WEAK PALEY–WIENER PROPERTY FOR COMPLETELY SOLVABLE LIE GROUPS

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We prove a weak Paley–Wiener property for completely solvable Lie groups, i.e. if the group Fourier transform of a measurable, bounded and compactly supported function vanishes on a set of positive Plancherel measure then the function itself vanishes almost everywhere on the group.

1. Introduction.

Let $G$ be a connected, simply connected, and completely solvable Lie group, with the Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of $\mathfrak{g}$. The equivalence classes of irreducible unitary representations $\hat{G}$ of $G$ is parametrized by the coadjoint orbits $\mathfrak{g}^*/G$ via the Kirillov-Bernat bijective map $K: \hat{G} \rightarrow \mathfrak{g}^*/G$. If $\rho \in \hat{G}$ and $\ell \in K(\rho)$, then there exists an analytic subgroup $H$ of $G$ and a unitary character $\chi$ of $H$, such that $\ell|_{\mathfrak{h}} = Id_{\chi}$, where $\mathfrak{h}$ is the Lie algebra of $H$. The induced representation $\rho = \text{Ind}_H^G \chi$ is irreducible. Moreover, $K$ is a bijection. The image on $\mathfrak{g}^*/G$ of a measure equivalent to Lebesgue measure gives a Plancherel measure on $\hat{G}$.

Let $\phi$ be a bounded, measurable and compactly supported function on $\mathbb{R}^n$. By the classical Paley–Wiener theorem, the Fourier transform $\hat{\phi}$ of $\phi$ extends to an entire function on $\mathbb{C}^n$. Using this we can conclude that if $\hat{\phi}$ vanishes on a set of positive Plancherel measure which is nothing but the Lebesgue measure, then $\hat{\phi}$ vanishes on the whole of $\mathbb{R}^n$. This in turn implies that $\phi = 0$ on $\mathbb{R}^n$.

In the same spirit, for a completely solvable Lie group we will think of the following as a weak Paley–Wiener property:

**Theorem.** Let $G$ be a connected, simply connected, and completely solvable Lie group, with the unitary dual $\hat{G}$. Let $\phi$ be a measurable, bounded, and compactly supported function (i.e $\phi \in L^\infty_c(G)$). Assume that there exists a subset $E \subset \hat{G}$ with positive Plancherel measure such that $\hat{\phi}_\rho = 0$ for all $\rho \in E$ where $\hat{\phi}_\rho$ is the group Fourier transform of $\phi$. Then $\phi = 0$ almost everywhere on $G$.

In [GG1] we proved, the same theorem for nilpotent Lie groups, by induction on the dimension of $G$. To prove the above theorem, also by using
induction on the dimension of $G$, we need a description of the dual space $\hat{G}$ of $G$ and an explicit Plancherel measure on $\hat{G}$. Here, we use the results of B.N. Currey [C], which are generalizations of the results of L. Pukanszky [Pu] on nilpotent Lie groups concerning the Plancherel measure and the Plancherel formula.

2. Preliminaries.

Let $G$ be a connected, simply connected, and completely solvable Lie group, with the Lie algebra $g$. Let $g^*$ be the dual of $g$. We fix a basis $B = \{X_1, \ldots, X_n\}$ of $g$, such that $g_i$ is spanned by the vectors $\{X_1, X_2 \cdots, X_j\}$, $1 \leq j \leq n$ and $g_0 = (0)$. As $G$ is completely solvable, there exists a chain of ideals

\[ 0 = g_0 \subset g_1 \subset \cdots \subset g_{n-1} \subset g_n = g \]

of $g$, such that the dimension of $g_i$ be $i$ for all $1 \leq i \leq n$. Let $B^* = \{X_1^*, \ldots, X_n^*\}$ be the dual basis of $g^*$. We fix a Lebesgue measure $dX$ on $g$, and a right Haar measure $dg$ on $G$ such that $d(\exp X) = j_G(X)dX$, where

\[ j_G(X) = \left| \det \left( \frac{1 - e^{-adX}}{adX} \right) \right|. \]

