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MODULES WITH NORMS WHICH TAKE VALUES IN A C*-ALGEBRA

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We consider modules E over a C*-algebra A which are equipped with a map into A_+ that has the formal properties of a norm. We completely determine the structure of these modules. In particular, we show that if A has no nonzero commutative ideals then every such E must be a Hilbert module. The commutative case is much less rigid: If $A = C_0(X)$ is commutative then E is merely isomorphic to the module of continuous sections of some bundle of Banach spaces over X. In general E will embed in a direct sum of modules of the preceding two types.

Let A be a C*-algebra, and let A_+ denote the set of positive elements of A. We define a **Finsler A-module** to be a left A-module E which is equipped with a map $\rho : E \to A_+$ such that

(1) the map $\|\cdot\|_E : x \mapsto \|\rho(x)\|$ is a Banach space norm on E; and

(2) $\rho(ax)^2 = a\rho(x)^2a^*$ for all $a \in A$ and $x \in E$.

If we use the convention $|b| = (bb^*)^{1/2}$ for $b \in A$, then condition (2) is equivalent to

$$\rho(ax) = |a\rho(x)|.$$

For A commutative this is the same as $\rho(ax) = |a|\rho(x)$, which is the usual form this sort of axiom takes in the commutative case. But this last version is not appropriate in the noncommutative case because $\rho(ax)$ is positive, while $|a|\rho(x)$, though a product of two positive elements, in general cannot be expected to be self-adjoint, let alone positive.

(Note that we have used the reverse of the usual definition of $|b|$. This is appropriate in the setting of left modules, while the usual convention is the best for right modules. For instance, A is a left Hilbert module over itself via the inner product $\langle a, b\rangle = ab^*$, and this makes it natural to take $|b^2| = \langle b, b\rangle = bb^*$. In any case, in the remainder of the paper we will use the notation $|b|$ only for normal elements b.)

If E is a Hilbert A-module then defining $\rho(x) = \langle x, x\rangle^{1/2}$ makes E a Finsler module; in particular, $\langle ax, ax\rangle = a\langle x, x\rangle a^*$, so condition (2) holds. This also helps to justify the specific form of this condition, on the grounds that any definition of an A-valued norm ought to include norms arising from Hilbert modules in this manner.

Indeed, in the commutative case Finsler modules are a natural generalization of Hilbert modules. To see this let X be a locally compact space and let $\mathcal{B} = \bigcup_{t \in X} H_t$ be a bundle of Hilbert spaces over X satisfying appropriate continuity properties. Then the set E of continuous sections (that is, continuous maps $f : X \to \mathcal{B}$ such that $f(t) \in H_t$ for $t \in X$) which vanish at infinity, is naturally a $C_0(X)$-module. Furthermore it has a $C_0(X)$-valued inner product defined by

$$\langle f, g\rangle(t) = \langle f(t), g(t)\rangle_{H_t}$$

for $t \in X$, hence is a Hilbert $C_0(X)$-module [16]. Conversely, every Hilbert $C_0(X)$-module is isomorphic to one of this form [21].

If we ask instead only that $\mathcal{B} = \bigcup_{t \in X} B_t$ be a bundle of Banach spaces over X, then the module of continuous sections now possesses a $C_0(X)$-valued norm

$$\rho(f)(t) = \|f(t)\|_{B_t}$$

rather than a $C_0(X)$-valued inner product. It is easy to see that this makes E a Finsler $C_0(X)$-module, and we prove conversely that (as an easy consequence of known facts) every Finsler $C_0(X)$-module is isomorphic to one of this form.

Thus, given the well-known conception of finitely generated projective modules over C*-algebras as "noncommutative vector bundles" and Hilbert modules as "noncommutative Hilbert bundles" ([18], [20]), it may appear that our Finsler modules might serve as the basis for a noncommutative version of Banach bundles. Now we mentioned above that every Hilbert A-module carries a natural Finsler structure. One might hope to construct non-Hilbert Finsler modules over many C*-algebras A by forming a suitable completion of the algebraic tensor product of A with a non-Hilbert Banach space. Surprisingly, we found that for "most" noncommutative C*-algebras, namely all those algebras A with no nonzero commutative ideals, every Finsler A-module must arise from a unique Hilbert A-module (Corollary 18). In comparison with the commutative situation just discussed, even with the case $A = \mathbf{C}$ (when E can be any Banach space), the noncommutative case is evidently far more rigid. From one standpoint this is merely a negative result which shows that A-valued norms are not interesting in the noncommutative case. On the other hand it may be viewed as a positive result about the robustness of the concept of Hilbert modules, a topic also explored in [8], and also as indirect evidence that operator modules are really the right noncommutative version of Banach bundles, a position we argue in Section 2. The Banach module properties of Hilbert modules have also been considered in [14].

Our terminology was chosen for the following reason. A natural example of a bundle of Hilbert spaces is given by the tangent bundle of a Riemannian manifold X. Here the vector space over a point $t \in X$ is simply the tangent space at t, and the fact that X is Riemannian means precisely that each tangent space has an inner product. Finsler geometry is an increasingly popular generalization of Riemannian geometry in which one requires only that each tangent space have a norm ([5], [9]). Thus Finsler geometry appears to involve Banach bundles in the same way that Riemannian geometry involves Hilbert bundles. We wish to thank David Blecher for pointing out this connection between Finsler manifolds and Banach bundles.

Section 1 contains preliminary general results. In Section 2 we establish connections between operator modules, Finsler modules, and Banach bundles in the commutative case. In Section 3 we consider the noncommutative case and obtain a complete description of the structure of an arbitrary Finsler module.

It is a pleasure to thank Charles Akemann for supplying a general C*-algebra fact, Theorem 4. This is a crucial result for our purposes and is also of independent interest.

1. Preliminaries.

In this section we collect some important general facts about Finsler modules. Aside from Akemann's result (Theorem 4) the material is fairly trivial.

