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CANONICAL ALMOST HERMITIAN STRUCTURES OF NORMAL BUNDLES AND APPLICATIONS TO KÄHLER FORMS

Po-Hsun Hsieh

For any (real) submanifold L of an almost Hermitian manifold (M, J, g, ω) ($\omega = g(J\cdot, \cdot)$), there is a canonical almost Hermitian structure $(\hat{J}, \hat{g}, \hat{\omega})$ ($\hat{\omega} = \hat{g}(\hat{J}\cdot, \cdot)$) on (the total space of) the normal bundle $L^\perp$. We have three main topics: (i) We investigate conditions under which $(L^\perp, \hat{J}, \hat{g})$ is Kähler or almost Kähler. (ii) If $\hat{\omega}$ is a symplectic form, then $\hat{\omega}$ is called the canonical symplectic form of $L^\perp$. We investigate conditions for two such canonical symplectic forms to be isomorphic. (iii) If (M, J, g) is Kähler, we investigate conditions under which ω and $\hat{\omega}$ are isomorphic: We obtain a single theorem which synthesizes, generalizes, and improves two of McDuff's theorems on Kähler forms of Kähler manifolds with certain nonpositive curvature.

1. Introduction

In this section we provide some motivation, define the canonical almost Hermitian structures of normal bundles, summarize the main results, and fix some notation.

First we consider:

Theorem 1.1 (McDuff [11]). (i) *Let M^{2n} be a complete Kähler manifold such that* $\exp : T_xM \to M$ *is a diffeomorphism for some $x \in M$. Suppose M has nonpositive radial sectional curvature with respect to $L := \{x\}$. Then the Kähler form of M is isomorphic to the usual symplectic form on $\mathbb{R}^{2n}$.*

(ii) *Let M be a nonpositively curved complete Kähler manifold with a totally geodesic Lagrangian submanifold L such that* $\exp : L^\perp \to M$ *is a diffeomorphism. Then the Kähler form of M is isomorphic to the canonical symplectic form of T^*L.*

Notice that two tensors A_1 and A_2 on manifolds M_1 and M_2 respectively are called isomorphic if there is a (C^∞) diffeomorphism from M_1 to M_2 that pushes A_1 forward to A_2. Is there a result similar to Theorem 1.1 if L is only an isotropic submanifold? A partial affirmative answer is Ciriza's isomorphism theorem [3]. In order to give a complete affirmative answer, we need to investigate whether the normal bundle $L^\perp$ has a natural symplectic form. We will see that $L^\perp$ does have a natural almost Hermitian structure $(\hat{J}, \hat{g}, \hat{\omega})$, and sometimes $\hat{\omega}$ is a symplectic form. See [10] for the definitions of almost Hermitian manifolds, fundamental 2-forms, etc.

Suppose (M, J, g, ω) is an almost Hermitian manifold and L is a (real) submanifold of M. The normal connection $\nabla^\perp$ determines a canonical metric (*Sasaki metric*) $\hat{g}$ on $L^\perp$ [2]. Clearly, there exists a unique almost complex structure $\hat{J}$ on (the total space of) $L^\perp$ such that $\hat{J}$ is parallel with respect to the Levi-Civita connection of $\hat{g}$ along each line of the form $\{tv | t \in \mathbb{R}\}$ $(0 \neq v \in L^\perp)$ and $\hat{J}_x = \exp^*(J|T_x M)$ for all $x \in L$. We call $(\hat{J}, \hat{g}, \hat{\omega})$ $(\hat{\omega} = \hat{g}(\hat{J}\cdot, \cdot))$ the *canonical almost Hermitian structure* of $L^\perp$ and $\hat{\omega}$ the *canonical fundamental* 2*-form* of $L^\perp$.

In Section 2, we present some more definitions, notation, and frequently used lemmas. Then in Sections 3, 4, and 5, we have three main topics respectively: (i) We investigate conditions under which $(L^\perp, \hat{J}, \hat{g})$ is Kähler or almost Kähler. (Theorem 3.3.) (ii) If $\hat{\omega}$ is a symplectic form, then $\hat{\omega}$ is called the *canonical symplectic form* of $L^\perp$. We investigate conditions for two such canonical symplectic forms to be isomorphic. (Theorem 4.1.) (iii) If (M, J, g) is Kähler, we investigate conditions under which ω and $\hat{\omega}$ are isomorphic: we obtain a single theorem which synthesizes, generalizes, and improves Theorem 1.1. (Theorem 5.1.)

The proof of (i) involves somewhat complicated calculations about certain vector fields and 1-forms, which are suitably decomposed. Also, (i) generalizes a theorem in Dombrowski [5] and Tondeur [13]: The canonical almost Kähler structure of TL is Kähler if and only if $(L, g|L)$ is flat.

The proof of (ii) mainly involves Moser's technique, the nice symmetry of $\hat{\omega}$, and the flow of a natural Liouville vector field.

The proof of Ciriza's isomorphism theorem mainly involves Ciriza's linearization theorem [4], some comparison results (Lemma 2.4 and (i) of Lemma 5.3), and Moser's technique. For (iii), our frame of proof of the preservation of the symplectic forms by f is similar to that of Ciriza's isomorphism theorem; but our more important technical contribution, the proof concerning the first jet of f, is carried out by considering the first jets of the various intermediate maps that compose to form f. The integrability (i.e. Kähler) condition and an O.D.E. comparison theorem are utilized in this aspect.

In Section 6 we give some examples. In particular we show that the Kähler form of any Kähler axial manifold is isomorphic to the usual symplectic form on $S^1 \times \mathbb{R}^{2n-1}$.

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2. Preliminaries.

In this section we give more definitions, notation, and frequently used lemmas.

Let M be a Riemannian manifold and L be a submanifold. The direct sum of the Levi-Civita connection ∇ of L and the normal connection $\nabla^\perp$ of $L^\perp$ is called the *combined connection* of L, and is always denoted by $\Box$. We use $R^\perp$ to denote the curvature of $\nabla^\perp$, $\tilde{\nabla}$ the Levi-Civita connection of M, and $\Diamond$ the Levi-Civita connection of $(L^\perp, \hat{g})$. We usually identify L with the zero section of a vector bundle over L, and write $T_LM = (TM)|L = TL \oplus L^\perp$ and $T_L(L^\perp) = (T(L^\perp))|L$. We also usually identify $T_L(L^\perp)$ with T_LM via the derivative of the exponential map. Hence $\hat{g}|T_L(L^\perp) = g|T_LM$.

The foot map $\pi : L^\perp \to L$ induces a map $\pi_* : T(L^\perp) \to TL \subset T_LM$. $\nabla^\perp$ determines a connection map $K : T(L^\perp) \to L^\perp \subset T_LM$ (e.g. [2]). Hence we have a map $S := \pi_* + K : T(L^\perp) \to T_LM$.

Definition 2.1. (i) Suppose $A \subset T_LM$, then $\hat{A} := S^{-1}(A) \subset T(L^\perp)$ is called the *Sasaki lift* of A (to $T(L^\perp)$).

(ii) If $B : T_LM \times \cdots \times T_LM \to \mathbb{R}$ and $C : T_LM \times \cdots \times T_LM \to T_LM$ are $C^\infty(L)$-linear, then their *Sasaki lifts* are defined as follows: $\hat{B} : T(L^\perp) \times \cdots \times T(L^\perp) \to \mathbb{R}$, $\hat{B}(\hat{X}_1, \cdots, \hat{X}_n) = B(X_1, \cdots, X_n)$ and $\hat{C} : T(L^\perp) \times \cdots \times T(L^\perp) \to T(L^\perp)$, $\hat{C}(\hat{X}_1, \cdots, \hat{X}_n) = (C(X_1, \cdots, X_n))\hat{}$, for all $X_i \in \Gamma(T_LM)$.

Remark 2.2. (i) If $A \subset TL$ (resp. $A \subset L^\perp$), then $\hat{A}$ is just the usual *horizontal* (resp. *vertical*) *lift* of A [2]. We use $\mathcal{H}$ to denote the horizontal lift of $TL \subset T_LM$ and $\mathcal{V}$ the vertical lift of $L^\perp \subset T_LM$. We have a $\hat{g}$-orthogonal decomposition $T(L^\perp) = \mathcal{H} \oplus \mathcal{V}$.

(ii) $\hat{J}$, $\hat{g}$, $\hat{\omega}$ are the Sasaki lifts of $J|T_LM$, $g|T_LM$, $\omega|T_LM$ respectively.

(iii) The operations of Sasaki lift and duality commute. More precisely, if we define $Z^* = \hat{g}(Z, \cdot)$ for any $Z \in T(L^\perp)$, then $(X^*)\hat{} = (\hat{X})^*$ for all $X \in T_LM$.

The reader should not confuse $\Diamond$ with $\nabla^\perp$ or $\Box$. The following lemma expresses $\Diamond$ completely in terms of ∇, $\nabla^\perp$, and $R^\perp$. Part (ii) of it follows from (1) and (2) with $\xi = 0$.

Lemma 2.3. (i) ([2, Lemma 6]). *Let* $X, Y \in \Gamma(TL)$ *and* $\Phi, \Psi \in \Gamma(L^\perp)$, *then at each point* $\xi \in L_x^\perp$ $(x \in L)$ *we have:*

$$\Diamond_{\hat{X}}\hat{Y}|_\xi = (\nabla_X Y|_x)^H - \frac{1}{2}(R^\perp(X_x, Y_x)\xi)^V, \tag{1}$$

$$\Diamond_{\hat{X}}\hat{\Psi}|_\xi = (\nabla_X^\perp \Psi|_x)^V + \frac{1}{2}(N^\perp(\xi, \Psi_x)X_x)^H, \tag{2}$$

$$\Diamond_{\hat{\Phi}}\hat{Y}|_\xi = \frac{1}{2}(N^\perp(\xi, \Phi_x)Y_x)^H, \Diamond_{\hat{\Phi}}\hat{\Psi}|_\xi = 0,$$

where H (*resp.* V) *denotes the* (*horizontal* (*resp. vertical*)) *lift and* $N^\perp$ *is defined by* $g(N^\perp(\mu,\nu)e, f) = g(R^\perp(e,f)\mu,\nu)$ *for all* $\mu,\nu \in L_x^\perp$ *and* $e, f \in T_xL$.