Let $\Delta$ be the modular function such that for all $g' \in G$, $d(gg') = \Delta(g')dg$. Let $O$ be a coadjoint orbit in $g^*$ and $\ell \in O$. The bilinear form $B_\ell : (X,Y) \rightarrow \ell([X,Y])$ defines a skew-symmetric and nondegenerate bilinear form on $g/\ell$. As the map $X \rightarrow X.\ell$ induces an isomorphism between $g/\ell$ and the tangent space of $O$ at $\ell$, the bilinear form $B_\ell$ defines a nondegenerate 2-form $\omega_\ell$ on this tangent space. If $2k$ is the dimension of $O$ we note that

\[ \beta_O = (2\pi)^{-k}(k!)^{-1}\omega \wedge \cdots \wedge \omega \quad (k \text{ times}) \]

the canonical measure on $O$. Lemma 3.2.2. in [DR], says us that there exists a nonzero rational function $\psi$ on $g^*$ such that $\psi(g.\ell) = \Delta(g)^{-1}\psi(\ell)$, $g \in G$, and $\ell \in g^*$. We fix one such $\psi$. There exists a unique measure $m_\psi$ on $g^*/G$ such that

\[ \int_{g^*} \phi(\ell)\psi(\ell)d\ell = \int_{g^*/G} \left( \int_O \phi(\ell)d\beta_O(\ell) \right) dm_\psi(O) \]

for all Borel functions $\phi$ on $g^*$.

B.N. Currey [C] gave an explicit description of the measure $m_\psi$ with the help of the coadjoint orbits $g^*/G$. We recall the theorem proved by B.N. Currey in [C] which is the essential tool to prove our Paley–Wiener property:

**Theorem 2.1.** Let $G$ be a connected, simply connected, and completely solvable Lie group. There exists a Zariski open subset $U$ of $g^*$, a subset $J = \{j_1 < j_2 < \cdots < j_{2k}\}$ of $\{1, 2, \cdots, n\}$, a subset $M = \{j_{r_1} < j_{r_2} < \cdots < j_{r_a}\}$ of $J$, for each $j$ in $M$ a real valued rational function $q_j$ (non
vanishing on \( U \)), and real analytic \( P_j, 1 \leq j \leq n \) functions in the variables \( w_1, w_2, \ldots, w_{2k}, \ell_1, \ell_2, \ldots, \ell_n \) such that the following hold.

1) If \( \epsilon \) denotes the number of elements in \( M \), for each \( \epsilon \in \{1, -1\}^a \), the set 
\[
U_\epsilon = \{ \ell \in U \mid \text{sign of } q_j(\ell) = \epsilon_m, 1 \leq m \leq a \}
\]
is a non empty open subset in \( g^* \).

2) Define \( V \subset \mathbb{R}^{2k} \) by \( V = \prod R_r \), where \( R_r = [0, \infty[ \) if \( j_r \in M \) and \( R_r = \mathbb{R} \) otherwise. Let \( \epsilon \in \{1, -1\}^a \) and for \( v \in V \), define \( ev \in \mathbb{R}^{2k} \) by \( (ev)_j = \epsilon_m v_j \) if \( j = j_m \in M \) and \( (ev)_j = v_j \) otherwise. Then for each \( \ell \in U_\epsilon \), the mapping \( v \rightarrow \sum_j P_j(\epsilon v, \ell) X_j^* \) is a diffeomorphism of \( V \) with the coadjoint orbit of \( \ell \).

3) Define \( W_D \) as the subspace spanned by the vectors \( \{ X_j^* \mid i \not\in J \} \) and \( W_M \) the subspace spanned by \( \{ X_j^* \mid j \in M \} \). Then the set 
\[
W = \{ \ell \in (W_D \oplus W_M) \cap U \mid |q_j(\ell)| = 1, j \in M \}
\]
is a cross-section for the coadjoint orbits \( U \). For each \( j \in M \) the rational function \( q_j \) is of the form \( q_j(\ell) = \ell_j + p_j(\ell_1, \ell_2, \ldots, \ell_{j-1}) \), where \( p_j \) is a rational function.

4) For each \( \ell \in U \), let \( \epsilon(\ell) \in \{1, -1\}^a \) such that \( \ell \in U_{\epsilon(\ell)} \). Then the mapping \( P : V \times W \to U \), defined by \( P(v, \ell) = \sum_j P_j(\epsilon(\ell) v, \ell) X_j^* \), is a diffeomorphism.

B.N. Currey [C] proved that \( m_\psi \) is a Plancherel measure on \( W \).

The idea is to compute the measure \( \psi(\ell)dl \) in terms of product measures on \( V \times W \) and then, using Lemma 1.3 of [C], we can read off \( m_\psi \) as a measure on \( W \). We have to determine coordinates for \( W \).

If the subset \( M \) of \( J \) is empty, then \( W = W_D \cap U \) and the coordinates for \( W \) are obtained by identifying \( W_D \) with \( \mathbb{R}^{n-2k} \), which is the parametrization of \( g^* \) in the nilpotent case. On the other hand, suppose that \( M \) is non empty, and \( a \) denotes the number of elements in \( M \). From [C], for each \( \epsilon \in \{1, -1\}^a \), there exists a non empty Zariski open subset \( U_\epsilon \) of \( U \) and \( U \) is the disjoint union of the sets \( U_\epsilon \). Let \( \epsilon \in \{1, -1\}^a \) and set \( W_\epsilon = W \cap U_\epsilon \). From [C], we have that 
\[
W_\epsilon = \{ \ell \in (W_D \oplus W_M) \cap U \mid \text{for each } j = j_m \in M, \ell_j = \epsilon_m - p_j(\ell_1, \ell_2, \ldots, \ell_{j-1}) \}
\]
where \( j \in M \) and \( p_j \) is a rational nonsingular function on \( U_\epsilon \).

