^*-algebra $C^*(E)$ is unital if and only if E^0 is finite.*

Proof. If E^0 is finite, $\sum_{v\in E^0} p_v$ is a unit for $C^*(E)$. If $E^0 = \{v_n\}_{n=1}^{\infty}$, then $q_n = \sum_{i=1}^{n} p_{v_i}$ is a strictly increasing approximate unit for $C^*(E)$. If $C^*(E)$ has a unit 1, then $q_n \to 1$ in norm, which forces $q_n = 1$ for large n; since $q_n p_{v_{n+1}} = 0$, this is impossible. □

2. Directed graphs with no loops.

Let E be a directed graph. A path $\alpha \in E^*$ with $|\alpha| > 0$ is a *loop based at* v, or a *return path for* v, if $s(\alpha) = r(\alpha) = v$; the loop is *simple* if the vertices $\{r(\alpha_i) : 1 \le i \le |\alpha|\}$ are distinct.

Proposition 2.1. *Suppose H is a subgraph of E with no exits* (*i.e.* $e \in E^1$, $s(e) \in H^0$ *imply* $e \in H^1$). *Then*

$$I := \overline{\operatorname{span}}\,\{s_\alpha s_\beta^* : \alpha, \beta \in E^*,\ r(\alpha) = r(\beta) \in H^0\}$$

is an ideal of $C^*(E)$ which is Morita equivalent to $B := \overline{\operatorname{span}}\,\{s_\alpha s_\beta^* : \alpha, \beta \in H^*\}$.

Proof. Because H has no exits, $r(\alpha) \in H^0$ and $\alpha\gamma' \in E^*$ imply $r(\gamma') \in H^0$. It therefore follows from (3) that when $r(\alpha) \in H^0$, every product $(s_\alpha s_\beta^*)(s_\gamma s_\delta^*)$ is either 0 or has the form $s_\mu s_\nu^*$ where $r(\mu) = r(\nu) \in H^0$. Thus I is indeed an ideal. The same argument shows that

$$X := \overline{\operatorname{span}}\,\{s_\alpha s_\beta^* : \alpha \in H^*, \beta \in E^* \text{ and } r(\alpha) = r(\beta) \in H^0\}$$

is a right ideal in $C^*(E)$, which satisfies $XX^* = B$ and $X^*X = I$. □

Corollary 2.2. *If v is a sink, then $I_v := \overline{\operatorname{span}}\,\{s_\alpha s_\beta^* : \alpha, \beta \in E^*, r(\alpha) = r(\beta) = v\}$ is a closed two-sided ideal in $C^*(E)$; I_v is isomorphic to the algebra $\mathcal{K}(\ell^2(E^*(v)))$, where $E^*(v) := \{\alpha \in E^* : r(\alpha) = v\}$ (which is non-empty, because $v \in E^*(v)$).*

Proof. If $r(\alpha) = v$, there are no paths of the form $\alpha\gamma'$, so the formulas (3) show that

$$(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) = \begin{cases} 0 & \text{unless } \beta = \gamma, \\ s_\alpha s_\delta^* & \text{if } \beta = \gamma. \end{cases}$$

Thus $\{s_\alpha s_\beta^* : r(\alpha) = r(\beta) = v\}$ is a family of matrix units, and $I_v \cong \mathcal{K}(\ell^2(E^*(v)))$. □

Corollary 2.3. *Suppose E is a finite graph with no loops, $v_1, \ldots, v_k$ are the sinks, and $n(v_i) := \#\{\alpha \in E^* : r(\alpha) = v_i\}$. Then*

$$C^*(E) = \bigoplus_{i=1}^{k} I_{v_i} \cong \bigoplus_{i=1}^{k} M_{n(v_i)}(\mathbf{C}).$$

Proof. Let $s_\alpha s_\beta^* \in C^*(E)$. If $r(\alpha) \neq v_i$ for some i, then $r(\alpha)$ is not a sink. Thus

$$s_\alpha s_\beta^* = \sum_{\{e : s(e) = r(\alpha)\}} s_\alpha s_e s_e^* s_\beta^*.$$

Since the graph is finite, and there are no loops, repeating this process must eventually realise $s_\alpha s_\beta^*$ as a finite sum of terms of the form $s_{\alpha\gamma} s_{\beta\gamma}^*$ where $r(\gamma) = v_i$ for some i. Thus the ideals I_{v_i} span $C^*(E)$.

