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COMPLETE RICCI FLAT KÄHLER METRIC ON $\mathbf{M}_I^n, \mathbf{M}_{II}^{2n}, \mathbf{M}_{III}^{4n}$

Tai-Chung Lee

We give the explicit formula of a complete Ricci flat Kähler metric on the complexification of S^n, CP^n and HP^n.

Introduction.

Stenzel [**S**] using the Lie group action and existence of special plurisubharmonic exhaustion function ρ proved that on the complexification of compact symmetric spaces of rank one, there always exist a complete Ricci flat Kähler metric (CRFK-metric) and the metric can be obtained by solving an O.D.E in terms of $u = \sqrt{\rho}$. In this note we will give the explicit formula for the CRFK-metric on the following Stein manifolds (i) $\mathbf{M}_I^n$ =hyperquadric in C^{n+1}, which is the complexification of S^n ,(ii) $\mathbf{M}_{II}^{2n} = CP^n \times CP^n - Q_\infty$ which is the complexification of CP^n and (iii) $\mathbf{M}_{III}^{4n} = Gr(2, 2n+2, C) - H_\infty$ which is the complexification of HP^n. Due to the complication of notation we will not work on the complexification of CaP^2 here (this is the only case left). Let us sketch what we will do in this note. To start with we will use the strictly plurisubharmonic function $\mathcal{N}$ from [**PW**] , since $\mathcal{N} = g(\rho)$ for some differentiable strictly increasing function g, we are dealing with the same equation as Stenzel [**S**] up to a change of variable. Assuming $f(\mathcal{N})$ is the potential function, all we have to do is to solve the Monge-Ampere equation $(\partial\bar{\partial} f(\mathcal{N}))^n = F\bar{F}$ for some holomorphic function F. Since the result is invariant under holomorphic change of coordinates, we will work in special local coordinates for our convenience. The equation turns out to be an O.D.E. in terms of $\mathcal{N}$. Once we get the formula for f' and f'' , we are done. We will identify the Kähler metric with its Kähler form in this note. Azad and Kobayashi prove the existence of CRFK-metric on the complexification of compact symmetric spaces of rank higher than 1 and for the general construction of CRFK-metric see Tian and Yau [**TY1**, **TY2**]. Before we go any further let us fix some convention and collect some formulas from linear algebra which will be used later on and we will sketch the proof of those lemmas for the reader's convenience . For any function $U(Z)$ of m complex variables $(Z_1, \cdots, Z_m)$ let ∂U be either the $(1,0)$ form or the corresponding row vector $(U_{Z_1}, \cdots, U_{Z_m})$ and let $\partial\bar{\partial} U$ be either the $(1,1)$ form or the complex Hessian matrix of U. If $x = (x_1, \dots, x_m), y = (y_1, \dots, y_m)$ then $x \cdot y = \sum_{j=1}^{m} x_j y_j$. $\bar{x} = (\bar{x}_1, \dots, \bar{x}_m)$ and x^t is the transpose of x. If $a = (a_1, \dots, a_m)$ and $b = (b_1, \dots, b_n)$, then $a^t \times b$ will be the $m \times n$ matrix $(a_i b_j)$.

Lemma 1. $\det(I_n + a^t \times b) = 1 + a \cdot b$ *and if* $1 + a \cdot b \neq 0$ *then*

$$(I_n + a^t \times b)^{-1} = \left(I_n - \frac{a^t \times b}{1 + a \cdot b}\right).$$

Proof. $\det(I_n + a^t \times b) = 1 + a \cdot b$ can be proved by induction on n. The inverse is trivial. □

Lemma 2. $\det(I_n + a^t \times b + c^t \times d) = (1 + a \cdot b)(1 + c \cdot d) - (a \cdot d)(c \cdot b)$.

Proof. Formally

$$(I_n + a^t \times b + c^t \times d) = (I_n + a^t \times b)(I_n + (I_n + a^t \times b)^{-1} c^t \times d)$$

then use Lemma 1. □

Lemma 3. *Let R be a non-singular $p \times p$ matrix and let S be a non-singular $q \times q$ matrix. If a, d are p-vectors and c, b are q-vectors. Let* $D = \begin{pmatrix} R & a^t \times b \\ c^t \times d & S \end{pmatrix}$, *then*