Recall that a Banach A-module is an A-module E that is simultaneously a Banach space and which satisfies $\|ax\| \leq \|a\|\|x\|$ for all $a \in A$ and $x \in E$.

Proposition 1. *Every Finsler A-module is a Banach A-module.*

Proof. By definition, $\|\cdot\|_E$ makes E a Banach space. We must therefore show that $\|ax\|_E \leq \|a\|\|x\|_E$ for all $a \in A$ and $x \in E$. This follows from condition (2):

$$\|ax\|_E^2 = \|\rho(ax)^2\| = \|a\rho(x)^2a^*\| \leq \|a\|^2\|x\|_E^2.$$

□

In the next result we observe that if A is commutative, every Finsler A-module has properties which make it look very much like a module with an A-valued norm. Of these, the first has a natural analog in Finsler condition (2), as we observed in the introduction.

The second property, a generalized triangle inequality, is far too strong in the noncommutative case (but see [**2**]), although we see in this proposition that if A is commutative it follows from the seemingly weaker assumption that $\|\cdot\|_E$ satisfies the triangle inequality. The latter is suitable in the noncommutative setting, and is already sufficient for the rather strong structure results to be given in Section 3. On the other hand, in Lemma 12 we give a kind of noncommutative generalization of this part of the proposition.

Proposition 2. *Let A be a C^*-algebra with center $Z(A)$, and let E be a Finsler A-module. Then ρ satisfies*

$$\rho(ax) = |a|\rho(x)$$

for all $a \in Z(A)$ and $x \in E$. If $A = C_0(X)$ is commutative then ρ satisfies

$$\rho(x+y) \leq \rho(x) + \rho(y)$$

for all $x, y \in E$.

Proof. If a belongs to the center of A then so does $|a|$, hence both commute with $\rho(x)^2$ and

$$\rho(ax)^2 = a\rho(x)^2 a^* = |a|^2\rho(x)^2 = |a|\rho(x)^2|a|;$$

taking square roots yields $\rho(ax) = |a|\rho(x)$. To prove the second statement suppose it is not the case and find a point $t \in X$ such that

$$\rho(x+y)(t) > \alpha > \rho(x)(t) + \rho(y)(t).$$

Let U be a compact neighborhood of t such that $\rho(x)(t') + \rho(y)(t') < \alpha$ for all $t' \in U$. Let $f \in C_0(X)$ satisfy $0 \leq f \leq 1$, $f|_{X-U} = 0$, and $f(t) = 1$. Then using the first part of this proposition we have

$$\begin{aligned}\|fx\|_E + \|fy\|_E &= \|\rho(fx)\| + \|\rho(fy)\| = \|f\rho(x)\| + \|f\rho(y)\| \leq \alpha \\ &< f(t)\rho(x+y)(t) = \rho(fx+fy)(t) \\ &\leq \|\rho(fx+fy)\| = \|fx+fy\|_E,\end{aligned}$$

contradicting the triangle inequality in E (Finsler condition (1)). This establishes the result. □

Next we observe that every Hilbert module gives rise to a Finsler module.

Proposition 3. *Let A be a C^*-algebra and E a Hilbert A-module. Then defining $\rho(x) = \langle x, x\rangle^{1/2}$ makes E a Finsler A-module.*

Proof. The fact that $\|\cdot\|_E$ is a complete norm is part of the definition of a Hilbert module. The second Finsler condition holds because

$$\rho(ax)^2 = \langle ax, ax\rangle = a\langle x, x\rangle a^* = a\rho(x)^2 a^*.$$

□

Finally, we come to the one substantive result in this section [**1**]. It is needed to show that ρ is continuous and uniquely determined by the scalar norm $\|\cdot\|_E$ (Corollaries 5 and 6), and also to show that Finsler modules can be factored (Lemma 12).

Theorem 4 (Akemann). *Let b and c be positive elements of a C^*-algebra A. Then*

$$\|b - c\| = \sup\{|\|aba\| - \|aca\|| : a \in A_+,\ \|a\| \leq 1\}.$$

Proof. The inequality $\geq$ is easy since

$$|\|aba\| - \|aca\|| \leq \|aba - aca\| \leq \|a\|\|b - c\|\|a\| \leq \|b - c\|$$

for any $a \in A_+$ with $\|a\| \leq 1$.

Now for the reverse inequality. Without loss of generality assume that $0 \leq b, c \leq 1$. By replacing A with $C^*(b, c)$ we can assume that A is separable. Assume that $\alpha = \|b-c\| > 0$. It now suffices to prove that there is a sequence $\{a_n\}$ in the positive unit ball of A such that

$$\lim_{n\to\infty} |\|a_n b a_n\| - \|a_n c a_n\|| = \alpha.$$

To prove this, observe that, exchanging b and c if necessary, there is a pure state f of A such that $f(b-c) = \alpha$. By Proposition 2.2 of [**3**], there is a sequence $\{a_n\}$ of positive norm 1 elements in A that excises f. This means that for any a in A, $\lim \|a_n a a_n - f(a)a_n^2\| = 0$. Taking $a = b$ and then $a = c$, we get

$$0 = \lim \|a_n b a_n - f(b)a_n^2\| \geq \lim |\|a_n b a_n\| - \|f(b)a_n^2\|| \geq 0$$

and

$$0 = \lim \|a_n c a_n - f(c)a_n^2\| \geq \lim |\|a_n c a_n\| - \|f(c)a_n^2\|| \geq 0.$$

Since $\|f(b)a_n^2\| = f(b)$ and $\|f(c)a_n^2\| = f(c)$, this means that $\lim \|a_n b a_n\| = f(b)$ and $\lim \|a_n c a_n\| = f(c)$. Since $f(b) - f(c) = \alpha$, we get $\lim(\|a_n b a_n\| - \|a_n c a_n\|) = \alpha$, as desired. □