Let \( \epsilon \in \{1, -1\}^a \). Then from [C], there is a Zariski open subset \( \Lambda_\epsilon \) of \( W_D \) and a rational function \( p_\epsilon : \Lambda_\epsilon \to W_M \) such that \( W_\epsilon = \text{graph}(p_\epsilon) \).

From [C], the projection of \( U_\epsilon \) into \( W_D \) parallel to \( W_J \) defines a diffeomorphism \( \pi_\epsilon \) of \( W_\epsilon \) with \( \Lambda_\epsilon \).

**Remark 2.2.** If \( G \) is nilpotent, then \( M \) is empty, \( U_\epsilon = U \), \( p_\epsilon = 0 \), and \( \Lambda_\epsilon = W = U \cap W_D \) is a open subset in \( W_D \).
Let $O_{\lambda, \epsilon}$ be the coadjoint orbit via $\pi_{\epsilon}^{-1}(\lambda)$ for $\lambda \in \Lambda_\epsilon$ and let $\beta_{\lambda, \epsilon}$ be the canonical measure on $O_{\lambda, \epsilon}$. Identify $W_D$ with $\mathbb{R}^{n-2k}$ via the basis $\{X_i^* \mid i \not\in J\}$ and let $d\lambda$ be the Lebesgue measure on $W_D$. If $W_D = \{0\}$ the measure $d\lambda$ is a point mass measure. This is the case for the “$ax + b$ group” (see the example, paragraph 5).

Define $\Theta_\epsilon : V \times \Lambda_\epsilon \rightarrow U_\epsilon$ by $\Theta_\epsilon(v, \lambda) = P(v, \pi_{\epsilon}^{-1}(\lambda))$. Then $\Theta_\epsilon$ is a diffeomorphism.

From 2.8 of [C], for any integrable function $F$ on $g^*/G$, we have

$$\int_{g^*/G} F(O) dm_\psi(O) = \sum_{\epsilon} \int_{\Lambda_\epsilon} F(O_{\lambda, \epsilon}) |\psi(\pi_{\epsilon}^{-1}(\lambda))| |Pf(\pi_{\epsilon}^{-1}(\lambda))|(2\pi)^{-2k} d\lambda$$

where $Pf(\pi_{\epsilon}^{-1}(\lambda))$ denotes the Pfaffian in $\pi_{\epsilon}^{-1}(\lambda)$.

Set $[\rho_{\lambda, \epsilon}] = K^{-1}(O_{\lambda, \epsilon})$ for $\epsilon \in \{1, -1\}^a$ and $\lambda \in \Lambda_\epsilon$. For each nonzero rational function $\psi$ on $g^*$ satisfying $\psi(g, \ell) = \Delta(g)^{-1}\psi(\ell)$ for $g \in G$ and $\ell \in g^*$, let $A_{\psi, \lambda, \epsilon}$ denote the semi-invariant operator of weight $\Delta$ for the irreducible representation $\rho_{\lambda, \epsilon}$ corresponding to the restriction of $\psi$ to $O_{\lambda, \epsilon}$ (this operator is constructed in [DR]).

In summary: Let $G$ be a connected, simply connected, and completely solvable Lie group. Let $\{X_1^*, X_2^*, \cdots, X_n^*\}$ be a Jordan-Hölder basis of $g^*$. Then, there is a finite collection of disjoint open subsets $U_\epsilon$ of $g^*$ and there is a subspace $W_D$ of $g^*$ such that for each $\epsilon$, $U_\epsilon$ is parametrized by a Zariski open subset $\Lambda_\epsilon$ of $W_D$, $\cup U_\epsilon$ is dense in $g^*$, and the complement of $\cup U_\epsilon$ has Plancherel measure zero. Let $\psi$ be a non empty rational function on $g^*$ such that $\psi(g, \ell) = \Delta(g)^{-1}\psi(\ell)$ for $g \in G$ and $\ell \in g^*$. For each $\epsilon$, there is a rational function $r_{\psi, \epsilon}$ on $W_D$ such that for any smooth compactly supported function $\phi$ on $G$, the function

$$\lambda \rightarrow \text{Tr}(A_{\psi, \lambda, \epsilon}^{-1/2}\rho_{\lambda, \epsilon}(\phi)A_{\psi, \lambda, \epsilon}^{-1/2}) r_{\psi, \epsilon}(\lambda)$$

