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NONCOMMUTATIVE JOINT DILATIONS AND FREE PRODUCT OPERATOR ALGEBRAS

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Let $\mathcal{A}_n$ ($n = 2, 3, \ldots$, or $n = \infty$) be the noncommutative disc algebra, and $\mathcal{O}_n$ (resp. $\mathcal{T}_n$) be the Cuntz (resp. Toeplitz) algebra on n generators. Minimal joint isometric dilations for families of contractive sequences of operators on a Hilbert space are obtained and used to extend the von Neumann inequality and the commutant lifting theorem to our noncommutative setting.

We show that the universal algebra generated by k contractive sequences of operators and the identity is the amalgamated free product operator algebra $\check{*}_{\mathbb{C}}\mathcal{A}_{n_i}$ for some positive integers $n_1, n_2, \ldots, n_k \geq 1$, and characterize the completely bounded representations of $\check{*}_{\mathbb{C}}\mathcal{A}_{n_i}$. It is also shown that $\check{*}_{\mathbb{C}}\mathcal{A}_{n_i}$ is completely isometrically imbedded in the "biggest" free product C^*-algebra $\check{*}_{\mathbb{C}}\mathcal{T}_{n_i}$ (resp. $\check{*}_{\mathbb{C}}\mathcal{O}_{n_i}$), and that all these algebras are completely isometrically isomorphic to some universal free operator algebras, providing in this way some factorization theorems.

We show that the free product disc algebra $\check{*}_{\mathbb{C}}\mathcal{A}_{n_i}$ is not amenable and the set of all its characters is homeomorphic to $\overline{(\mathbb{C}^{n_1})_1} \times \cdots \times \overline{(\mathbb{C}^{n_k})_1}$.

An extension of the Naimark dilation theorem to free semigroups is considered. This is used to construct a large class of positive definite operator-valued kernels on the unital free semigroup on n generators and to study the class $\mathcal{C}_\rho$ ($\rho > 0$) of ρ-contractive sequences of operators.

The dilation theorems are also used to extend the operatorial trigonometric moment problem to the free product C^*-algebras $\check{*}_{\mathbb{C}}\mathcal{T}_{n_i}$ and $\check{*}_{\mathbb{C}}\mathcal{O}_{n_i}$.

1. Introduction and preliminaries.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. We identify $M_m(B(\mathcal{H}))$, the set of $m \times m$ matrices with entries from $B(\mathcal{H})$, with $B(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{\text{m-times}})$. Thus we have a natural C^*-norm on $M_m(B(\mathcal{H}))$. If X is an operator space, i.e., a linear subspace of $B(\mathcal{H})$, we consider $M_m(X)$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. The appropriate morphisms between operator spaces are the completely bounded maps [**Ar**], [**P2**], [**Pi**]. Let X, Y be operator spaces and $u : X \to Y$ be a linear map. Define $u_m : M_m(X) \to M_m(Y)$ by

$$u_m([x_{ij}]) = [u(x_{ij})] .$$

We say that u is completely bounded (*cb* in short) if

$$\|u\|_{cb} = \sup_{m\geq 1} \|u_m\| < \infty.$$

If $\|u\|_{cb} \leq 1$ (resp. u_m is an isometry for any $m \geq 1$) then u is completely contractive (resp. isometric), and if u_m is positive for all m, then u is called completely positive.

Let $n = 2, 3, \ldots,$ and $v_1, v_2, \ldots, v_n$ be isometries satisfying

$$v_i^* v_j = \delta_{ij} 1. \tag{1.1}$$

We proved in [**Po4**] that $\text{Alg}(1, v_1, \ldots, v_n)$, the closed non-selfadjoint algebra generated by $1, v_1, \ldots, v_n$, is completely isometrically isomorphic to the noncommutative disc algebra $\mathcal{A}_n$ [**Po2**]. Let us recall [**Cu**] that the Cuntz algebra $\mathcal{O}_n$ (resp. Toeplitz algebra $\mathcal{T}_n$) is uniquely defined as the C^*-algebra generated by n isometries satisfying (1.1) and $\sum_{i=1}^n v_i v_i^* = 1$ (resp. $\sum_{i=1}^n v_i v_i^* < 1$). If $n = \infty$, only the condition (1.1) is required to define $\mathcal{A}_\infty$, $\mathcal{O}_\infty$, and $\mathcal{T}_\infty$. If $n = 1$, we set $\mathcal{A}_1 := A(\mathbf{D})$, the classical disc algebra, $\mathcal{O}_1 := C(\mathbf{T})$, and $\mathcal{T}_1 := C^*(S)$, the C^*-algebra generated by the unilateral shift S acting on the Hardy space $H^2(\mathbf{T})$ (see [**C**]). Since $\mathcal{A}_n$ is completely isometrically imbedded in $\mathcal{O}_n$ (resp. $\mathcal{T}_n$) one can view the noncommutative disc algebra as a "non-selfadjoint Cuntz algebra" as well as an "analytic Toeplitz algebra". Let us mention that $\mathcal{A}_n$ is the universal algebra generated by a row contraction and the identity [**Po6**], and if $n \neq m$ then $\mathcal{A}_n$ is not Banach isomorphic to $\mathcal{A}_m$ [**Po4**]. For a concrete realization of $\mathcal{A}_n$ and $\mathcal{T}_n$ see [**Po2**], [**Po4**].

In Section 2 we obtain a minimal joint isometric dilation for any sequence of contractive sequences of operators on a Hilbert space, extending in this way the Schaffer's construction [**Sc**], [**SzF2**] and also some results from [**F**], [**Bu**], [**Po1**], [**DSz**]. Other dilation theorems are obtained using some results from [**Ar**], [**Bo**], [**S**], [**Po1**], [**Po2**], [**Po3**], and [**Po4**]. Some consequences of these joint isometric dilations are considered in this paper.

The free product of C^*-algebras has been studied by many authors ([**Av**], [**Bo**], [**BP**], [**V**], etc.) but still remains misterious. We consider here the so-called "biggest" free product of operator algebras [**BP**].

Inequalities of von Neumann type are considered in Section 2 and Section 3, extending some results from [**vN**], [**A**], [**Boz**], [**Po2**], [**Po3**], [**Po4**], [**Po7**]. Let $k \geq 1$ and $n_1, \ldots, n_k \geq 1$ be fixed positive integers, and $\{x_{ij}\}_{\substack{i=1,2,\ldots,k \\ j=1,2,\ldots,n_i}}$,

$\{y_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ be noncommuting indeterminates satisfying the relation

$$y_{i\alpha}x_{i\beta} = \delta_{\alpha\beta}1, \quad \text{for any } i=1,2,\dots,k, \text{ and } \alpha,\beta = 1,2,\dots,n_i. \tag{1.2}$$

Let $\mathcal{P}$ be the set of all reduced polynomials in these indeterminates, i.e., each monomial is in reduced form according to the relation (1.2). Let $(T_{i1},\dots,T_{in_i})$, $i=1,2,\dots,k$, be contractive sequences of operators on a Hilbert space $\mathcal{H}$, i.e.,

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* \le I_{\mathcal{H}}, \qquad i=1,2,\dots,k.$$

For each polynomial $p = p(1,\{x_{ij}\},\{y_{ij}\}) \in \mathcal{P}$ let us define the operator $p(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\})$ acting on the Hilbert space $\mathcal{H}$. We prove in Section 3 the following extension of the von Neumann inequality

$$\|p(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\})\| \le \|p\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}}, \tag{1.3}$$

where $p\in\mathcal{P}$ is viewed as an element in the free product C^*-algebra $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$ (see Section 3). On the other hand, for any polynomial $q(1,\{x_{ij}\})$ in noncommuting indeterminates $\{x_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$, we prove that

$$\|q(I_{\mathcal{H}},\{T_{ij}\})\| \le \|q\|_{\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}} = \|q\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}} \le \|q\|_{\check{*}_{\mathbf{C}}C(\mathbf{T}_r)}, \tag{1.4}$$

where $\mathbf{T}_r=\mathbf{T}$, the circle group, $r=1,2,\dots,n_1+n_2+\cdots+n_k$, and $q(1,\{x_{ij}\})$ is seen as an element of $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$, $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$, and $\check{*}_{\mathbf{C}}C(\mathbf{T}_r)$, respectively. If a sequence of operators $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$ satisfies the relation

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* = I_{\mathcal{H}}, \quad i=1,2,\dots,k,$$

then the inequality (1.4) is extended to

$$\|q(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\})\| \le \|q\|_{\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}} \le \|q\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}}, \tag{1.5}$$

for any polynomial $p(1,\{x_{ij}\},\{y_{ij}\})\in\mathcal{P}$.

Let $\mathcal{A}_{n_i}$ $(i=1,2,\dots,k)$ be the noncommutative disc algebra and $\check{*}_{\mathbf{C}}\mathcal{A}_{n_i}$ be the amalgamated (over the identity) free product operator algebra. We prove in Section 3 that the universal algebra generated by k contractive sequences of operators and the identity is the free product disc algebra $\check{*}_{\mathbf{C}}\mathcal{A}_{n_i}$, for some integers $n_1,\dots,n_k\ge 1$. Moreover, using Paulsen's result [**P1**], we give a complete characterization of the completely bounded (resp. contractive) representations of $\check{*}_{\mathbf{C}}\mathcal{A}_{n_i}$. We shall prove that $\check{*}_{\mathbf{C}}\mathcal{A}_{n_i}$ is completely isometrically imbedded in $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$ (resp. $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$). On the other hand, it is proved that all these algebras are completely isometrically isomorphic to some free operator algebras of type $OA(\Delta,\mathcal{R})$, considered by Blecher [**B**]. This identification together with the internal characterization of the matrix norm on a universal algebra [**B**], [**BP**] lead to factorization theorems of type considered in [**B**], [**BP**], [**Po6**].

In Section 4 we shall show that the set of all characters (multiplicative functionals) on $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ is homeomorphic to $\overline{(\mathbf{C}^{n_1})_1}\times\cdots\times\overline{(\mathbf{C}^{n_k})_1}$

and that the first cohomology group of $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ with coefficients in $\mathbf{C}$ is isomorphic to the additive group $\mathbf{C}^{n_1+\cdots+n_k}$. In particular, this shows that the free product disc algebra is not amenable.

In Section 5 we consider an extension of the Naimark dilation theorem [**N**] to free semigroups and construct a large class of positive definite operator-valued kernels on the unital free semigroup on n generators. As an application, we define the class $\mathcal{C}_\rho$ $(\rho > 0)$ of ρ-contractive sequences of operators and prove, using the results from the preceeding sections, that any sequence of class $\mathcal{C}_\rho$ is simultaneously similar to a sequence of class $\mathcal{C}_1$, extending in this way the classical result of Sz.-Nagy and Foiaş [**SzF1**] (see also [**Po5**]).

In Section 6, using some joint dilation theorems from Section 1, we extend the operatorial trigonometric moment problem [**Ak**], [**Po5**] to the free product C^*-algebras $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$ and $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$.

Let us remark that in the particular case when $n_1 = n_2 = \cdots = n_k = n$ one can obtain joint dilations and universal algebras associated to $k \times n$ operator matrices $[T_{ij}]$ with contractive rows. On the other hand, let us mention that if $k, n_1, \ldots, n_k$ are infinite all the results of this paper hold true in a slightly adapted version.

2. Joint minimal isometric dilations.

Let $k \geq 1$ and $n_1, \ldots, n_k \geq 1$ be fixed positive integers. For each $i = 1, 2, \ldots, k$, let $(T_{i1}, \ldots, T_{in_i})$ be a contractive sequence of operators on a Hilbert space $\mathcal{H}$, i.e.,

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* \leq I_{\mathcal{H}}.$$

In what follows we extend the noncommutative dilation theorem [**Po1**] to our setting. The following result also subsumes the isometric dilation theorems from [**SzF2**], [**F**], [**Bu**], and [**DSz**].

Theorem 2.1. *Let $(T_{i1}, \ldots, T_{in_i})$, $i = 1, 2, \ldots, k$, be contractive sequences of operators on a Hilbert space $\mathcal{H}$. Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and contractive sequences $(V_{i1}, \ldots, V_{in_i})$, $i = 1, 2, \ldots, k$, of isometries on $\mathcal{K}$ with the following properties:*

(i) $V_{ij}^*|_{\mathcal{H}} = T_{ij}^* \quad (i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, n_i)$;
(ii) $P_{\mathcal{K}\ominus\mathcal{H}}V_{i_1j_1}\mathcal{K} \perp P_{\mathcal{K}\ominus\mathcal{H}}V_{i_2j_2}\mathcal{K}$ *if* $i_1 \neq i_2$, $j_1 = 1, 2, \ldots, n_{i_1}$, *and* $j_2 = 1, 2, \ldots, n_{i_2}$;
(iii) $\mathcal{K} = \mathcal{H} \bigvee V_{i_1j_1}\cdots V_{i_pj_p}\mathcal{H}$ (*any finite product in* V_{ij} *is considered*).

Moreover, the joint isometric dilation satisfying these properties is uniquely determined up to an isomorphism.