On the other hand, suppose $r(\alpha) = v_i = r(\beta)$ and $r(\gamma) = v_j = r(\delta)$. Then the absence of paths of the form $\beta\gamma'$ and $\gamma\beta'$ implies that $(s_\alpha s_\beta^*)(s_\gamma s_\delta^*) = 0$ unless $v_i = v_j$. Thus $I_{v_i} I_{v_j} = 0$ unless $i = j$, and we have a direct sum decomposition. For each i, there are $n(v_i)$ distinct paths α with $r(\alpha) = v_i$; thus by 2.2 we have $I_{v_i} \cong M_{n(v_i)}(\mathbf{C})$. □

Theorem 2.4. *A directed graph E has no loops if and only if $C^*(E)$ is an AF algebra.*

Proof. First suppose that E has no loops. We have to prove that every finite set of elements of $C^*(E)$ can be approximated by elements lying in a finite-dimensional subalgebra. Since the elements $s_\alpha s_\beta^*$ span a dense subspace of $C^*(E)$, it is enough to show that each finite set of such elements lies in a finite-dimensional subalgebra. So suppose F is a finite set of pairs $(\alpha, \beta) \in E^* \times E^*$ satisfying $r(\alpha) = r(\beta)$. Let G be the finite subgraph of E consisting of all edges e occurring in the paths $\{\alpha, \beta : (\alpha, \beta) \in F\}$, and all their vertices $r(e)$, $s(e)$. Let H be the subgraph obtained by adding to G all edges f such that $s(f) = s(e)$ for some $e \in G^1$, and the ranges of such edges. Since each vertex emits only finitely many edges, H is also a finite subgraph of E. For each vertex $v \in H^0$, either all edges $e \in E^1$ with $s(e) = v$ lie in H^1, or none do; thus for those with $\{e \in H^1 : s(e) = v\} \neq \emptyset$, the Cuntz-Krieger relation

$$p_v = \sum_{\{e\in H^1 : s(e)=v\}} s_e {s_e}^*$$

follows from the corresponding relation in $C^*(E)$. Thus $\{p_v, s_e : v, e \in H\}$ is a Cuntz-Krieger family for the graph H, and the universal property of $C^*(H)$ gives a homomorphism of $C^*(H)$ onto the subalgebra of $C^*(E)$ generated by $\{p_v, s_e : v, e \in H\}$. Since E has no loops, neither does H, and Corollary 2.3 implies that $C^*(H)$ and its image in $C^*(E)$ are finite dimensional. Since each $s_\alpha s_\beta^*$ with $(\alpha, \beta) \in F$ lies in this image, this proves that $C^*(E)$ is AF.

Next, suppose that E has a loop α with $|\alpha| \geq 1$. Then either α has an exit or it does not. If α has an exit, then without loss of generality we may assume that this occurs at $v = s(\alpha)$. If $f \neq \alpha_1$ satisfies $s(f) = v$, then because the ranges of the partial isometries s_f and s_{α_1} are orthogonal, we have

$$p_v = s_\alpha^* s_\alpha \sim s_\alpha s_\alpha^* \leq s_{\alpha_1} s_{\alpha_1}^* < s_{\alpha_1} s_{\alpha_1}^* + s_f s_f^* \leq p_v,$$

and so p_v is an infinite projection. A projection in an AF algebra is equivalent to one in a finite-dimensional subalgebra, and hence cannot be infinite; thus $C^*(E)$ cannot be AF.

If α does not have any exits, we may as well assume that α is a simple loop. Let $v = s(\alpha)$. Then by 2.1

$$I_v = \overline{\operatorname{span}}\,\{s_\gamma s_\delta^* : \gamma, \delta \in E^*, r(\gamma) = r(\delta) = v\}$$

is a two-sided ideal in $C^*(E)$. If

$$B_\alpha = \overline{\operatorname{span}}\{s_\gamma {s_\delta}^* : \gamma, \delta \in E^*, r(\gamma) = s(\gamma) = s(\delta) = v\},$$

then an argument similar to the proof of 2.1 shows that B_α is Morita equivalent to I_v. We claim that B_α is generated by a unitary with full spectrum. Since $s_\alpha s_\alpha^* = s_\alpha^* s_\alpha = p_v = 1_{B_\alpha}$, s_α is unitary in B_α. Moreover, if $x = s_\gamma s_\delta^* \in B_\alpha$, then $\gamma = \alpha^n$, $\delta = \alpha^m$ for some n, m, and $x = (s_\alpha)^{n-m}$; thus s_α generates B_α.