Corollary 5. *Let E be a Finsler module over a C^*-algebra A. Then $\rho : E \to A_+$ is continuous from $\|\cdot\|_E$ to $\|\cdot\|$.*

Proof. Let $x_n, x \in E$ and set $b_n = \rho(x_n)^2$, $b = \rho(x)^2$. Suppose $\|x_n - x\|_E \to 0$ and let $C = \sup\{\|x_n\|_E\}$. Then for any a in the positive unit ball of A we have

$$\begin{aligned} \big|\|ab_n a\| - \|aba\|\big| &= \big|\|\rho(ax_n)^2\| - \|\rho(ax)^2\|\big| \\ &= \big|\|ax_n\|_E^2 - \|ax\|_E^2\big| \\ &= \big(\|ax_n\|_E + \|ax\|_E\big)\big|\|ax_n\|_E - \|ax\|_E\big| \\ &\le 2C\|a(x_n - x)\|_E \\ &\le 2C\|x_n - x\|_E. \end{aligned}$$

Thus Theorem 4 implies

$$\|\rho(x_n)^2 - \rho(x)^2\| = \|b_n - b\| \le 2C\|x_n - x\|_E \to 0,$$

so that we have $\rho(x_n)^2 \to \rho(x)^2$. Finally, by [**22**, Proposition 4.10], taking square roots implies that $\rho(x_n) \to \rho(x)$. Thus, $x_n \to x$ in E implies $\rho(x_n) \to \rho(x)$ in A. □

Corollary 6. *Let A be a C^*-algebra and E an A-module. Suppose $\rho, \rho' : E \to A_+$ are two functions both of which make E a Finsler module and which induce the same norm $\|\cdot\|_E$. Then $\rho = \rho'$.*

Proof. Suppose $\rho(x)^2 \ne \rho'(x)^2$ for some $x \in E$. Then $\|\rho(x)^2 - \rho'(x)^2\| \ne 0$ and so Theorem 4 implies that there exists $a \in A_+$ with $\|a\| \le 1$ such that $\|a\rho(x)^2 a\| \ne \|a\rho'(x)^2 a\|$. Hence

$$\|ax\|_E^2 = \|\rho(ax)^2\| \ne \|\rho'(ax)^2\| = \|ax\|_E^2,$$

a contradiction. □

A similar statement about Hilbert modules, to the effect that the scalar norm determines the A-valued inner product, was given in [**8**]. This also follows from Corollary 6 since the A-valued inner product is uniquely determined via polarization from the Finsler norm that it gives rise to via Proposition 3.

2. Banach bundles.

A need for a noncommutative version of Banach bundles arises in the theory of noncommutative metrics. This happens in the following way. First of all, if X is a Riemannian manifold then the cotangent bundle is a Hilbert bundle, as mentioned in the introduction. For X any metric space there is a corresponding construction [**11**] which involves Banach bundles, and this may be regarded as an integrated version of the cotangent bundle construction [**23**]. This construction of de Leeuw actually "encodes" the metric structure of X in a manner so robust as to suggest that a notion of a noncommutative metric could be based on a noncommutative version of the set-up [**23**]. To describe this noncommutative scheme one needs a noncommutative version of the notion of a Banach bundle.

On the basis of examples it has become clear that in describing noncommutative metrics, Hilbert modules are sufficient for situations in which one has "noncommutative Riemannian structure" ([**19**], [**24**]), but more generally one needs operator modules ([**25**], [**26**]). Thus our first goal here is to show how in the commutative case operator modules correspond to Banach bundles, which suggests that general operator modules may be viewed as noncommutative Banach bundles. Modules associated to Banach bundles have been thoroughly studied and so our results in this section are fairly easy consequences of known facts.

Before proceeding we must introduce a distinction emphasized in [**12**], between **(F) Banach bundles** and **(H) Banach bundles**. These are the bundle notions which respectively correspond to the concepts of continuous fields of Banach spaces [**15**] and uniform fields of Banach spaces [**10**]. In brief, the topology interacts with the norm in such a way that the fiberwise norm of a continuous section of an (F) Banach bundle is continuous, while in an (H) Banach bundle it need only be semicontinuous.

We also need the following definitions. Let $A = C(X)$ be a unital commutative C*-algebra. By an **abelian operator A-module** (see [**13**]) we mean a Banach A-module E for which there exists a commutative C*-algebra B together with an isometric embedding $\pi : E \to B$ and a $*$-isomorphic embedding $\varphi : A \to B$, such that

$$\pi(ax) = \varphi(a)\pi(x)$$

for $a \in A$ and $x \in E$. An **A-convex A-module** [**12**] is a Banach A-module E which satsifies

$$\|fx + gy\| \leq \max(\|x\|, \|y\|)$$

for any $x, y \in E$ and any positive $f, g \in C(X)$ such that $f + g = 1$.

Theorem 7. *Let $A = C(X)$ be a unital commutative C^*-algebra and let E be a Banach A-module. The following are equivalent:*

(a) *E is an abelian operator A-module.*

(b) *E is an A-convex A-module.*

(c) *There is an (H) Banach bundle over X of which E is isomorphic to the module of continuous sections.*

Proof. (a) $\Rightarrow$ (b). Let $B = C_0(Y)$ be a commutative C*-algebra and suppose A and E are embedded in B. Then the A-convex inequality is trivially checked at each $t \in Y$:

$$|f(t)x(t) + g(t)y(t)| \leq (f(t) + g(t))\max(\|x\|, \|y\|) \leq \max(\|x\|, \|y\|),$$

hence $\|fx + gy\| \leq \max(\|x\|, \|y\|)$.

(b) $\Rightarrow$ (c). This is ([**12**], Theorem 2.5).