Proof. For each $i = 1, 2, \ldots, k$, let us consider the operator matrix $T_i := [T_{i1} \ldots T_{in_i}]$ and let D_{T_i} be the defect operator defined on $\oplus_{j=1}^{n_i}\mathcal{H}$ by $D_{T_i} =$

$(I - T_i^* T_i)^{1/2}$. Let

$$\mathcal{D} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_k,$$

where $\mathcal{D}_i = \overline{D_{T_i}(\oplus_{j=1}^{n_i} \mathcal{H})}$. Let $n = n_1 + n_2 + \cdots + n_k$ and consider the full Fock space [**E**]

$$F^2(H_n) = \mathbf{C}1 \oplus \oplus_{m\geq 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$. For each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, let $S_{ij} \in B(F^2(H_n))$ be the left creation operator with e_{ij}, i.e.,

$$S_{ij}\xi = e_{ij} \otimes \xi, \ \xi \in F^2(H_n).$$

Consider the operator $D_{ij} : \mathcal{H} \to F^2(H_n) \otimes \mathcal{D}$ defined by

$$\begin{aligned} &D_{ij}h \\ &= 1 \otimes \left(\underbrace{0 \oplus \cdots \oplus 0}_{i-1 \text{ times}} \oplus D_{T_i}(\underbrace{0, \ldots, 0}_{j-1 \text{ times}}, h, \underbrace{0, \ldots, 0}_{n_i - j \text{ times}}) \oplus \underbrace{0 \oplus \cdots \oplus 0}_{k-i \text{ times}} \right) \oplus 0 \oplus 0 \\ &\quad + \cdots \end{aligned}$$

for any $h \in \mathcal{H}$. Consider the Hilbert space

$$\mathcal{K} = \mathcal{H} \oplus \left(F^2(H_n) \otimes \mathcal{D} \right). \tag{2.1}$$

For each $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, n_i$, we define the operator V_{ij} on $\mathcal{K}$ by

$$V_{ij}(h \oplus (\xi \otimes d)) = T_{ij}h \oplus (D_{ij}h + (S_{ij} \otimes I_{\mathcal{D}})(\xi \otimes d)),$$

for any $h \in \mathcal{H}, \ \xi \in F^2(H_n)$, and $d \in \mathcal{D}$. One can see that

$$V_{ij} = \begin{bmatrix} T_{ij} & 0 \\ D_{ij} & S_{ij} \otimes I_{\mathcal{D}} \end{bmatrix} \tag{2.2}$$

with respect to the decomposition (2.1). It follows that

$$V_{ij}^* V_{ij} = \begin{bmatrix} T_{ij}^* T_{ij} + D_{ij}^* D_{ij} & D_{ij}^*(S_{ij} \otimes I_{\mathcal{D}}) \\ (S_{ij}^* \otimes I_{\mathcal{D}}) D_{ij} & S_{ij}^* S_{ij} \otimes I_{\mathcal{D}} \end{bmatrix}.$$

Using the definition of D_{ij}, an easy computation shows that $T_{ij}^* T_{ij} + D_{ij}^* D_{ij} = I_{\mathcal{H}}$ and $(S_{ij}^* \otimes I_{\mathcal{D}}) D_{ij} = 0$. Since $S_{ij}^* S_{ij} = I$, it follows that $V_{ij}^* V_{ij} = I_{\mathcal{K}}$. According to the relation (2.2), it is clear that $V_{ij}^*|_{\mathcal{H}} = T_{ij}^*$. If $i = 1, 2, \ldots, k$ is fixed and $\alpha, \beta = 1, 2, \ldots, n_i, \ \alpha \neq \beta$, then one can similarily prove that

$V_{i\alpha}^* V_{i\beta} = 0$. This shows that $(V_{i1}, \ldots, V_{in_i})$ is a contractive sequence of isometries. On the other hand, we have

$$P_{\mathcal{K}\ominus\mathcal{H}}^{\mathcal{K}} V_{ij} = \begin{bmatrix} 0 & 0 \\ D_{ij} & S_{ij} \otimes I_{\mathcal{D}} \end{bmatrix}.$$

According to the definition of the operators D_{ij}, and since $\{S_{ij}\}$ are isometries with orthogonal ranges, one can infer that

$$P_{\mathcal{K}\ominus\mathcal{H}} V_{i_1 j_1}\mathcal{K} \perp P_{\mathcal{K}\ominus\mathcal{H}} V_{i_2 j_2}\mathcal{K}$$

if $i_1 \neq i_2,\ j_1 = 1, 2, \ldots, n_{i_1},\ j_2 = 1, 2, \ldots, n_{i_2}$.

Let us verify that $\{V_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ is the minimal isometric dilation of $\{T_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$. Consider $\mathcal{H}_1 := \mathcal{H} \bigvee \left(\bigvee_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} V_{ij}\mathcal{H}\right)$ and

$$\mathcal{H}_q := \mathcal{H}_{q-1} \bigvee \left(\bigvee_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} V_{ij}\mathcal{H}_{q-1}\right), \quad \text{if } q \geq 2.$$

It is easy to see that $\mathcal{H}_1 = \mathcal{H} \oplus (\mathbf{C}1 \otimes \mathcal{D})$ and

$$\mathcal{H}_q = \mathcal{H} \oplus \left(\mathbf{C}1 \oplus \oplus_{m=1}^{q-1} H_n^{\otimes m}\right) \otimes \mathcal{D}, \quad \text{if } q \geq 2.$$

Clearly we have $\mathcal{H}_q \subset \mathcal{H}_{q+1}$ and

$$\bigvee_{q=1}^{\infty} \mathcal{H}_q = \mathcal{H} \oplus (F^2(H_n) \otimes \mathcal{D}).$$

Hence, and according to (2.1), we infer that

$$\mathcal{K} = \mathcal{H} \bigvee V_{i_1 j_1} \cdots V_{i_p j_p}\mathcal{H}.$$

Let us show that the minimal isometric dilation $\{V_{ij}\}$ of $\{T_{ij}\}$ is unique up to a unitary operator. Following the classical case, it is enough to show that the inner product

$$L := \langle V_{i_1 j_1} \cdots V_{i_p j_p} h,\ V_{\alpha_1 \beta_1} \cdots V_{\alpha_q \beta_q} h' \rangle,$$

$(h, h' \in \mathcal{H})$, depends only on the operators T_{ij} $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i)$. We can assume that $(i_1, j_1) \neq (\alpha_1, \beta_1)$. If $i_1 = \alpha_1$ and $j_1 \neq \beta_1$ then $V_{\alpha_1 \beta_1}^* V_{i_1 j_1} = 0$, hence $L = 0$. If $i_1 \neq \alpha_1$ then $P_{\mathcal{K}\ominus\mathcal{H}} V_{i_1 j_1}\mathcal{K} \perp P_{\mathcal{K}\ominus\mathcal{H}} V_{\alpha_1 \beta_1}\mathcal{K}$. Therefore,

$$\begin{aligned} L &= \langle (P_{\mathcal{H}} + P_{\mathcal{K}\ominus\mathcal{H}}) V_{i_1 j_1} \cdots V_{i_p j_p} h,\ (P_{\mathcal{H}} + P_{\mathcal{K}\ominus\mathcal{H}}) V_{\alpha_1 \beta_1} \cdots V_{\alpha_q \beta_q} h' \rangle \\ &= \langle P_{\mathcal{H}} V_{i_1 j_1} \cdots V_{i_1 j_p} h,\ P_{\mathcal{H}} V_{\alpha_1, \beta_1} \cdots V_{\alpha_q \beta_q} h' \rangle \\ &= \langle T_{i_1 j_1} \cdots T_{i_p j_p} h,\ T_{\alpha_1 \beta_1} \cdots T_{\alpha_q \beta_q} h' \rangle. \end{aligned}$$

Let $\{V'_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ be another minimal isometric dilation of $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ on a Hilbert space $\mathcal{K}' \supset \mathcal{H}$. Setting

$$U\left(\sum_{\text{finite}} V_{i_1j_1}\cdots V_{i_pj_p}h_{i_1j_1,\dots,i_pj_p}\right) = \sum_{\text{finite}} V'_{i_1j_1}\cdots V'_{i_pj_p}h_{i_1j_1,\dots,i_pj_p}$$

with $h_{i_1j_1,\dots,i_pj_p}\in\mathcal{H}$, we define an isometric operator. Since the isometric dilations are minimal, the operator U can be extended by continuity to a unitary from $\mathcal{K}$ to $\mathcal{K}'$. The proof is complete. □

Let $\{x_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ and $\{y_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ be noncommuting indeterminates satisfying the relation

$$y_{i\alpha}x_{i\beta} = \delta_{\alpha\beta}1, \quad \text{for any } i=1,2,\dots,k, \text{ and } \alpha,\beta=1,2,\dots,n_i.$$

Let $\mathcal{P}$ be the set of all reduced polynomials in these indeterminates, i.e., each monomial is in reduced form according to the above mentioned relation.

The following version of von Neumann's inequality [**vN**] holds.

Corollary 2.2. *For every polynomial* $p(1,\{x_{ij}\},\{y_{ij}\})\in\mathcal{P}$ *and* $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset \mathcal{B}(\mathcal{H})$ *such that*

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* \le I_{\mathcal{H}}, \quad i=1,2,\dots,k,$$

we have

$$\|p(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\})\| \le \sup \|p(I,\{V_{ij}\},\{V_{ij}^*\})\|, \tag{2.3}$$

where the supremum is taken over all contractive sequences of isometries $(V_{i1},\dots,V_{in_i})$ $(i=1,2,\dots,k)$ *on a Hilbert space.*

Proof. Let $\{V_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset \mathcal{B}(\mathcal{K})$ be the minimal joint isometric dilation of $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ in the sense of Theorem 2.1. Using the properties of this dilation, one can prove that

$$P_{\mathcal{H}}V_{i_1j_1}\cdots V_{i_pj_p}V_{r_1q_1}^*\cdots V_{r_mq_m}^*|_{\mathcal{H}} = T_{i_1j_1}\cdots T_{i_pj_p}T_{r_1q_1}^*\cdots T_{r_mq_m}^*,$$

and

$$P_{\mathcal{H}}V_{i_1j_1}^*\cdots V_{i_pj_p}^*V_{r_1q_1}\cdots V_{r_mq_m}|_{\mathcal{H}} = T_{i_1j_1}^*\cdots T_{i_pj_p}^*T_{r_1q_1}\cdots T_{r_mq_m}$$

if $i_p \ne r_1$. Now, using these relations, one can see that, for any polynomial $p(1,\{x_{ij}\},\{y_{ij}\})\in\mathcal{P}$,

$$P_{\mathcal{H}}p(I_{\mathcal{K}},\{V_{ij}\},\{V_{ij}^*\})|_{\mathcal{H}} = p(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\}).$$

Hence, we deduce (2.3). This completes the proof. □

We can apply [**Ar**, Theorem 1.3.1] to our setting in order to get the following commutant lifting theorem for $C^*(\{T_{ij}\})'$.

Theorem 2.3. *Let* $\{T_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} \subset \mathcal{B}(\mathcal{H})$ *be such that* $(T_{i1},\ldots,T_{in_i})$ *is contractive for each* $i=1,2,\ldots,k$, *and let* $\{V_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} \subset \mathcal{B}(\mathcal{K})$ *be its minimal isometric dilation. If* $X \in C^*(\{T_{ij}\})'$ *then there is a unique* $\tilde{X} \in C^*(\{V_{ij}\})' \cap \{P_{\mathcal{H}}\}'$ *such that* $P_{\mathcal{H}}\tilde{X}|_{\mathcal{H}} = X$, *where* $P_{\mathcal{H}}$ *is the orthogonal projection from* $\mathcal{K}$ *onto* $\mathcal{H}$. *Moreover, the map* $X \to \tilde{X}$ *is a* $*$*-isomorphism.*

A particular case which can be proved directly is the following. The proof is similar to [**BrJ**, Lemma 6.2], so we omit it.

Corollary 2.4. *If* $U \in C^*(\{T_{ij}\})'$ *is a unitary then it has a unitary extension* $\tilde{U} \in C^*(\{V_{ij}\})'$. *Moreover this extension is unique.*

Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and let $\mathcal{A} *_{\mathbf{C}} \mathcal{B}$ be their algebraic free product amalgamated over the identity, which is a $*$-algebra. For $x \in \mathcal{A} *_{\mathbf{C}} \mathcal{B}$ define

$$\|x\| = \sup\{\|\pi(x)\|\},$$

where the supremum is taken over all $*$-representations of $\mathcal{A} *_{\mathbf{C}} \mathcal{B}$. Let us mentioned that all $*$-representations of $\mathcal{A} *_{\mathbf{C}} \mathcal{B}$ are in one-to-one correspondence with pairs of $*$-representations of $\mathcal{A}$ and $\mathcal{B}$, which act on the same Hilbert space. The "biggest" free product of $\mathcal{A}$ and $\mathcal{B}$ is the completion of $\mathcal{A} *_{\mathbf{C}} \mathcal{B}$ in this norm, and is denoted by $\mathcal{A} \check{*}_{\mathbf{C}} \mathcal{B}$ (see [**Av**]).