(ii) $\exp^*(\Box) = \Diamond|T_L(L^\perp)$. (exp *is defined in a neighborhood of* L *in* $L^\perp$.)

Now let (M, J, g, ω) be an almost Hermitian manifold and L be a submanifold. (i) L is called a *JP submanifold* (or *J-parallel submanifold* to emphasize J) of M if $J|T_LM$ is parallel with respect to $\Box$. (ii) We define the (singular) *primary complex bundle* $D = TL \cap J(TL)$ and *secondary complex bundle* $E = L^\perp \cap J(L^\perp)$, and use D' (resp. E') to represent the g-orthogonal complement of D (resp. E) in TL (resp. $L^\perp$). Notice that $\hat{D} \oplus \hat{D}' = \mathcal{H}$ and $\hat{E}' \oplus \hat{E} = \mathcal{V}$ are $\hat{g}$-orthogonal decompositions. (iii) Recall that L is called a *CR-submanifold* if D has constant rank and $J(D') = E'$, and *totally real* (or equivalently, *isotropic*) if in addition $D = 0$.

We call a submanifold L of Riemannian manifold M a *post* if the normal exponential map is defined on all of $L^\perp$ and is a diffeomorphism from $L^\perp$ onto M, and a *geo-post* if it is in addition totally geodesic. Now suppose L is a post of M. Let $L_s^\perp = \{X \in L^\perp| |X| = s\}$ and $T_s = \exp(L_s^\perp)$, where $s > 0$. (i) The elements of $T(T_s)$ are called spherical vectors, and the vectors orthogonal to T_s radial vectors. By the *radial sectional curvature* we mean the sectional curvature of a plane spanned by a radial vector and a spherical vector. (ii) For every $s, t > 0$, we have a diffeomorphism $T_{t,s} : T_s \to T_t$, $\exp(sv) \mapsto \exp(tv)$ for all v in the unit normal bundle of L. L is called *pre-increasing* if there exists a positive constant c such that $|dT_{t,s}(v)| \geq c|v|$ for all $v \in T(T_s)$ and $0 < s \leq t$, and *increasing* if in addition $c = 1$. The following generalization of Rauch's comparison theorem is routine to prove.

Lemma 2.4. *Suppose a Riemannian manifold has nonpositive radial sectional curvature with respect to a geo-post* L. *Then* L *is an increasing post.*

The rest of this section is a preparation for applying Moser's technique, which will be used in Sections 4, 5, and 6.

By the (*dilation*) *norm* $|\alpha|$ of a covariant n-tensor α on (M, g) we mean $\sup_{|Y_i|=1} |\alpha(Y_1, \ldots, Y_n)|$. We say α has *sublinear growth* if $|\alpha| \leq Ar + B$, where r is the distance function on M with respect to a fixed point of M, and A and B are some fixed constants. If $n = 2$, by the *first inf norm* $|\alpha|_{1-\inf}$ of α we mean $\inf_{|Y|=1} |\alpha(Y, \cdot)|$. The following lemma can be proved by the techniques in [12].

Lemma 2.5. *Let (M, g) be a complete Riemannian manifold and ω, ω' be symplectic forms on M such that $\omega - \omega' = d\alpha$, where α has sublinear growth. Define $\omega_t = (1-t)\omega + t\omega'$. Suppose there exists a positive constant c such that $|\omega_t|_{1-\inf} \geq c$ for all $t \in [0,1]$. Let f_t be the flow of the vector field X_t defined by $\omega_t(X_t, \cdot) = \alpha$, $t \in [0,1]$. Then $(f_1)^*(\omega') = \omega$.*

Recall that, if A is a vector bundle over a manifold M, then the *position vector field* of A is the unique vector field on (the total space of) A that generates the action $(t, v) \to e^t v$ for all $t \in \mathbb{R}$ and $v \in A$. We usually denote it by P_A.

Lemma 2.6. *Let M be a Riemannian manifold with a pre-increasing post L. Let γ be a closed 2-form of bounded global norm on M. Suppose $\gamma|TL = 0$. Identify $L^\perp$ and M via* exp. *Define $P = P_{L^\perp}$ and $P_t = P/t$, $t \in (0,1]$. By* [9, p. 120, Lemma 2], *$\gamma = d\lambda$, where $\lambda = \int_0^1 (\phi_t)^*(P_t \lrcorner \gamma) dt$, where ϕ_t is the flow of P_t (in fact, $\phi_t(\xi) = t\xi$). We claim λ has sublinear growth.*

Proof. Suppose $|\gamma| \leq c$. There exists a constant $c' \geq 1$ such that $|(\phi_t)_*(Y)| \leq c'|Y|$ for all $Y \in T(L^\perp)$. Therefore, if $Y \in T_\xi(L^\perp)$, then

$$\begin{aligned} (3) \qquad \lambda_\xi(Y) &= \int_0^1 \gamma((\phi_t)_* P_t, (\phi_t)_* Y)\, dt = \int_0^1 \gamma\left(t \cdot \frac{\xi}{t}, (\phi_t)_* Y\right)\, dt, \\ |\lambda_\xi(Y)| &\leq \int_0^1 c|\xi|\, |(\phi_t)_* Y|\, dt \leq c|\xi| c' |Y|, \end{aligned}$$

where $\xi \in L^\perp$ has been identified with the corresponding element in $T_{t\xi} L^\perp$ for each t in the usual way. This completes the proof. □

Unless otherwise stated, all given manifolds and submanifolds are assumed to be C^∞ and connected, and all given maps are assumed to be C^∞.

3. Fundamental properties of the canonical almost Hermitian structures.

In this section we study some fundamental properties of the previously introduced canonical almost Hermitian structures. We indicate why J-parallelism is important in this regard, and determine the conditions for the canonical almost Hermitian structures to be Kähler or almost Kähler.

If N is a Riemannian manifold, then TN has an induced canonical almost Kähler structure [5]. The following result implies that, with regard to this structure, a normal bundle can be seen as a generalization of a tangent bundle.

Proposition 3.1. *Let (M, J, g) be an almost Hermitian manifold and L be a Lagrangian submanifold of M. Let $(TL, \check{J}, \check{g})$ be the canonical almost Kähler structure induced by $(L, g|L)$. Then:*

(i) *There exists a unique vector bundle isomorphism $f : L^{\perp} \to TL$ over* Id *such that $T_x f : T_x(L^{\perp}) \to T_x(TL)$ is unitary for all $x \in L$.*

(ii) *f is an isometry if and only if L is a JP submanifold of M. Moreover, in this case, f is a unitary isometry.*

Proof. Part (i) is trivial by the definitions $\check{g}(X, Y) = g(\pi_* X, \pi_* Y) + g(KX, KY)$ and $\check{J}(h \oplus v) = (-v) \oplus h$. Let $\check{\nabla}$ denote the Levi-Civita connection of the $(TL, \check{g})$.

First suppose f is an isometry. Then $f_*(\Diamond) = \check{\nabla}$. Since L is a JP submanifold of $(TL, \check{J}, \check{g})$, L is a JP submanifold of M by (ii) of Lemma 2.3.

Now suppose L is a JP submanifold of M. Let γ be a smooth curve in L and $X \in \Gamma(T_\gamma L)$ be a vector field along γ parallel with respect to ∇. Since L is a JP submanifold of M, $\hat{J}(X)$ is a horizontal curve in $L^{\perp}$. But $\check{J}(X)$ is also a horizontal curve in TL. Hence $f_*(\Diamond) = \check{\nabla}$, and thus $f_*(\hat{J}) = \check{J}$. Hence f is an isometry and is unitary. ∎

For convenience, we call L *normal flat* if $\nabla^{\perp}$ is flat.

Proposition 3.2. *Let L be a submanifold of almost Hermitian manifold (M, J, g).*

(i) *If $(L^{\perp}, \hat{J}, \hat{g})$ is Kähler, then L is J-parallel.*

(ii) *If L is J-parallel and normal flat, then $(L^{\perp}, \hat{J}, \hat{g})$ is Kähler.*

Proof. Recall that if (N, J, g) is an almost Hermitian manifold, then it is Kähler if and only if J is parallel with respect to the Levi-Civita connection. Therefore, (i) follows directly from (ii) of Lemma 2.3; and to prove (ii), it suffices to show that $\hat{J}$ is parallel with respect to $\Diamond$. Locally $L^{\perp}$ is isometric to and thus identified as $L' \times \mathbb{R}^m$, where L' is an open submanifold of L. Suppose $\gamma = (\gamma_1, \gamma_2) : [0, 1] \to L' \times \mathbb{R}^m$ is a smooth path. Define two paths α_1 and α_2 in $L' \times \mathbb{R}^m$: $\alpha_1(t) = (\gamma_1(t), \gamma_2(0))$, $\alpha_2(t) = (\gamma_1(1), \gamma_2(t))$. Parallel translation along γ is the same as that along α_1 followed by that along α_2. We easily see $\hat{J}$ is parallel along both α_1 and α_2. ∎

The previous two propositions provide strong motivation for us to concentrate on JP submanifolds. The following is the main theorem of this section.

Theorem 3.3. *Let (M, J, g) be an almost Hermitian manifold and L be a JP CR-submanifold. Then:*

(i) *$(L^\perp, \hat{J}, \hat{g})$ is Kähler if and only if L is normal flat.*

(ii) *$(L^\perp, \hat{J}, \hat{g})$ is almost Kähler if and only if $R^\perp(TL, D)L^\perp = R^\perp(D', D')E = 0$. In particular, if L is a JP isotropic submanifold, then $(L^\perp, \hat{J}, \hat{g})$ is almost Kähler if and only if L is secondary flat (i.e., $\nabla^\perp|E$ is flat).*

Remark 3.4. (i) Since L is JP, we easily see that $\nabla^\perp$ can be decomposed as $\nabla^\perp = \nabla^\perp|E' \oplus \nabla^\perp|E$.

(ii) Clearly, this theorem implies that if L is a JP almost complex submanifold of M, then $(L^\perp, \hat{J}, \hat{g})$ is Kähler if and only if almost Kähler.