on $\Lambda_\epsilon$ is Lebesgue integrable. For any such $\phi$ we have

$$\phi(\epsilon) = \sum_{\epsilon} \int_{\Lambda_\epsilon} \text{Tr}(A_{\psi, \lambda, \epsilon}^{-1/2}\rho_{\lambda, \epsilon}(\phi)A_{\psi, \lambda, \epsilon}^{-1/2}) r_{\psi, \epsilon}(\lambda) d\lambda$$

where $r_{\psi, \epsilon}(\lambda) = \psi(\pi_{\epsilon}^{-1}(\lambda))Pf(\pi_{\epsilon}^{-1}(\lambda))(2\pi)^{-2k}$.


We consider two cases:

First case: We suppose that $g^\ell \subset g_{n-1}$ for all $\ell \in W_\epsilon$ i.e. all the general position orbits are saturated with respect to $g_{n-1}$. We can choose a basis of $g$ in which the first $n-1$ vectors of the basis

$$\{X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell)\}$$
for \( \ell \in W_\epsilon \) depends on \( \ell \), the \( X_i(\ell) \) are in \( g^{\ell_j}_j \) for certain \( j \) with \( \ell_j = \ell|_{g_j} \), and \( g^{\ell_j}_j = \{ X \in g_j | ad^*X \ell_j = 0 \} \). As \( g^\ell \subset g_{n-1} \), the last vector of the basis does not depend on \( \ell \). Let

\[
\mathcal{B}_W(\ell) = \{ X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell), \: X_n \}
\]

be one such basis of \( g \).

Remark that the index set \( J_1 \) for \( G_{n-1} \) is equal to \( J \backslash \{ j_1, j_2, \ldots, j_{r_1} \} \) and that \( M_1 = \{ j_{r_2}, \ldots, j_{r_{a_1}} \} \) is a subset of \( J_1 \). For each \( \epsilon_1 \in \{ 1, -1 \}^{a_1} \), the set \( U_{\epsilon_1} \) is a nonempty open subset of \( g^{\epsilon_1}_{n-1} \). Denote \( W_{D_1} \) the subspace spanned by \( \{ X^*_j \mid i \notin J_1 \} \) in \( g^{\epsilon_1}_{n-1} \). Then, we have \( W_{D_1} = W_D \oplus R X^*_{j_1} \) and \( W_{M_1} \) is the subspace spanned by \( \{ X^*_j \mid j \in M_1 \} \).

Set \( W_{\epsilon_1} = W_1 \cap U_{\epsilon_1} \) where

\[
W_1 = \{ \ell_1 \in (W_{D_1} \oplus W_{M_1}) \cap U_1 \mid |q_j(\ell_1)| = 1, j \in M_1 \}.
\]

Now, by the corresponding theory for \( G_{n-1} \) we have a Zariski open subset \( \Lambda_{\epsilon_1} \) of \( W_{D_1} \) and a rational function \( p_{\epsilon_1} : \Lambda_{\epsilon_1} \rightarrow W_{M_1} \) such that \( W_{\epsilon_1} = \text{graph}(p_{\epsilon_1}) \).

Remark that \( a_1 = a - 1 \). In fact there is a case where \( a_1 = a \). If we start with any chain of ideals \( 0 = g_0 \subset g_1 \subset \cdots \subset g_i \subset \cdots \subset g_{n-1} \subset g_n = g \), to avoid this case it suffices to choose a chain in such a manner that the chain passes through the nil-radical of \( g \) when \( g \) is non nilpotent. Also \( \epsilon_1 \) is obtained by deleting an element from \( \epsilon \). Let \( A_{\epsilon_1} \) denote the projection of \( \Lambda_{\epsilon_1} \) on \( g^{\epsilon_1}_{n-1} \), and \( A^\prime_{\epsilon_1} \) denote the projection of \( \Lambda_{\epsilon_1} \) on \( g^{\epsilon_1}_{n-1} \).

The measure on \( W_{\epsilon_1} \) is

\[
d\mu_1(\pi^{-1}_{\epsilon_1}(\lambda_1)) = \sum_{\epsilon_1 \in \{ 1, -1 \}^{a_1}} (2\pi)^{(2k-2)}\psi_1(\pi^{-1}_{\epsilon_1}(\lambda_1))Pf(\pi^{-1}_{\epsilon_1}(\lambda_1))d\lambda_1
\]

where \( Pf(\pi^{-1}_{\epsilon_1}(\lambda_1))^2 = \det(\pi^{-1}_{\epsilon_1}(\lambda_1)([X_i, X_j])_{i,j \in J_1}) \) with \( \pi^{-1}_{\epsilon_1}(\lambda_1) = \pi^{-1}_{\epsilon_1}(\lambda)|_{g^{\epsilon_1}_{n-1}} \) and \( \psi_1 \) is a non empty rational function on \( g^{\epsilon_1}_{n-1} \) such that we have \( \psi_1(h, \ell_1) = \Delta(h)^{-1}\psi_1(\ell_1) \). Remark that, \( g^{\epsilon_{n-1}} = g^\ell \oplus R X_{j_1}, \quad [X_i, X_j] \in g_{n-1} \) for \( i, j \) in \( J_1 \), and \( \ell([X_{j_1}, g_{n-1}]) = 0 \).