Theorem 2.5. *For each* $i=1,2,\ldots,k$, *let* $(T_{i1},\ldots,T_{in_i})$ *be a contractive sequence of operators on a Hilbert space* $\mathcal{H}$. *Then there exists a Hilbert space* $\mathcal{K} \supset \mathcal{H}$ *and contractive sequences* $(V_{i1},\ldots,V_{in_i})$ $(i=1,2,\ldots,k)$ *of isometries on* $\mathcal{K}$ *such that*

$$V_{i1}V_{i1}^* + \cdots + V_{in_i}V_{in_i}^* = I_{\mathcal{K}} \quad (i=1,2,\ldots,k)$$

and

$$p(I_{\mathcal{H}},\{T_{ij}\}) = P_{\mathcal{H}}p(I_{\mathcal{K}},\{V_{ij}\})|_{\mathcal{H}}$$

for any polynomial p *in noncommuting indeterminates* $\{x_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$.

Proof. Consider the case $k \geq 2$. For each $i=1,2,\ldots,k$, let $\sigma_{i1},\ldots,\sigma_{in_i}$ be a system of generators for the Cuntz algebra $\mathcal{O}_{n_i}$. We proved in [**Po4**] that the Banach algebras $Alg(1,\sigma_{i1},\ldots,\sigma_{in_i})$ and the noncommutative disc algebra $\mathcal{A}_{n_i}$ are completely isometrically isomorphic. According to the noncommutative von Neumann inequality [**Po2**], [**Po4**], we infer that the map

$$\Phi_i : Alg(1,\sigma_{i1},\ldots,\sigma_{in_i}) \to B(\mathcal{H})$$

defined by

$$\Phi(p(1,\sigma_{i1},\ldots,\sigma_{in_i})) = p(I_{\mathcal{H}},T_{i1},\ldots,T_{in_i})$$

is a completely contractive homomorphism.

Using the extension theorem of Arveson [**Ar**] we infer that there is a completely positive linear map $\Psi_i : \mathcal{O}_{n_i} \to B(\mathcal{H})$ such that $\Psi_i|_{Alg(1,\sigma_{i1},\ldots,\sigma_{in_i})} =$

Φ_i. Now using Boca's result [**Bo**], there is a common completely positive extension

$$\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k : \mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k} \to B(\mathcal{H})$$

with

$$(\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k)(p(1, \{\sigma_{ij}\})) = p(I_{\mathcal{H}}, \{T_{ij}\}),$$

where $p(1, \{\sigma_{ij}\}) \in \mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k}$ is any polynomial in $1, \sigma_{i1}, \ldots, \sigma_{in_i}$ $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i)$. According to Stinespring's theorem [**S**],

$$(\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k)(p(1, \{\sigma_{ij}\})) = P_{\mathcal{H}} \pi(p(1, \{\sigma_{ij}\}))|_{\mathcal{H}}$$

for any $p(1, \{\sigma_{ij}\}) \in \mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k}$, where π is a $*$-representation of $\mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, and $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. In particular, we have

$$p(I_{\mathcal{H}}, \{T_{ij}\}) = P_{\mathcal{H}} p(I_{\mathcal{K}}, \{\pi(\sigma_{ij})\})|_{\mathcal{H}}.$$

Notice that $(\pi(\sigma_{i1}), \ldots, \pi(\sigma_{in_i}))$ is a sequence of isometries such that

$$\pi(\sigma_{i1})\pi(\sigma_{i1})^* + \cdots + \pi(\sigma_{in_i})\pi(\sigma_{in_i})^* = I_{\mathcal{K}},\ i = 1, 2, \ldots, k.$$

Denote $V_{ij} = \pi(\sigma_{ij})$, $i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i$. This completes the proof when $k \geq 2$. The case $k = 1$ can be treated similarly (see also [**Bo**]). ☐

Let $\mathcal{P}_x$ be the set of all polynomials in noncommuting indeterminates x_{ij} $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i)$. Notice that $\mathcal{P}_x \subset \mathcal{P}$.

Corollary 2.6. *For every polynomial* $p(1, \{x_{ij}\}) \in \mathcal{P}_x$ *and* $\{T_{ij}\}_{\substack{i=1,2,\ldots,k \\ j=1,2,\ldots,n_i}} \subset B(\mathcal{H})$ *such that* $(T_{i1}, \ldots, T_{in_i})$ *is contractive for any* $i = 1, 2, \ldots, k$,

$$\|p(I_{\mathcal{H}}, \{T_{ij}\})\| \leq \sup \|p(I_{\mathcal{K}}, \{V_{ij}\})\| \tag{2.4}$$

where the supremum is taken over all sequences of isometries $(V_{i1}, \ldots, V_{in_i})$ $(i = 1, 2, \ldots, k)$ *on a Hilbert space* $\mathcal{K}$ *such that*

$$V_{i1}V_{i1}^* + \cdots + V_{in_i}V_{in_i}^* = I_{\mathcal{K}}, \quad i = 1, 2, \ldots, k.$$

Let us remark that in the particular case when $n_1 = n_2 = \cdots = n_k = 1$ one obtains Bozejko's version [**Boz**] of von Neumann's inequality [**vN**]. On the other hand, in the particular case when $k = 1,\ n_1 = n$ we find a version of the noncommutative von Neumann inequality obtained in [**Po2**].

Theorem 2.7. *Let* $(T_{i1}, \ldots, T_{in_i})$, $i = 1, 2, \ldots, k$, *be contractive sequences of operators on a Hilbert space* $\mathcal{H}$ *such that*

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* = I_{\mathcal{H}}, \quad i = 1, 2, \ldots, k. \tag{2.5}$$

Then there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ *and contractive sequences* $(V_{i1}, \ldots, V_{in_i})$, $i = 1, 2, \ldots, k$, *of isometries on* $\mathcal{K}$ *with the following properties:*

(i) $V_{i1}V_{i1}^* + \cdots + V_{in_i}V_{in_i}^* = I_{\mathcal{K}} \quad (i = 1, 2, \ldots k)$;
(ii) $V_{ij}^*|_{\mathcal{H}} = T_{ij}^* \quad (i = 1, 2, \ldots, k,\ j = 1, 2, \ldots, n_i)$;

(iii) *For any polynomial* $p(1, \{x_{ij}\}, \{y_{ij}\}) \in \mathcal{P}$,

$$P_{\mathcal{H}} p(I_{\mathcal{K}}, \{V_{ij}\}, \{V_{ij}^*\})|_{\mathcal{H}} = p(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\});$$

(iv) $\mathcal{K} = \mathcal{H} \bigvee V_{i_1 j_1} \cdots V_{i_p j_p} \mathcal{H}$ (*any finite product in* V_{ij} *is considered*).

Moreover, the joint isometric dilation satisfying these properties is uniquely determined up to an isomorphism.

Proof. Let $k \geq 2$ and $i \in \{1, 2, \ldots, k\}$ be fixed. Since (2.5) holds, according to [**Po1**, Proposition 2.5], there is a Hilbert space $\mathcal{K}_i \supset \mathcal{H}$ and a contractive sequence $(W_{i1}, \ldots, W_{in_i})$ of isometries on $\mathcal{K}_i$ having the following properties:

(i) $W_{i1} W_{i1}^* + \cdots + W_{in_i} W_{in_i}^* = I_{\mathcal{K}_i}$;
(ii) $W_{ij}^*|_{\mathcal{H}} = T_{ij}^* \quad (j = 1, 2, \ldots, n_i)$.

Therefore, for any polynomial $p(1, \{x_{ij}\}, \{y_{ij}\}) \in \mathcal{P}$, we have

$$P_{\mathcal{H}} p(I_{\mathcal{K}_i}, \{W_{ij}\}, \{W_{ij}^*\})|_{\mathcal{H}} = p(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\}). \tag{2.6}$$

For each $i = 1, 2, \ldots, k$, let $\sigma_{i1}, \ldots, \sigma_{in_i}$ be a system of generators of the Cuntz algebra $\mathcal{O}_{n_i}$. Since the Cuntz algebra does not depend on the generators [**Cu**], and using (2.6), we infer that the map $\phi_i : \mathcal{O}_{n_i} \to B(\mathcal{H})$ defined by

$$\phi_i(p(1, \{\sigma_{ij}\}, \{\sigma_{ij}^*\})) = p(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\})$$

is completely contractive, hence completely positive. Using Boca's result [**Bo**], there is a common completely positive extension

$$\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k : \mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k} \to B(\mathcal{H})$$

with

$$(\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k)(p(1, \{\sigma_{ij}\}, \{\sigma_{ij}^*\})) = p(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\}),$$

for any $p(1, \{x_{ij}\}, \{y_{ij}\}) \in \mathcal{P}$. According to Stinespring's theorem [**S**],

$$(\Psi_1 *_{\mathbf{C}} \cdots *_{\mathbf{C}} \Psi_k)(p(1, \{\sigma_{ij}\}, \{\sigma_{ij}^*\})) = P_{\mathcal{H}} p(I_{\mathcal{K}}, \{\pi(\sigma_{ij})\}, \{\pi(\sigma_{ij}^*)\})|_{\mathcal{H}},$$

where π is a $*$-representation of $\mathcal{O}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{O}_{n_k}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$, and $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. Denote $V_{ij} = \pi(\sigma_{ij})$ $(i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i)$. Now, it is easy to see that $\{V_{ij}\}$ satisfies all the properties stated in the theorem. Let us just mention that the property (ii) follows from the relation

$$P_{\mathcal{H}} V_{ij} V_{ij}^*|_{\mathcal{H}} = (P_{\mathcal{H}} V_{ij}|_{\mathcal{H}})(P_{\mathcal{H}} V_{ij}^*|_{\mathcal{H}}),$$

using an argument from [**A**].

The uniqueness is a consequence of Stinespring's theorem. Notice that the case $k = 1$ can be treated similarly. This completes the proof. □

Let us remark that, in the setting of Theorem 2.7, one can obtain a commutant lifting theorem similar to Theorem 2.3.

Corollary 2.8. *If* $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$ *satisfies the relation* (2.5), *then for any polynomial* $p(1,\{x_{ij}\},\{y_{ij}\}) \in \mathcal{P}$,

$$\|p(I_{\mathcal{H}},\{T_{ij}\},\{T_{ij}^*\})\| \le \sup \|p(I_{\mathcal{K}},\{V_{ij}\},\{V_{ij}^*\})\|, \tag{2.7}$$

where the supremum is taken over all sequences of isometries $(V_{i1},\dots,V_{in_i})$ $(i=1,2,\dots,k)$ *on a Hilbert space* $\mathcal{K}$ *such that*

$$V_{i1}V_{i1}^* + \cdots + V_{in_i}V_{in_i}^* = I_{\mathcal{K}}, \quad i=1,2,\dots,k.$$

Let us mention here a particular case. A hereditary polynomial in $2n$ noncommuting indeterminates $\{x_i\},\{y_i\}$ $(i=1,2,\dots,n)$ has the form

$$p(1,\{x_i\},\{y_i\}) = a_0 + \sum a_{i_1\cdots j_q} x_{i_1}\cdots x_{i_p} y_{j_1}\cdots y_{j_q},$$

where $a_0, a_{i_1\cdots j_q} \in \mathbf{C}$.

Corollary 2.9. *If* $\{T_i\}_{i=1}^n \subset B(\mathcal{H})$ *such that*

$$T_1T_1^* + \cdots T_nT_n^* = I_{\mathcal{H}}, \tag{2.8}$$

then for any hereditary polynomial $p(1,\{x_i\},\{y_i\})$

$$\|p(I_{\mathcal{H}},\{T_i\},\{T_i^*\})\| \le \|p(1,\{\sigma_i\},\{\sigma_i^*\})\|_{\mathcal{O}_n}, \tag{2.9}$$

where $\{\sigma_i\}_{i=1}^n$ *is a system of generators for the Cuntz algebra* $\mathcal{O}_n$.

Let us remark that, under the condition (2.8), the inequality (2.9) is sharper than the one obtained in [**Po3**].

3. Free product operator algebras and their representations.

We need a few definitions from [**B**]. Let Γ be a set, and let $n:\Gamma \to \mathbb{N}$ be a function with $n(\gamma) = n_\gamma$. Let Λ be a set of variables (or formal symbols) x_{ij}^γ, one variable for each $\gamma \in \Gamma$ and each i,j, $1 \le i,j \le n_\gamma$. We call these matrix entry variables, or quantum variables. Let $\mathcal{F}$ be the free associative algebra on Λ. Let $\mathcal{R}$ be a set of polynomial identities $P = 0$ in the variables in Λ. Regard $\mathcal{R}$ as subset of $\mathcal{F}$. Take a quotient of $\mathcal{F}$ by the ideal generated by $\mathcal{R}$.