To see that s_α has full spectrum, let $J = \{(\gamma, \delta) : s(\delta) = r(\delta) = v\} \subseteq k_E$, and $\mathcal{H} = \ell^2(J)$, with orthonormal basis $\{e_{(\gamma,\delta)}\}$. For $f \in E^1$ and $v \in E^0$ define $S_f, P_v \in \mathcal{B}(\mathcal{H})$ by

$$S_f e_{(\gamma,\delta)} = \begin{cases} e_{(f\gamma,\delta)} & \text{if } r(f) = s(\gamma) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad P_v e_{(\gamma,\delta)} = \begin{cases} e_{(\gamma,\delta)} & \text{if } v = s(\gamma) \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{S_f, P_v\}$ is a Cuntz-Krieger E-family on $\mathcal{H}$. Because $e_{(\gamma\alpha,\delta\alpha)} = e_{(\gamma,\delta)}$, $\mathcal{H}_\alpha := S_\alpha \mathcal{H}$ is spanned by $\{e_{(\alpha^n,v)}, e_{(v,\alpha^n)} : n \geq 0\}$, where $\alpha^0 := v$. Since

$$S_\alpha e_{(\alpha^n,v)} = e_{(\alpha^{n+1},v)} \text{ for } n \geq 0, \quad \text{and} \quad S_\alpha e_{(v,\alpha^n)} = e_{(v,\alpha^{n-1})} \text{ for } n \geq 1,$$

the action of S_α on $\mathcal{H}_\alpha$ is conjugate to the shift on $\ell^2(\mathbf{Z})$, and hence has full spectrum.

We have now shown that $C^*(E)$ has an ideal which is Morita equivalent to an algebra $B_\alpha = C^*(s_\alpha) \cong C(T)$ which is not AF, and so $C^*(E)$ cannot be AF. □

Remark 2.5. (1) Although it was not necessary for the above argument, the homomorphism π of $C^*(H)$ into $C^*(E)$ is always injective. To see this, just note that by 1.2 each projection p_v is nonzero, and hence none of the ideals I_v can be in the kernel of π. Since each ideal is simple, and $C^*(H)$ is the direct sum of such ideals, we deduce that ker $\pi = \{0\}$, as claimed.

This observation means that, by arbitrarily increasing the set F of allowable pairs (α, β), we can obtain a specific description of $C^*(E)$ as an increasing union of finite-dimensional C^*-algebras of the form $C^*(H)$.

(2) The representation constructed at the end of the proof can be viewed as a representation of $C^*(\mathcal{G}_E)$ induced from a 1-dimensional representation of $C_0(E^\infty)$: let $x = \alpha\alpha\ldots \in E^\infty$, take $\mathcal{H} = \ell^2(s^{-1}(x))$ and let $\mathcal{G}_E$ act on $\mathcal{H}$ by multiplication.

3. Directed graphs with sufficiently many loops.

In this section E will be a locally finite graph with no sinks, and $\mathcal{G}_E$ such that $C^*(E) \cong C^*(\mathcal{G}_E)$ (see Remark 1.3). We aim to show using the ideas of [2, §2] that, if E has enough loops, then $C^*(\mathcal{G}_E)$ is purely infinite: the graph E satisfies our analogue (L) of Cuntz and Krieger's condition (I) precisely when the groupoid $\mathcal{G}_E$ is essentially free in the sense of [2, Definitions 1.1.2].

Let $\mathcal{G}$ be a locally compact groupoid $\mathcal{G}$ with range and source maps r, s and unit space $\mathcal{G}^{(0)}$. The *isotropy group* of $u \in \mathcal{G}^{(0)}$ is the set $\mathcal{G}(u) = r^{-1}(u) \cap s^{-1}(u) \subset \mathcal{G}$, which turns out to be a group. As in [2], we say $\mathcal{G}$ is *essentially free* if the set of points with trivial isotropy is dense in $\mathcal{G}^{(0)}$. (Warning: this need not be the same as the definition in [11] if the groupoid is not minimal, but is consistent with the definitions of [2, 7].) A subset B of $\mathcal{G}$ is a *bisection* (or $\mathcal{G}$-set in [10, Definition I.1.10]) if r and s are one-to-one on B; if $\mathcal{G}$ is r-discrete, then $\mathcal{G}$ has a basis of open bisections. For an open bisection B of an r-discrete groupoid, the map $\alpha_B : x \mapsto s(xB)$ is a homeomorphism of $r(B)$ onto $s(B)$.