$$\det D = (\det R)(\det S)(1 - (dR^{-1}a^t)(bS^{-1}c^t)).$$

In case

$$1 - (dR^{-1}a^t)(bS^{-1}c^t) \neq 0$$

then

$$D^{-1} = \begin{pmatrix} R^{-1} + \frac{(bS^{-1}c^t)(R^{-1}a^t \times dR^{-1})}{1-(dR^{-1}a^t)(bS^{-1}c^t)} & -\frac{R^{-1}a^t \times bS^{-1}}{1-(dR^{-1}a^t)(bS^{-1}c^t)} \\ -\frac{S^{-1}c^t \times dR^{-1}}{1-(dR^{-1}a^t)(bS^{-1}c^t)} & S^{-1} + \frac{(dR^{-1}a^t)(S^{-1}c^t \times bS^{-1})}{1-(dR^{-1}a^t)(bS^{-1}c^t)} \end{pmatrix}.$$

Proof. Since

$$D = \begin{pmatrix} R & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} I_p & R^{-1}a^t \times b \\ S^{-1}c^t \times d & I_q \end{pmatrix}$$

the general case can be reduced to the special case $\begin{pmatrix} I_p & a^t \times b \\ c^t \times d & I_q \end{pmatrix}$.

$$\det \begin{pmatrix} I_p & a^t \times b \\ c^t \times d & I_q \end{pmatrix} = \det \begin{pmatrix} I_p & 0 \\ c^t \times d & I_q - (a \cdot d)c^t \times b \end{pmatrix} = 1 - (a \cdot d)(c \cdot b)$$

the last equality follows from Lemma 1. Since

$$\begin{pmatrix} I_p & a^t \times b \\ c^t \times d & I_q \end{pmatrix} \begin{pmatrix} I_p & -a^t \times b \\ -c^t \times d & I_q \end{pmatrix} = \begin{pmatrix} I_p - (c \cdot b)a^t \times d & 0 \\ 0 & I_q - (a \cdot d)c^t \times b \end{pmatrix}$$

we have

$$\begin{aligned} &\begin{pmatrix} I_p & a^t \times b \\ c^t \times d & I_q \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I_p & -a^t \times b \\ -c^t \times d & I_q \end{pmatrix} \begin{pmatrix} I_p - (c \cdot b)a^t \times d & 0 \\ 0 & I_q - (a \cdot d)c^t \times b \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I_p + \frac{(c\cdot b)a^t \times d}{1-(c\cdot b)(a\cdot d)} & \frac{-a^t \times b}{1-(c\cdot b)(a\cdot d)} \\ \frac{-c^t \times d}{1-(c\cdot b)(a\cdot d)} & I_q + \frac{(a\cdot d)c^t \times b}{1-(c\cdot b)(a\cdot d)} \end{pmatrix}. \end{aligned}$$

□

Lemma 4. $\det \begin{pmatrix} pI_n & qI_n \\ rI_n & sI_n \end{pmatrix} = (ps - qr)^n$ *and if* $ps - qr \neq 0$ *then*

$$\begin{pmatrix} pI_n & qI_n \\ rI_n & sI_n \end{pmatrix}^{-1} = \frac{1}{ps - qr} \begin{pmatrix} sI_n & -qI_n \\ -rI_n & pI_n \end{pmatrix}.$$

Proof. Similar to 2×2 case. □

Lemma 5. *Let A be a $n \times n$ hermitian matrix . If* $\det A > 0$ *and A is positive definite on a $(n-1)$-dimensional subspace, then A is positive definite.*

Proof. Since A is positive definite on a subspace of dimension $(n-1)$, A has at least $n-1$ positive eigenvalues. Also because $\det A > 0$, all eigenvalues of A are positive. □

$\mathbf{M}_I^n$.

Let $\mathbf{M}_I^n = \{(z_0, z_1, \ldots, z_n) \in C^{n+1} : \sum_0^n z_i^2 = 1\}$ and let $\mathcal{N}$ be the restriction of $\sum_0^n |z_j|^2$ to $\mathbf{M}_I^n$ in this section. Assume $f \circ \mathcal{N}$ is the potential function we are looking for. Let's compute $\det \partial\overline{\partial} f(\mathcal{N})$ in local coordinates $Z = (z_1, \ldots, z_n)$ and $z_0 = \sqrt{1 - \sum_1^n z_j^2}$.

$$\partial f(\mathcal{N}) = f' \left(\bar{Z} - \frac{\bar{z}_0}{z_0} Z \right),$$

$$\partial\bar{\partial}f(\mathcal{N}) = f'\left(I_n + \frac{Z^t \times \bar{Z}}{|z_0|^2}\right) + f''\left(\bar{Z} - \frac{\bar{z}_0}{z_0}Z\right)^t \times \overline{\left(\bar{Z} - \frac{\bar{z}_0}{z_0}Z\right)},$$
$$\det\partial\bar{\partial}f(\mathcal{N}) = \frac{1}{|z_0|^2}\left[\mathcal{N}(f')^n + (\mathcal{N}^2-1)(f')^{n-1}f''\right].$$