**Remark 3.1.** For \( \ell \in W_\epsilon \), let \( A(\ell) = (\ell([X_i, X_j])_{i,j \in J} \) be the skew-symmetric matrix.

\[
A(\ell) = \begin{pmatrix}
0 & \cdots & 0 & \ell([X_n, X_{j_1}]) \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & A_{n-1}(\ell) & \ddots \\
\ell([X_{j_1}, X_n]) & \cdots & \cdots & * \\
\end{pmatrix}
\]

where \( A_{n-1}(\ell) = \ell([X_i, X_{j_1})_{i,j \in J_1} \).

Then: \( \det A(\ell)^{\frac{1}{2}} = |\ell([X_{j_1}, X_n])|(|\det A_{n-1}(\ell)|^{\frac{1}{2}} \).

That is, \( Pf(\ell) = \ell([X_{j_1}, X_n])Pf(\ell_{n-1}) \) where \( \ell_{n-1} = \ell|_{g_{n-1}} \).
Lemma 3.2. We suppose that \( g^\ell \subset g_{n-1} \) for all \( \ell \in W_\epsilon \). Let \( \psi \) be a non empty rational function on \( g^* \) such that \( \psi(x,\ell) = \Delta(x)^{-1}\psi(\ell) \) for all \( \ell \in W_\epsilon \) and \( x \in G \). Then:

i. \( \psi(\ell) = \psi(\ell') \) for \( \ell' \in \ell + g_{n-1}^* \).

ii. Let \( \ell_1 \in g_{n-1}^* \) and \( \tilde{\ell}_1 \) be an extension of \( \ell_1 \) to \( g^* \). By taking \( \psi_1(\tilde{\ell}_1) = \psi(\ell_1) \) we obtain a rational function \( \psi_1 \) on \( g_{n-1}^* \) verifying \( \psi(\ell_1) = \Delta G_{n-1}(h)^{-1}\psi_1(\tilde{\ell}_1) \) for \( h \in G_{n-1} \) and \( \ell_1 \in W_\epsilon \).

\[ \text{Proof.} \] We have \( G^\ell \subset G^\ell_{n-1} \) for \( \ell \in W_\epsilon \) hence the stabilizer of \( \ell_{n-1} \in g_{n-1}^* \) in \( G \) is also equal to \( G^\ell_{n-1} \).

Let \( \ell' = \ell + \gamma \) where \( \gamma \in g_{n-1}^* \). Then \( \ell' = a.\ell \) with \( a \in G^\ell_{n-1} \), hence we have that \( \psi(\ell') = \psi(a.\ell) = \Delta(a)^{-1}\psi(\ell) \). We have to verify that \( \Delta(a) = 1 \) if \( a \in G^\ell_{n-1} \). But, \( \Delta(a) = \Delta G_{n-1}(a) \) since \( G_{n-1} \) is normal in \( G \). Moreover, \( G_{n-1}/G^\ell_{n-1} \) has an invariant measure, so we have \( \Delta G_{n-1}(a) = \Delta G^\ell_{n-1}(a) \).

It suffices to see that \( G^\ell_{n-1} \) is abelian since, the orbit of \( \ell_1 \) is of maximal dimension (see [B2], Chapter II). Hence \( \psi(\ell') = \psi(\ell) \) which allows us to define \( \psi_1 \).

For all \( h \in G_{n-1} \) and \( \ell_1 \in g_{n-1}^* \) we have

\[ \psi_1(h.\ell_1) = \psi(h.\tilde{\ell}_1) = \psi(h.\ell_1) = \Delta G(h)^{-1}\psi_1(\tilde{\ell}_1) = \Delta G_{n-1}(h)^{-1}\psi_1(\ell_1). \]

\[ \square \]

We express the measure \( d\mu_1 \) on \( W_\epsilon \) in terms of local coordinates on \( g_{n-1}^* \). From the above remark and the Lemma we have that

\[ d\mu_1 = \sum_{\epsilon_1 \in \{1, -1\}^{n_1}} \frac{(2\pi)^{2k-2}}{\psi_1(\pi^{-1}_\epsilon(\lambda_1))} \frac{1}{Pf(\pi^{-1}_\epsilon(\lambda_1))} d\lambda_1 \]

\[ = \left( \sum_{\epsilon'} \frac{(2\pi)^{2k-2} \pi^{-1}_\epsilon(\lambda)(X_{j_1}, X_{n})}{Pf(\pi^{-1}_\epsilon(\lambda))} \frac{1}{\psi(\pi^{-1}_\epsilon(\lambda))} d\lambda \right) dX_{j_1}^* \]

where \( \epsilon' \) describes a part of \( \{1, -1\}^{n_1} \).