Part (ii) of Proposition 3.1 and part (i) of Theorem 3.3 imply:

Corollary 3.5 ([**5**, **13**])**.** *Let (N, h) be a Riemannian manifold. Then the induced canonical almost Kähler structure on TN is Kähler if and only if (N, h) is flat.*

Needing lengthy calculation later, we also introduce some simplification of notation in the following lemma, which follows easily from Lemma 2.3.

Lemma 3.6 (and notation)**.** *Let (M, J, g) be an almost Hermitian manifold and L be a JP CR-submanifold.*

(i) *Fix an $x \in L$. We can choose a local orthonormal basis $\{P_a, Q_a, U_i, V_i, X_r, Y_r\}$ for $T_L M$ about x such that $Q_a = J(P_a)$, $V_i = J(U_i)$, $Y_r = J(X_r)$, and (locally) $P_a, Q_a \in \Gamma(D)$, $U_i \in \Gamma(D')$, $V_i \in \Gamma(E')$, $X_r, Y_r \in \Gamma(E)$ are parallel with respect to $\Box$ at x. We also use P^a ($P^a : T_L M \to \mathbb{R}$) to denote the dual of P_a with respect to g, i.e., $P^a = g(P_a, \cdot)$. Q^a, U^i, V^i, X^r, Y^r are similarly defined.*

(ii) *Fix a $\xi \in L^\perp_x$. For each $h_1, h_2 \in T_x L$ and $v \in L^\perp_x$, define $(h_1, h_2, v) = g(R^\perp(h_1, h_2)\xi, v)$. Let A, B be $\mathbb{R}$-linear combinations of P_a, Q_a, U_i and F, G be $\mathbb{R}$-linear combinations of V_i, X_r, Y_r. Let $\hat{B}^*$ and $\hat{G}^*$ denote the $\hat{g}$-dual of $\hat{B}$ and $\hat{G}$ respectively. When there is no risk of confusion, we will omit the sign $|_\xi$. We have*

$$2 \Diamond_{\hat{A}} \hat{B}|_\xi = [-(A, B, V_k)\hat{V}_k - (A, B, X_t)\hat{X}_t - (A, B, Y_t)\hat{Y}_t]|_\xi,$$
$$2 \Diamond_{\hat{A}} \hat{G}|_\xi = (A, P_c, G)\hat{P}_c + (A, Q_c, G)\hat{Q}_c + (A, U_k, G)\hat{U}_k,$$
$$2 \Diamond_{\hat{F}} \hat{B}|_\xi = (B, P_c, F)\hat{P}_c + (B, Q_c, F)\hat{Q}_c + (B, U_k, F)\hat{U}_k,$$

$$2\,\Diamond_{\hat{F}}\,\hat{G}|_\xi = 0.$$

Similar formulas for $2\,\Diamond_{\hat{A}}\,\hat{B}^*|_\xi$, $2\,\Diamond_{\hat{A}}\,\hat{G}^*|_\xi$, $2\,\Diamond_{\hat{F}}\,\hat{B}^*|_\xi$, *and* $2\,\Diamond_{\hat{F}}\,\hat{G}^*|_\xi$ *also hold by duality. For example,* $2\,\Diamond_{\hat{A}}\,\hat{B}^*|_\xi = -(A,B,V_k)\hat{V}^k - (A,B,X_t)\hat{X}^t - (A,B,Y_t)\hat{Y}^t$.

The following proposition is the main ingredient of the proof of (i) of Theorem 3.3. We will use the notation $\mathcal{H}^* := \{Z^*|Z \in \mathcal{H}\}$ and $\mathcal{V}^* := \{Z^*|Z \in \mathcal{V}\}$.

Proposition 3.7. *Let* (M,J,g) *be an almost Hermitian manifold and* L *be a JP CR-submanifold.*

(i) $\hat{J}$ *is parallel along horizontal directions (i.e.* $\Diamond_Z \hat{J} = 0$ *for all* $Z \in \mathcal{H}$*) if and only if the following equations are satisfied:*

$$R^\perp(TL,D')E = R^\perp(TL,D)E' = 0, \tag{4}$$

$$g(R^\perp(TL,Jd)L^\perp, Je) = g(R^\perp(TL,d)L^\perp, e)\forall d \in \Gamma(D), e \in \Gamma(E), \tag{5}$$

$$g(R^\perp(TL,Je')L^\perp, Jd') = -g(R^\perp(TL,d')L^\perp, e')\forall d' \in \Gamma(D'), e' \in \Gamma(E'). \tag{6}$$

(ii) $\hat{J}$ *is parallel along vertical directions (i.e.* $\Diamond_Z \hat{J} = 0$ *for all* $Z \in \mathcal{V}$*) if and only if the following equations are satisfied:*

$$R^\perp(TL,D')L^\perp = 0, \tag{7}$$

$$g(R^\perp(Jd_1,Jd_2)L^\perp, L^\perp) = g(R^\perp(d_1,d_2)L^\perp, L^\perp)\forall d_1, d_2 \in \Gamma(D). \tag{8}$$

Proof. Notice that (4) and (7) are respectively equivalent to (9)-(10) and (11):

$$g(R^\perp(TL,D')L^\perp, E) = 0, \tag{9}$$

$$g(R^\perp(TL,D)L^\perp, E') = 0, \tag{10}$$

$$g(R^\perp(TL,D')L^\perp, L^\perp) = 0. \tag{11}$$

Fix a $\xi \in L^\perp$. We have $\hat{J} = \hat{Q}_b \otimes \hat{P}^b - \hat{P}_b \otimes \hat{Q}^b + \hat{V}_j \otimes \hat{U}^j - \hat{U}_j \otimes \hat{V}^j + \hat{Y}_s \otimes \hat{X}^s - \hat{X}_s \otimes \hat{Y}^s$.

(Proof of part (i).) Let A be an $\mathbb{R}$-linear combination of P_a, Q_a, U_i. We can write $2\Diamond_{\hat{A}}\hat{J}|_\xi = J_1 + J_2 + J_3, +J_4$, where J_1, J_2, J_3, J_4 are the $\mathcal{H}\otimes\mathcal{H}^*, \mathcal{H}\otimes\mathcal{V}^*, \mathcal{V}\otimes\mathcal{H}^*, \mathcal{V}\otimes\mathcal{V}^*$ components of $2\Diamond_{\hat{A}}\hat{J}$ respectively. By Lemma 3.6, we have $J_1 = 2(\Diamond_{\hat{A}}\hat{V}_j \otimes \hat{U}^j - \hat{U}_j \otimes \Diamond_{\hat{A}}\hat{V}^j) = J_{11} + J_{12}$, where

$$J_{11} = (A,P_c,V_j)\hat{P}_c \otimes \hat{U}^j + (A,Q_c,V_j)\hat{Q}_c \otimes \hat{U}^j - \hat{U}_j \otimes (A,P_c,V_j)\hat{P}^c \\ -\hat{U}_j \otimes (A,Q_c,V_j)\hat{Q}^c,$$

$$J_{12} = [(A, U_k, V_j) - (A, U_j, V_k)]\hat{U}_k \otimes \hat{U}^j.$$

Clearly $J_{11} = 0$ for all A, ξ if and only if (10) holds, and $J_{12} = 0$ for all A, ξ if and only if (6) holds. Similarly, $J_2 = 2(\hat{Q}_b \otimes \lozenge_{\hat{A}} \hat{P}^b - \hat{P}_b \otimes \lozenge_{\hat{A}} \hat{Q}^b + \lozenge_{\hat{A}} \hat{Y}_s \otimes \hat{X}^s - \lozenge_{\hat{A}} \hat{X}_s \otimes \hat{Y}^s) = J_{21} + J_{22} + J_{23}$, where

$$\begin{aligned} J_{21} &= -\hat{Q}_b \otimes (A, P_b, V_k)\hat{V}^k + \hat{P}_b \otimes (A, Q_b, V_k)\hat{V}^k, \\ J_{22} &= \hat{P}_b \otimes (A, Q_b, X_t)\hat{X}^t + (A, P_c, Y_s)\hat{P}_c \otimes \hat{X}^s + \hat{P}_b \otimes (A, Q_b, Y_t)\hat{Y}^t \\ &\quad -(A, P_c, X_s)\hat{P}_c \otimes \hat{Y}^s - \hat{Q}_b \otimes (A, P_b, X_t)\hat{X}^t + (A, Q_c, Y_s)\hat{Q}_c \otimes \hat{X}^s \\ &\quad -\hat{Q}_b \otimes (A, P_b, Y_t)\hat{Y}^t - (A, Q_c, X_s)\hat{Q}_c \otimes \hat{Y}^s, \\ J_{23} &= (A, U_k, Y_s)\hat{U}_k \otimes \hat{X}^s - (A, U_k, X_s)\hat{U}_k \otimes \hat{Y}^s. \end{aligned}$$

Clearly $J_{21} = 0$ for all A, ξ if (10) holds, $J_{22} = 0$ for all A, ξ if (5) holds, and $J_{23} = 0$ if (9) holds. Similarly, $J_3 = 2(\lozenge_{\hat{A}} \hat{Q}_b \otimes \hat{P}^b - \lozenge_{\hat{A}} \hat{P}_b \otimes \hat{Q}_b + \hat{Y}_s \otimes \lozenge_{\hat{A}} \hat{X}^s - \hat{X}_s \otimes \lozenge_{\hat{A}} \hat{Y}^s) = J_{31} + J_{32} + J_{33}$, where

$$\begin{aligned} J_{31} &= -(A, Q_b, V_k)\hat{V}_k \otimes \hat{P}^b + (A, P_b, V_k)\hat{V}_k \otimes \hat{Q}^b, \\ J_{32} &= [(A, Q_b, X_t) + (A, P_b, Y_t)][\hat{X}_t \otimes \hat{P}^b + \hat{Y}_t \otimes \hat{Q}^b] \\ &\quad +[(A, P_b, X_t) - (A, Q_b, Y_t)][\hat{Y}_t \otimes \hat{P}^b + \hat{X}_t \otimes \hat{Q}^b], \\ J_{33} &= \hat{Y}_s \otimes (A, U_k, X_s)\hat{U}^k - \hat{X}_s \otimes (A, U_k, Y_s)\hat{U}^k. \end{aligned}$$

Clearly $J_{31} = 0$ for all A, ξ if and only if (10) holds, $J_{32} = 0$ for all A, ξ if and only if (5) holds, and $J_{33} = 0$ for all A, ξ if and only if (9) holds. Similarly, $J_4 = 2(\hat{V}_j \otimes \lozenge_{\hat{A}} \hat{U}^j - \lozenge_{\hat{A}} \hat{U}_j \otimes \hat{V}^j) = J_{41} + J_{42}$, where

$$\begin{aligned} J_{41} &= -\hat{V}_j \otimes (A, U_j, V_k)\hat{V}^k + (A, U_j, V_k)\hat{V}_k \otimes \hat{V}^j, \\ J_{42} &= -\hat{V}_j \otimes (A, U_j, X_t)\hat{X}^t - \hat{V}_j \otimes (A, U_j, Y_t)\hat{Y}^t + (A, U_j, X_t)\hat{X}_t \otimes \hat{V}^j \\ &\quad +(A, U_j, Y_t)\hat{Y}_t \otimes \hat{V}^j. \end{aligned}$$

Clearly $J_{41} = 0$ for all A, ξ if (6) holds, and $J_{42} = 0$ for all A, ξ if (9) holds.