Example 3.1. For $\alpha, \beta \in E^*$ with $r(\alpha) = r(\beta)$, the set $B := Z(\alpha, \beta)$ is a bisection of $\mathcal{G}_E$, with $\alpha_B : Z(\alpha) \to Z(\beta)$ given by $\alpha_B(\alpha z) = \beta z$ (see [6, Proposition 2.6]).

Lemma 3.2. *A unit $x \in E^\infty = \mathcal{G}_E^{(0)}$ has non-trivial isotropy if and only if x is eventually periodic.*

Proof. Just note that x is eventually periodic with period k iff $(x, k, x) \in \mathcal{G}_E$. ☐

Recall from [6] that the directed graph E is *cofinal* if for every $x \in E^\infty$ and $v \in E^0$, there exists $\alpha \in E^*$ with $s(\alpha) = v$ and $r(\alpha) = s(x_n)$ for some n. Let V_0 be the set of vertices in E^0 with no return paths, V_1 the set of vertices with precisely one simple return path, and $V_2 := E^0 \setminus (V_0 \cup V_1)$. The graph E satisfies Condition (I) if every vertex v connects to a vertex in V_2; we shall say that E satisfies Condition (L) if every loop has an exit.

Lemma 3.3. *Let E be a directed graph. If E^0 is finite, then* (L) *is equivalent to* (I)*. If E^0 is infinite, then* (L) *is weaker than* (I)*.*

Proof. Suppose that E^0 is finite and that E satisfies (L). Let $v \in E^0$. Since every path $x \in E^\infty$ starting at v must pass through some vertex infinitely often, v connects to a loop. By hypothesis every loop has an exit, so v connects via a finite path β to a vertex w with a return path α which has an exit at w. Let $f \in E^1$ satisfy $s(f) = w$ but $f \neq \alpha_1$. Let $v' = r(f)$, and consider any infinite path x' starting at v'. If x' visits any vertex on the paths α, β, then w has two distinct return paths, and E satisfies (I). If x' does not visit any vertex on α or β, then we are back in the original situation with v' instead of v but with fewer vertices to choose from. Since the number of vertices is finite this process must terminate, and hence E satisfies (I).

Now suppose that E satisfies (I). If $\alpha \in E^*$ is a loop without an exit, then $w = r(\alpha_1)$ connects only to vertices in V_1, which contradicts (I). Hence every loop has an exit, and E satisfies (L). To see that (L) is weaker, note that the directed graph

$$\cdots \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$$

trivially satisfies satisfies (L), but does not satisfy (I). ☐

Lemma 3.4. *The groupoid $\mathcal{G}_E$ is essentially free if and only if E satisfies* (L)*.*

Proof. Suppose that E satisfies (L). By [6, Corollary 2.2] the cylinder sets $Z(\alpha)$ for $\alpha \in E^*$ form a basis for the topology of $\mathcal{G}_E^{(0)}$. By 3.2, it is enough to show that every such set contains an aperiodic path, and hence enough to show that every vertex $v \in E^0$ is the source of an aperiodic path. If v connects via $\alpha \in E^*$ to a vertex $w \in V_2$, then the technique of [6, Proposition 6.3] gives a path x which is aperiodic and hence has no isotropy. Hence we may assume that v does not connect to V_2.

We now construct a path starting at v which does not pass through the same vertex twice. For v and every vertex w reachable from v, we associate an edge $\gamma(w)$ which does not form part of a loop wherever this is possible. Specifically, if $w \in V_0$, choose for $\gamma(w)$ any edge e with $s(e) = w$; if $w \in V_1$, and w emits two edges $e, f \in E^1$, then choose for $\gamma(w)$ an edge f which is not on the return path for w; if w emits only one edge e, choose $\gamma(w) = e$. Now define $x \in E^\infty$ recursively by setting $x_1 := \gamma(v)$, and $x_i := \gamma(r(x_{i-1}))$ for $i \geq 2$. To see that x does not pass through the same vertex twice, suppose there is a vertex w such that $s(x_n) = w = r(x_m)$ for some $m \geq n$. Then every vertex u on $\alpha := (x_n, \dots, x_m)$ is in V_1, and if there were an exit from u, it would have been taken. Hence the return path α for w has no exits, which contradicts the premise that E satisfies (L). In particular, we deduce that x is an aperiodic path starting at v.