This measure \( W_\epsilon \subset g_{n-1}^* \) is a Plancherel measure on \( \widehat{G}_{n-1} \), the unitary dual of \( G_{n-1} \).

For \( \ell \in W_\epsilon \), \( \rho_\ell = \rho_{\lambda, \epsilon} \) is an induced representation of \( G \), where \( \ell_{n-1} = \ell|_{g_{n-1}} \) and \( \rho_{\lambda, \epsilon} = \rho_{\lambda, \ell_{n-1}} \) is a representation of \( G_{n-1} \). Let \( \mathcal{C}^\infty(G, \rho) \) be the set of \( f \in \mathcal{C}^\infty(G) \) with compact support modulo \( G_{n-1} \) such that \( f(hg) = (\rho_{\ell_{n-1}}(h))f(g) \) for all \( h \in G_{n-1}, g \in G \).

For all \( \phi \in \mathcal{C}^\infty(G) \) and \( \rho_\ell \in \widehat{G} \) such that \( \ell \in W_\epsilon \), the group Fourier transform is defined by

\[ \widehat{\phi}_{\rho_\ell} = \int_G \phi(g)\rho_\ell(g)dg. \]
Set \(\ell^t = Ad^*(\exp(-tX))\ell\). Remark that
\[
\rho_{t^*}(g) = \rho_t(\exp(tX).g.\exp(-tX)).
\]
Choose \(X \in g\setminus g_{n-1}\). For all \(s, t\ in \mathbb{R}\), the action of \(\phi \in C_c^\infty(G)\) on \(f \in \mathcal{H}_{\rho_t}\) gives us
\[
(\hat{\phi}_{t^*})f(\exp(tX)) = \int_G \phi(g)\rho_t(g)f(\exp(tX))dg.
\]
As the induced representation acts by right translation on \(f \in \mathcal{H}_{\rho_t}\), we have
\[
(\hat{\phi}_{t^*}f)(\exp(tX)) = \int_G \phi(g)f(\exp(tX).g)dg
\]
\[
= \int \int \int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(sX))dhds
\]
\[
= \int \int \int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(-tX).\exp(tX).\exp(sX))dhds
\]
\[
= \int \int \int_{G_{n-1}} \phi(h.\exp(sX))f(\exp(tX).h.\exp(-tX).\exp(t+s)X)dhds
\]
\[
= \int \int \int_{G_{n-1}} \phi(h.\exp(sX))\rho_{(t^*)_{n-1}}(h)f(\exp(t+s)X)dhds
\]
\[
= \int \int \int_{G_{n-1}} \phi^s(h)\rho_{(t^*)_{n-1}}(h)f(\exp(t+s)X)dhds
\]
\[
= \int \int_{G_{n-1}} (\hat{\phi}_{(t^*)_{n-1}}^s) f(\exp(t+s)X)ds
\]
where \(\phi^s(h) = \phi(h.\exp(sX))\).
For all \(\alpha \in \mathbb{R}\) we set \(f_\alpha(h.\exp(sX)) = e^{i\alpha s}f(h.\exp(sX))\). We have \(f_\alpha \in \mathcal{H}_{\rho_t}\), since \(f\) is in \(\mathcal{H}_{\rho_t}\).
Let \(\ker \rho_t\) denote the kernal of \(\rho_t\) in \(C^*(G)\), the \(C^*\)- algebra of the group \(G\). If \(\phi \in \ker \rho_t\), then, from the above calculations, for all \(f \in \mathcal{H}_{\rho_t}\) we have
\[
0 = \int \int_{G_{n-1}} (\hat{\phi}_{(t^*)_{n-1}}^s) f_\alpha(\exp(s+t)X)ds
\]
\[
= \int \int_{G_{n-1}} e^{i\alpha(s+t)}\hat{\phi}_{(t^*)_{n-1}}^s f(\exp(s+t)X)ds
\]
\(\forall \alpha \in \mathbb{R}\),
which implies that \(\hat{\phi}_{(t^*)_{n-1}}^s = 0\) for all \(s \in \mathbb{R}\). Conversely, for all \(s\) and \(t\) in \(\mathbb{R}\), if \(\hat{\phi}_{(t^*)_{n-1}}^s = 0\) we have \(\hat{\phi}_{t^*} = 0\) which implies that \(\phi \in \ker \rho_t\). We have established an equivalence
\[
\phi \in \ker \rho_t \iff (\hat{\phi}_{(t^*)_{n-1}}^s = 0 \forall s, t).
\]
Second case: If all the general position orbits are not saturated with respect to \( g_{n-1} \), we can choose a basis of \( g \) in such a way that the last vector of the basis \( X_n \) does not depend on \( \ell \) and \( X_n(\ell) \in g^\ell \). Let
\[
\mathfrak{B}_W(\ell) = \{X_1(\ell), \ldots, X_r(\ell), \ldots, X_m(\ell), \ldots, X_{n-1}(\ell), X_n(\ell)\}
\]
be one such basis of \( g \) in which the \( X_i(\ell) \) are in \( g^\ell_j \) for certain \( j \) with \( \ell_j = \ell|_{g_j} \).