(Proof of part (ii).) Let F be an $\mathbb{R}$-linear combination of V_i, X_r, Y_r. We write $2 \lozenge_{\hat{F}} \hat{J}|_\xi = K_1 + K_2 + K_3 + K_4$, where K_1, K_2, K_3, K_4 are the $\mathcal{H} \otimes \mathcal{H}^*, \mathcal{H} \otimes \mathcal{V}^*, \mathcal{V} \otimes \mathcal{H}^*, \mathcal{V} \otimes \mathcal{V}^*$ components of $2\lozenge_{\hat{F}} \hat{J}$ respectively. We have $K_1 = 2(\lozenge_{\hat{F}} \hat{Q}_b \otimes \hat{P}^b + \hat{Q}_b \otimes \lozenge_{\hat{F}} \hat{P}^b - \lozenge_{\hat{F}} \hat{P}_b \otimes \hat{Q}^b - \hat{P}_b \otimes \lozenge_{\hat{F}} \hat{Q}^b) = K_{11} + K_{12}$, where

$$\begin{aligned} K_{11} &= [(P_b, Q_c, F) + (Q_b, P_c, F)][\hat{P}_c \otimes \hat{P}^b + \hat{Q}_b \otimes \hat{Q}^c] \\ &\quad +[(Q_b, Q_c, F) - (P_b, P_c, F)][\hat{Q}_c \otimes \hat{P}^b + \hat{P}_c \otimes \hat{Q}^b], \\ K_{12} &= (Q_b, U_k, F)\hat{U}_k \otimes \hat{P}^b + \hat{Q}_b \otimes (P_b, U_k, F)\hat{U}^k - (P_b, U_k, F)\hat{U}_k \otimes \hat{Q}^b \\ &\quad -\hat{P}_b \otimes (Q_b, U_k, F)\hat{U}^k. \end{aligned}$$

Clearly $K_{11} = 0$ for all F, ξ if and only if (8) holds, and $K_{12} = 0$ for all F, ξ if (11) holds. An easy calculation yields $K_2 = K_3 = 0$ for all F, ξ if and only if (11) holds. By Lemma 3.6, $K_4 = 0$ automatically. □

Now we are ready for:

Proof of Theorem 3.3. (Proof of part (i).) The backward direction is a special case of Proposition 3.2. Now suppose that $(L^\perp, \hat{J}, \hat{g})$ is Kähler. By (5) and (8), $g(R^\perp(d_1, Jd_2)\xi, Je) = g(R^\perp(d_1, d_2)\xi, e) = g(R^\perp(Jd_1, Jd_2)\xi, e)$ for all $d_1, d_2 \in \Gamma(D), \xi \in \Gamma(L^\perp), e \in \Gamma(E)$. Therefore,

$$\begin{aligned} g(R^\perp(d_1, d_2)\xi, Je) &= g(R^\perp(Jd_1, Jd_2)\xi, Je) = g(R^\perp(Jd_1, d_2)\xi, e) \\ &= -g(R^\perp(d_2, Jd_1)\xi, e) = g(R^\perp(d_2, d_1)\xi, Je) = -g(R^\perp(d_1, d_2)\xi, Je), \end{aligned}$$

which implies $g(R^\perp(D, D)L^\perp, E) = 0$, i.e. $R^\perp(D, D)E = 0$. This, together with (4) and (7), implies that L is normal flat.

(Proof of part (ii).) Notice that the equation $R^\perp(TL, D)L^\perp = R^\perp(D', D')E = 0$ is equivalent to the following two ones:

$$g(R^\perp(TL, D)L^\perp, L^\perp) = 0, \tag{12}$$

$$g(R^\perp(D', D')L^\perp, E) = 0. \tag{13}$$

If G is an $\mathbb{R}$-linear combination of V_i, X_r, Y_r, then

$$\begin{aligned} 2d\hat{G}^*|_\xi &= 2(\hat{P}^a \wedge \Diamond_{\hat{P}_a}\hat{G}^* + \hat{Q}^a \wedge \Diamond_{\hat{Q}_a}\hat{G}^* + \hat{U}^i \wedge \Diamond_{\hat{U}_i}\hat{G}^* + \hat{V}^k \wedge \Diamond_{\hat{V}_k}\hat{G}^* \\ &\quad + \hat{X}^t \wedge \Diamond_{\hat{X}_t}\hat{G}^* + \hat{Y}^t \wedge \Diamond_{\hat{Y}_t}\hat{G}^*) \\ &= 2(P_a, Q_c, G)\hat{P}^a \wedge \hat{Q}^c + 2(P_a, U_k, G)\hat{P}^a \wedge \hat{U}^k \\ &\quad + 2(Q_a, U_k, G)\hat{Q}^a \wedge \hat{U}^k + (P_a, P_c, G)\hat{P}^a \wedge \hat{P}^c \\ &\quad + (Q_a, Q_c, G)\hat{Q}^a \wedge \hat{Q}^c + (U_i, U_k, G)\hat{U}^i \wedge \hat{U}^k. \end{aligned} \tag{14}$$

If B is an $\mathbb{R}$-linear combination of P_a, Q_a, U_i, a similar calculation yields $2d\hat{B}^*|_\xi = 0$. Therefore, letting $\hat{\omega}$ denote the fundamental 2-form of $(L^\perp, \hat{J}, \hat{g})$, we have

(15)

$$\begin{aligned} 2d\hat{\omega}|_\xi &= -2\hat{U}^j \wedge d\hat{V}^j + 2(d\hat{X}^s) \wedge \hat{Y}^s - 2\hat{X}^s \wedge d\hat{Y}^s \\ &= -\hat{U}^j \wedge [2(P_a, Q_c, V_j)\hat{P}^a \wedge \hat{Q}^c + 2(P_a, U_k, V_j)\hat{P}^a \wedge \hat{U}^k \\ &+ 2(Q_a, U_k, V_j)\hat{Q}^a \wedge \hat{U}^k \\ &+ (P_a, P_c, V_j)\hat{P}^a \wedge \hat{P}^c + (Q_a, Q_c, V_j)\hat{Q}^a \wedge \hat{Q}^c + (U_i, U_k, V_j)\hat{U}^i \wedge \hat{U}^k] \\ &- \hat{Y}^s \wedge [2(P_a, Q_c, X_s)\hat{P}^a \wedge \hat{Q}^c + 2(P_a, U_k, X_s)\hat{P}^a \wedge \hat{U}^k \end{aligned}$$

$$
\begin{aligned}
&+ 2(Q_a, U_k, X_s)\hat{Q}^a \wedge \hat{U}^k \\
&+ (P_a, P_c, X_s)\hat{P}^a \wedge \hat{P}^c + (Q_a, Q_c, X_s)\hat{Q}^a \wedge \hat{Q}^c + (U_i, U_k, X_s)\hat{U}^i \wedge \hat{U}^k] \\
&- \hat{X}^s \wedge [2(P_a, Q_c, Y_s)\hat{P}^a \wedge \hat{Q}^c + 2(P_a, U_k, Y_s)\hat{P}^a \wedge \hat{U}^k \\
&+ 2(Q_a, U_k, Y_s)\hat{Q}^a \wedge \hat{U}^k \\
&+ (P_a, P_c, Y_s)\hat{P}^a \wedge \hat{P}^c + (Q_a, Q_c, X_s)\hat{Q}^a \wedge \hat{Q}^c + (U_i, U_k, Y_s)\hat{U}^i \wedge \hat{U}^k].
\end{aligned} \tag{16}
$$

Equations (14) and (15) imply that $2d\hat{\omega}|_\xi = 0$ if and only if $-\hat{U}^j \wedge d\hat{V}^j|_\xi = (d\hat{X}^s) \wedge \hat{Y}^s|_\xi = -\hat{X}^s \wedge d\hat{Y}^s|_\xi = 0$. Therefore, by (16), $d\hat{\omega} = 0$ implies that (12) and (13) hold. Hence it remains to prove the converse.