We define a semi-norm on $M_n(\mathcal{F})$ by

$$\|[u_{ij}]\|_\Lambda = \sup\{\|[\pi(u_{ij})]\|\} \tag{3.1}$$

where the supremum is taken over all algebra representations π of $\mathcal{F}/_{\mathcal{R}}$ on a separable Hilbert space satisfying the condition $\|[\pi(x_{ij}^\gamma)]\| \le 1$ for all γ. This later matrix is indexed on rows by i and on columns by j, for all $1 \le i,j \le n_\gamma$.

Now, quotient by nullspace of this semi-norm to obtain an operator algebra. The completion of this space is denoted by $OA(\Lambda,\mathcal{R})$. This is called the free operator algebra on Λ with relations $\mathcal{R}$ (see [**B**]).

Let Δ_{xy} have the identity e and also contain the ordinary variables $\{x_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ and $\{y_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$, and let $\mathcal{R}_{xy}$ be the relations

$$y_{i\alpha}x_{i\beta} = \delta_{\alpha\beta}e, \quad \text{for any } i = 1,2,\dots,k \text{ and } \alpha,\beta = 1,2,\dots,n_i.$$

Form the universal algebra $OA(\Delta_{xy}, \mathcal{R}_{xy})$.

Theorem 3.1. *The universal algebra $OA(\Delta_{xy}, \mathcal{R}_{xy})$ is completely isometrically isomorphic to the amalgamated (over the identity) free product C^*-algebra $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$.*

Proof. According to (3.1), for any polynomials $p_{rs}(e, \{x_{ij}\}, \{y_{ij}\})$, $1 \le r,s \le m$, we have

$$\|[p_{rs}(e,\{x_{ij}\},\{y_{ij}\})]\|_{\Delta_{xy}} = \sup\{\|[p(I_{\mathcal{H}},\{A_{ij}\},\{B_{ij}\})]\|\}, \tag{3.2}$$

where the supremum is taken for all contractions $A_{ij}, B_{ij} \in B(\mathcal{H})$ $(i = 1,2,\dots,k;\ j = 1,2,\dots,n_i)$ satisfying the relations

$$B_{i\alpha}A_{i\beta} = \delta_{\alpha\beta}I_{\mathcal{H}}, \quad \text{for any } i = 1,2,\dots,k;\ \alpha,\beta = 1,2,\dots,n_i.$$

Under the above conditions, one can prove that $A_{i\beta}^* = B_{i\beta}$ (see [**Po6**]) and consequently $(A_{i1},\dots,A_{in_i})$ is a contractive sequence of isometries for each $i = 1,2,\dots,k$. Therefore, the relation (3.2) becomes

$$\|[p_{rs}(e,\{x_{ij}\},\{y_{ij}\})]\|_{\Delta_{xy}} = \sup\{\|[p(I,\{V_{ij}\},\{V_{ij}^*\})]\|\}, \tag{3.3}$$

where the supremum is taken for all contractive sequences of isometries $(V_{i1},\dots,V_{in_i})$ $(i = 1,2,\dots,k)$ acting on a Hilbert space.

On the other hand, the $*$-representations π of $*_{\mathbf{C}}\mathcal{T}_{n_i}$ are in one-to-one corespondence with k-tuples $\pi_1,\dots,\pi_k$ of $*$-representations of $\mathcal{T}_{n_1},\dots,\mathcal{T}_{n_k}$, respectively, on the same Hilbert space, i.e.,

$$\pi|_{\mathcal{T}_{n_i}} = \pi_i, \quad i = 1,2,\dots,k. \tag{3.4}$$

According to [**Po3**], the $*$-representations $\pi_i : \mathcal{T}_{n_i} \to B(\mathcal{K})$ are in one-to-one corespondence with the contractive sequences of isometries $(V_{i1},\dots,V_{in_i})$ such that $\pi_i(S_{ij}) = V_{ij}$ and $\pi_i(1) = I_{\mathcal{K}}$, where $S_{i1},\dots,S_{in_i}$ is a system of generators of the Toeplitz C^*-algebra $\mathcal{T}_{n_i}$. Therefore,

$$\begin{aligned}\|[p_{rs}(e,\{x_{ij}\},\{y_{ij}\})]\|_{\Delta_{xy}} &= \sup\|[p_{rs}(I,\{\pi_i(S_{ij})\},\{\pi_i(S_{ij})^*\})]\| \\ &= \sup\|[\pi(p_{rs}(I,\{S_{ij}\},\{S_{ij}^*\}))]\|,\end{aligned}$$

where the supremum is taken over all $*$-representations π of $*_{\mathbf{C}}\mathcal{T}_{n_i}$ such that (3.4) holds. This shows that $OA(\Delta_{xy}, \mathcal{R}_{xy})$ is completely isometrically isomorphic to $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$, and the proof is complete. □

The internal characterization of the matrix norm on a universal algebra $OA(\Lambda, \mathcal{R})$ (see [**B**], [**BP**]) leads to the following factorization theorem.

Theorem 3.2. *If* $P = [p_{rs}]_{m\times m}$ *is a matrix of polynomials in* $e, \{x_{ij}\},\{y_{ij}\}$ *then,* $\|P\|_{\Delta_{xy}} < 1$ *if and only if there is a positive integer* t *such that*

$$P = A_0 D_1 A_1 D_2 \cdots D_t A_t, \tag{3.5}$$

where A_ℓ $(\ell = 0, 1, \ldots, t)$ *are scalar matrices (with a finite number of nonzero entries), each* $\|A_\ell\| < 1$*, and each* D_ℓ *is diagonal matrix with* e, x_{ij}, y_{ij} $(i \in \{1, 2, \ldots, k\}$ *and* $j \in \{1, 2, \ldots, n_i\})$ *as the diagonal entries.*

Blecher and Paulsen defined in [**BP**] the free product with amalgamation over $\mathbf{C}$ in the category consisting of unital operator algebras as objects and completely contractive homomorphisms as morphisms. For each $i = 1, 2, \ldots, k$, let $\mathcal{A}_{n_i}$ be the noncommutative disc algebra on n_i-generators [**Po4**], and let $\mathcal{A}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{A}_{n_k}$ be the amalgamated free product operator algebra. This is the unique unital algebra which has the following universal property: there are unital completely isometric imbeddings

$$\chi_i : \mathcal{A}_{n_i} \to \mathcal{A}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{A}_{n_k} \quad (i = 1, 2, \ldots, k)$$

such that the images of $\mathcal{A}_{n_i}$ $(i = 1, 2, \ldots, k)$ under χ_i $(i = 1, 2, \ldots, k)$ generate $\mathcal{A}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{A}_{n_k}$, and if for each $i = 1, 2, \ldots, k$, π_i is a unital completely contractive homomorphism from $\mathcal{A}_{n_i}$ into an operator algebra $\mathcal{C}$, then there is a unique unital completely contractive homomorphism

$$\pi_1 * \cdots * \pi_k : \mathcal{A}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{A}_{n_k} \to \mathcal{C}$$

with $(\pi_1 * \cdots * \pi_k) \circ \chi_i = \pi_i \quad (i = 1, 2, \ldots, k)$.

Let us denote by $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ the closed subalgebra generated by the variables $e, \{x_{ij}\}_{\substack{i=1,2,\ldots,k \\ j=1,2,\ldots,n_i}}$ in $OA(\Delta_{xy}, \mathcal{R}_{xy})$.

Theorem 3.3. *The amalgamated free product operator algebra* $\mathcal{A}_{n_1} \check{*}_{\mathbf{C}} \cdots \check{*}_{\mathbf{C}} \mathcal{A}_{n_k}$ *is completely isometrically isomorphic to the operator algebra* $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$.

Proof. It is enough to prove that the algebra $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ has the above-mentioned universal property. Let $\{S_{i1}, \ldots, S_{in_i}\}$ be a system of generators of $\mathcal{A}_{n_i}$ $(i = 1, 2, \ldots, k)$. According to the von Neumann inequality (see [**Po2**], [**Po4**]), one can easily see that the homomorphism

$$\chi_i : \mathcal{A}_{n_i} \to OA_+(\Delta_{xy}, \mathcal{R}_{xy})$$

defined by $\chi_i(S_{ij}) = x_{ij} \quad (j = 1, 2, \ldots, n_i)$ is a unital completely isometric imbedding, and $\chi_i(\mathcal{A}_{n_i}) \quad (i = 1, 2, \ldots, k)$ generate $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$. Moreover, if $\mathcal{C}$ is an operator algebra and

$$\pi_i : \mathcal{A}_{n_i} \to \mathcal{C} \quad (i = 1, 2, \ldots, k)$$

are unital completely contractive homomorphism, then according to [**Po4**], there exists $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset \mathcal{C}$ such that $(T_{i1},\dots,T_{in_i})$ is a contractive sequence of operators for each $i=1,2,\dots,k$, and $\pi_i(S_{ij})=T_{ij}$. Define

$$(\pi_1 * \cdots * \pi_k)(p(e,\{x_{ij}\})) = p(I,\{T_{ij}\})$$

for any polynomial in $e, \{x_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$. Taking into account the definition of $OA_+(\Delta_{xy},\mathcal{R}_{xy})$, Theorem 3.1, and the von Neumann inequality (2.3), we infer that $\pi_1*\cdots*\pi_k$ is contractive and can be uniquely extended to a unital contractive homomorphism

$$\pi_1 * \cdots * \pi_k : OA_+(\Delta_{xy},\mathcal{R}_{xy}) \to \mathcal{C}$$

such that $(\pi_1*\cdots*\pi_k)(\chi_i(S_{ij}))=T_{ij}$. The proof is complete. □

Using Theorem 3.1 one can deduce the following.

Corollary 3.4. *The operator algebra $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ is completely isometrically embedded in the C^*-algebra $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$.*

Remark 3.5. The result from Theorem 3.3 can be also obtained using the results from [**Bo**], [**BP**], and [**Po4**].

Let $\mathcal{P}_x$ be the set of all polynomials in the noncommuting indeterminates $\{x_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$. Notice that $\mathcal{P}_x\subset\mathcal{P}$ ($\mathcal{P}$ was introduced in Section 1), and any $p\in\mathcal{P}$ can be viewed as an element in $OA(\Delta_{xy},\mathcal{R}_{xy})$.

Theorem 3.6. *Let $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ be in $B(\mathcal{H})$. Then $(A_{i1},\dots,A_{in_i})$ is a contractive sequence of operators for each $i=1,2,\dots,k$, if and only if the map*

$$\Phi:\mathcal{P}_x\subset OA_+(\Delta_{xy},\mathcal{R}_{xy})\to B(\mathcal{H})$$

defined by $\Phi(p(e,\{x_{ij}\}))=p(I_{\mathcal{H}},\{A_{ij}\})$ is a completely contractive homomorphism.

Proof. Assume that $(A_{i1},\dots,A_{in_i})$ is contractive for each $i=1,2,\dots,k$. According to Theorem 2.1, there is a Hilbert space $\mathcal{K}\supset\mathcal{H}$ and contractive sequences $(V_{i1},\dots,V_{in_i})$ of isometries on $\mathcal{K}$ such that

$$V_{ij}^*|_{\mathcal{H}}=A_{ij}^*,\quad i=1,2,\dots,k;\ j=1,2,\dots,n_i. \tag{3.6}$$

The map $\Psi: OA(\Delta_{xy},\mathcal{R}_{xy})\to C^*(\{V_{ij}\})$ defined by

$$\Psi(p(e,\{x_{ij}\},\{y_{ij}\}))=p(I_{\mathcal{K}},\{V_{ij}\},\{V_{ij}^*\}),$$

where $p(e,\{x_{ij}\},\{y_{ij}\})\in\mathcal{P}$, is completely contractive (see Theorem 3.1). Therefore, $\|\Psi\|_{cb}\le 1$. According to (3.6), we have

$$\begin{aligned}\Phi(p(e,\{x_{ij}\})) &= p(I_{\mathcal{K}},\{A_{ij}\}) = P_{\mathcal{H}}p(I_{\mathcal{K}},\{V_{ij}\})|_{\mathcal{H}}\\ &= P_{\mathcal{H}}\Psi(p(e,\{x_{ij}\}))|_{\mathcal{H}},\end{aligned}$$

for any $p(e,\{x_{ij}\}) \in OA_+(\Delta_{xy},\mathcal{R}_{xy})$. Therefore, $\|\Phi\|_{cb} \leq \|\Psi\|_{cb} \leq 1$.