Let f be a solution of

$$\mathcal{N}(f')^n + (\mathcal{N}^2-1)(f')^{n-1}f'' = 1 \tag{1}$$

then

$$((\mathcal{N}^2-1)^{\frac{n}{2}}(f')^n)' = n(\mathcal{N}^2-1)^{\frac{n}{2}-1},$$
$$f' = \frac{\left(n\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)^{\frac{1}{n}}}{(\mathcal{N}^2-1)^{\frac{1}{2}}},$$
$$f'' = \frac{\left(n\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)^{\frac{1}{n}-1}\left((\mathcal{N}^2-1)^{\frac{n}{2}} - n\mathcal{N}\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)}{(\mathcal{N}^2-1)^{\frac{3}{2}}}.$$

Proposition 1. *Let* $\mathbf{M}_I^n = \{(z_0, z_1, \ldots, z_n) \in C^{n+1} : \sum_0^n z_i^2 = 1\}$ *and let* $\mathcal{N} = \sum_0^n |z_j|^2$. *Then*

$$\frac{\left(n\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)^{\frac{1}{n}-1}\left((\mathcal{N}^2-1)^{\frac{n}{2}} - n\mathcal{N}\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)}{(\mathcal{N}^2-1)^{\frac{3}{2}}}\partial\mathcal{N}\wedge\bar{\partial}\mathcal{N}$$
$$+\frac{\left(n\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)^{\frac{1}{n}}}{(\mathcal{N}^2-1)^{\frac{1}{2}}}\partial\bar{\partial}\mathcal{N}$$

defines a CRFK-metric on $\mathbf{M}_I^n$.

Proof. Since $f' = \frac{\left(n\int_1^{\mathcal{N}}(s^2-1)^{\frac{n}{2}-1}ds\right)^{\frac{1}{n}}}{(\mathcal{N}^2-1)^{\frac{1}{2}}} > 0$, the Kähler form is positive definite on $\langle\partial\mathcal{N}\rangle^\perp$ and $\det\partial\bar{\partial}f(\mathcal{N}) = \frac{1}{|z_0|^2} > 0$. By Lemma 5 the Kähler form is positive. The completeness of this metric can be proved by calculation using formula (3.24) of [**PW**]. □

Note.

(1) Equation (1) also appear in [**S**];

(2) For $n = 2$ this metric is called the Eguchi-Hanson metric.

$\mathbf{M}_{II}^{2n}$.

In this section let $(Z,W) = (z_0,\dots,z_n,w_0,\dots,w_n)$ be the homogeneous coordinate on $CP^n \times CP^n$,$A(Z,W) = \sum_{0\le j,k\le n} |z_j w_k|^2$, $B(Z,W) = \sum_0^n z_i w_i$, $Q_\infty = \{(Z,W) : B(Z,W) = 0\}$ and $\mathcal{N}(Z,W) = \frac{A(Z,W)}{B(Z,W)\overline{B(Z,W)}}$. Let $\mathbf{M}_{II}^{2n} = CP^n \times CP^n - Q_\infty$. We are going to work in inhomogeneous coordinates $(1,z_1,\dots,z_n,1,w_1,\dots,w_n)$ so $A = (1+|z|^2)(1+|w|^2)$ and $B = 1+z\cdot w$ where $z = (z_1,\dots,z_n)$ and $w = (w_1,\dots,w_n)$.

Since

$$\partial f(\mathcal{N}) = f'\left(\frac{\partial A}{B\bar{B}} - \frac{A}{B\bar{B}}\frac{\partial B}{B}\right)$$