Let R denote the curvature of $(L, g|L)$. If $\xi \in E'$, then $R^\perp(h_1, h_2)\xi = R(h_1, h_2)J\xi$ for all $h_1, h_2 \in T_xL$ by J-parallelism. Therefore (if $\operatorname{rank}(D') \geq 3$),

$$
\begin{aligned}
&\quad (U_1, U_2, V_3) + (U_2, U_3, V_1) + (U_3, U_1, V_2) \\
&= g(R^\perp(U_1, U_2)\xi, V_3) + g(R^\perp(U_2, U_3)\xi, V_1) + g(R^\perp(U_3, U_1)\xi, V_2) \\
&= g(JR^\perp(U_1, U_2)\xi, JV_3) + g(JR^\perp(U_2, U_3)\xi, JV_1) + g(JR^\perp(U_3, U_1)\xi, JV_2) \\
&= -g(R(U_1, U_2)J\xi, U_3) - g(R(U_2, U_3)J\xi, U_1) - g(R(U_3, U_1)J\xi, U_2) \\
&= 0 \quad \text{when} \quad \xi \in E',
\end{aligned}
$$

which implies $-U^j \wedge (U_i, U_k, V_j)U^i \wedge U^k = 0$ when $\xi \in E'$. Therefore, by (16), Equations (12) and (13) imply $d\hat{\omega}|_\xi = 0$ when $\xi \in E'$. Notice that (13) is equivalent to $g(R^\perp(D', D')E, L^\perp) = 0$. Therefore, by (16) again, (12) and (13) also imply $d\hat{\omega}|_\xi = 0$ when $\xi \in E$. Therefore, by linearity, (12) and (13) imply $d\hat{\omega}|_\xi = 0$ when $\xi \in E + E' = L^\perp$. □

4. Canonical symplectic forms.

For convenience, by a *peculiar submanifold* of an almost Hermitian submanifold M we mean a JP CR-submanifold L of M such that the canonical fundamental 2-form of $L^\perp$ is a symplectic form. As before, we use D_i to represent the primary complex bundle of L_i, $\hat{\omega}_i$ the canonical symplectic form of $L_i^\perp$, etc. The main theorem of this section is

Theorem 4.1. *Let L_1, L_2 be peculiar submanifolds of almost Hermitian manifolds $(M_1, J_1, g_1, \omega_1)$, $(M_2, J_2, g_2, \omega_2)$ respectively. Suppose $f : L_1^\perp \to L_2^\perp$ is a vector bundle isomorphism such that*

$$
f(E_1) = E_2,\ f_*(\hat{D}_1) = \hat{D}_2,\ \omega_2(f(X), f(Y)) = \omega_1(X, Y) \forall X, Y \in \Gamma(L_1^\perp). \tag{17}
$$

Then $\hat{\omega}_1$ is isomorphic to $\hat{\omega}_2$. If, in addition,

$$f(\nabla_v^{\perp,1}\xi) = \nabla_{f_*(v)}^{\perp,2} f \circ \xi \circ (f^{-1}|L_2) \tag{18}$$

for all $v \in D_1', \xi \in \Gamma(E_1)$, then $f_*(\hat{\omega}_1) = \hat{\omega}_2$.

Remark 4.2 (i) Notice that $f \circ \xi \circ (f^{-1}|L_2)$ is just the push-forward of ξ by f. Hence (18) just means that f preserves the E-parts of the normal connections along D'-directions.

(ii) Notice that we do not assume that g_1 or g_2 is complete.

(iii) The reader can get a clearer picture of this theorem by considering the isotropic case: $D_1 = D_2 = 0$, which implies $D_i' = TL_i$. We will present a handy corollary of this case.

The following lemma indicates that the canonical fundamental 2-form has nice symmetry.

Lemma 4.3. *Suppose L is a JP CR-submanifold of an almost Hermitian manifold. Let a, b be positive real numbers, $\hat{\omega}_D = \hat{\omega}|\hat{D}$, $\hat{\omega}_C = \hat{\omega}|\hat{D}' + \hat{E}'$, and $\hat{\omega}_E = \hat{\omega}|\hat{E}$. Let $\mathcal{M}_a^{E'}$ be the multiplication by a on E', $\mathcal{M}_{\sqrt{b}}^{E}$ be the multiplication by $\sqrt{b}$ on E, and $f = \mathcal{M}_a^{E'} \oplus \mathcal{M}_{\sqrt{b}}^{E}$. Then $f^*(\hat{\omega}) = \hat{\omega}_D \oplus a\hat{\omega}_C \oplus b\hat{\omega}_E$.*

Proof. First notice that

$$\hat{\omega} = \hat{\omega}_D \oplus \hat{\omega}_C \oplus \hat{\omega}_E. \tag{19}$$

Since L is JP, it is easy to see that $\nabla^\perp = \nabla^\perp|E' \oplus \nabla^\perp|E$, $\mathcal{M}_a^{E'}$ preserves $\nabla^\perp|E'$, and $\mathcal{M}_{\sqrt{b}}^{E}$ preserves $\nabla^\perp|E$. The lemma then follows easily. □

The following handy corollary will be used in Example 6.3.

Corollary 4.4. *Let L_1, L_2 be peculiar isotropic submanifolds of almost Hermitian manifolds $(M_1, J_1, g_1, \omega_1)$, $(M_2, J_2, g_2, \omega_2)$ respectively. Suppose f is a vector bundle isomorphism from E_1 to E_2 such that $\omega_2(f(X), f(Y)) = \omega_1(X, Y)$ for all $X, Y \in \Gamma(E_1)$. Then $\hat{\omega}_1$ is isomorphic to $\hat{\omega}_2$. If, in addition, $f_*(\nabla^{\perp,1}|E_1) = \nabla^{\perp,2}|E_2$, then $\hat{\omega}_1$ is isomorphic to $\hat{\omega}_2$ through a vector bundle isomorphism from $L_1^\perp$ to $L_2^\perp$ over $f|L_1$.*

Proof. By (19), $\hat{\omega}_1|D_1' + E_1'$ is nondegenerate. Also notice that D_1' is Lagrangian for $\hat{\omega}_1|D_1' + E_1'$. Similar remarks hold for $\hat{\omega}_2$. Hence there exists a unique vector bundle isomorphism h from E_1' to E_2' over the map $f|L_1$

such that $h_*(\hat{\omega}_1|D_1'+E_1') = \hat{\omega}_2|D_2'+E_2'$. By considering the map $h \oplus f$, this corollary follows from Theorem 4.1. □

Now we turn to:

Proof of Theorem 4.1. We consider $\omega' := \hat{\omega}_1|\hat{D}_1 \oplus 2\hat{\omega}_1|(\hat{D}_1'+\hat{E}_1') \oplus 4\hat{\omega}_1|\hat{E}_1$, $\omega'' := \hat{\omega}_2|\hat{D}_2 \oplus 2\hat{\omega}_2|(\hat{D}_2'+\hat{E}_2') \oplus 4\hat{\omega}_2|\hat{E}_2$, and $\omega^* := f^*(\omega'')$. By Lemma 4.3,

$$h^*(\hat{\omega}_1) = \omega', \tag{20}$$

where $h : L_1^\perp \to L_1^\perp$ is the multiplication by 2. A similar relation holds for $\hat{\omega}_2$ and ω''. By (17), we easily see

$$f(E_1') = E_2'. \tag{21}$$

Therefore, by (17) again, we have

$$\omega^*|\hat{D}_1 = \omega'|\hat{D}_1, \quad \omega^*|\hat{D}_1 \times \hat{F}_1 = \omega^*|\hat{F}_1 \times \hat{D}_1 = 0, \tag{22}$$

where $\hat{F}_1 = \hat{D}_1' \oplus \hat{E}_1' \oplus \hat{E}_1$. Let $\hat{\Pi}_1 \in \Gamma(T(L_1^\perp))$ be defined by $\hat{\Pi}_1 := P_{E_1'} + (1/2)P_{E_1}$. Define $\hat{\Pi}_2$ similarly. By (17) and (21) we have

$$f^*(\hat{\Pi}_2) = \hat{\Pi}_1. \tag{23}$$

An easy calculation yields that, with respect to the decomposition $T(L_1^\perp) = \hat{D}_1 \oplus \hat{F}_1$,

$$\mathcal{L}_{\hat{\Pi}_1}\omega' = 0 \oplus (\omega'|\hat{F}_1), \tag{24}$$

where $\mathcal{L}$ denotes the Lie derivative. We also have a similar formula for $\mathcal{L}_{\hat{\Pi}_2}\omega''$. Define $\gamma = \omega^* - \omega'$. Equations (21)-(24) imply

$$\mathcal{L}_{\hat{\Pi}_1}\gamma = 0 \oplus (\gamma|\hat{F}_1), \tag{25}$$

$$\gamma|\hat{D}_1 \times \hat{F}_1 = \gamma|\hat{F}_1 \times \hat{D}_1 = \gamma|\hat{D}_1 \times \hat{D}_1 = 0. \tag{26}$$

By J-parallelism D_1' is parallel with respect to the Levi-Civita connection of L_1. Frobenius' theorem then implies that D_1' is integrable. Let N be a leaf of D_1', $\mu = \gamma|(L_1^\perp|N)$, and $\xi = \hat{\Pi}_1|(L_1^\perp|N)$. Let $u^1, \ldots, u^k$ be a local coordinate system for N, $V_1, \ldots, V_k$ be a local orthonormal frame for $D_1'|N$, and $W_{2k+1}, \ldots, W_{2n}$ be a local orthonormal frame for $E_1'|N$. Also let $v^1, \ldots, v^k, w^{2k+1}, \ldots, w^{2n}$ be the local vector bundle coordinates of $(L_1^\perp|N)$ associated with $V_1, \ldots, V_k, W_{2k+1}, \ldots, W_{2n}$. The flow $\{f_t\}(t \in \mathbb{R})$ of ξ is given by

$$f_t(u^1, \ldots, u^k, v^1, \ldots, v^k, w^{2k+1}, \ldots, w^{2n})$$

$$= (u^1, \dots, u^k, e^t v^1, \dots, e^t v^k, e^{t/2} w^{2k+1}, \dots, e^{t/2} w^{2n}). \tag{27}$$

Hence the differential $T(f_t)$ is given by the diagonal matrix $\mathrm{diag}(I_k, e^t I_k, e^{t/2} I_{2n-2k})$. Express μ in terms of $u^1, \dots, u^k$, $v^1, \dots, v^k$, $w^{2k+1}, \dots, w^{2n}$. Then clearly we can decompose μ as $\mu = \mu_0 + \mu_{1/2} + \mu_1 + \mu_{3/2} + \mu_2$ such that $f_t^*(\mu_\beta) = e^{\beta t}\mu_\beta$, where μ_0 is a ($C^\infty(L_1^\perp, \mathbb{R})$-linear) combination of $du^i \wedge du^j$, $\mu_{1/2}$ of $du^i \wedge dw^j$, μ_1 of $du^i \wedge dv^j$ and $dw^i \wedge dw^j$, $\mu_{3/2}$ of $dv^i \wedge dw^j$, and μ_2 of $dv^i \wedge dv^j$. Now let c be a non-constant integral curve of ξ. The equation $\mathcal{L}_\xi \mu = \mu$ (which follows from (25) and (26)) implies $\frac{d}{dt} f_t^*[\mu(c(t))] = \mu(c(t))$. Routine calculation then yields $\mu_\beta(c(t)) = e^{(1-\beta)t}\mu_\beta(c(0))$. By (27), $c(t)$ approaches N as t approaches $-\infty$. Since μ vanishes on N, we conclude $\mu_1 = \mu_{3/2} = \mu_2 = 0$. Hence,

$$\mu = \mu_0 + \mu_{1/2} = \sum_{i<j} h_{ij} du^i \wedge du^j + \sum h'_{ij} du^i \wedge dw^j, \tag{28}$$

where h_{ij} and h'_{ij} are C^∞ real functions on $L_1^\perp$.