Conversely, suppose $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$ such that map

$$\Phi : \mathcal{P}_x \subset OA_+(\Delta_{xy},\mathcal{R}_{xy}) \to B(\mathcal{H})$$

defined by $\Phi(p(e,\{x_{ij}\})) = p(I_{\mathcal{H}},\{A_{ij}\})$ is completely contractive. In particular, for each $i = 1,2,\dots,k$, we have

$$\left\| \begin{bmatrix} A_{i1} & A_{i2} & \cdots & A_{in_i} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} x_{i1} & x_{i2} & \cdots & x_{in_i} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\|_{\Delta_x} \leq 1.$$

Hence, $\left\| \sum_{i=1}^{n_i} A_{ij}A_{ij}^* \right\| \leq 1$ for each $i = 1,2,\dots,k$. This completes the proof. □

The above theorem and Theorem 3.3 show also that the universal algebra generated by a finite number of contractive sequences of operators on a Hilbert space and the identity is completely isometrically isomorphic to the amalgamated free product operator algebra $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ for some integers $n_1,\dots,n_k \geq 1$, in the following sense. Given any contractive sequences $(T_{i1},\dots,T_{in_i})\quad (i = 1,2,\dots,k)$ of operators on a Hilbert space $\mathcal{H}$, there is a completely contractive homomorphism

$$\Phi : \mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k} \to \mathcal{B}(\mathcal{H})$$

such that $\Phi(1) = 1$ and $\Phi(x_{ij}) = T_{ij}$ for any $i = 1,2,\dots,k$ and $j = 1,2,\dots,n_i$. Moreover, this property characterizes $\mathcal{A}_n\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ up to unital complete isometric isomorphism.

Similarly to the proof of Theorem 3.6, one can prove the following result.

Theorem 3.7. *Let $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$. Then $(A_{i1},\dots,A_{in_i})$ is a contractive sequence of operators for each $i = 1,2,\dots,k$, if and only if the map*

$$\Psi : \mathcal{P} \subset OA(\Delta_{xy},\mathcal{R}_{xy}) \to B(\mathcal{H})$$

defined by

$$\Psi(p(\{x_{ij}\},\{y_{ij}\})) = p(\{A_{ij}\},\{A_{ij}^*\})$$

is completely positive.

Now, using Theorem 3.1 and Theorem 3.7, we infer the following extension of the von Neumann inequality [**vN**], [**Po2**], [**Po3**].

Corollary 3.8. *If $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$ such that*

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* \leq I_{\mathcal{H}}, \quad i = 1,2,\dots,k,$$

then for any $p \in \mathcal{P} \subset OA(\Delta_{xy}, \mathcal{R}_{xy})$,

$$\|p(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\})\| \leq \|p\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}},$$

where p *is viewed as an element of free product* C^**-algebra* $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$.

Using Theorem 3.6 and a well-known result of Paulsen ([**P1**], [**Po2**]), one can easily infer the following.

Theorem 3.9. *Let* $\{A_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} \subset B(\mathcal{H})$. *The following statements are equivalent.*

(i) *The map* $\Phi : OA_+(\Delta_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$ *defined by*

$$\Phi(p(e, \{x_{ij}\})) = p(I, \{A_{ij}\})$$

is completely bounded.

(ii) *There is a sequence* $\{T_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} \subset B(\mathcal{H})$ *such that* $(T_{i1}, \ldots, T_{in_i})$ $(i = 1, 2, \ldots, k)$ *is contractive, and an invertible operator* S *satisfying*

$$A_{ij} = S^{-1}T_{ij}S, \quad \text{for any } \ i = 1, 2, \ldots, k;\ j = 1, 2, \ldots, n_i.$$

Let $\Delta_{x'y'}$ have the identity e and also contain the ordinary variables $\{x'_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ and $\{y'_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$, and let $\mathcal{R}_{x'y'}$ be the relations

$$y'_{i\alpha}x'_{i\beta} = \delta_{\alpha\beta}e \ \text{ for any } i = 1, 2, \ldots, k \text{ and } \alpha, \beta = 1, 2, \ldots, n_i,$$

and

$$x'_{i1}y'_{i1} + \cdots + x'_{in_i}y'_{in_i} = e, \quad i = 1, 2, \ldots, k.$$

Form the universal algebra $OA(\Delta_{x'y'}, \mathcal{R}_{x'y'})$. One can prove that

$$OA(\Delta_{x'y'}, \mathcal{R}_{x'y'}) = \check{*}_{\mathbf{C}}\mathcal{O}_{n_i}.$$

The proof is similar to that of Theorem 3.1, so we will omit it. Let us denote by $OA_+(\Delta_{x'y'}, \mathcal{R}_{x'y'})$ the closed subalgebra generated by $e, \{x'_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ in $OA(\Delta_{x'y'}, \mathcal{R}_{x'y'})$. Using Theorem 2.5, we can deduce the following.

Corollary 3.10. *If* $\{T_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}} \subset B(\mathcal{H})$ *such that* $(T_{i1}, \ldots, T_{in_i})$ *is contractive for each* $i = 1, 2, \ldots, 1$, *then the map*

$$\Phi : OA_+(\Delta_{x'y'}, \mathcal{R}_{x'y'}) \to \mathcal{B}(\mathcal{H})$$

defined by $\Phi(p(e, \{x'_{ij}\})) = p(I_{\mathcal{H}}, \{T_{ij}\})$ *is a completely contractive homomorphism.*

Due to the universal property of $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$, one can deduce the following.

Theorem 3.11. *The amalgamated free product operator algebra* $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ *is completely isometrically isomorphic to* $OA_+(\Delta_{x'y'}, \mathcal{R}_{x'y'})$.

Now, the inequality (1.4) announced in Section 1, follows from Corollary 3.8 and Theorem 3.11.

Corollary 3.12. *If* $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in B(\mathcal{H})$ *such that*

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* \leq I_{\mathcal{H}}, \quad i = 1, 2, \dots, k,$$

then for any polynomial $q(1, \{x_{ij}\}) \in \mathcal{P}$,

$$\|q(I_{\mathcal{H}}, \{T_{ij}\})\| \leq \|q\|_{\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}} = \|q\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}} \leq \|q\|_{\check{*}_{\mathbf{C}}C(\mathbf{T}_r)}, \tag{3.7}$$

where $\mathbf{T}_r = \mathbf{T}$ $(r = 1, 2, \dots, n_1 + \cdots n_k)$, *and* $q(1, \{x_{ij}\})$ *is seen as an element of* $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$, $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$, *and* $\check{*}_{\mathbf{C}}C(\mathbf{T}_r)$, *respectively.*

The inequality

$$\|q(I_{\mathcal{H}}, \{T_{ij}\})\| \leq \|q\|_{\check{*}_{\mathbf{C}}C(\mathbf{T}_r)} \tag{3.8}$$

was proved by Bozejko in [**Boz**] (see also [**Bo**]) and follows also from Corollary 2.6. Notice that, in our setting, the inequality (3.7) is sharper than (3.8).

Corollary 3.13. *The operator algebra* $\mathcal{A}_{n_1}\check{*}_{\mathbf{C}}\cdots\check{*}_{\mathbf{C}}\mathcal{A}_{n_k}$ *is completely isometrically imbedded in the* C^**-algebra* $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$.

Using Theorem 2.7 and Corollary 3.8 one can infer the following version of the von Neumann inequality.

Corollary 3.14. *If* $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in B(\mathcal{H})$ *such that*

$$T_{i1}T_{i1}^* + \cdots + T_{in_i}T_{in_i}^* = I_{\mathcal{H}}, \quad i = 1, 2, \dots, k,$$

then for any polynomial $q(1, \{x_{ij}\}, \{y_{ij}\}) \in \mathcal{P}$,

$$\|q(I_{\mathcal{H}}, \{T_{ij}\}, \{T_{ij}^*\})\| \leq \|q\|_{\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}} \leq \|q\|_{\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}},$$

where $q(1, \{x_{ij}, \{y_{ij}\}\})$ *is seen as an element of* $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$ *and* $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$, *respectively.*

Let Δ_x have the entries in the row matrices $[x_{i1}, \dots, x_{in_i}]$ $(i = 1, 2, \dots, k)$ (so there are some relations forcing the other entries to be zero) and also an identity e (i.e., $x_{ij}e = ex_{ij}$ for $i = 1, 2, \dots, k$ and $j = 1, 2, \dots, n_i$). Consider the universal algebra $OA(\Delta_x, \mathcal{R}_x)$.

Theorem 3.15. *The universal algebra* $OA(\Delta_x, \mathcal{R}_x)$ *is completely isometrically isomorphic to* $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$.

Proof. According to (3.1), for any polynomials $p_{rs}(e, \{x_{ij}\}), 1 \leq r, s \leq m$, we have

$$\|[p_{rs}(e, \{x_{ij}\})]\|_{\Delta_x} = \sup\{\|[p_{rs}(I_{\mathcal{H}}, \{T_{ij}\})]\|\},$$

where the sumpremum is taken for all $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset \mathcal{B}(\mathcal{H})$ such that $(T_{i1},\dots,T_{in_i})$ $(i=1,2,\dots,k)$ is contractive. According to Theorem 2.1, we infer that

$$\|[p_{rs}(e,\{x_{ij}\})]\|_{\Delta_x} = \sup\{\|[p_{rs}(I,\{V_{ij}\})]\|\},$$

where the supremum is taken for all contractive sequences of isometries $(V_{i1},\dots,V_{in_i})$ $(i=1,2,\dots,k)$ acting on a Hilbert space. Using the relation (3.3) we deduce that

$$\|[p_{rs}(e,\{x_{ij}\})]\|_{\Delta_x} = \|[p_{rs}(e,\{x_{ij}\})]\|_{\Delta_{xy}},$$

which completes the proof. □

Since $\check{*}_{\mathbf{C}}\mathcal{A}_{n_i}$ and $\check{*}_{\mathbf{C}}\mathcal{O}_{n_i}$ are universal algebras of type $OA(\Lambda,\mathcal{R})$, one can obtain factorizations of type (3.5) in a similar manner.

On the other hand, let us remark that all the von Neumann inequalities presented in this section can be easily extended to matrices.

4. Characters on free product disc algebras and cohomology.

Let $\lambda = \{\lambda_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ be a sequence of complex numbers such that

$$|\lambda_{i1}|^2 + \cdots + |\lambda_{in_i}|^2 \le 1 \quad \text{for each } i=1,2,\dots,k,$$

and define the "evaluation" functional

$$\Phi_\lambda : \mathcal{P}_x \to \mathbf{C}; \quad \Phi_\lambda(p(e,\{x_{ij}\})) = p(1,\{\lambda_{ij}\}),$$

where $\mathcal{P}_x$ is the set of all polynomials $p(e,\{x_{ij}\}) \in OA_+(\Delta_{xy},\mathcal{R}_{xy})$. According to Theorem 3.7, we have

$$|p(1,\{\lambda_{ij}\})| \le \|p(e,\{x_{ij}\})\|_{OA_+(\Delta_{xy},\mathcal{R}_{xy})}.$$

Hence, Φ_λ has a unique extension to $OA_+(\Delta_{xy},\mathcal{R}_{xy})$ Therefore Φ_λ is a character on $OA_+(\Delta_{xy},\mathcal{R}_{xy})$. Let $M_{OA_+(\Delta_{xy},\mathcal{R}_{xy})}$ be the set of all characters of $OA_+(\Delta_{xy},\mathcal{R}_{xy})$ and let

$$\Psi : \overline{(\mathbf{C}^{n_1})}_1 \times \overline{(\mathbf{C}^{n_2})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1 \to M_{OA_+(\Delta_{xy},\mathcal{R}_{xy})}$$

be defined by $\Psi(\lambda) = \Phi_\lambda$, where $\lambda = \{\lambda_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$.

Theorem 4.1. *The map Ψ is a homeomorphism of $\overline{(\mathbf{C}^{n_1})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1$ onto $M_{OA_+(\Delta_{xy},\mathcal{R}_{xy})}$.*

Proof. Let us show that Ψ is one-to-one. If $\lambda = \{\lambda_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ and $\mu = \{\mu_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ are in $E_{n_1,\dots,n_k} := \overline{(\mathbf{C}^{n_1})}_1 \times \cdots \times \overline{(\mathbf{C}^{n_k})}_1$, then $\Psi(\lambda)=\Psi(\mu)$ implies

$$\lambda_{ij} = \Phi_\lambda(x_{ij}) = \Phi_\mu(x_{ij}) = \mu_{ij}$$

for any $i = 1, 2, \dots .k,\ j = 1, 2, \dots, n_i$. Therefore $\lambda = \mu$. Now, assume that $\Phi : OA_+(\Delta_{xy}, \mathcal{R}_{xy}) \to \mathbf{C}$ is a character. Setting $\Phi(x_{ij}) = \lambda_{ij} \in \mathbf{C}$ we have

$$\Phi(p(\{x_{ij}\})) = p(\{\lambda_{ij}\}),$$

for any $p(\{x_{ij}\}) \in OA_+(\Delta_{xy}, \mathcal{R}_{xy})$. Since Φ is a character it follows that it is completely contractive. Applying Theorem 3.7 when $A_{ij} = \lambda_{ij} I_{\mathbf{C}},\ i = 1, 2, \dots, k,\ j = 1, 2, \dots, n_i$, we infer that $\{\lambda_{ij}\} \in E_{n_1,\dots,n_k}$.

On the other hand, the identity

$$\Phi(p(\{x_{ij}\})) = p(\lambda_{ij}) = \Phi_\lambda(p(\{x_{ij}\}))$$

proves that $\Phi = \Phi_\lambda$ on the subset $\mathcal{P}_x$ which is dense in $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$. Hence $\Phi = \Phi_\lambda$. Since both $E_{n_1,\dots,n_k}$ and $M_{OA_+(\Delta_{xy}, \mathcal{R}_{xy})}$ are compact Hausdorff spaces and Ψ is one-to-one and onto, to complete the proof it suffices to show that Φ is continuous.