then

$$\begin{aligned}
\partial\overline{\partial} f(\mathcal{N}) &= f''\left(\frac{\partial A}{B\bar{B}} - \frac{A}{B\bar{B}}\frac{\partial B}{B}\right)^t \times \overline{\left(\frac{\partial A}{B\bar{B}} - \frac{A}{B\bar{B}}\frac{\partial B}{B}\right)} \\
&\quad + f'\Bigg(\frac{\partial\overline{\partial} A}{B\bar{B}} - \left(\frac{\partial A}{B\bar{B}}\right)^t \times \overline{\left(\frac{\partial B}{B}\right)} - \left(\frac{\partial B}{B}\right)^t \times \overline{\left(\frac{\partial A}{B\bar{B}}\right)} \\
&\quad + \frac{A}{B\bar{B}}\left(\frac{\partial B}{B}\right)^t \times \overline{\left(\frac{\partial B}{B}\right)}\Bigg) \\
&= \frac{f'}{B\bar{B}}\Bigg[\partial\overline{\partial} A - \frac{1}{A}(\partial A)^t \times \bar{\partial} A \\
&\quad + \left(\frac{f''}{f'B\bar{B}} + \frac{1}{A}\right)\left(\partial A - A\frac{\partial B}{B}\right)^t \times \overline{\left(\partial A - A\frac{\partial B}{B}\right)}\Bigg].
\end{aligned}$$

Hence

(2)

$$\begin{aligned}
&\det \partial\overline{\partial} f(\mathcal{N}) \\
&= \left(\frac{f'}{B\bar{B}}\right)^{2n} \det \partial\overline{\partial} A \det\Bigg[I_{2n} - \frac{1}{A}(\partial\overline{\partial} A)^{-1}\partial A^t \times \bar{\partial} A \\
&\quad + \left(\frac{f''}{f'B\bar{B}} + \frac{1}{A}\right)(\partial\overline{\partial} A)^{-1}\left(\partial A - A\frac{\partial B}{B}\right)^t \times \overline{\left(\partial A - A\frac{\partial B}{B}\right)}\Bigg] \\
&= \left(\frac{f'}{B\bar{B}}\right)^{2n} \det \partial\overline{\partial} A\Bigg[\left(1 - \frac{1}{A}\bar{\partial} A(\partial\overline{\partial} A)^{-1}\partial A^t\right) \\
&\quad \cdot \left(1 + \left(\frac{f''}{f'B\bar{B}} + \frac{1}{A}\right)\overline{\left(\partial A - A\frac{\partial B}{B}\right)}(\partial\overline{\partial} A)^{-1}\left(\partial A - A\frac{\partial B}{B}\right)^t\right)
\end{aligned}$$

$$+\left(\frac{1}{A}\overline{\left(\partial A - A\frac{\partial B}{B}\right)}(\partial\overline{\partial}A)^{-1}\partial A^t\left(\frac{f''}{f'B\bar{B}}+\frac{1}{A}\right)\right.$$
$$\left.\left.\cdot\,\bar{\partial}A(\partial\overline{\partial}A)^{-1}\left(\partial A - A\frac{\partial B}{B}\right)^t\right)\right].$$

The last equality comes from Lemma 2. Now we are going to compute $\det\partial\overline{\partial}A$ and $(\partial\overline{\partial}A)^{-1}$.

$$\partial\overline{\partial}A = \begin{pmatrix}(1+|w|^2)I_n & \bar{z}^t\times w\\ \bar{w}^t\times z & (1+|z|^2)I_n\end{pmatrix}.$$

To simplify notation let $X=\frac{1}{1+|z|^2+|w|^2}$, then by Lemma 3

$$\det\partial\overline{\partial}A = (1+|z|^2)^{n-1}(1+|w|^2)^{n-1}\frac{1}{X}$$

and

$$(\partial\overline{\partial}A)^{-1} = \begin{pmatrix}\frac{I_n+X|w|^2\bar{z}^t\times z}{1+|w|^2} & -X\bar{z}^t\times w\\ -X\bar{w}^t\times z & \frac{I_n+X|z|^2\bar{w}^t\times w}{1+|z|^2}\end{pmatrix}.$$

Hence

$$\begin{aligned}
(\partial\overline{\partial}A)^{-1}(\partial A)^t &= X\begin{pmatrix}(1+|z|^2)\bar{z}^t\\(1+|w|^2)\bar{w}^t\end{pmatrix},\\
(\partial\overline{\partial}A)^{-1}(\partial B)^t &= \begin{pmatrix}\frac{w^t-X(B-1)\bar{z}^t}{1+|w|^2}\\ \frac{z^t-X(B-1)\bar{w}^t}{1+|z|^2}\end{pmatrix},\\
(\bar{\partial}A)(\partial\overline{\partial}A)^{-1}(\partial A)^t &= AX(|z|^2+|w|^2) = A(1-X),\\
(\bar{\partial}A)(\partial\overline{\partial}A)^{-1}(\partial B)^t &= z\cdot w(1+X) = (B-1)(1+X),\\
(\bar{\partial}B)(\partial\overline{\partial}A)^{-1}(\partial A)^t &= \overline{z\cdot w}(1+X) = (\bar{B}-1)(1+X),\\
(\bar{\partial}B)(\partial\overline{\partial}A)^{-1}(\partial B)^t &= \frac{|z|^2}{1+|z|^2}+\frac{|w|^2}{1+|w|^2}\\
&\quad - X|B-1|^2\left(\frac{1}{1+|z|^2}+\frac{1}{1+|w|^2}\right).
\end{aligned}$$