By looking at the determinants for $\omega_t := (1-t)\omega^* + t\omega'$ in suitable coordinates, we conclude, by (22), (26), and (28), that ω_t is a symplectic form for all $t \in [0,1]$. Let $\lambda = \hat{\Pi}_1 \lrcorner (\omega^* - \omega')$. Then $\omega^* - \omega' = d\lambda$. Now apply Moser's technique as in Lemma 2.5. In particular we define X_t by $\omega_t(X_t, .) = \lambda$. By (22), (26), and (28), X_t is tangent to the fibers of $L^\perp$. With (27) and (28), a routine calculation yields that X_t has (uniform) sublinear growth (cf. the proof of [**3**, Proposition 3.1]). Hence, ω^* and ω' are isomorphic. By (20) and its analogue for $\hat{\omega}_2$, this completes the proof of the first part of the theorem.

Now assume that (18) holds. To prove the second part of the theorem, it suffices to show $\mu = 0$, in view of (20) and its analogue for $\hat{\omega}_2$. We let $\hat{U}_i$, $\hat{V}_i$, $\hat{W}_i$ denote the Sasaki lift of $\frac{\partial}{\partial u^i}|L$, V_i, W_i respectively. By (ii) of Theorem 3.3, $\nabla^{\perp,1}|E_1$ is flat. Therefore, we can choose u^i, V_i, v^i, W_i, w^i as before, requiring that W_i are parallel with respect to $\nabla^{\perp,1}$. This implies $\frac{\partial}{\partial u^i} = \hat{U}_i + \sum a_{ij}\hat{V}_j$, where $a_{ij} = 0$ at $v^1 = \cdots = v^k = w^{2k+1} = \cdots w^{2n} = 0$. This in turn implies

$$du^i = \sum b_{ij}\hat{U}_j^* + c_{ij}\hat{V}_j^*, \tag{29}$$

where a_{ij} are independent of $v^1, \dots, v^k, w^{2k+1}, \dots w^{2n}$, and $b_{ij} = 0$ at $v^1 = \cdots = v^k = w^{2k+1} = \cdots w^{2n} = 0$. (The symbol * refers to the metric dual with respect to $\hat{g}_1$. Hence, $\hat{U}_j^* = \hat{g}_1(\hat{U}_j, \cdot)$, etc.) We also have

$$dw^i = \hat{W}_i^*. \tag{30}$$

Now notice that (18) implies $\gamma|((\hat{D}'_1 \times \hat{E}_1)|(L_1^\perp|N)) = \gamma|((\hat{E}_1 \times \hat{D}'_1)|(L_1^\perp|N)) = 0$. Equations (28)–(30) then force $h'_{ij} = 0$. The closedness of μ then implies that h_{ij} are independent of $v^1, \dots, v^k$, $w^{2k+1}, \dots, w^{2n}$. But this means $h_{ij} = 0$, since μ vanishes on N. Hence $\mu = 0$. □

5. Generalization of McDuff's results.

Throughout this section, we use the symbol $\mathcal{J}_p^k f$ to denote the k-th jet of a smooth map f at the point p. This section is devoted to prove:

Theorem 5.1. *Let M be a complete Kähler manifold with an isotropic geo-post L such that the canonical fundamental 2-form of $L^\perp$ is a symplectic form. Suppose M has nonpositive radial sectional curvature with respect to L. Then the Kähler form of M is isomorphic to the canonical symplectic form on $L^\perp$ by a diffeomorphism f satisfying $\mathcal{J}_p^1(f) = \mathcal{J}_p^1(\exp^{-1})$ for all $p \in L$.*

Remark 5.2 (i) Since every totally geodesic submanifold of M is JP (Example 6.1), by (ii) of Theorem 3.3, the condition "the canonical fundamental 2-form of $L^\perp$ is a symplectic form" can be equivalently changed to "L is secondary flat."

(ii) The canonical symplectic form of T^*L is isomorphic to the canonical symplectic form $\check{g}(\check{J}\cdot,\cdot)$ of TL [**5**]. Hence Theorem 5.1 generalizes Theorem 1.1, in view of (ii) of Proposition 3.1 and (ii) of Theorem 3.3.

If L is a post of (M, J, g), then $\rho(x) := |\exp^{-1}(x)|$ is the g-distance from $x \in M$ to L. Hence ρ^2 is C^∞ and we can define a 2-form $\omega_\rho := -dJd\rho^2$.

Lemma 5.3. *Let (M, J, g) be a Kähler manifold and L be an isotropic geo-post. Suppose M has nonpositive radial sectional curvature with respect to L. Then:*

(i) *Let $g_\rho = \omega_\rho(\cdot, J\cdot)$. Then (M, J, g_ρ) is a Kähler manifold and there exists a positive number ϵ such that $g_\rho \geq \epsilon g$.*

(ii) *Define a vector field Π on M by $\Pi \lrcorner \omega_\rho = -Jd\rho^2$. Let $\hat{\Pi} = P_{E'} + \frac{1}{2}P_E$ and $\omega_{24} = 2\hat{\omega}_C \oplus 4\hat{\omega}_E$. (See Lemma 4.3 and notice $D = 0$.) Let $\Pi^* = \exp^*(\Pi)$ and $\omega_\rho^* = \exp^*(\omega_\rho)$. Then for every $p \in L$, $\mathcal{J}_p^1(\hat{\Pi}) = \mathcal{J}_p^1(\Pi^*)$ and $\mathcal{J}_p^1(\omega_{24}) = \mathcal{J}_p^1(\omega_\rho^*)$.*

(iii) *Define $\mathcal{M}_2 := exp_*(\mathcal{M}_2^{L^\perp})$, where $\mathcal{M}_2^{L^\perp}$ is the multiplication by 2 on $L^\perp$. Then there exists a (C^∞) diffeomorphism f from M to M such that $f^*(\omega) = \omega_\rho$ and $\mathcal{J}_p^1(f) = \mathcal{J}_p^1(\mathcal{M}_2)$ for all $p \in L$.*

Proof. (Proof of parts (i) and (ii).) Part (i) and $\mathcal{J}_p^1(\hat{\Pi}) = \mathcal{J}_p^1(\Pi^*)$ follow from the proofs of [**11**, Lemma 3.1] and [**11**, Lemma 3.4] with slight modification. Hence it remains to prove $\mathcal{J}_p^1(\omega_{24}) = \mathcal{J}_p^1(\omega_\rho^*)$.

Fix $p \in L$. Let $u^1, \dots, u^k$ be a normal coordinate system for L around p, defined on a normal ball $\mathcal{B} \subset L$. For each i, let $(V_i)_p = J(\frac{\partial}{\partial u^i}|_p) \in E'_p$ and $V_i \in \Gamma(E'|\mathcal{B})$ parallel translations (with respect to $\nabla^\perp$) of $(V_i)_p$

along the radial geodesics from p in $\mathcal{B}$. Let $X_1, \dots, X_m \in \Gamma(E|\mathcal{B})$ and $Y_1 = JX_1, \dots, Y_m = JX_m \in \Gamma(E|\mathcal{B})$ be orthonormal sections of $E|\mathcal{B}$ which are parallel along the radial geodesics from p in $\mathcal{B}$. We use coordinates $(u^1, \dots, u^k,\ v^1, \dots, v^k,\ x^1, \dots, x^m,\ y^1, \dots, y^m)$ to represent the element $[\sum v^i V_i + \sum (x^i X_i + y^i Y_i)]_x$, where $x = (u^1, \dots, u^k)$. For simplicity, we identify $L^\perp$ with M via exp. All the Christoffel symbols $\tilde{\Gamma}^A_{BC}$ of $\tilde{\nabla}$ with respect to this coordinate system vanish at p.

We use $r(x) = |x|$ to denote the natural Euclidean distance of $x \in M$ to p associated with the coordinates. $J^A_B = (J^A_B)_p + O(r^2)$ because of the the vanishing of the Christoffel symbols. Therefore we have, at the point $(u^i, v^j, x^\alpha, y^\beta)$ near p, $-Jd\rho^2 = (-2v^i, 0, -2y^\alpha, 2x^\beta) + O(r^2)$. Since $\mathcal{J}^1_p(\hat{\Pi}) = \mathcal{J}^1_p(\Pi^*)$, we also have $\Pi = (0, v^j, \frac{1}{2}x^\alpha, \frac{1}{2}y^\beta)^t + O(r^2)$. Hence we have a matrix equation $((0, v^j, \frac{1}{2}x^\alpha, \frac{1}{2}y^\beta) + O(r^2))((\omega_\rho)_{AB}) = (-2v^i, 0, -2y^\alpha, 2x^\beta) + O(r^2)$. Comparison of both sides yields

$$\omega_\rho = ((\omega_\rho)_{AB}) = \begin{pmatrix} * & 2I_k & 0 & 0 \\ -2I_k & 0 & 0 & 0 \\ 0 & 0 & 0 & 4I_m \\ 0 & 0 & -4I_m & 0 \end{pmatrix} + O(r^2),$$

where $*$ denotes some unknown matrix. The closedness of ω then implies $* = O(r^2)$. Hence, with $\mathbb{M}$ representing the above matrix (with $* = 0$ and without $O(r^2)$), it remains to prove

$$\omega_{24} = \mathbb{M} + O(r^2) \quad \text{w.r.t. frame } \left\{ \frac{\partial}{\partial u^i}, \hat{V}_j, \hat{X}_\alpha, \hat{Y}_\beta \right\}. \tag{31}$$

We remark that $\hat{V}_j = \frac{\partial}{\partial v^j}$, $\hat{X}_\alpha = \frac{\partial}{\partial x^\alpha}$, and $\hat{Y}_\beta = \frac{\partial}{\partial y^\beta}$.