(c) $\Rightarrow$ (a). Let $\mathcal{B} = \bigcup_{t \in X} B_t$ be an (H) Banach bundle over X and let E be the module of continuous sections of $\mathcal{B}$. Let

$$Y = \{(t, v) : t \in X, v \in B_t^*, \|v\| \leq 1\}$$

where B_t^* is the dual Banach space to B_t. Then $A = C(X)$ embeds in $B = l^\infty(Y)$ by setting $\varphi(f)(t, v) = f(t)$, and E embeds in B by setting $\pi(x)(t, v) = v(x(t))$. The module structure is preserved by these embeddings, for

$$\pi(fx)(t, v) = v(fx(t)) = v(f(t)x(t)) = f(t)v(x(t)) = \varphi(f)\pi(x)(t, v).$$

Thus E is an abelian operator A-module. □

We view Theorem 7 as justifying the idea that general operator modules are "noncommutative Banach bundles." Note that the equivalence of parts (a) and (c) easily extends to the case where $A = C_0(X)$ is nonunital, since any Banach module over A is also a Banach module over the unitization of A. But now part (c) will involve a Banach bundle over the one-point compactification of X.

Next we prove a similar fact about Finsler $C_0(X)$-modules.

Theorem 8. *Let $A = C_0(X)$ be a commutative C^*-algebra and let E be a Banach A-module. The following are equivalent:*

(a) *There exists a map $\rho : E \to A_+$ (necessarily unique) which induces the given norm on E and makes E into a Finsler A-module.*

(b) *There is an* (F) *Banach bundle over* X *of which* E *is isomorphic to the module of continuous sections vanishing at infinity.*

Proof. (a) $\Rightarrow$ (b). For X compact (b) follows from page 48 of [**12**] and Proposition 2. If X is locally compact but not compact, let X^+ be its one-point compactification; then E is also a Finsler module over $C(X^+)$. Hence E is isomorphic to the module of continuous sections of some (F) Banach bundle over X^+. But since $\rho(x) \in C_0(X)$ for all $x \in E$, the fiber over ∞ must be trivial. So E is also isomorphic to the module of continuous sections vanishing at infinity of the restriction of the bundle to X.

(b) $\Rightarrow$ (a). We have already described the construction of ρ in the introduction, namely $\rho(x)(t) = \|x(t)\|_{B_t}$ for any section $x : X \to \mathcal{B}$. This is a continuous function of t precisely by the definition of an (F) Banach bundle, and it satisfies the Finsler conditions because it satisfies them fiberwise. Uniqueness of ρ was Corollary 6. ☐

Finally, we show that if A is a commutative von Neumann algebra then abelian operator modules and Finsler modules coincide. We say that a Banach $L^\infty(X)$-module E has the L^∞ **norm property** if

$$\|x\| = \max(\|px\|, \|(1-p)x\|)$$

for any $x \in E$ and any projection $p \in L^\infty(X)$.

Theorem 9. *Let* $A = L^\infty(X)$ *be a commutative von Neumann algebra and let* E *be a Banach* A*-module. The following are equivalent:*

(a) E *is an abelian operator* A*-module.*

(b) *There exists a map* $\rho : E \to A_+$ (*necessarily unique*) *which induces the given norm on* E *and makes* E *into a Finsler* A*-module.*

(c) E *satisfies the* L^∞ *norm property.*

Proof. (a) $\Rightarrow$ (b). Let A and E be embedded in a commutative C*-algebra B; without loss of generality suppose B has a unit and A is embedded unitally. (Otherwise replace B by $1_A B$. This does not alter the embedding of E in B since $x = 1_A x$ for all $x \in E$.) For each $x \in E$ define

$$\rho(x) = \inf\{f \in A : |x| \leq f\} = \inf\{f \in A : |x| \leq f \text{ and } \|f\| = \|x\|\}.$$

The two infima are equal since everything in the first set dominates something in the second set. Since A is unitally embedded in B, the second set contains $f = \|x\| \cdot 1$, so it is nonempty, bounded, and self-adjoint; therefore its infimum exists. This also shows that $\|\rho(x)\| \leq \|x\|$, and conversely, as

$|x| \leq f$ implies $\|x\| \leq \|f\|$, we see that $\|\rho(x)\| = \|x\|$. So ρ does induce the original norm on E. This automatically implies Finsler condition (1), and condition (2) in the form $\rho(ax) = |a|\rho(x)$ is easy:

$$\rho(ax) = \inf\{f \in A : |ax| \leq f\} = |a| \inf\{f \in A : |x| \leq f\} = |a|\rho(x).$$

Again, uniqueness of ρ follows from Corollary 6.

(b) $\Rightarrow$ (c). Let p be a projection in $L^\infty(X)$, let $q = 1 - p$, and let $x \in E$. Then

$$\rho(x) = \rho((p+q)x) = (p+q)\rho(x) = p\rho(x) + q\rho(x) = \rho(px) + \rho(qx).$$

Since $\rho(px) = p\rho(x)$ and $\rho(qx) = q\rho(x)$ have disjoint support, we get

$$\|x\|_E = \|\rho(x)\| = \max(\|\rho(px)\|, \|\rho(qx)\|) = \max(\|px\|_E, \|qx\|_E)$$

as desired.

(c) $\Rightarrow$ (a). We assume the L^∞ norm property and prove that E is A-convex; this suffices by Theorem 7.

Thus let $x, y \in E$ and let $f, g \in L^\infty(X)$ be positive functions such that $f + g = 1$. Let $\varepsilon > 0$. Partition X into measurable subsets $X_1, \ldots, X_n$ such that f and g each vary by less than $\varepsilon' = \varepsilon/(\|x\| + \|y\|)$ on each X_j. Let p_j be the characteristic function of X_j.