Now suppose that E does not satisfy condition (L), so there is a vertex $v \in E^0$ and a return path $\alpha \in E^*$ for v without an exit. Then the only path starting at v is $x = \alpha\alpha\dots$, so $Z(\alpha) = \{x\}$, and x has isotropy group isomorphic to $\mathbf{Z}$. Thus there is an open set of elements with non-trivial isotropy, and $\mathcal{G}_E$ is not essentially free. □

To justify our claimed analogy between conditions (L) and (I), we prove a uniqueness theorem for $C^*(E)$ along the lines of [5, Theorem 2.13]. For this, we need the following adaptation of [2, Proposition 2.4]:

Lemma 3.5. *Let $\mathcal{G}$ be an r-discrete essentially free groupoid such that $\mathcal{G}^{(0)}$ has a base of compact open sets, and H a hereditary subalgebra of $C_r^*(\mathcal{G})$. Then there is a non-zero partial isometry $v \in C_r^*(\mathcal{G})$ such that $v^*v \in H$ and $vv^* \in C_0(\mathcal{G}^{(0)})$.*

Proof. It is enough to do this when H is the hereditary subalgebra generated by a single positive element a. By rescaling, we can assume that $\|P(a)\| = 1$, where $P : C_r^*(\mathcal{G}) \to C_0(\mathcal{G}^{(0)})$ is the faithful conditional expectation given by restriction (see [10, p. 104], [2, p. 6]).

Choose $b \in C_r^*(\mathcal{G})^+ \cap C_c(\mathcal{G})$ such that $\|a - b\| < \frac{1}{4}$. Then $b_0 = P(b)$ satisfies $\|b_0\| > \frac{3}{4}$, and $b_1 = b - b_0$ has its compact support K contained in $\mathcal{G} \backslash \mathcal{G}^{(0)}$. Let $U := \{\gamma \in \mathcal{G}^{(0)} : b_0(\gamma) > \frac{3}{4}\}$. By [2, Lemma 2.3], there is an open subset V of U such that $r^{-1}(V) \cap s^{-1}(V) \cap K = \emptyset$. Since $\mathcal{G}$ has a basis of compact open sets, there is a nonempty compact open subset W of V; let $f = \chi_W$. Since $(fb_1f)(\gamma) = f \circ r(\gamma) f \circ s(\gamma) b_1(\gamma)$, we see that $fbf = fb_0f$. Since f is a projection, we have $fbf = fb_0f \geq \frac{3}{4}f^2 = \frac{3}{4}f$, and so $faf \geq fbf - \frac{1}{4}f \geq \frac{1}{2}f$. It then follows that faf is invertible in fAf. We denote by c its inverse, and put $v := c^{1/2} f a^{1/2}$. We have $vv^* = f \in C_0(\mathcal{G}^{(0)})$, which in particular implies that v is a partial isometry; since $v^*v = a^{1/2} fcf a^{1/2} \leq \|c\| a$, it belongs to the hereditary subalgebra H generated by a, as required. □

Corollary 3.6. *Let $\mathcal{G}$ be an r-discrete essentially free groupoid such that $\mathcal{G}^{(0)}$ has a base of compact open sets. If $\pi : C_r^*(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ is a representation, and π is faithful on $C_0(\mathcal{G}^{(0)})$ then π is faithful.*

Proof. If $\ker \pi \neq 0$, the Proposition gives a non-zero partial isometry $v \in C_r^*(\mathcal{G})$ such that $v^*v \in \ker \pi$ and $vv^* \in C_0(\mathcal{G}^{(0)})$. But $\pi(v^*v) = 0$ implies $\pi(vv^*) = 0$, which is impossible since π is faithful on $C_0(\mathcal{G}^{(0)})$. □

Theorem 3.7. *Let E be a locally finite directed graph which has no sinks and satisfies condition* (L). *Suppose B is a C^*-algebra generated by a Cuntz-Krieger E-family $\{S_e : e \in E^1\}$ with all S_e non-zero. Then there is an isomorphism π of $C^*(E)$ onto B such that $\pi(s_e) = S_e$.*