Suppose that $\lambda^\alpha = \{\lambda_{ij}^\alpha\} (\alpha \in J)$ is a net in $E_{n_1,\dots,n_k}$ such that $\lim_{\alpha\in J} \lambda^\alpha = \lambda = \{\lambda_{ij}\}$. Since $\sup_{\alpha\in J} \|\Phi_{\lambda^\alpha}\| \le 1$ and $\mathcal{P}_x$ is dense in $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ and since

$$\lim_{\alpha\in J} \Phi_{\lambda^\alpha}(p(\{x_{ij}\})) = \lim_{\alpha\in J} p(\{\lambda_{ij}\}) = \Phi_\lambda(p(\{x_{ij}\}))$$

for every $p(\{x_{ij}\}) \in \mathcal{P}_x$, it follows that Ψ is continuous. The proof is complete. □

Let us remark that in the particular case when $k = 1,\ n_1 = n$ we get $M_{\mathcal{A}_n} = (\overline{\mathbf{C}^n})_1$ ($\mathcal{A}_n$ is the noncommutative disc algebra [**Po2**]), result that was obtained in [**Po4**].

Let A be a complex Banach algebra with unit, X be a Banach A-bimodule, and X' be the dual Banach A-bimodule (see [**BD**]). We need to recall from [**BD**] a few definitions.

A bounded X-derivation is a bounded linear mapping D of A into X such that

$$D(ab) = (Da)b + a(Db), \quad \text{for any } a, b \in A.$$

The set of all bounded X-derivations is denoted by $Z^1(A, X)$. For each $x \in X$ let us define $\delta_x : A \to X$ by $\delta_x(a) = ax - xa$. We call δ_x an inner X-derivation, and denote by $B^1(A, X)$ the set of all inner X-derivations. The quotient space $Z^1(A, X)/B^1(A, X)$ is called the first cohomology group of A with coefficients in X, and it is denoted by $H^1(A, X)$. A Banach algebra A is said to be amenable if $H^1(A, X') = \{0\}$ for every Banach A-bimodule X.

It is clear that $\mathbf{C}$, the set of all complex numbers, is a Banach $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$-bimodule under the module multiplication

$$\lambda \cdot f(\{x_{ij}\}) = f(\{x_{ij}\}) \cdot \lambda = \lambda f(\{0\})$$

for each $f(\{x_{ij}\}) \in OA_+(\Delta_{xy}, \mathcal{R}_{xy})$. According to the von Neumann inequality (3.5), we infer that $|\lambda \cdot f(\{x_{ij}\})| \le |\lambda| \|f(\{x_{ij}\})\|$, for any $\lambda \in \mathbf{C}$ and $f(\{x_{ij}\}) \in OA_+(\Delta_{xy}, \mathcal{R}_{xy})$.

Since the proof of the following theorem is a straightforward extension of [**Po4**, Theorem 4.1], we will omit it.

Theorem 4.2. *The first cohomology group of $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ with complex coefficients is isomorphic to the additive group $\mathbf{C}^{n_1+n_2+\cdots+n_k}$.*

Since $\mathbf{C}$ is a dual bimodule we infer the following.

Corollary 4.3. *The free product operator algebra $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ is not amenable.*

5. Positive definite operator-valued kernels on free semigroups.

Let $\mathbf{F}_n^+$ be the unital semigroup on n generators. A positive definite kernel on $\mathbf{F}_n^+$ is a map

$$K : \mathbf{F}_n^+ \times \mathbf{F}_n^+ \to B(\mathcal{H})$$

with the property that $K(\sigma, \omega) = K(\omega, \sigma)^*$, $(\sigma, \omega \in \mathbf{F}_n^+)$ and

$$\sum_{i,j=1}^{k} \langle K(\sigma_i, \sigma_j) h_j, h_i \rangle \ge 0$$

for any $k \in \mathbf{N}$, for any $h_1, \dots, h_k \in \mathcal{H}$, and $\sigma_1, \dots, \sigma_k \in \mathbf{F}_n^+$. A kernel K on $\mathbf{F}_n^+$ is called Toeplitz if $K(e, e) = I_{\mathcal{H}}$ and

$$K(\alpha\sigma, \alpha\omega) = K(\sigma, \omega) \quad \text{for any } \alpha, \sigma, \omega \in \mathbf{F}_n^+$$

(see [**Po5**] for a particular case). We say that K has a Naimark dilation if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\{V_\sigma\}_{\sigma \in \mathbf{F}_n^+}$ a semigroup of isometries on $\mathcal{K}$, i.e., $V_\sigma V_\omega = V_{\sigma\omega}$ $(\sigma, \omega \in \mathbf{F}_n^+), V_e = I_{\mathcal{K}}$, such that

$$K(\sigma, \omega) = P_{\mathcal{H}} V_\sigma^* V_\omega|_{\mathcal{H}} \quad \text{for any } \sigma, \omega \in \mathbf{F}_n^+,$$

where $P_{\mathcal{H}}$ is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. The Naimark dilation is called minimal if $\mathcal{K} = \bigvee_{\sigma \in \mathbf{F}_n^+} V_\sigma \mathcal{H}$.

The following result is an extension of the Naimark dilation [**N**], [**SzF2**] to free semigroups. The proof is similar to that of Theorem 2.1 from [**Po5**], so we will omit it. However, let us point out that in [**Po5**] we considered just a particular Toeplitz kernel. Here, we have a more general setting.

Theorem 5.1. *A Toeplitz kernel on $\mathbf{F}_n^+$ is positive definite if and only if it admits a minimal Naimark dilation. Moreover, the minimal Naimark dilation is unique up to an isomorphism.*

Let $\mathbf{F}_{n_i}^+$ $(i = 1, 2, \dots, k)$ be the unital free semigroup on n_i generators: $s_{i1}, s_{i2}, ..., s_{in_i}$, and let e be the neutral element. Then $\mathbf{F}_n^+ := \mathbf{F}_{n_1}^+ * \cdots * \mathbf{F}_{n_k}^+$ is the unital free semigroup on $n = n_1 + \cdots + n_k$ generators. If $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$, then for each $\sigma = s_{i_1 j_1} \cdots s_{i_p j_p} \in \mathbf{F}_n^+$ denote $T_\sigma := T_{i_1 j_1} \cdots T_{i_p j_p}$, and $T_\sigma := I_{\mathcal{H}}$ if $\sigma = e$.

For any $\sigma, \omega \in \mathbf{F}_n^+$, let us denote by $\operatorname{gld}(\sigma, \omega)$ the greatest left common divisor of them. Therefore,

$$\sigma = \operatorname{gld}(\sigma,\omega)\alpha \quad \text{and} \quad \omega = \operatorname{gld}(\sigma,\omega)\beta \quad \text{for some} \quad \alpha, \beta \in \mathbf{F}_n^+, \tag{5.1}$$

and $\operatorname{gld}(\alpha, \beta) = e$. Notice that to each pair $(\sigma, \omega) \in \mathbf{F}_n^+ \times \mathbf{F}_n^+$ corresponds a unique pair $(\alpha, \beta) \in \mathbf{F}_n^+ \times \mathbf{F}_n^+$ with the above mentioned properties.

Let us define the kernel $K_c : \mathbf{F}_n^+ \times \mathbf{F}_n^+ \to B(\mathcal{H})$ by $K_c(\sigma, \omega) = 0$ if $\alpha \neq e, \beta \neq e$, and both words α, β start with some generators of the same semigroup $\mathbf{F}_{n_i}^+$, for some $i = 1, 2, \dots, k$, and $K_c(\sigma, \omega) = T_\alpha^* T_\beta$ otherwise. It is clear that K_c is a Toeplitz kernel. Notice also that if $j_1 \neq j_2$, then

$$K_c(s_{ij_1}\sigma, s_{ij_2}\omega) = 0 \tag{5.2}$$

for any $\sigma, \omega \in \mathbf{F}_n^+$.

Theorem 5.2. *Let $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \subset B(\mathcal{H})$. Then $(T_{i1}, \dots, T_{in_i})$ is a contractive sequence of operators for each $i = 1, 2, \dots, k$ if and only if the Toeplitz kernel $\mathcal{K}_c$ is positive definite.*

Proof. Suppose that for each $i = 1, 2, \dots, k$ the sequence $(T_{i1}, \dots, T_{in_i})$ is contractive. According to Theorem 2.1, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and contractive sequences $(V_{i1}, \dots, V_{in_i})$ $(i = 1, 2, \dots, k)$ of isometries on $\mathcal{K}$ such that $V_{ij}^*|_{\mathcal{H}} = T_{ij}^*$ $(i = 1, \dots, k;\ j = 1, \dots, n_i)$ and $P_{\mathcal{K}\ominus\mathcal{H}} V_{i_1 j_1}\mathcal{K} \perp P_{\mathcal{K}\ominus\mathcal{H}} V_{i_2 j_2}\mathcal{K}$ if $i_1 \neq i_2$, $j_1 = 1, 2, \dots, n_{i_1}$, and $j_2 = 1, 2, \dots, n_{i_2}$. According to the definition of the Toeplitz kernel K_c, for any finitely supported sequence $\{h_\omega\}_{\omega \in \mathbf{F}_n^+} \subset \mathcal{H}$ we have

$$\begin{aligned}
\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle K_c(\sigma,\omega) h_\omega, h_\sigma \rangle &= \sideset{}{^*}\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle T_\alpha^* T_\beta h_\omega, h_\sigma \rangle \\
&= \sideset{}{^*}\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle P_{\mathcal{H}} V_\beta h_\omega, P_{\mathcal{H}} V_\alpha h_\sigma \rangle \\
&= \sideset{}{^*}\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle (P_{\mathcal{H}} + P_{\mathcal{K}\ominus\mathcal{H}}) V_\beta h_\omega, (P_{\mathcal{H}} + P_{\mathcal{K}\ominus\mathcal{H}}) V_\alpha h_\sigma \rangle \\
&= \sideset{}{^*}\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle V_\beta h_\omega, V_\alpha h_\sigma \rangle = \sideset{}{^*}\sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle V_\alpha^* V_\beta h_\omega, h_\sigma \rangle \\
&= \sum_{\sigma,\omega \in \mathbf{F}_n^+} \langle V_\sigma^* V_\omega h_\omega, h_\sigma \rangle = \left\| \sum_{\sigma \in \mathbf{F}_n^+} V_\sigma h_\sigma \right\|^2 \geq 0,
\end{aligned}$$

where $\sum^*$ is taken over all $\sigma, \omega \in \mathbf{F}_n^+$ such that $K_c(\sigma,\omega) = T_\alpha^* T_\beta$ (see the definition of K_c). This proves that the Toeplitz kernel K_c is positive definite.

Conversely, assume that K_c is positive definite. According to Theorem 5.1, there exists a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and $\{V_\sigma\}_{\sigma\in\mathbf{F}_n^+}$ a semigroup of isometries on $\mathcal{K}$ such that

$$K_c(\sigma,\omega) = P_{\mathcal{H}} V_\sigma^* V_\omega|_{\mathcal{H}} \quad \text{for any } \sigma,\omega \in \mathbf{F}_n^+.$$

Since the relation (5.2) holds, we infer that for each $i = 1, \ldots, k$, the sequence of isometries $(V_{i1}, \ldots, V_{in_i})$ is contractive (see [Po5]). Since

$$T_{s_{ij}} = K_c(e, s_{ij}) = P_{\mathcal{H}} V_{s_{ij}}|_{\mathcal{H}}$$

for each $i = 1, \ldots, k$, we have

$$\sum_{j=1}^{n_i} \|T_{ij}^* h\|^2 \le \sum_{j=1}^{n_i} \|V_{ij}^* h\|^2 \le \|h\|^2 \quad \text{for any } h \in \mathcal{H}.$$

This shows that $(T_{i1}, \ldots, T_{in_i})$ is a contractive sequence of operators for each $i = 1, 2, \ldots, k$. The proof is complete. □

Let us remark that in the particular case when $k = 1$ and $n_1 = n$ we find again Corollary 2.2 from [**Po5**]. In the particular case when $n_1 = \cdots = n_k = 1$ we obtain the following.

Corollary 5.3. *Let $\{T_1, \ldots, T_k\} \subset B(\mathcal{H})$. Then $\{T_1, \ldots, T_k\}$ is a sequence of contractions if and only if the Toeplitz kernel*

$$K : \mathbf{F}_k^+ \times \mathbf{F}_k^+ \to B(\mathcal{H})$$

defined by $K(\sigma,\omega) = K(\alpha,\beta) = T_\alpha^* T_\beta$, where $\sigma = \mathrm{gld}(\sigma,\omega)\alpha$ and $\omega = \mathrm{gld}(\sigma,\omega)\beta$, is positive definite.