Let $\hat{U}_i \in \Gamma(L^\perp|\mathcal{B})$ be defined by the following three rules: (a). $(\hat{U}_i)_p = (\frac{\partial}{\partial u^i})_p$. (b). For all $q \in \mathcal{B}$, $(\hat{U}_i)_q$ is the parallel translation (with respect to ∇) of $(\hat{U}_i)_p$ along the radial geodesic connecting p to q. (c). For every $\xi \in L^\perp|\mathcal{B}$ with foot $q \in \mathcal{B}$, $(\hat{U}_i)_\xi$ is the Sasaki lift of $(\hat{U}_i)_q$ to ξ. By the J-parallelism of L, we have

$$\omega_{24} = \mathbb{M} \quad \text{w.r.t. frame } \{\hat{U}_i, \hat{V}_j, \hat{X}_\alpha, \hat{Y}_\beta\}. \tag{32}$$

Now the equation for parallel translation $\dot{z}^a + \sum \tilde{\Gamma}^a_{ij} \dot{u}^i \dot{u}^j = 0$ implies $(\hat{U}_i)_q = (\frac{\partial}{\partial u^i})_q + O(r^2) \quad (q \in \mathcal{B}$ only). Also, if K denotes the connection map from $T(L^\perp)$ to $L^\perp$ induced by $\nabla^\perp$, then

$$K\left(\left(\frac{\partial}{\partial u^i} \right)_{(u^1, \dots, u^k, v^1, \dots, v^k, x^1, \dots, x^m, y^1, \dots, y^m)} \right)$$

$$= \nabla^{\perp}_{(\frac{\partial}{\partial u^i})_{(u^1,\ldots,u^k,0,\ldots,0,\ldots,0,\ldots)}} \left(\sum v^{i'} V_{i'} + \sum x^{\alpha'} X_{\alpha'} + \sum y^{\beta'} Y_{\beta'}\right)$$
$$= \sum v^{i'} \tilde{\Gamma}^{k+i''}_{i,k+i'} V^{i''} + \sum x^{\alpha'} \tilde{\Gamma}^{2k+\alpha''}_{i,2k+\alpha'} X_{\alpha''} + \sum y^{\beta'} \tilde{\Gamma}^{2k+m+\beta''}_{i,2k+m+\beta'} Y_{\beta''} = O(r^2).$$

Hence $\hat{U}_i = \frac{\partial}{\partial u^i} + O(r^2)$. Hence (31) follows from (32).

(Proof of part (iii).) Let $\gamma = \omega_\rho - \omega$, $\omega_t = (1-t)\omega_\rho + t\omega$ ($t \in [0,1]$). By part (i), there exists a positive constant $\epsilon \leq 1$ such that the first inf norm $|\omega_t|_{1-\inf} \geq \epsilon$ for all $t \in [0,1]$. Hence Lemma 2.4 implies $\gamma = d\lambda$, where λ is given as in Lemma 2.6. Define vector field X_t by $\omega_t(X_t, \cdot) = \lambda$. By Lemma 2.5, $(f_1)^*(\omega) = \omega_\rho$, where f_t is the flow of X_t. Define $f = f_1$. Then it remains to show that $\mathcal{J}^1_p(f) = \mathcal{J}^1_p(\mathcal{M}_2)$ for all $p \in L$.

Fix $p \in L$. By part (ii), $\omega_\rho = \sum 2du^i \wedge dv^i + \sum 4dx^\alpha \wedge dy^\alpha + O(r^2)$. Also, $\omega = \sum du^i \wedge dv^i + \sum dx^\alpha \wedge dy^\alpha + O(r^2)$. Therefore,

$$\gamma = \sum du^i \wedge dv^i + \sum 3dx^\alpha \wedge dy^\alpha + O(r^2), \tag{33}$$

$$\omega_t = \sum (2-t) du^i \wedge dv^i + \sum (4-3t) dx^\alpha \wedge dy^\alpha + O(r^2). \tag{34}$$

Let $Z \in T_\xi(L^\perp)$ and $\xi \in L^\perp$. We will use $\mathcal{H}Z$ to denote the $\{\frac{\partial}{\partial u^i}\}$-component of Z and $\mathcal{V}Y$ to denote the $\left\{\frac{\partial}{\partial v^j}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial y^\alpha}\right\}$-component of Z. By (3) and (33),

$$\lambda_\xi(Z) = \int_0^1 \gamma(\xi, (\phi_t)_* Z)\, dt$$
$$= \int_0^1 \left(\sum du^i \wedge dv^i + \sum 3dx^\alpha \wedge dy^\alpha + O(r^2)\right)(\xi, \mathcal{H}Z + t\mathcal{V}Z)\, dt$$
$$= \int_0^1 \left\{\left[\sum -v^i du^i(\mathcal{H}Z) + \sum t[3x^\alpha dy^\alpha(\mathcal{V}Z) - 3y^\alpha dx^\alpha(\mathcal{V}Z)]\right] + O(r^2)\right\} dt$$
$$= \sum -v^i du^i(\mathcal{H}Z) + \sum \frac{3x^\alpha dy^\alpha(\mathcal{V}Z) - 3y^\alpha dx^\alpha(\mathcal{V}Z)}{2} + O(r^2),$$

which implies

$$\lambda = \sum -v^i du^i + \sum \frac{3x^\alpha dy^\alpha - 3y^\alpha dx^\alpha}{2} + O(r^2). \tag{35}$$

Equations (34) and (35) imply $X_t = Y_t + O(r^2)$, where $Y_t = \sum \frac{v^i}{2-t}\frac{\partial}{\partial v^i} + \sum \left[\frac{3x^\alpha}{8-6t}\frac{\partial}{\partial x^\alpha} + \frac{3y^\alpha}{8-6t}\frac{\partial}{\partial y^\alpha}\right]$. Let h_t denote the flow of Y_t. Then $h_1(u^i, v^j, x^\alpha, y^\beta) = (u^i, 2v^j, 2x^\alpha, 2y^\beta)$, which implies $\mathcal{J}^1_p(h_1) = \mathcal{J}^1_p(\mathcal{M}_2)$. Hence it remains to prove $\mathcal{J}^1_p(h_1) = \mathcal{J}^1_p(f_1)$.

Since $f_t(0) = h_t(0) = 0$ for all $t \in [0,1]$, there exists a neighborhood $\mathcal{N}$ of 0 in M and a positive constant A such that $|f_t(x)| \leq A|x|$, $|h_t(x)| \leq A|x|$, and $|f_t(x) - h_t(x)| \leq A|x|$ for all $x \in \mathcal{N}$ and $t \in [0,1]$. By adjusting A and $\mathcal{N}$ if necessary, we can then assume $|X_t(f_t(x)) - Y_t(f_t(x))| \leq A|x|^2$ and

$|Y_t(f_t(x)) - Y_t(h_t(x))| \le A|f_t(x) - h_t(x)|$ for all $x \in \mathcal{N}$ and $t \in [0,1]$. Now fix $x \in \mathcal{N}$. Define $H(t) = f_t(x) - h_t(x)$. We have $|\dot{H}(t)| = |X_t(f_t(x)) - Y_t(h_t(x))| \le A|x|^2 + A|H(t)|$. Consider the O.D.E.: $\dot{s} = A|x|^2 + As$ with initial condition $s(0) = 0$. We have $s(1) = |x|^2(e^A - 1)$. A comparison theorem of O.D.E. [8, p. 32, Corollary 6.2] implies $|H(1)| \le s(1) = O(|x|^2)$. This means $\mathcal{J}_p^1(H(1)) = 0$ and thus $\mathcal{J}_p^1(f_1) = \mathcal{J}_p^1(h_1)$. □

Now we are ready for:

Proof of Theorem 5.1. First notice that L is J-parallel (see Example 6.2).

We use the notation in Lemma 5.3. In view of Lemma 4.3 (with $a = \sqrt{b} = 2$) and (iii) of Lemma 5.3, it suffices to find a (C^∞) diffeomorphism $h : L^\perp \to L^\perp$ such that $h^*(\omega_\rho^*) = \omega_{24}$ and $\mathcal{J}_p^1(h) = \mathcal{J}_p^1(\mathrm{Id})$.

Fix a positive integer K. Since $\mathcal{J}_p^1(\hat{\Pi}) = \mathcal{J}_p^1(\Pi^*)$ for all $p \in L$, by Ciriza's linearization theorem [4, Theorem 1.1], for every $p \in L$, there exist neighborhoods N_1 and N_2 of p in $L^\perp$ and a C^K diffeomorphism $\phi_{K,p} : N_1 \to N_2$ fixing $L \cap N_1$ pointwise such that $(\phi_{K,p})^*(\Pi^*) = \hat{\Pi}$. Since $\hat{\Pi} = P_{E'} + \frac{1}{2}P_E$, such $\phi_{K,p}$ is unique if we require $\mathcal{J}_q^1(\phi_{K,p}) = \mathcal{J}_q^1(\mathrm{Id})$ for all $q \in L^\perp \cap N_1$. Therefore, there exist neighborhoods $\mathcal{N}_1$ and $\mathcal{N}_2$ of L in $L^\perp$ and a C^K diffeomorphism $h_K : \mathcal{N}_1 \to \mathcal{N}_2$ such that $(h_K)^*(\Pi^*) = \hat{\Pi}$ and $\mathcal{J}_p^1(h_K) = \mathcal{J}_p^1(\mathrm{Id})$ for all $p \in L$.