Proof. Because E is locally finite and has no sinks, [6, Theorem 4.2] says that $(C^*(E), s_e)$ is isomorphic to $(C^*(\mathcal{G}_E), 1_{Z(e,r(e))})$. Further, because $\mathcal{G}_E$ is amenable [6, Corollary 5.5], we have $C^*(\mathcal{G}_E) \cong C_r^*(\mathcal{G}_E)$. The universal property of $C^*(E) \cong C_r^*(\mathcal{G}_E)$ says there is a homomorphism π of $C_r^*(\mathcal{G}_E)$ onto B with $\pi(1_{Z(e,r(e))}) = S_e$, and we then have $\pi(1_{Z(\alpha,\beta)}) = S_\alpha S_\beta^*$ for all α, β; in particular, $\pi(1_{Z(\alpha)}) = S_\alpha S_\alpha^*$ for the projections $1_{Z(\alpha)}$ which span $C_0(\mathcal{G}^{(0)})$. For each n, the projections $\{1_{Z(\alpha)} : |\alpha| = n\}$ are mutually orthogonal, and span a finite-dimensional C^*-subalgebra A_n of $C_0(\mathcal{G}^{(0)})$. Similarly, the projections $S_\alpha S_\alpha^*$ are mutually orthogonal, and because all the S_e are non-zero, all the $S_\alpha S_\alpha^*$ are non-zero. Thus the representation π is faithful on A_n for each n. Since $C_0(\mathcal{G}^{(0)}) = \overline{\cup A_n}$, it follows from, for example, [1, Lemma 1.3] that π is faithful on $C_0(\mathcal{G}^{(0)})$. The result now follows from Corollary 3.6. □

Following [2, Definition 2.1], we say that an r-discrete groupoid $\mathcal{G}$ is *locally contracting* if for every non-empty open subset U of $\mathcal{G}^{(0)}$, there are an open subset V of U and an open bisection B with $\overline{V} \subset s(B)$ and $\alpha_{B^{-1}}(\overline{V})$ a proper subset of V.

Lemma 3.8. *If every vertex in E connects to a vertex which has a return path with an exit, then the groupoid $\mathcal{G}_E$ is locally contracting.*

Proof. If U is a non-empty open subset of $\mathcal{G}_E^{(0)}$, then by definition of the topology of $\mathcal{G}_E^{(0)} = E^\infty$ [6, Corollary 2.2], there exists $\alpha \in E^*$ such that $Z(\alpha) \subset U$. By hypothesis there is a finite path β such that $s(\beta) = r(\alpha)$ and $r(\beta)$ has a return path κ with an exit. Then $\overline{Z(\alpha\beta)} = Z(\alpha\beta) = s(Z(\alpha\beta\kappa, \alpha\beta))$, and $\alpha_{Z(\alpha\beta,\alpha\beta\kappa)}(Z(\alpha\beta)) = Z(\alpha\beta\kappa)$; because κ has an exit, $Z(\alpha\beta\kappa)$ is a proper subset of $Z(\alpha\beta)$, and taking $V := Z(\alpha\beta)$, $B := Z(\alpha\beta\kappa, \alpha\beta)$ proves the result. □

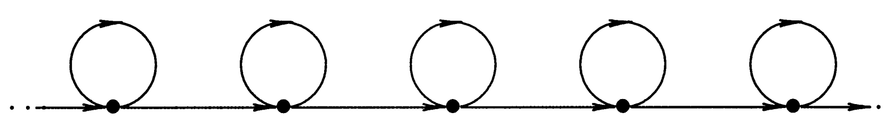
Theorem 3.9. *Let E be a locally finite directed graph with no sinks. Then $C^*(E)$ is purely infinite if and only if every vertex connects to a loop and E satisfies condition* (L).

Proof. Suppose first that E satisfies (L) and that every vertex connects to a loop. Then $\mathcal{G}_E$ is essentially free by 3.4, and locally contracting by 3.8, so that $C_r^*(\mathcal{G}_E)$ is purely infinite by [2, Proposition 2.4]. Since $\mathcal{G}_E$ is amenable, $C^*(E) = C^*(\mathcal{G}_E) = C_r^*(\mathcal{G}_E)$.

If E does not satisfy condition (L), then there is a loop without an exit, and the argument in the second last paragraph of the proof of 2.4 shows that $C^*(E)$ has an ideal which is Morita equivalent to a commutative C^*-algebra, and is therefore not purely infinite.