Plugging each piece into (2) gives, after a long calculation,

$$\det\partial\overline{\partial}f(\mathcal{N}) = \left(\frac{1}{B\bar{B}}\right)^{n+1}[(2\mathcal{N}-1)\mathcal{N}^{n-1}(f')^{2n}+2(\mathcal{N}-1)\mathcal{N}^n(f')^{2n-1}f''].$$

Let f be the solution of

$$(2\mathcal{N}-1)\mathcal{N}^{n-1}(f')^{2n} + 2(\mathcal{N}-1)\mathcal{N}^{n}(f')^{2n-1}f'' = 1$$

then

$$\begin{aligned} n\mathcal{N}^{n-1}(\mathcal{N}-1)^{n-1}(2\mathcal{N}-1)(f')^{2n} \\ +2n\mathcal{N}^{n}(\mathcal{N}-1)^{n}(f')^{2n-1}f'' &= n(\mathcal{N}-1)^{n-1}, \\ [\mathcal{N}^{n}(\mathcal{N}-1)^{n}(f')^{2n}]' &= [(\mathcal{N}-1)^{n}]', \\ \mathcal{N}^{n}(f')^{2n} &= 1, \\ f' &= \frac{1}{\mathcal{N}^{\frac{1}{2}}}, \\ f'' &= -\frac{1}{2}\frac{1}{\mathcal{N}^{\frac{3}{2}}}. \end{aligned}$$

Proposition 2. *Let* $\mathbf{M}_{II}^{2n} = CP^{n} \times CP^{n} - CP_{\infty}^{2n}$ *and let*

$$\mathcal{N}(Z,W) = \frac{\sum_{0\le j,k\le n} |z_j w_k|^2}{\left|\sum_0^n z_i w_i\right|^2}.$$

Then

$$-\frac{1}{2}\frac{1}{\mathcal{N}^{\frac{3}{2}}}\partial\mathcal{N}\wedge\bar{\partial}\mathcal{N} + \frac{1}{\mathcal{N}^{\frac{1}{2}}}\partial\bar{\partial}\mathcal{N}$$

defines a CRFK-metric on $\mathbf{M}_{II}^{2n}$*.*

Proof. Similar to the proof of Proposition 1. □

$\mathbf{M}_{III}^{4n}$.

Let $Gr(2, 2n+2, C)$ be the complex Grassmann manifold of 2-planes in C^{2n+2} through the origin and let $(Z,W) = (z_1, \dots, z_{2n+2}, w_1, \dots, w_{2n+2})$ be the homogeneous coordinates on it. In this section let

$$A(Z,W) = \sum_{1\le j<k\le 2n+2} |z_j w_k - z_k w_j|^2,$$

$$B(Z,W) = \sum_{j=1}^{n+1} z_{2j}w_{2j-1} - z_{2j-1}w_{2j}$$

and let

$$H_\infty = \{(Z,W) : B(Z,W) = 0\}.$$

Let $\mathbf{M}_{III}^{4n} = Gr(2, 2n+2, C) - H_\infty$. We will work in inhomogeneous coordinates $(z_1, \ldots, z_{2n}, 1, 0, w_1, \ldots, w_{2n}, 0, 1)$ and we will use $z = (z_1, \ldots, z_{2n})$, $w = (w_1, \ldots, w_{2n})$. Then $A = (1+|z|^2)(1+|w|^2) - (z \cdot \bar{w})(\bar{z} \cdot w)$, $B = \sum_{j=1}^{n} z_{2j} w_{2j-1} - z_{2j-1} w_{2j} - 1$ and let $\mathcal{N} = \frac{A}{B\bar{B}}$ be our exhaustion function. The Monge-Ampere operator will be