Let $\mathcal{C}_\rho$ ($\rho > 0$) denote the set of all sequences $\{A_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ of operators on a Hilbert space $\mathcal{H}$ for which there exists a sequence of isometries $\{V_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that

$$\sum_{i=1}^{n_i} V_{ij} V_{ij}^* \le I_{\mathcal{H}}$$

for each $i = 1, 2, \ldots, k$, and

$$A_{i_1 j_1} \cdots A_{i_m j_m} = \rho P_{\mathcal{H}} V_{i_1 j_1} \cdots V_{i_m j_m}|_{\mathcal{H}},$$

for any $i_q \in \{1, 2, \ldots, k\}$, $j_q \in \{1, 2, \ldots, n_{i_q}\}$, $q \in \{1, 2, \ldots, m\}$ and $m \ge 1$. Let $\mathcal{K}_\rho : \mathbf{F}_n^+ \times \mathbf{F}_n^+ \to B(\mathcal{H})$ be the Toeplitz kernel defined by $K_\rho(e,e) = I_{\mathcal{H}}$ and

$$K_\rho(\sigma,\omega) = \frac{1}{\rho} K_c(\sigma,\omega)$$

if $\sigma \in \mathbf{F}_n^+\backslash\{e\}$ or $\omega \in \mathbf{F}_n^+\backslash\{e\}$, where K_c is the Toeplitz kernel associated to $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ (see the definition following Theorem 5.1).

Applying Theorem 5.2, we infer the following.

Theorem 5.4. $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in \mathcal{C}_\rho$ *if and only if the Toeplitz kernel* $\mathcal{K}_\rho$ *is positive definite.*

One can prove that the class $\mathcal{C}_\rho$ $(0 < \rho < \infty)$ increases with ρ, i.e., $\mathcal{C}_\rho \subset \mathcal{C}_{\rho'}$ and $\mathcal{C}_\rho \neq \mathcal{C}_\rho'$ for $0 < \rho < \rho' < \infty$ (see [**Po5**] for a particular case).

The von Neumann inequality (3.7) can be extended, in an apropriate form, to the class $\mathcal{C}_\rho$.

Theorem 5.5. *If* $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in \mathcal{C}_\rho$ $(\rho > 0)$, *then for any polynomial* $p(e, \{x_{ij}\}) \in OA_+(\Delta_{xy}, \mathcal{R}_{xy})$,

$$\|p(I, \{A_{ij}\})\| \leq \|(1-\rho)p(e,\{0\}) + \rho p(e, \{x_{ij}\})\|_{OA_+(\Delta_{xy},\mathcal{R}_{xy})}. \tag{5.3}$$

Corollary 5.6. *Let* $p(\{x_{ij}\}) \in OA_+(\Delta_{xy}, \mathcal{R}_{xy})$ *be such that* $p(\{0\}) = 0$ *and*

$$\|p(\{x_{ij}\})\|_{OA_+(\Delta_{xy},\mathcal{R}_{xy})} \leq 1.$$

If $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in \mathcal{C}_\rho$ $(\rho > 0)$ *then* $p(\{A_{ij}\}) \in \mathcal{C}_\rho$ *(in the classical sense).*

A sequence of operators $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ is called simultaneously similar to a sequence $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}}$ if there is an invertible operator X such that $A_{ij} = XT_{ij}X^{-1}$ for any $i = 1,2,\dots,k, j = 1,2,\dots,n_i$.

In what follows we consider an extension of the result of Sz.-Nagy and Foiaş [**SzF1**] and also [**Po5**].

Theorem 5.7. *Any sequence* $\{A_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in \mathcal{C}_\rho$ $(\rho > 0)$ *is simultaneously similar to a sequence* $\{T_{ij}\}_{\substack{i=1,2,\dots,k\\ j=1,2,\dots,n_i}} \in \mathcal{C}_1$.

Proof. The inequality (5.3) can be extended to matrices. One can easily prove that, for any polynomials $p_{rs}(e, \{x_{ij}\})$, $1 \leq r, s \leq m$,

$$\|[p_{rs}(I, \{A_{ij}\})]\| \leq (|1-\rho| + \rho)\|[p_{rs}(e, \{x_{ij}\})]\|_{\Delta_{xy}}.$$

This shows that the map $\Phi : \mathcal{P}_x \to B(\mathcal{H})$ defined by

$$\Phi(p(e, \{x_{ij}\})) = p(I, \{A_{ij}\})$$

can be extended to a completely bounded homomorphism of the free product disc algebra $OA_+(\Delta_{xy}, \mathcal{R}_{xy})$. Now, according to Theorem 3.9, the result follows. □

In the particular case when $k = 1$, $n_1 = 1$ we find again the classical result of Sz.-Nagy and Foiaş [**SzF1**].

6. Trigonometric moment problem for some free product C^*-algebras.

As in the previous section, let $\mathbf{F}_{n_i}^+ \quad (i = 1, 2, \dots, k)$ be the unital free semigroup on n_i generators: $g_{i1}, g_{i2}, \dots, g_{in_i}$, and let e be the neutral element. Then $\Lambda := \mathbf{F}_{n_1}^+ * \cdots * \mathbf{F}_{n_k}^+$ is the unital free semigroup on $n = n_1 + \cdots + n_k$ generators.

For each $i = 1, 2, \dots, k$, let $\{x_{ij}\}_{\substack{i=1,2,\dots,k \\ j=1,2,\dots,n_i}}$ and $\{y_{ij}\}_{\substack{i=1,2,\dots,k \\ j=1,2,\dots,n_i}}$ be ordinary variables satisfying the relation

$$y_{i\alpha} x_{i\beta} = \delta_{\alpha,\beta} e \tag{6.1}$$

for any $i = 1, 2, \dots, k$, and $\alpha, \beta = 1, 2, \dots, n_i$. For each $\sigma = g_{i_1 j_1} \cdots g_{i_p j_p} \in \Lambda$ let $\tilde{\sigma} := g_{i_p j_p} \cdots g_{i_1 j_1}$, $x_\sigma = x_{i_1 j_1} \cdots x_{i_p j_p}$, and $y_\sigma = y_{i_1 j_1} \cdots y_{i_p j_p}$. If $\sigma = e$ (the neutral element in Λ) then we set $x_e = y_e := e$ (the neutral element in Δ_{xy} (see Section 3)).

If π is a representation of the universal algebra $OA(\Lambda_{xy}, \mathcal{R}_{xy})$ on $B(\mathcal{K})$ then, according to Theorem 3.1, it is determined by contractive sequences of isometries $(S_{i1}, \dots, S_{in_i}) \quad (i = 1, 2, \dots, k)$ on the same Hilbert space $\mathcal{K}$ such that $\pi(x_{ij}) = S_{ij}$, $\pi(y_{ij}) = S_{ij}^*$ and $\pi(e) = I_{\mathcal{K}}$. Notice that for each $\sigma \in \Lambda$ we have $\pi(x_\sigma) = S_\sigma$, $\pi(y_{\tilde{\sigma}}) = S_\sigma^* = \pi(y_\sigma)^*$. According to the relation (6.1), $y_{\tilde{\sigma}} x_\omega \quad (\sigma, \omega \in \Lambda \backslash \{e\})$ is a reduced word if and only if there exist $i_1, i_2 \in \{1, 2, \dots, k\}, i_1 \neq i_2$, such that ω (resp. σ) starts, in its unique representation, with a generator of $\mathbf{F}_{n_{i_1}}^+$ (resp. $\mathbf{F}_{n_{i_2}}^+$).

Define the following subsets of $\Lambda \times \Lambda$:

$$\begin{aligned}
\Gamma_1 &= \{(e, \sigma) : \sigma \in \Lambda\}; \\
\Gamma_2 &= \{(\omega, e) : \omega \in \Lambda\}; \\
\Gamma_3 &= \{(\omega, \sigma) : \omega, \sigma \in \Lambda \backslash \{e\} \text{ and } y_{\tilde{\sigma}} x_\omega \text{ is a reduced word}\}; \\
\Gamma &= \Gamma_1 \cup \Gamma_2 \cup \Gamma_3.
\end{aligned}$$

Notice that if $(\omega, \sigma) \in \Gamma$ then $(\sigma, \omega) \in \Gamma$. On the other hand, if $k = 1$, then $\Gamma = \Gamma_1 \cup \Gamma_2$. Let $\{A_{(\sigma,\omega)}\}_{(\sigma,\omega) \in \Gamma}$ be a sequence of operators in $B(\mathcal{H})$ such that $A_{(\sigma,\omega)} = A_{(\omega,\sigma)}^*$ for any $(\sigma, \omega) \in \Gamma$, and $A_{(e,e)} = I_{\mathcal{H}}$.

For any $\sigma, \omega \in \Lambda$ let us denote by $\operatorname{gld}(\sigma, \omega)$ the greatest left common divisor of them. Therefore,

$$\sigma = \operatorname{gld}(\sigma, \omega) \alpha \quad \text{and} \quad \omega = \operatorname{gld}(\sigma, \omega) \beta$$

for some $\alpha, \beta \in \Lambda$ with $\operatorname{gld}(\alpha, \beta) = e$. We associate to the sequence of operators $\{A_{(\sigma,\omega)}\}_{(\sigma,\omega) \in \Gamma}$ the kernel $K_A : \Lambda \times \Lambda \to B(\mathcal{H})$ defined by

$$K_A(\sigma, \omega) := \begin{cases} A_{(\alpha,\beta)} & \text{if } (\alpha, \beta) \in \Gamma \\ 0 & \text{if } (\alpha, \beta) \notin \Gamma. \end{cases}$$

It is easy to see that $K_A(e,e) = I_{\mathcal{H}}$ and $K_A(\alpha\sigma, \alpha\omega) = K_A(\sigma, \omega)$ for any $\alpha, \sigma, \omega \in \Lambda$, i.e., K_A is a Toeplitz kernel. Notice that if $i = 1, 2, \ldots, k$; $j_1, j_2 \in \{1, 2, \ldots, n_i\}$ with $j_1 \neq j_2$ then

$$K_A(g_{ij_1}\sigma, g_{ij_2}\omega) = 0 \tag{6.2}$$

for any $\sigma, \omega \in \Lambda$. Define the operator matrix

$$M_m = [K_A(\sigma, \omega)]_{|\sigma| \leq m, |\omega| \leq m}, \tag{6.3}$$

where $|\sigma|$ stands for the length of $\sigma \in \mathbf{F}_n^+$. Denote

$$\Gamma_m := \{(\alpha, \beta) \in \Gamma : |\alpha| \leq m, |\beta| \leq m\}, \ m = 1, 2, \ldots.$$

In what follows we extend the operatorial trigonometric moment problem [**Ak**] (see also [**Po5**]) to the free product C^*-algebra $\check{*}_{\mathbf{C}}\mathcal{T}_{n_i}$.

Theorem 6.1. *Let $\{A_{(\alpha,\beta)}\}_{(\alpha,\beta)\in\Gamma_m}$ be a sequence of operators in $B(\mathcal{H})$ such that $A_{(e,e)} = I_{\mathcal{H}}$ and $A_{(\alpha,\beta)} = A^*_{(\beta,\alpha)}$ for any $(\alpha, \beta) \in \Gamma_m$. If*

$$\mu : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$$

is a completely positive linear map such that

$$\mu(y_{\tilde{\alpha}} x_\beta) = A_{(\alpha,\beta)}$$

for any $(\alpha, \beta) \in \Gamma_m$, then M_m is positive.

Conversely, if M_m is positive then there is a completely positive linear map $\mu : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$ such that $\mu(y_{\tilde{\alpha}} x_\beta) = A_{(\alpha,\beta)}$ for any $(\alpha, \beta) \in \Gamma_{m-1}$.

Proof. Assume that $\mu : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$ is a completely positive linear map such that

$$\mu(y_{\tilde{\alpha}} x_\beta) = A_{(\alpha,\beta)} \quad \text{for any } (\alpha, \beta) \in \Gamma_m. \tag{6.4}$$

According to Stinespring's theorem [**S**], there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a representation $\pi : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$ such that

$$\mu(f) = P_{\mathcal{H}} \pi(f)|_{\mathcal{H}}, \quad f \in OA(\Lambda_{xy}, \mathcal{R}_{xy}). \tag{6.5}$$

Let $K : \Lambda \times \Lambda \to B(\mathcal{H})$ be the kernel defined by $K(\sigma, \omega) = P_{\mathcal{H}} \pi(y_{\tilde{\sigma}} x_\omega)|_{\mathcal{H}}$ for any $\sigma, \omega \in \Lambda$. It is easy to see that $K(e,e) = I_{\mathcal{H}}$, $K(\omega, \sigma) = K(\sigma, \omega)^*$ and $K(\alpha\sigma, \alpha\omega) = K(\sigma, \omega)$ for any $\alpha, \sigma, \omega \in \Lambda$. Since for any finitely supported sequence $\{h_\omega\}_{\omega\in\Lambda} \subset \mathcal{H}$,

$$\begin{aligned}
\sum_{\sigma,\omega\in\Lambda} \langle K(\sigma, \omega) h_\omega, h_\sigma \rangle &= \sum_{\sigma,\omega\in\Lambda} \langle P_{\mathcal{H}} \pi(y_{\tilde{\sigma}} x_\omega) h_\omega, h_\sigma \rangle \\
&= \sum_{\sigma,\omega\in\Lambda} \langle \pi(x_\omega) h_\omega, \pi(y_\sigma) h_\sigma \rangle = \sum_{\omega\in\Lambda} \|\pi(x_\omega) h_\omega\|^2 \geq 0,
\end{aligned}$$

we infer that K is a positive definite Toeplitz kernel.