Notice that $d(\rho^2)(X) = g_\rho(X, \Pi)$ for all $X \in \Gamma(TM)$. Hence Π is transverse to the level surfaces $\rho = \text{constant} \neq 0$. Lemma 5.3 implies the g-norm of Π has sublinear growth with respect to g. Hence, Π is complete, and for every $x \in M$, $f_t(x)$ has a unique limit in L as t approaches $-\infty$, where f_t is the flow of Π. Similar remarks apply to $\hat{\Pi}$. Therefore, h_K can be uniquely (C^K) extended to all of $L^\perp$ and still satisfies $(h_K)^*(\Pi^*) = \hat{\Pi}$ and $\mathcal{J}_p^1(h_K) = \mathcal{J}_p^1(\mathrm{Id})$ for all $p \in L$. But now by the uniqueness of each h_K in a neighborhood of L, $h_1 = h_2 = h_3 = \cdots$. This means h_1 is C^∞. Let $h = h_1$. Then $h^*(\Pi^*) = \hat{\Pi}$ and

$$\mathcal{J}_p^1(h) = \mathcal{J}_p^1(\mathrm{Id}) \tag{36}$$

for all $p \in L$. Hence it remains to show $h^*(\omega_\rho^*) = \omega_{24}$.

We easily see $\mathcal{L}_{\Pi^*}\omega_\rho^* = \omega_\rho^*$ and $\mathcal{L}_{\hat{\Pi}}\omega_{24} = \omega_{24}$. Define $\mu = \omega_{24} - h^*(\omega_\rho^*)$. Then $\mathcal{L}_{\hat{\Pi}}\mu = \mu$. By (36) and $\mathcal{J}_p^1(\omega_{24}) = \mathcal{J}_p^1(\omega_\rho^*)$ (Lemma 5.3), we have

$$\mathcal{J}_p^1(\mu) = 0 \tag{37}$$

for all $p \in L$. Now we use the coordinates $u^1, \ldots, u^k$, $v^1, \ldots, v^k$, $w^{2k+1}, \ldots, w^{2n}$ as in the proof of Theorem 4.1. We still have (28): $\mu = \sum h_{ij} du^i \wedge du^j + \sum h'_{ij} du^i \wedge dw^j$. With the aid of (37), arguments similar to those in the proof of Theorem 4.1 yield $\mu = 0$. Hence $h^*(\omega_\rho^*) = \omega_{24}$. □

6. Examples.

In this section we give some examples and applications.

Example 6.1. Let M be a Kähler manifold. A CR-submanifold of M is JP if and only if its D and E are parallel (with respect to $\Box$). In particular, every complex or Lagrangian submanifold of M is JP; an isotropic submanifold of M is JP if and only if its E is parallel. Every totally geodesic submanifold of M is also JP. All these are routine to prove.

Example 6.2. Let $(\mathbb{C}H^n, J, g)$ be the complex hyperbolic space. $\mathbb{R}H^m$ (viewed as a submanifold of $\mathbb{C}H^n$, $0 \leq m \leq n$) is not normal flat if $m \geq 2$. On the other hand, since $\Box$ agrees with $\tilde{\nabla}$ along $\mathbb{R}H^m$, the formula for Riemannian curvature tensor $\tilde{R}$ of $\mathbb{C}H^n$,

$$\begin{aligned}&\tilde{R}_{XY}Z\\&= \frac{1}{4}[g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ],\end{aligned}$$

implies that $\mathbb{R}H^m$ is secondary flat. $\mathbb{R}H^m$ is also an isotropic geo-post [1]. Therefore, Theorem 5.1 is applicable to this case in view of (i) of Remark 5.2.

The usual symplectic form on $\mathbb{R} \times \mathbb{R}^{2n-1}$ is just the usual one on $\mathbb{R}^{2n} = \mathbb{R} \times \mathbb{R}^{2n-1}$. The usual symplectic form on $S^1 \times \mathbb{R}^{2n-1}$ is just $d\theta \wedge dy^1 + dx^2 \wedge dy^2 + \cdots dx^n \wedge dy^n$. The following example complements (i) of Theorem 1.1.

Example 6.3. Let (M^{2n}, J, g) be a complete Kähler manifold. Suppose M admits a 1-dimensional geo-post L such that M has nonpositive radial sectional curvature with respect to L. Then the Kähler form of M is isomorphic to the usual symplectic form on $L \times \mathbb{R}^{2n-1}$. In particular, the Kähler form of any Kähler axial manifold (i.e. a Kähler manifold that is also an axial manifold [6]) is isomorphic to the usual symplectic form on $S^1 \times \mathbb{R}^{2n-1}$. The proof is as follows:

Let $L \times \mathbb{R}^{2n-1}$ be equipped with the usual Kähler structure. Also let $L_1 = L_2 = L$, $M_1 = M$, and $M_2 = L \times \mathbb{R}^{2n-1}$. By Theorem 5.1, the Kähler form of M_i is isomorphic to $\hat{\omega}_i$. Consider $(E_i, \omega_i|E_i)$, viewed as a symplectic vector bundle over L_i in the canonical way. By the last paragraph of [14, p. 32] and the remark following it, we can show that $(E_1, \omega_1|E_1)$ and $(E_2, \omega_2|E_2)$ are isomorphic. Therefore, Corollary 4.4 implies the main claim of this example. The particular case then follows, since every axial manifold admits a geo-post diffeomorphic to S^1 [6, Corollary 6.16].

If (M, J, g, ω) and (M', J', g', ω') are almost Hermitian manifolds and there exist a diffeomorphism $f : M' \to M$ and a positive constant c such that $(f_*\omega')(X, JX) \geq c\omega(X, JX)$ for all $X \in TN$, then we say (M', J', g') *dominates* (M, J, g) *symplectically via* f. While Examples 6.2 and 6.3 involve nonpositive curvature, the following one does not.

Example 6.4. (i) Suppose (M, J, g, ω) is a complete almost Kähler manifold, and L is peculiar post such that exp is a quasi-isometry and (M, J, g) dominates $(L^\perp, \hat{J}, \hat{g})$ symplectically via $\exp^{-1}$. Then the Kähler form of M is isomorphic to the canonical symplectic form of $L^\perp$ via a diffeomorphism f fixing L pointwise. Proof: Clearly L is an increasing geo-post of $(L^\perp, \hat{g})$. Also, $\exp^*(\omega)$ has bounded $\hat{g}$-norm. Hence, with the aid of Lemma 2.6, the result follows from Lemma 2.5.

In view of Greene-Wu's quasi-isometry theorem [**7**, p. 56, Theorem C], if L is a post of a Riemannian manifold M such that M is asymptotically flat with respect to L in certain sense, then exp is a quasi-isometry.

(ii) If $(M, \{x\}, J, g)$ is a Kählerian model [**15**] and the corresponding exponential map is a quasi-isometry, then (M, J, g) dominates $(\{x\}^\perp, \hat{J}, \hat{g})$ symplectically via $\exp^{-1}$. The proof is as follows:

Let γ be a unit-speed radial geodesic from x. Choose orthonormal vectors $e_1, \ldots, e_{2n}$ of T_xM such that $e_1 = \gamma'(0)$ and $J(e_i) = e_{i+1}$ if i is odd. All the subspaces span$\{e_1, e_2\}$, span$\{e_1, e_3\}$, $\ldots$, span$\{e_1, e_{2n}\}$ are totally geodesic with respect to g^* [**15**, Theorem A]. ($g^* := \exp^*(g)$, $J^* := \exp^*(J)$.) Thus there exists a positive number c such that $p_t^*(e_i) = s(t, i)\hat{p}_t(e_i)$ for some $s(t, i) \geq c$ for all i and t. (Here $\hat{p}_t$ (resp. p_t^*) is the parallel translation from $\gamma(0)$ to $\gamma(t)$ along γ with respect to the metric $\hat{g}$ (resp. g^*).) Hence, letting $\hat{e}_i$ denote the Sasaki lift of e_i and adjusting c if necessary, we have $g^*(\hat{e}_i, \hat{e}_j) = c_i\delta_{ij}$, $J^*(\hat{e}_i) = b_i\hat{e}_{i+1}$ if i is odd, $J^*(\hat{e}_i) = -b_i\hat{e}_{i-1}$ if i is even, for some $b_i, c_i \geq c$. An easy calculation then yields $g^*(\hat{J}\sum a_i\hat{e}_i, J^*\sum a_i\hat{e}_i) \geq c^2\sum a_i^2 = c^2\hat{g}(\sum a_i\hat{e}_i, \sum a_i\hat{e}_i)$.

References

[1] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1-49.

[2] A.A. Borisenko and A.L. Yampol'skii, *On the Sasaki metric of the normal bundle of a submanifold in a Riemannian space*, Math. USSR Sbornik, **62**(**1**) (1989), 157-175.

[3] E. Ciriza, *On special submanifolds in symplectic geometry*, Diff. Geom. Appl., **3** (1993), 91-99.

[4] ———, *The local structure of a Liouville vector field*, Amer. J. Math., **115** (1993), 735-747.

[5] P. Dombrowski, *On the geometry of the tangent bundle*, J. Reine Angew Math., **210** (1962), 73-88.

[6] P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math., **46** (1973), 45-109.

[7] R.E. Greene and H. Wu, *Function Theory on Manifolds which Possess a Pole*, LNM 699, Springer-Verlag, Berlin, 1979.

[8] J.K. Hale, *Ordinary Differential Equations*, 2nd ed., Krieger, Malabar, 1980.

[9] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhäuser, Basel, 1994.

[10] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. 2, Interscience Publishers John Wiley & Sons, Inc., New York-London-Sydney, 1969.

[11] D. McDuff, *The symplectic structure of Kähler manifolds of non-positive curvature*, J. Diff. Geom., **28** (1988), 467-475.

[12] J. Moser, *On volume elements on a manifold*, Trans. Amer. Math. Soc., **120** (1965), 286-294.

[13] Ph. Tondeur, *Structure presque Kählerienne naturelle sur le fibré des vecteurs covariants d'une variété riemanienne*, C. R. Acad. Sci., **254** (1962), 407-408.

[14] I. Vaisman, *Symplectic Geometry and Secondary Characteristic Classes*, Birkhäuser, Boston, 1987.

[15] Y. Watanabe and H. Mori, *Geometry on unitary-symmetric Kähler manifolds*, Math. J. Toyama Univ., **16** (1993), 135-193.

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University of Maryland
College Park, MD 20742
E-mail address: phh@math.umd.edu