(3)

$$\begin{aligned}
&\det \partial\overline{\partial} f(\mathcal{N}) \\
&= \left(\frac{f'}{B\bar{B}}\right)^{4n} \det \partial\overline{\partial} A \Bigg[\left(1 - \frac{1}{A} \bar{\partial} A (\partial\overline{\partial} A)^{-1} \partial A^t\right) \\
&\quad \cdot \left(1 + \left(\frac{f''}{f' B\bar{B}} + \frac{1}{A}\right) \overline{\left(\partial A - A\frac{\partial B}{B}\right)} (\partial\overline{\partial} A)^{-1} \left(\partial A - A\frac{\partial B}{B}\right)^t\right) \\
&\quad + \left(\frac{1}{A} \overline{\left(\partial A - A\frac{\partial B}{B}\right)} (\partial\overline{\partial} A)^{-1} \partial A^t \left(\frac{f''}{f' B\bar{B}} + \frac{1}{A}\right)\right. \\
&\quad \left. \cdot \bar{\partial} A (\partial\overline{\partial} A)^{-1} \left(\partial A - A\frac{\partial B}{B}\right)^t\right) \Bigg].
\end{aligned}$$

To compute $\det \partial\overline{\partial} A$ and $(\partial\overline{\partial} A)^{-1}$ we have

$$\partial\overline{\partial} A = \begin{pmatrix} (1+|w|^2) I_{2n} - \bar{w}^t \times w & \bar{z}^t \times w - (\bar{z} \cdot w) I_{2n} \\ \bar{w}^t \times z - (z \cdot \bar{w}) I_{2n} & (1+|z|^2) I_{2n} - \bar{z}^t \times z \end{pmatrix} = M_1 M_2 M_3,$$

where

$$\begin{aligned}
M_1 &= \begin{pmatrix} (1+|w|^2) I_{2n} & -(\bar{z} \cdot w) I_{2n} \\ -(z \cdot \bar{w}) I_{2n} & (1+|z|^2) I_{2n} \end{pmatrix}, \\
M_2 &= \begin{pmatrix} I_{2n} - \bar{w}^t \times \frac{(1+|z|^2) w - (\bar{z} \cdot w) z}{A} & 0 \\ 0 & I_{2n} - \bar{z}^t \times \frac{(1+|w|^2) z - (z \cdot \bar{w}) w}{A} \end{pmatrix}, \\
M_3 &= \begin{pmatrix} I_{2n} & (\bar{z}^t + \frac{\bar{z} \cdot w}{1+|z|^2} \bar{w}^t) \times \frac{(1+|z|^2) w - (\bar{z} \cdot w) z}{A} \\ \left(\bar{w}^t + \frac{z \cdot \bar{w}}{1+|w|^2} \bar{z}^t\right) \times \frac{(1+|w|^2) z - (z \cdot \bar{w}) w}{A} & I_{2n} \end{pmatrix}.
\end{aligned}$$

By Lemmas 1, 2, 3 and 4 we have

$$\begin{aligned}
\det M_1 &= A^{2n}, \\
\det M_2 &= \frac{(1+|z|^2)(1+|w|^2)}{A^2}, \\
\det M_3 &= \frac{1+|z|^2+|w|^2}{(1+|z|^2)(1+|w|^2)},
\end{aligned}$$

and

$$M_1^{-1} = \frac{1}{A}\begin{pmatrix}(1+|z|^2)I_{2n} & (\bar{z}\cdot w)I_{2n} \\ (z\cdot\bar{w})I_{2n} & (1+|w|^2)I_{2n}\end{pmatrix},$$

$$M_2^{-1} = \begin{pmatrix} I_{2n} + \bar{w}^t \times \frac{(1+|z|^2)w-(\bar{z}\cdot w)z}{1+|z|^2} & 0 \\ 0 & I_{2n} + \bar{z}^t \times \frac{(1+|w|^2)z-(z\cdot\bar{w})w}{1+|w|^2}\end{pmatrix},$$

$$M_3^{-1} = \begin{pmatrix}\Gamma_{z\bar{z}} & \Gamma_{z\bar{w}} \\ \Gamma_{w\bar{z}} & \Gamma_{w\bar{w}}\end{pmatrix},$$

where

$$\Gamma_{z\bar{z}} = I_{2n} + \frac{|w|^2((1+|z|^2)\bar{z}^t + (\bar{z}\cdot w)\bar{w}^t)}{1+|z|^2+|w|^2} \times \frac{(1+|w|^2)z-(z\cdot\bar{w})w}{A},$$

$$\Gamma_{w\bar{w}} = I_{2n} + \frac{|z|^2((1+|w|^2)\bar{w}^t + (z\cdot\bar{w})\bar{z}^t)}{1+|z|^2+|w|^2} \times \frac{(1+|z|^2)w-(\bar{z}\cdot w)z}{A},$$

$$\Gamma_{z\bar{w}} = -\frac{(1+|w|^2)((1+|z|^2)\bar{z}^t + (\bar{z}\cdot w)\bar{w}^t)}{1+|z|^2+|w|^2} \times \frac{(1+|z|^2)w-(\bar{z}\cdot w)z}{A},$$

$$\Gamma_{w\bar{z}} = -\frac{(1+|z|^2)((1+|w|^2)\bar{w}^t + (z\cdot\bar{w})\bar{z}^t)}{1+|z|^2+|w|^2} \times \frac{(1+|w|^2)z-(z\cdot\bar{w})w}{A}.$$