Fix j and let $\alpha, \beta \in \mathbf{R}^+$ satisfy $\alpha + \beta = 1$ and

$$\|p_j(f - \alpha 1)\|, \|p_j(g - \beta 1)\| \leq \varepsilon'.$$

Then

$$\begin{aligned}\|p_j(fx + gy)\| &\leq \|p_j((f - \alpha 1)x + (g - \beta 1)y)\| + \|p_j(\alpha x + \beta y)\| \\ &\leq \varepsilon'\|x\| + \varepsilon'\|y\| + \alpha\|x\| + \beta\|y\| \\ &\leq \varepsilon + \max(\|x\|, \|y\|).\end{aligned}$$

But then the L^∞ norm property implies that

$$\begin{aligned}\|fx + gy\| &= \max(\|p_1(fx + gy)\|, \ldots, \|p_n(fx + gy)\|) \\ &\leq \varepsilon + \max(\|x\|, \|y\|),\end{aligned}$$

which in the limit $\varepsilon \to 0$ establishes that E is A-convex. □

A simple example of an abelian operator module which is not a Finsler module in the C* case is given by $A = C([0,1])$ and $E = L^\infty([0,1])$.

3. The noncommutative case.

Lemma 10. *Let A be a C*-algebra. Then A has a unique maximal commutative ideal I and it may be obtained as the intersection of the kernels of all irreducible representations of A of dimension greater than 1. Moreover, I is contained in the center of A.*

Proof. Let I be the intersection of the kernels of all irreducible representations of A of dimension greater than 1. If J is any ideal of A then any irreducible representation of A either annihilates J or restricts to an irreducible representation of J ([4], Theorem 1.3.4). As no commutative C*-algebra has irreducible representations of dimension greater than 1, it follows that I contains every commutative ideal.

Now let $a \in I$ and $b \in A$. Then $ab - ba$ is annihilated by every homomorphism into $\mathbf{C}$, and it also belongs to I, hence it is annihilated by every irreducible representation of A. Thus $ab - ba = 0$, and we conclude that $I \subset Z(A)$. In particular, I is commutative. □

Recall that if A, B, and D are C*-algebras, and if homomorphisms $\varphi : A \to D$ and $\psi : B \to D$ are given, then the C*-algebra $A \oplus_D B$ is defined as

$$A \oplus_D B = \{(a, b) \in A \oplus B : \varphi(a) = \psi(b)\}.$$

We use the same notation for modules, Banach spaces, etc.

Lemma 11. *Let A and I be as in Lemma 10. Then every multiplicative linear functional on I extends uniquely to a multiplicative linear functional on A. The intersection of the kernels of these functionals is an ideal J of A with the properties that $I \cap J = 0$ and A/J is commutative. We have $A \cong C_0(X) \oplus_{C_0(Y)} B$ where $X = \mathrm{Prim}(A/J)$, $B = A/I$, and $Y = \mathrm{Prim}(A/(I+J))$.*

Proof. Let $\widehat{I}$ denote the spectrum of I. Every $\omega \in \widehat{I}$ extends uniquely to a multiplicative linear functional ε_ω on A by Theorem 1.3.4 of [4]. Let $J = \bigcap_{\omega \in \widehat{I}} \ker(\varepsilon_\omega)$.

Since the range of each ε_ω is $\mathbf{C}$, it follows that each $\ker(\varepsilon_\omega)$ contains the commutator ideal of A. Hence so does J, so that A/J is commutative. For every nonzero element of I there exists an $\omega \in \widehat{I}$ which does not annihilate it, so that $I \cap J = 0$.

Define $\varphi : A \to C_0(X) \oplus_{C_0(Y)} B$ by $\varphi(a) = (a + J, a + I)$. Then φ is clearly a $*$-homomorphism, and it is injective because $I \cap J = 0$. To see that φ is surjective, let $b, c \in A$ satisfy $b + (I + J) = c + (I + J)$. Then $c - b \in I + J$ and so by the natural isomorphism $J = J/(I \cap J) \cong (I+J)/I$ there exists $b' \in J$ such that $b'+I = (c-b)+I$. Then $(b+b')+J = b+J$ and $(b+b')+I = c+I$, that is, $\varphi(b+b') = (b+J, c+I)$. □

Note that J can also be described in the following way. If U is the open subset of $\mathrm{Prim}(A)$ corresponding to I, then J corresponds to the interior of the complement of U.

For subsets B of a C*-algebra A and F of a Banach A-module E, we denote by BF the closed linear span of all products ax with $a \in B$ and $x \in F$.

Lemma 12. *Let E be a Finsler module over a C*-algebra A, let I be an ideal of A, let $B = A/I$, let $\pi : A \to B$ be the quotient map, and let $\rho' = \pi \circ \rho$. Then $IE = \ker(\rho')$, E/IE is a B-module, and ρ' descends to a B-valued Finsler norm on E/IE.*

Proof. It is clear that E/IE is naturally a B-module. If $a \in I$ and $x \in E$ then $\rho(ax)^2 = a\rho(x)^2a^* \in I$, hence $\rho(ax) \in I$, that is, $ax \in \ker(\rho')$. This shows that $IE \subset \ker(\rho')$. Conversely, if $x \in \ker(\rho')$ then $\rho(x) \in I$, and so there exists a sequence $\{e_n\}$ of positive elements of I such that $e_n\rho(x) \to \rho(x)$. We claim that

$$\rho(x - e_nx)^2 = \rho(x)^2 - \rho(x)^2e_n - e_n\rho(x)^2 + e_n\rho(x)^2e_n.$$

To see this, let b be the left side and c the right side. Then for any $a \in A$ we have

$$aca^* = (a - ae_n)\rho(x)^2(a^* - e_na^*) = \rho((a - ae_n)x)^2 = a\rho(x - e_nx)^2a^* = aba^*.$$

Thus, $a(b-c)a^* = 0$ for all $a \in A$, whence $b = c$ as claimed. It now follows that $\rho(x - e_nx)^2 \to 0$. Thus $\|x - e_nx\|_E \to 0$ and so $x \in IE$. We have therefore shown that $IE = \ker(\rho')$.

