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AN APPLICATION OF EINSTEIN KAHLER METRICS TO PROPER HOLOMORPHIC MAPPINGS BETWEEN PSEUDOCONVEX DOMAINS

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This paper applies the Cheng-Yau Einstein Kähler metric to prove a partial result on a conjecture concerning the compactness of the family of proper holomorphic mappings between strongly pseudoconvex domains.

1. Introducton.

In this paper we would like to deal with the following conjectural generalization of [**5**]:

Let D_1 and D_2 be two bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n, n \geq 2$. Then both D_1 and D_2 are biholomorphic to an euclidean ball iff $P(D_1, D_2)$ is a non-compact subset in $C(D_1, D_2)$ relative to compact-open topology, where $P(D_1, D_2) =$ the set of all proper holomorphic mappings from D_1 to D_2, $C(D_1, D_2) =$ the set of all continuous mappings from D_1 to D_2.

The method via comparison of intrinsic measures in [**3**] can yield the following result which is weaker than the above statement.

Theorem A [**3**]. *Let D_1 and D_2 be two bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n, n \geq 2$. Then both D_1 and D_2 are biholomorphic to an euclidean ball iff there exist a sequence $\{f_i\} \subset P(D_1, D_2)$, which is compactly divergent in $C(D_1, D_2)$ (i.e., for any $x \in D_1, \{f_i(x)\}$ converge to a boundary point of D_2).*

However, it will be shown here that a curvature argument of the the invariant Einstein-Kahler metric constructed by S.Y. Cheng and S.T. Yau [**2**] can gain the following result. This is the principal observation of this note.

Theorem B. *Let D_1 and D_2 be two bounded strongly pseudoconvex domains in with smooth boundary $\mathbb{C}^n, n \geq 2$. If $P(D_1, D_2)$ is a non-compact subset in $C(D_1, D_2)$, then both D_1 and D_2 are covered holomorphically by the euclidean ball.*

In the proof of Theorem B, the use of Cheng-Yau metric seems to be indispensible. There is difficulty for us to conclude this result using either the method of comparison of intrinsic measures [3] or the Bergman metric. This is thus far the only proof we know.

2. Proof of Theorem B.

The argument goes as follows. If $P(D_1, D_2)$ is noncompact, then there is a sequence $\{f_i\}$ in $P(D_1, D_2)$ which has no subsequence converging to an element of $P(D_1, D_2)$. By a normal al family argument there exists a subsequence denoted by $\{f_i\}$ again, converges on compacta on D_1 to a holomorphic mapping $f : D_1 \to \mathbb{C}^n$. There are two possibilities for f.

(1) $f(D_1)$ lies on ∂D_2. Since D_2 is strongly pseudoconvex, f is thus a constant map which brings the whole D_1 to one point on ∂D_2. Hence, both D_1 and D_2 are biholomorphic to B_n by invoking Theorem A (also see the remark 2 after the proof).

(2) f is a non-proper holomorphic map from D_1 into D_2. This is the situation we have to handle.

Let us assume $\{f_i\}$ converges on compacta to a non-proper holomorphic map $f : D_1 \to D_2$. Consequently, there exists a sequence $\{x_k\}$ in D_1 converging to a boundary point $p \in \partial D_1$ such that $\{y_k = f(x_k)\}$ converges to an interior point $q \in D_2$. Since f_i is proper, $S_i = \{f_i^{-1}(f_i(x_1))\}, x_1 \in D_1$, is a set consisting of finitely many points in D_1. We denote by t_i the diameter of S_i relative to the Kobayashi metric $d^K_{D_1}$ on D_1.

There are two cases of this sequence of numbers $\{t_i\}$.

Case 1. Suppose a subsequence of $\{t_i\}$ tends to infinity as i grows. By Pincuk's Theorem [4], all proper holomorphic mappings between s.p.c. domains are unbranching. Therefore they are all covering projections. For any point $s \in D_2$ it is easy to see, through the path lifting property of covering space and the triangle inequality of $d^K_{D_j}$, that the following inequality is true.

$$\begin{pmatrix}\text{Diameter of } S_i \\ \text{relative to } d^K_{D_1}\end{pmatrix} \leq 2\ d^K_{D_2}(y_1, s) + \begin{pmatrix}\text{Diameter of } \{f_i^{-1}(s)\} \\ \text{relative to } d^K_{D_1}\end{pmatrix}$$

where $d^K_{D_j}$ = the Kobayashi metric on $D_j, j = 1, 2$. This implies that the diameter of $\{f_i^{-1}(s)\}$ also tends to infinity as i gets large. Thus $\{f_i^{-1}(s)\}$ contains a point arbitrarily close to ∂D_1 for sufficiently large i.

(For the properties of the Kobayaski metric needed here one can consult "Function Theory of Several Complex Variables" by S. Krantz and "Hyperbolic manifolds and holomorphic mappings" by S. Kobayaski.)

We denote by ds_1^2 and ds_2^2 the Cheng-Yau Kahler Einestein metrics [2] on D_1 and D_2 respectively. Since f_i is proper, $f_i^*(ds_2^2)$ is complete. Thus, by the uniqueness of the Cheng-Yau metric we have $ds_1^2 = f_i^*(ds_2^2)$ for each i.

Therefore f_i is a local isometry which preserves the holomorphic sectional curvature. Since the holomorphic sectional curvatures of the Cheng-Yau metric is asymptotically equal to a negative constant around a strongly pseudoconvex boundary, we have proved that (D_2, ds_2^2) and (D_1, ds_1^2) are a complete Kahler metric with a constant negative holomorphic sectional curvature. A well-known uniformization theorem thus implies that both D_1 and D_2 are covered holomorphically by the ball.