To simplify the notation we will let $X = \frac{1}{1+|z|^2+|w|^2}$. From the above result we have

$$\det \partial\overline{\partial}A = A^{2n}\frac{(1+|z|^2)(1+|w|^2)}{A^2}\frac{1+|z|^2+|w|^2}{(1+|z^2)(1+|w|^2)} = \frac{A^{2n-2}}{X},$$

and

$$(\partial\overline{\partial}A)^{-1} = \begin{pmatrix}\Omega_{z\bar{z}} & \Omega_{z\bar{w}} \\ \Omega_{w\bar{z}} & \Omega_{w\bar{w}}\end{pmatrix}$$

where

$$\Omega_{z\bar{z}} = \frac{(1+|z|^2)}{A}\Bigg[I_{2n} + \left(\bar{w}^t - \frac{z\cdot\bar{w}}{1+|z|^2}\bar{z}^t\right)\times w + X(|w|^2\bar{z}^t - (\bar{z}\cdot w)\bar{w}^t)\times\left(z + \frac{z\cdot\bar{w}}{1+|z|^2}w\right)\Bigg],$$

$$\Omega_{w\bar{w}} = \frac{(1+|w|^2)}{A}\Bigg[I_{2n} + \left(\bar{z}^t - \frac{\bar{z}\cdot w}{1+|w|^2}\bar{w}^t\right)\times z + X(|z|^2\bar{w}^t - (z\cdot\bar{w})\bar{z}^t)\times\left(w + \frac{\bar{z}\cdot w}{1+|w|^2}z\right)\Bigg],$$

$$\Omega_{z\bar{w}} = \frac{(\bar{z}\cdot w)}{A}\left[I_{2n} + \left(\bar{z}^t - \frac{\bar{z}\cdot w}{1+|w|^2}\bar{w}^t\right)\times z\right]$$

$$
\begin{aligned}
&+\frac{X}{A}(|z|^2(\bar z\cdot w)\bar w^t-(1+|z|^2)(1+|w|^2)\bar z^t)\times\left(w+\frac{\bar z\cdot w}{1+|w|^2}z\right),\\
\Omega_{w\bar z}={}&\frac{(z\cdot\bar w)}{A}\left[I_{2n}+\left(\bar w^t-\frac{z\cdot\bar w}{1+|z|^2}\bar z^t\right)\times w\right]\\
&+\frac{X}{A}(|w|^2(z\cdot\bar w)\bar z^t-(1+|z|^2)(1+|w|^2)\bar w^t)\times\left(z+\frac{z\cdot\bar w}{1+|z|^2}w\right).
\end{aligned}
$$

Using the relations

$$
\begin{aligned}
&(B_z\cdot w)=0,\quad (B_w\cdot z)=0,\quad (B_z\cdot z)=B+1,\quad (B_w\cdot w)=B+1,\\
&(B_z\cdot\overline{B_w})=-(\bar z\cdot w),\quad (\overline{B}_z\cdot B_w)=-(z\cdot\bar w)
\end{aligned}
$$

we have

$$
\begin{aligned}
&(\partial\overline{\partial}A)^{-1}(\partial A)^t=X\begin{pmatrix}(1+|z|^2)\bar z^t+(\bar z\cdot w)\bar w^t\\(z\cdot\bar w)\bar z^t+(1+|w|^2)\bar w^t\end{pmatrix},\\
&(\partial\overline{\partial}A)^{-1}(\partial B)^t\\
&=\frac{1}{A}\begin{pmatrix}(1+|z|^2)B_z^t+(\bar z\cdot w)B_w^t-X(B+1)((1+|z|^2)\bar z^t+(\bar z\cdot w)\bar w^t)\\(z\cdot\bar w)B_z^t+(1+|w|^2)B_w^t-X(B+1)((z\cdot\bar w)\bar z^t+(1+|w|^2)\bar w^t)\end{pmatrix}.
\end{aligned}
$$