Next we show that $\|\rho'(\cdot)\|$ satisfies the triangle inequality. For suppose this fails and

$$\|\rho'(x+y)\| > \|\rho'(x)\| + \|\rho'(y)\|$$

for some $x, y \in E$. Then there is a pure state f on B such that $f(\rho'(x+y)^2) = \alpha^2$, $f(\rho'(x)^2) = \beta^2$, and $f(\rho'(y)^2) = \gamma^2$ with $\alpha = \|\rho'(x+y)\|$, $\beta \leq \|\rho'(x)\|$, and $\gamma \leq \|\rho'(y)\|$ (hence $\alpha > \beta + \gamma$). Letting $f' = f \circ \pi$, we get that f' is a pure state on A and $f'(\rho(x+y)^2) = \alpha^2$, $f'(\rho(x)^2) = \beta^2$, $f'(\rho(y)^2) = \gamma^2$.

By Proposition 2.2 of [3], there exists a net $\{a_\lambda\}$ of positive norm one elements of A such that

$$\lim \|a_\lambda b a_\lambda - f'(b)a_\lambda^2\| = 0$$

for any $b \in A$. It follows that $\|a_\lambda b a_\lambda\| \to f'(b)$ for all b. In particular with $b = \rho(x+y)^2$ we have

$$\|\rho(a_\lambda x + a_\lambda y)^2\| = \|a_\lambda \rho(x+y)^2 a_\lambda\| \to \alpha^2,$$

hence

$$\|a_\lambda x + a_\lambda y\|_E = \|\rho(a_\lambda x + a_\lambda y)\| \to \alpha,$$

and similarly $\|a_\lambda x\|_E \to \beta$ and $\|a_\lambda y\|_E \to \gamma$. Since $\alpha > \beta + \gamma$ this contradicts the triangle inequality in E. We conclude that $\|\rho'(\cdot)\|$ satisfies the triangle inequality.

We must now show that ρ' descends to E/IE, that is, we must prove that $\rho'(x) = \rho'(x+y)$ for any $x \in E$ and $y \in IE$. The triangle inequality just established implies that for any $a \in A_+$ we have

$$\begin{aligned}\|\rho'(ax)\| = \|\rho'(ax)\| - \|\rho'(ay)\| &\le \|\rho'(ax+ay)\| \\ &\le \|\rho'(ax)\| + \|\rho'(ay)\| = \|\rho'(ax)\|,\end{aligned}$$

whence $\|\rho'(ax)\| = \|\rho'(ax+ay)\|$. If $b \in B_+$, then there is $a \in A_+$ such that $\pi(a) = b$, so that

$$\|b\rho'(x)^2 b\| = \|\rho'(ax)\|^2 = \|\rho'(ax+ay)\|^2 = \|b\rho'(x+y)^2 b\|.$$

Theorem 4 now implies that $\rho'(x) = \rho'(x+y)$, showing that ρ' does descend to E/IE. □

Lemma 13. *Retain the notation of Lemma 12. Suppose ρ' satisfies the parallelogram law*

$$\rho'(x+y)^2 + \rho'(x-y)^2 = 2\rho'(x)^2 + 2\rho'(y)^2$$

for x, $y \in E$. Then E/IE is a Hilbert B-module for a unique B-valued inner product which gives rise to ρ'.

Proof. We begin by showing that the polarization formula

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(x + i^k y)^2$$

defines a $\mathbf{C}$-sesquilinear map $\langle\cdot,\cdot\rangle : E \times E \to B$ which satisfies $\langle x, y\rangle^* = \langle y, x\rangle$ and $\rho'(x)^2 = \langle x, x\rangle$ for all $x, y \in E$.

First, we have

$$\langle x, x\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'((1+i^k)x)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k (1+i^k)\rho'(x)^2(1+i^{-k}) = \rho'(x)^2.$$

Next, observe that $\rho'(i^k z)^2 = i^k \rho'(z)^2 i^{-k} = \rho'(z)^2$, whence

$$\begin{aligned}\langle y, x\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(y + i^k x)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(i^k(x + i^{-k} y))^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^{-k} \rho'(x + i^k y)^2 = \langle x, y\rangle^*.\end{aligned}$$

This proves one of the claims and also shows that to prove **C**-sesquilinearity we need only check **C**-linearity in the first variable. This is done by exactly the same argument that one uses in the scalar case (e.g. see [6]), but we include this argument for completeness.

For $u, v, z \in E$ the parallelogram law gives

$$\rho'((u + i^k z) + v)^2 + \rho'((u + i^k z) - v)^2 = 2\rho'(u + i^k z)^2 + 2\rho'(v)^2,$$

that is,

$$\rho'((u + v) + i^k z)^2 + \rho'((u - v) + i^k z)^2 = 2\rho'(u + i^k z)^2 + 2\rho'(v)^2.$$

Multiplying by $i^k/4$ and summing over k yields

$$(*) \qquad \langle u + v, z\rangle + \langle u - v, z\rangle = 2\langle u, z\rangle.$$

Substituting $v = u$ then shows that $\langle 2u, z\rangle = 2\langle u, z\rangle$, using the fact that

$$\langle 0, z\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(i^k z)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(z)^2 = 0.$$

So replacing $2\langle u, z\rangle$ with $\langle 2u, z\rangle$ in ($*$) and substituting $u = (x + y)/2$ and $v = (x - y)/2$, we get

$$\langle x, z\rangle + \langle y, z\rangle = \langle x + y, z\rangle.$$

This proves additive linearity.