In particular, the matrix $[K(\sigma,\omega)]_{|\sigma|\le m,|\omega|\le m}$ is positive. According to (6.4) and (6.5), it is a routine to show that $K(\sigma,\omega)=K_A(\sigma,\omega)$ for any $\sigma,\omega\in\Gamma$ with $|\sigma|\le m, |\omega|\le m$. Therefore, the matrix M_m is positive.

Conversely, assume that the matrix M_m is positive. Let $\mathcal{K}_m^0$ be the Hilbert space of all sequences of the form $\{h_\sigma\}_{\substack{\sigma\in\Lambda\\|\sigma|\le m}}$ $(h_\sigma\in\mathcal{H})$ with the bilinear form $\langle\cdot,\cdot\rangle$ on $\mathcal{K}_m^0$ defined by

$$\langle\{h_\omega\}_{|\omega|\le m},\{h'_\sigma\}_{|\sigma|\le m}\rangle=\sum_{\substack{\omega,\sigma\in\Lambda\\|\omega|,|\sigma|\le m}}\langle K_A(\sigma,\omega)h_\omega,h'_\sigma\rangle_{\mathcal{H}}.$$

Since M_m is positive $\langle\cdot,\cdot\rangle$ is positive semi-definite. Consider

$$\mathcal{N}_m=\{k\in\mathcal{K}_m^0:\langle k,k\rangle=0\}$$

and $\mathcal{K}_m^0/\mathcal{N}_m$. Let $\mathcal{K}_m$ be the Hilbert space obtained by completing $\mathcal{K}_m^0/\mathcal{N}_m$ with the induced inner product.

Let $\mathcal{X}^0$ be the subspace of $\mathcal{K}_m^0$ defined by

$$\mathcal{X}^0=\{\{h_\sigma\}\in\mathcal{K}_m^0:\ h_\sigma=0 \text{ for } |\sigma|=m\}$$

and let $\mathcal{X}=\mathcal{X}^0/_{\mathcal{N}_m}\subset\mathcal{K}_m$. For each generator g_{ij} $(i=1,2,\ldots,k; j=1,2,\ldots,n_i)$ of Λ let $V_{ij}:\mathcal{X}\to\mathcal{K}_m$ be defined by

$$V_{ij}\left(\{h_\sigma\}\right)=\{\delta_{g_{ij}\sigma}(t)h_\sigma\}_{|t|\le m}. \tag{6.6}$$

Define also $T_{ij}:\mathcal{X}\to\mathcal{X}$ by $T_{ij}^*=V_{ij}^*|_{\mathcal{X}}$. Embed $\mathcal{H}$ in $\mathcal{K}_m$ by setting $h=\{\delta_e(t)h\}_{|t|\le m}$. This identification is allowed since it preserves the linear and metric structure of $\mathcal{H}$.

For any $(\sigma,\omega)\in\Gamma_{m-1}$ and $h,h'\in\mathcal{H}$ we have

$$\begin{aligned}\langle P_{\mathcal{H}}T_\sigma^*T_\omega h,h'\rangle_{\mathcal{H}}&=\langle T_\omega h,T_\sigma h'\rangle_{\mathcal{K}_m}\\&=\langle P_{\mathcal{X}}(\{\delta_\omega(t)h\}_{|t|\le m}),P_{\mathcal{X}}(\{\delta_\sigma(s)h'\}_{|s|\le m})\rangle_{\mathcal{K}_m}\\&=\langle\{\delta_\omega(t)h\}_{|t|\le m},\{\delta_\sigma(s)h'\}_{|s|\le m}\rangle_{\mathcal{K}_m}\\&=\sum_{\substack{s,t\in\Lambda\\|s|,|t|\le m}}\langle K_A(s,t)\delta_\omega(t)h,\delta_\sigma(s)h'\rangle_{\mathcal{H}}\\&=\langle K_A(\sigma,\omega)h,h'\rangle_{\mathcal{H}}.\end{aligned}$$

Therefore, $K_A(\sigma,\omega)=P_{\mathcal{H}}T_\sigma^*T_\omega|_{\mathcal{H}}$ for any $(\sigma,\omega)\in\Gamma_{m-1}$. Let us remark that this relation holds, in fact, for any $(\sigma,\omega)\in\Gamma_m$ such that either $|\sigma|\le m-1$ or $|\omega|\le m-1$.

Let us show that for each $i=1,2,\ldots,k$,

$$T_{i1}T_{i1}^*+\cdots+T_{in_i}T_{in_i}^*\le I_{\mathcal{X}}. \tag{6.7}$$

Since $\sum_{j=1}^{n_i}T_{ij}T_{ij}^*=P_{\mathcal{X}}\sum_{j=1}^{n_i}V_{ij}V_{ij}^*|_{\mathcal{X}}$, it is enough to prove that $(V_{i1},\ldots,V_{in_i})$ is a sequence of isometries with orthogonal ranges. According to (6.6) and

(6.2), for $j_1 \neq j_2$; $j_1, j_2 \in \{1, 2, \ldots, n_i\}$, we have

$$\begin{aligned}&\langle V_{ij_1}\{h_\omega\}, V_{ij_2}\{h'_\sigma\}\rangle_{\mathcal{K}_m}\\&= \sum_{|\sigma|,|\omega|\leq m-1} \langle K_A(g_{ij_2}\sigma, g_{ij_1}\omega)h_\omega,\ h'_\sigma\rangle = 0.\end{aligned}$$

Since K_A is a Toeplitz kernel one can similarly prove that $V_{ij}(j = 1, 2, \ldots, n_i)$ are isometries. Therefore $\sum_{j=1}^{n_i} V_{ij}V_{ij}^* \leq I_{\mathcal{K}_m}$. Hence, and using the definition of T_{ij} we infer the relation (6.7).

Let $\{W_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ be the minimal isometric dilation of $\{T_{ij}\}_{\substack{i=1,2,\ldots,k\\ j=1,2,\ldots,n_i}}$ on a Hilbert space $\mathcal{K} \supset \mathcal{X}$ (see Theorem 2.1). Since

$$P_{\mathcal{K}\ominus\mathcal{X}}W_{i_1j_1}\mathcal{K} \perp P_{\mathcal{K}\ominus\mathcal{X}}W_{i_2j_2}$$

if $i_1 \neq i_2$, $j_1 = 1, 2, \ldots, n_{i_1}$ and $j_2 = 1, 2, \ldots, n_{i_2}$, and $W_{ij_1}\mathcal{K} \perp W_{ij_2}\mathcal{K}$ for any $i = 1, 2, \ldots, k$ and $j_1, j_2 \in \{1, 2, \ldots, n_i\}$ such that $j_1 \neq j_2$, a simple computation shows that $T_\sigma^*T_\omega = P_\mathcal{X}W_\sigma^*W_\omega|_\mathcal{X}$, for any $(\sigma, \omega) \in \Gamma_m$. Therefore, for any $(\sigma, \omega) \in \Gamma_{m-1}$ we have

$$K_A(\sigma, \omega) = P_\mathcal{H}T_\sigma^*T_\omega|_\mathcal{H} = P_\mathcal{H}W_\sigma^*W_\omega|_\mathcal{H}. \tag{6.8}$$

Define $\mu : OA(\Lambda_{xy}, R_{xy}) \to B(\mathcal{H})$ by

$$\mu(f) = P_\mathcal{H}\pi(f)|_\mathcal{H} \tag{6.9}$$

where $\pi : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{K})$ is a representation determined by $\pi(e) = I_\mathcal{K}$, $\pi(x_{ij}) = W_{ij}$ and $\pi(y_{ij}) = W_{ij}^*$. Thus, μ is a completely positive linear map. On the other hand, using the relations (6.8) and (6.9), we infer that

$$\mu(y_{\tilde{\alpha}}x_\beta) = P_\mathcal{H}W_\alpha^*W_\beta|_\mathcal{H} = K_A(\alpha, \beta) = A_{(\alpha,\beta)} \tag{6.10}$$

for any $(\alpha, \beta) \in \Gamma_{m-1}$, which completes the proof. □

Notice that if $k = 1$ then the relation (6.10) is true for any $(\alpha, \beta) \in \Gamma_m$. Let us remark that in the particular case when $k = 1$ and $n_1 = n$ we have $\Gamma_3 = \emptyset$, $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Lambda = \mathbf{F}_n^+$, and we find again Theorem 3.1 from [**Po5**].

Corollary 6.2. *Let $\{A_{(\sigma,\omega)}\}_{(\sigma,\omega)\in\Gamma}$ be a sequence of operators in $B(\mathcal{H})$ such that $A_{(\sigma,\omega)} = A_{(\omega.\sigma)}^*$ for any $(\sigma, \omega) \in \Gamma$ and $A_{(e,e)} = I_\mathcal{H}$.*

Then, there is a completely positive linear map $\mu : OA(\Lambda_{xy}, \mathcal{R}_{xy}) \to B(\mathcal{H})$ such that $\mu(y_{\tilde{\alpha}}x_\beta) = A_{(\alpha,\beta)}$ for any $(\alpha, \beta) \in \Gamma$ if and only if the Toeplitz kernel K_A is positive definite.

Let us recall that $OA(\Delta_{x'y'}, \mathcal{R}_{x'y'})$ is completely isometrically isomorphic to the free product C^*-algebra $\check{*}_\mathbf{C}\mathcal{O}_{n_i}$.

Using Theorem 2.5 and Arverson's extension theorem [**Ar**], one can easily adapt the last part of the proof of Theorem 6.1 to obtain the following results.

Corollary 6.3. *If the operator matrix M_m is positive definite, then there is a completely positive map*

$$\psi : OA(\Delta_{x'y'}, \mathcal{R}_{x'y'}) \to B(\mathcal{H})$$

such that $\psi(x_\sigma) = A_{(e,\sigma)}$ for any $\sigma \in \Lambda, |\sigma| \le m$.

Notice that in the particular case when $k = 1$ the converse of the above corollary also holds. Therefore, in the particular case when $k = 1, n_1 = n$ we have $\Lambda = \mathbf{F}_n^+$ and we infer the following trigonometric moment problem for the Cuntz algebra $\mathcal{O}_n$.

Let $v_1, v_2, \ldots, v_n$ be a system of generators of the Cuntz algebra $\mathcal{O}_n$. Let $\mathbf{F}_n^+$ be the unital free semigroup on n generators: $g_1, g_2, \ldots, g_n$. For each $\sigma = g_{i_1} \cdots g_{i_p} \in \mathbf{F}_n^+$ denote $v_\sigma = v_{i_1} \cdots v_{i_p}$ and $v_e = 1$.

Corollary 6.4. *Let $\{B_{(\sigma)}\}_{\sigma \in \mathbf{F}_n^+}$ be a sequence of operators in $B(\mathcal{H})$ with $B_{(e)} = I_{\mathcal{H}}$. Then, there is a completely positive linear map $\mu : \mathcal{O}_n \to B(\mathcal{H})$ such that $\mu(v_\sigma) = B_{(\sigma)}, \sigma \in \mathbf{F}_n^+$, if and only if the Toeplitz kernel $K : \mathbf{F}_n^+ \times \mathbf{F}_n^+ \to B(\mathcal{H})$ defined by $K(e,e) = I_{\mathcal{H}}$ and*

$$K(\sigma, \omega) = \begin{cases} B_{(\tau)}^* & \text{if } \sigma = \omega\tau \text{ for some } \tau \in \mathbf{F}_n^+ \\ B_{(\tau)} & \text{if } \omega = \sigma\tau \text{ for some } \tau \in \mathbf{F}_n^+ \\ 0 & \text{otherwise} \end{cases}$$

is positive definite.

In the particular case when $n = 1$ we find again the classical operatorial trigonometric moment problem [**Ak**].

Corollary 6.5. *Given the operators $A_k \in B(\mathcal{H}), k = 0, 1, \ldots, m$ $(A_0 = I)$, there exists a positive linear map $\mu : C(\mathbf{T}) \to B(\mathcal{H})$ such that $\mu(e^{ikt}) = A_k$, $k = 0, 1, \ldots, m$, if and only if the block matrix*

$$\begin{bmatrix} I_{\mathcal{H}} & A_1 & \cdots & A_m \\ A_1^* & I_{\mathcal{H}} & \cdots & A_{m-1} \\ \vdots & \vdots & & \vdots \\ A_m^* & A_{m-1}^* & \cdots & I_{\mathcal{H}} \end{bmatrix}$$

built up on the given operators $\{A_k\}_{k=1}^m$ is positive.

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