Case 2. Let us assume the sequence $\{t_i\}$ are bounded above by a finite positive constant M (i.e. $t_i \le M < +\infty$ for all i). Let us fix two points x_1 and x_k from our sequence $\{x_k\} \subset D_1$. For each covering $f_i : D_1 \to D_2$, we can make use of the path-lifting property to find a point $z \in \{f_i^{-1}(f_i(x_1))\} = S_i$ such that

$$d_{D_1}^K(z, x_k) = d_{D_2}^K(f_i(x_1), f_i(x_k)).$$

By triangle inequality, one has $d_{D_1}^K(x_1, x_k) \le d_{D_1}^K(x_1, z) + d_{D_1}^K(z, x_k)$. We notice that $d_{D_1}^K(x_1, z) \le M$, because $z \in \{f_i^{-1}(f_i(x_1))\}$ and $t_i \le M$, where t_i = diameter of $\{f_i^{-1}(f_i(x_1))\}$ with respect to $d_{D_1}^K$. Combining all these inequalities we reach the conclusion, for each i and k, $d_{D_1}^K(z, x_k) \le M + d_{D_2}^K(f_i(x_1), f_i(x_k))$. Allowing i grows, we have $\{f_i(x_1)\} \to f(x_1) = y_1$, $\{f_i(x_k)\} \to f(x_k) = y_k$. This yields, for each fixed k,

$$d_{D_1}^K(x_1, x_k) \le M + d_{D_2}^K(y_1, y_k) \ .$$

Finally, letting $k \to \infty$, we have $d_{D_1}^K(x_1, x_k) \to \infty$ because $\{x_k\}$ is approaching the boundary ∂D_1. On the other hand, $\mathrm{Lim}_{k\to\infty} d_{D_2}^K(y_1, y_k)$ will be a finite number because $\{y_k\}$ approaches an interior point $q \in D_2$ as a limit. This gains a contradiction to the assumption that f is non-proper.

This completes the whole proof.

Remark 1. There is an alternate differential geometric proof for the possibility (2) in the proof of Theorem B.

As before we know that $f_i : (D_1, ds_a^2) \to (D_2, ds_2^2)$ is a local isometry and $\{f_i\}$ converge on compacta to $f : D_1 \to D_2$. Thus f is also a local isometry. The following is a known fact.

Lemma. *Let $\varphi : S_1 \to S_2$ be a local isometry from a complete Riemannian manifold S_1 onto another Riemannian manifold S_2. Then φ is a covering map.*

(For a proof one can consult, e.g. DoCarmo, Differential geometry of curves and surfaces in $\mathbb{R}^3$, p. 386.)

We can conclude $f : D_1 \to D_2$ is a covering from the lemma if f is shown to be surjective. Let $q = f(p)$ be a fixed point in D_2 with $p \in D_1$. Suppose

s is an arbitrary point in D_2. By the path lifting property of the covering $f_i : D_1 \to D_2$, one can find a sequence of points $\{p_i\} \subset D_1$ s.t. $\{d^K_{D_1}(p, p_i)\}$ converge to the fixed number $d^K_{D_2}(q, s)$, where $p_i \in \{f_i^{-1}(s)\}$. Thus $\{p_i\}$ must lie in a compact subset in D_1 because $d^K_{D_1}$ is a complete metric. A subsequence of $\{p_i\}$ will converge to a point $r \in D_1$ satisfying the property $f(r) = s$.

Therefore, for any point $s \in D_2, \{f^{-1}(s)\}$ contains a point arbitrarily close to the boundary of D_1 because f is non-proper. Since Cheng-Yau metrics are invariant under the covering map f, we can conclude (D_1, ds_1^2) and (D_2, ds_2^2) both have constant negative holomorphic curvature, using the fact that the holomorphic curvature of the Einstein Kahler metric is asymptotically close to a negative constant around the boundary of a strongly pseudoconvex domain. This would imply both D_1 and D_2 are covered holomorphically by a euclidean ball.

Remark 2. An argument using the Cheng-Yau metric can also furnish a direct proof of possibility (1) that $f(D_1)$ lies on ∂D_2 without invoking Theorem A. Let $x \in D_1$ be an arbitrary point. Then a subsequence of $\{f_i(x)\}$ will converge to a boundary point of D_2. For the same reason that Cheng-Yau metrics are invariant under coverings as in our proof, one can prove (D_1, ds_1^2) and (D_2, ds_2^2) both have constant negative holomorphic curvature. Thus they are covered by the ball.

Nevertheless there seems a difficulty to conclude both D_1 and D_2 are actually biholomorphic to the ball without using the comparison method of intrinsic measures employed in [**3**].

3. Some discussions.

It is clear from the proofs that our problem mentioned in the introduction depends on the truth on the following generalization of H. Cartan's lemma from biholomorphisms to proper holomorphic mappings.

Conjecture. *Let D_1 and D_2 be two bounded strongly pseudoconvex domains with smooth boundary in $\mathbb{C}^n, n \geq 2$. Then a sequence of proper holomorphic mappings can never be convergent on compacta to a non-proper holomorphic mapping $f : D_1 \to D_2$ (see* [**1**] *for a detailed discussions).*

From our results we know that one only has to consider the case when both D_1 and D_2 are covered holomorphically by the euclidean ball.

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