Now for $\alpha \in \mathbf{R}$ define $f(\alpha) = \langle \alpha x, y\rangle \in A$. The map $\alpha \mapsto \alpha x + i^k y$ is continuous since E is a Banach module, so Corollary 5 implies that the map $\alpha \mapsto \rho'(\alpha x + i^k y)^2$ is continuous. It follows that f is continuous. As we also have $f(\alpha + \beta) = f(\alpha) + f(\beta)$, it follows that $f(\alpha) = \alpha f(1)$ for all $\alpha \in \mathbf{R}$, that is, $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$. Finally, direct calculation shows that

$$\begin{aligned}\langle ix, y\rangle &= \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(ix + i^k y)^2 = \frac{1}{4}\sum_{k=0}^{3} i^k \rho'(x + i^{k-1} y)^2 \\ &= \frac{1}{4}\sum_{k=0}^{3} i^{k+1} \rho'(x + i^k y)^2 = i\langle x, y\rangle.\end{aligned}$$

So additive linearity implies that

$$\langle(\alpha+i\beta)x,y\rangle=\langle\alpha x,y\rangle+\langle i\beta x,y\rangle=\alpha\langle x,y\rangle+i\beta\langle x,y\rangle=(\alpha+i\beta)\langle x,y\rangle.$$

This completes the proof of **C**-linearity in the first variable, hence of **C**-sesquilinearity by the comment made earlier.

Let $y\in IE$, so that $\langle y,y\rangle=\rho'(y)^2=0$. For any positive linear functional f on B, $f(\langle\cdot,\cdot\rangle)$ is a **C**-valued positive semidefinite sesquilinear form, hence it satisfies the Cauchy-Schwartz inequality. In particular

$$|f(\langle x,y\rangle)|^2\leq f(\langle x,x\rangle)f(\langle y,y\rangle)=0.$$

It follows that $\langle x,y\rangle=0$ for all $x\in E$, and from this we conclude that $\langle\cdot,\cdot\rangle$ descends to E/IE.

We now know that $\langle x+IE,y+IE\rangle=\langle x,y\rangle$ defines a B-valued inner product on E/IE that satisfies $\langle x+IE,y+IE\rangle^*=\langle y+IE,x+IE\rangle$ and $\rho'(x)^2=\langle x+IE,x+IE\rangle$ and is **C**-sesquilinear. To complete the proof that E/IE is a Hilbert B-module we must prove B-sesquilinearity. To do this fix $x,y\in E$ and consider the **C**-sesquilinear forms $\{\cdot,\cdot\},\{\cdot,\cdot\}':A\times A\to B$ defined by $\{a,b\}=\pi(a)\langle x,y\rangle\pi(b)^*$ and $\{a,b\}'=\langle ax,by\rangle$. For any $a\in A$ we have

$$\begin{aligned}\{a,a\}&=\frac{1}{4}\sum_{k=0}^{3}i^k\pi(a)\rho'(x+i^ky)^2\pi(a)^*=\frac{1}{4}\sum_{k=0}^{3}i^k\pi(a\rho(x+i^ky)^2a^*)\\&=\frac{1}{4}\sum_{k=0}^{3}i^k\rho'(ax+i^kay)^2=\{a,a\}'.\end{aligned}$$

It follows that $\{\cdot,\cdot\}=\{\cdot,\cdot\}'$ by polarization. Thus

$$\pi(a)\langle x+IE,y+IE\rangle\pi(b)^*=\langle\pi(a)(x+IE),\pi(b)(y+IE)\rangle$$

for all $a,b\in A$ and $x,y\in E$, and this shows that $\langle\cdot,\cdot\rangle$ is B-sesquilinear.

Finally, $\langle\cdot,\cdot\rangle$ is unique by polarization. □

The following lemma is a much simpler relative of Lemma 6.7.1 of [17]. The proof is simple enough that we give it anyway.

Lemma 14. *Let $A\subset B(H)$ be a C*-algebra irreducibly represented on a Hilbert space H of dimension greater than 1. Let $\xi\in H$. Then there exist $a,b\in A$ such that $a=a^*$, $a\xi=\xi$, $ba=0$, and $bb^*=a^2$.*

Proof. Let $\zeta\in H$ be orthogonal to ξ and satisfy $\|\zeta\|=\|\xi\|$. By the Kadison Transitivity Theorem ([17], Theorem 2.7.5) we can find self-adjoint $c,s\in A$

such that $c\xi = \xi$, $c\zeta = 0$, $s\xi = \zeta$, and $s\zeta = \xi$. Choose continuous functions $f, g : \mathbf{R} \to [0,1]$ with $f(1) = 1$, $g(0) = 1$, and $fg = 0$. Set $b = f(c)sg(c)$. Note that $b\zeta = \xi$ and $b^*\xi = \zeta$, so $bb^*\xi = \xi$. Set $a = (bb^*)^{1/2}$. Then $fg = 0$ implies $b^2 = 0$ implies $ba(ba)^* = b^2(b^*)^2 = 0$, so $ba = 0$. □

Lemma 15. *Let E be a Finsler A-module, let I be the maximal commutative ideal of A, and let $B = A/I$. Then E/IE is a Hilbert B-module.*

Proof. Let $\pi : A \to B$ be the natural projection and let $\rho' = \pi \circ \rho$. By Lemma 13 it will suffice to show that ρ' satisfies the parallelogram law. Let $x, y \in E$, let $\varphi : A \to B(H)$ be an irreducible representation on a Hilbert space of dimension greater than 1, and let $\xi \in H$.

By Lemma 14 there exist $a, b \in A$ such that the elements $\tilde{a} = \varphi(a)$ and $\tilde{b} = \varphi(b)$ satisfy $\tilde{a} = \tilde{a}^*$, $\tilde{a}\xi = \xi$, $\tilde{b}\tilde{b}^* = \tilde{a}^2$, and $\tilde{b}\tilde{a} = 0$. Letting $\tilde{\rho} = \varphi \circ \rho$ and using the fact proved in Lemma 12 that $\tilde{\rho}$ descends to $E/\ker(\tilde{\rho}) = E/\ker(\varphi)E$ (and $ba, bb^* - a^2 \in \ker(\varphi)$) we have

$$\begin{aligned}\langle \tilde{\rho}(x \pm y)^2\xi, \xi\rangle &= \langle \tilde{a}^2\tilde{\rho}(x \pm y)^2\tilde{a}^2\xi, \xi\rangle = \langle \tilde{\rho}(a^2(x \pm y))^2\xi, \xi\rangle \\ &= \langle \tilde{\rho}((a \pm b)(ax + b^*y))^2\xi, \xi\rangle \\ &= \langle (\tilde{a} \pm \tilde{b})\tilde{\rho}(ax + b^*y)^2(\tilde{a} \pm \tilde{b}^*)\xi, \xi\rangle.\end{aligned}$$