**Lemma 3.3.** Assume that \( g^\ell \not\subset g_{n-1} \) for all \( \ell \in W_e \). Let \( \psi \) be a non empty rational function on \( g^\ast \) such that \( \psi(x, \ell) = \Delta(x)^{-1}\psi(\ell) \) for all \( \ell \in W_e \) and \( x \in G \). Let \( \ell_1 \in g^*_{n-1} \) and \( \tilde{\ell}_1 \) be an extension of \( \ell_1 \) to \( g^\ast \). By letting \( \psi_1(\ell_1) = \psi(\tilde{\ell}_1) \) we obtain a rational function \( \psi_1 \) on \( g^*_{n-1} \) satisfying \( \psi_1(h, \ell_1) = \Delta(h)^{-1}\psi_1(\ell_1) \) for all \( h \in G_{n-1} \).

**Proof.** For all \( \ell \in g^\ast \) and \( \alpha \in \mathbb{R} \) we have \( \ell_\alpha = \ell + \alpha X_n^\ast \) and \( g^\ast = g^*_{n-1} \oplus \mathbb{R} X_n^\ast \). For all \( h \in G_{n-1} \), we have \( h, \ell_\alpha = h, \ell + \alpha X_n^\ast \) since \( G.X_n^\ast = X_n^\ast \). By choosing \( \alpha = 0 \), we have \( \ell_0 = \ell + 0X_n^\ast \) and \( h, \ell_0 = h, \ell \). Hence, \( \psi_1(\ell_1) = \psi(\tilde{\ell}_1) \) and
\[
\psi_1(h, \ell_1) = \psi(h, \ell_1) = \Delta(h)^{-1}\psi(\ell_1) = \Delta_{G_{n-1}}(h)^{-1}\psi_1(\ell_1).
\]

Remark that the set of indices \( J_1 \) for \( G_{n-1} \) is equal to \( J \). In this case as \( g^\ell = g^{n-1} + \mathbb{R} X_n \) we have \( W_D = W_{D_1} + \mathbb{R} X_n \), where \( W_{D_1} \) is the subspace of \( g^*_{n-1} \) corresponding to \( W_D \) in \( g^\ast \). Moreover, \( \Lambda_\epsilon = \Lambda_{\epsilon_2} + \mathbb{R} X^\ast \). The Plancherel measure over \( \tilde{G} \) can be written as;
\[
d\mu(\ell) = \sum_\epsilon (2\pi)^{2k} \frac{1}{\psi(\pi_{\epsilon^{-1}}(\lambda))} \frac{1}{Pf(\pi_{\epsilon^{-1}}(\lambda))} dX_1^\ast \cdots dX_{n-2k-1}^\ast dX_n^\ast
\]
\[
= \left( \sum_{\epsilon_2} (2\pi)^{2k} \frac{1}{\psi_1(\pi_{\epsilon_2^{-1}}(\lambda_1))} \frac{1}{Pf(\pi_{\epsilon_2^{-1}}(\lambda_1))} dX_1^\ast \cdots dX_{n-2k-1}^\ast \right) dX_n^\ast
\]
\[
= d\mu_1 \times dX_n^\ast.
\]