Now suppose that E satisfies (L), and that there is a vertex $v \in E^0$ which does not connect to a loop. Let H be the subgraph of E formed by those vertices and edges which can be reached from v. Since H has no exits, $\{s_e : e \in H^1\}$ is a Cuntz-Krieger H-family, and by 3.7 there is an isomorphism π of $C^*(H)$ onto $\overline{\operatorname{span}}\{s_\alpha s_\beta^* : \alpha, \beta \in H^*\}$, which is a hereditary subalgebra of $C^*(E)$ by 2.1. As H has no loops, we can deduce from 2.4 that $C^*(H)$ is AF. Hence $C^*(E)$ contains a hereditary subalgebra which is AF, and $C^*(E)$ cannot be purely infinite. $\square$

Example 3.10. Consider the following directed graph E:



Though E does not satisfy (I), E satisfies (L), and so $C^*(E)$ is purely infinite by 3.9. It is not simple — indeed, it has infinitely many ideals. Label the horizontal edges in E by $\{e_n : n \in \mathbf{Z}\}$. By removing one e_n we obtain two graphs R_n (the component to the right) and L_n (the component to the left). The arguments of [6, Theorem 6.6] show that there is an ideal J_n which is Morita equivalent to $C^*(R_n)$, and has quotient $C^*(E)/J_n$ isomorphic to $C^*(L_n)$. For $r \geq 1$, the ideals J_{n-r}/J_n form a composition series for $C^*(L_n)$, whose subquotients are Morita equivalent to $C(\mathbf{T})$. Hence $C^*(L_n)$ is a type I C^*-algebra. If E_n is the subgraph of E formed by adding e_n and $r(e_n)$ to L_n, then by 2.2 $C^*(E_n)$ is an extension of $C^*(L_n)$ by the compacts, and hence is also type I. Because E satisfies (L), we can use 3.7 to express $C^*(E)$ as the increasing union of $C^*(E_n)$, and deduce that $C^*(E)$ is an inductive limit of type I C^*-algebras.

We finish by formally stating our dichotomy:

Corollary 3.10. *Let E be a locally finite graph which has no sinks, is cofinal, and satisfies condition* (L)*. Then $C^*(E)$ is simple, and*

(i) *if E has no loops, then $C^*(E)$ is AF;*

(ii) *if E has a loop, $C^*(E)$ is purely infinite.*

Proof. If E is cofinal and satisfies condition (L), then E satisfies condition (K) of [6, §6], and so $C^*(E)$ is simple by [6, Corollary 6.8]. Property (i) follows from 2.4. For (ii), note that by cofinality, every vertex connects to the loop. Thus the hypotheses of 3.9 are satisfied, and $C^*(E)$ is purely infinite. □

References

[1] S. Adji, M. Laca, M. Nilsen and I. Raeburn, *Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc. Amer Math. Soc., **122** (1994), 1133-1141.

[2] C. Anantharaman-Delaroche, *Purely infinite C^*-algebras arising from dynamical systems*, preprint, Université d'Orléans, 1996.

[3] A. an Huef and I. Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergod. Th. and Dyn. Sys., **17** (1997), 611-624.

[4] J. Cuntz, *A class of C^*-algebras and topological Markov chains* II: *Reducible chains and the Ext-functor for C^*-algebras*, Invent. Math., **63** (1981), 25-40.

[5] J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math., **56** (1980), 251-268.

[6] A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids and Cuntz-Krieger algebras*, J. Funct. Anal., to appear.

[7] M. Laca and J. Spielberg, *Purely infinite C^*-algebras from boundary actions of discrete groups*, J. Reine Angew. Math., to appear.

[8] M.H. Mann, I. Raeburn and C.E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bull. Austral. Math. Soc., **46** (1992), 225-243.

[9] D. Pask and I. Raeburn, *On the K-theory of Cuntz-Krieger algebras*, Publ. RIMS Kyoto Univ., **32** (1996), 415-443.

[10] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, **793**, Springer-Verlag, 1980.

[11] ———, *The ideal structure of groupoid crossed product C^*-algebras*, J. Operator Theory, **25** (1991), 3-36.

Received October 16, 1996 and revised April 14, 1997.

University of Nevada
Reno, NV 89557-0045
E-mail address: alex@unr.edu

and

The University of Newcastle
NSW 2308, Australia
E-mail address: davidp@maths.newcastle.edu.au
iain@maths.newcastle.edu.au