Adding yields

$$\begin{aligned}\big\langle \big[\tilde{\rho}(x + y)^2 + \tilde{\rho}(x - y)^2\big]\xi, \xi\big\rangle &= \big\langle \big[2\tilde{a}\tilde{\rho}(ax + b^*y)^2\tilde{a} + 2\tilde{b}\tilde{\rho}(ax + b^*y)^2\tilde{b}^*\big]\xi, \xi\big\rangle \\ &= \big\langle \big[2\tilde{\rho}(a(ax + b^*y))^2 + 2\tilde{\rho}(b(ax + b^*y))^2\big]\xi, \xi\big\rangle \\ &= \big\langle \big[2\tilde{\rho}(a^2x)^2 + 2\tilde{\rho}(a^2y)^2\big]\xi, \xi\big\rangle \\ &= \big\langle \tilde{a}^2\big[2\tilde{\rho}(x)^2 + 2\tilde{\rho}(y)^2\big]\tilde{a}^2\xi, \xi\big\rangle \\ &= \big\langle \big[2\tilde{\rho}(x)^2 + 2\tilde{\rho}(y)^2\big]\xi, \xi\big\rangle.\end{aligned}$$

Let $c = \rho(x+y)^2 + \rho(x-y)^2 - 2\rho(x)^2 - 2\rho(y)^2$, so we have $\langle c\xi, \xi\rangle = 0$. As ξ was arbitrary we get $\varphi(c) = 0$, and since this is true for all irreducible representations φ of dimension greater than 1 we conclude that $c \in I$. This implies that $\pi(c) = 0$ and so ρ' satisfies the parallelogram law, as desired. □

Lemma 16. *Let $A = B_1 \oplus_D B_2$, with $\varphi_i : B_i \to D$ surjective.*

(1) *Let F_1 and F_2 be Finsler modules over B_1 and B_2, let H be a Finsler module over D, and let $\psi_i : F_i \to H$ be continuous linear maps inducing Finsler module isomorphisms $\overline{\psi}_i : F_i/\ker(\varphi_i)F_i \to H$. Then $E = F_1 \oplus_H F_2$ is a Finsler module over A, with the module structure $(b_1, b_2)(x_1, x_2) = (b_1x_1, b_2x_2)$ and $\rho_E(x_1, x_2) = (\rho_{F_1}(x_1), \rho_{F_2}(x_2))$.*

(2) *Let E be a Finsler module over A. Let $\pi_i : A \to B_i$ be the projection maps, and set $F_i = E/\ker(\pi_i)E$ and $H = E/\ker(\varphi_1 \circ \pi_1)E = E/\ker(\varphi_2 \circ \pi_2)E$. Then there is a canonical isomorphism $E \cong F_1 \oplus_H F_2$.*

Proof. (1) This is a straightforward calculation, and is omitted.

(2) We have $\varphi_1 \circ \pi_1 = \varphi_2 \circ \pi_2$ by the definition of $B_1 \oplus_D B_2$. Since φ_1 and φ_2 are surjective, so are π_1 and π_2. Therefore Lemma 12 shows that F_i is a Finsler module over B_i and H is a Finsler module over D. It is immediate that the map $x \mapsto (x + \ker(\pi_1)E, x + \ker(\pi_2)E)$ is a homomorphism of A-modules, and easy to check that it intertwines the Finsler norms. Since it intertwines the Finsler norms, it must be injective, and it is surjective by the argument used at the end of the proof of Lemma 11. □

Theorem 17. *Let A be a C^*-algebra and as in Lemma 11 write $A \cong C_0(X) \oplus_{C_0(Y)} B$, where $C_0(X) = A/J$, $B = A/I$, and $C_0(Y) = A/(I + J)$. If E_0 is a Hilbert module over $C_0(Y)$, E_1 is a Finsler module over $C_0(X)$, E_2 is a Hilbert module over B, and $\psi_i : E_i \to E_0$ are continuous linear maps inducing Finsler module isomorphisms*

$$\overline{\psi}_1 : E_1/[(I + J)/J]E_1 \to E_0 \quad \textit{and} \quad \overline{\psi}_2 : E_2/[(I + J)/I]E_2 \to E_0,$$

then $E_1 \oplus_{E_0} E_2$ is a Finsler module over A. Conversely, every Finsler module over A arises in this way.

Proof. If in the statement we merely require E_0 and E_2 to be Finsler modules, then this is just the previous lemma. However, Lemma 15 implies that E_2 is necessarily a Hilbert module, and it follows that $E_0 \cong E_2/[(I + J)/I]E_2$ is also automatically a Hilbert module. □

Corollary 18. *Let A be a C^*-algebra. Then the class of Finsler A-modules equals the class of Hilbert A-modules if and only if A has no nonzero commutative ideals. In particular this holds if A is simple with $\dim(A) > 1$, approximately divisible [**7**], or a von Neumann algebra with no abelian summand.*

Proof. We use the notation of Theorem 17. If A has no commutative ideals then $I = 0$, $J = A$, $C_0(X) = C_0(Y) = 0$, and $B = A$. Thus Theorem 17 identifies the Finsler modules over A with the Hilbert modules over $B = A$. Conversely, if I is a nontrivial commutative ideal of A then $C_0(X) \neq 0$. Choose a non-Hilbert Banach space V, and take E_1 to be the module of continuous maps $\widehat{I} \to V$ which vanish at infinity. Letting $E_2 = E_0 = 0$, Theorem 17 produces a Finsler A-module; but it is not a Hilbert A-module because the norm ρ does not satisfy the parallelogram law. □

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