And then

$$
\begin{aligned}
(\bar\partial A)(\partial\overline{\partial}A)^{-1}(\partial A)^t&=A\frac{|z|^2+|w|^2}{1+|z|^2+|w|^2}=A(1-X),\\
(\bar\partial B)(\partial\overline{\partial}A)^{-1}(\partial A)^t&=(\bar B+1)\frac{2+|z|^2+|w|^2}{1+|z|^2+|w|^2}=(\bar B+1)(1+X),\\
(\bar\partial A)(\partial\overline{\partial}A)^{-1}(\partial B)^t&=(B+1)\frac{2+|z|^2+|w|^2}{1+|z|^2+|w|^2}=(B+1)(1+X),\\
(\bar\partial B)(\partial\overline{\partial}A)^{-1}(\partial B)^t&=\frac{1}{A}\left(2A-\frac{1}{X}-1-(B+1)(\bar B+1)(1+X)\right).
\end{aligned}
$$

Plug each piece into (3) gives, after a long calculation,

$$
\det\partial\overline{\partial}f(\mathcal{N})=\left(\frac{1}{B\bar B}\right)^{2n+2}[(2\mathcal{N}-1)\mathcal{N}^{2n-2}(f')^{4n}+2(\mathcal{N}-1)\mathcal{N}^{2n-1}(f')^{4n-1}f''].
$$

Let f be the solution of

$$
(2\mathcal{N}-1)\mathcal{N}^{2n-2}(f')^{4n}+2(\mathcal{N}-1)\mathcal{N}^{2n-1}(f')^{4n-1}f''=1
$$

then

(4)
$$\begin{aligned}
&\left(\frac{(\mathcal{N}-1)^{2n}\mathcal{N}^{2n}}{2n}(f')^{4n}\right)' \\
&= (\mathcal{N}-1)^{2n-1}\mathcal{N}[(2\mathcal{N}-1)\mathcal{N}^{2n-2}(f')^{4n} + 2(\mathcal{N}-1)\mathcal{N}^{2n-1}(f')^{4n-1}f''] \\
&= (\mathcal{N}-1)^{2n-1}\mathcal{N}.
\end{aligned}$$

Integrate both sides of (4), we got

$$\begin{aligned}
\frac{(\mathcal{N}-1)^{2n}\mathcal{N}^{2n}}{2n}(f')^{4n} &= \frac{(\mathcal{N}-1)^{2n+1}}{2n+1} + \frac{(\mathcal{N}-1)^{2n}}{2n}, \\
(f')^{4n} &= \left(\frac{1}{\mathcal{N}}\right)^{2n}\frac{2n\mathcal{N}+1}{2n+1}, \\
f' &= \left(\frac{1}{\mathcal{N}}\right)^{\frac{1}{2}}\left(\frac{2n\mathcal{N}+1}{2n+1}\right)^{\frac{1}{4n}}, \\
f'' &= -\left(\frac{1}{\mathcal{N}}\right)^{\frac{3}{2}}\frac{(2n-1)\mathcal{N}+1}{2(2n+1)}\left(\frac{2n+1}{2n\mathcal{N}+1}\right)^{1-\frac{1}{4n}}.
\end{aligned}$$

Proposition 3. *Let* $\mathbf{M}^{4n}_{III} = Gr(2, 2n+2, C) - H_\infty$ *and let*

$$\mathcal{N}(Z,W) = \frac{\sum_{1\le j<k\le 2n+2}|z_j w_k - z_k w_j|^2}{\left|\sum_{j=1}^{n+1} z_{2j}w_{2j-1} - z_{2j-1}w_{2j}\right|^2}.$$

Then

$$-\left(\frac{1}{\mathcal{N}}\right)^{\frac{3}{2}}\frac{(2n-1)\mathcal{N}+1}{2(2n+1)}\left(\frac{2n\mathcal{N}+1}{2n+1}\right)^{\frac{1}{4n}-1}\partial\mathcal{N}\wedge\bar\partial\mathcal{N} + \left(\frac{1}{\mathcal{N}}\right)^{\frac{1}{2}}\left(\frac{2n\mathcal{N}+1}{2n+1}\right)^{\frac{1}{4n}}\partial\bar\partial\mathcal{N}$$

defines a CRFK-metric on $\mathbf{M}^{4n}_{III}$.

Proof. Similar to the proof of Proposition 1. □

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Department of Mathematics
National Tsing Hua University
Hsin-Chu, Taiwan 30043
ROC
E-mail address: tclee@math.nthu.edu.tw