For \( \ell = \pi_{\epsilon^{-1}}(\lambda) \in W_e \), and \( \alpha \in \mathbb{R} \) we let \( \ell_\alpha = \ell + \alpha X^\ast \). Hence, \( \ell_\alpha(X) = \ell(X) + \alpha \) and \( \rho t_\alpha = \rho \ell \otimes \chi_\alpha \) with \( \chi_\alpha(h, \exp(sX)) = e^{\iota \alpha s} \) for all \( h \in G_{n-1} \).
The restriction of $\rho_{\ell_\alpha}$ to $G_{n-1}$ is irreducible and equivalent to $\rho_{\ell_{n-1}}$ for all $\alpha \in \mathbb{R}$. For all $\xi, \eta \in H$, we have

$$\langle \hat{\phi}_{\rho_{\ell_\alpha}} \xi, \eta \rangle = \int_G \langle \rho_{\ell_\alpha}(g) \xi, \eta \rangle \phi(g) \, dg$$

$$= \int_G \langle \rho_{\ell} \otimes \chi_\alpha(g) \xi, \eta \rangle \phi(g) \, dg$$

$$= \int_{\mathbb{R}} \int_{G_{n-1}} \langle \rho_{\ell} \otimes \chi_\alpha(\exp(sX) . h) \xi, \eta \rangle \phi(\exp(sX) . h) \, dh \, ds$$

$$= \int_{\mathbb{R}} \int_{G_{n-1}} \langle e^{i\alpha s} \rho_{\ell}((\exp(sX)) \rho_{\ell_{n-1}}(h)) \xi, \eta \rangle \phi(\exp(sX) . h) \, dh \, ds$$

$$= \int_{\mathbb{R}} e^{i\alpha s} \langle \rho_{\ell}(\exp(sX)) \hat{\phi}^s_{\rho_{\ell_{n-1}}} \xi, \eta \rangle \, ds$$

where $\phi^s(h) = \phi(\exp(sX) . h)$. Hence we have expressed $\hat{\phi}_{\rho_{\ell_\alpha}}$ with the help of $\hat{\phi}^s_{\rho_{\ell_{n-1}}}$.

**4. Weak Paley–Wiener Property.**

**Theorem 4.1.** Let $G$ be a connected, simply connected, and completely solvable Lie group with the unitary dual $\hat{G}$, and let $\phi$ be a bounded, measurable and compactly supported function (i.e. $\phi \in L^\infty_c(G)$). Assume that there is a subset $E \subset \hat{G}$ with positive Plancherel measure such that $\hat{\phi}_{\rho_{\ell_\alpha}} = 0$ for all $\rho \in E$, where $\hat{\phi}_{\rho}$ is the group Fourier transform of $\phi$. Then $\phi = 0$ almost everywhere on $G$.

**Proof.** We proceed by induction on the dimension $n$ of $G$. The result is true if the dimension of $G$ is one, since $G \cong \mathbb{R}$. Assume that the result is true for all groups of dimension $n-1$. We can assume that $E$ is contained in $W_\epsilon$ (it suffices to take $E$ as the finite union of $E \cap W_\epsilon$).

**First case:** $g^\ell \subset g_{n-1}$ for all $\ell \in W_\epsilon$. Let $\phi \in C^\infty_c(G)$. By hypothesis, for all $\rho_{\ell}$, such that $\ell \in E$ we have $0 = \hat{\phi}_{\rho_{\ell}}$; we will show that $\phi = 0$ almost everywhere on $G$.

Notice that for all $\epsilon_1 \in \{-1, 1\}^{\alpha_n}$, the associated set $\Lambda_{\epsilon_1}$ corresponds to two sets $\Lambda_{\epsilon_+}$ and $\Lambda_{\epsilon_-}$, $\epsilon_+ \in \{-1, 1\}^n$ in $W_D$. If $\Lambda_{\epsilon_+}'$ and $\Lambda_{\epsilon_-}'$ are the projections of $\Lambda_{\epsilon_+}$ and $\Lambda_{\epsilon_-}$ on $g_{n-1}$ such that $\Lambda_{\epsilon_+}' = (\exp \mathbb{R}X). \Lambda_{\epsilon_+}' \cup (\exp \mathbb{R}X). \Lambda_{\epsilon_-}'$ and $T_{\ell} = \{\exp tX. \ell_{n-1} \mid t \in \mathbb{R}\}$ are contained in the projection of $\Lambda_{\epsilon_+}$ or in $\Lambda_{\epsilon_-}$. $\Lambda_{\epsilon_1}'$ is a Zariski open set in $\Lambda_{\epsilon_1}$.

From paragraph 3 we have that

$$\phi \in \ker \rho_{\ell} \iff \left( \hat{\phi}^s_{\rho_{\ell}(t)_{n-1}} = 0 \forall s, t \right).$$
By hypothesis, $\hat{\phi}_{\rho_t} = 0$ for all $\ell \in E$ and from the above equivalence we have
$$\hat{\phi}_{\rho_{(t)} n_{-1}}^{s} = 0$$
for all $s, t$ in $\mathbb{R}$. This relation tells us that a set $A$ contained in $\Lambda_{t_{+}} \cup \Lambda_{t_{-}}$ has positive Plancherel measure if and only if the set $\cup_{\rho_{t} \in A T_{\ell}}$ has positive Plancherel measure in $\Lambda_{t_{1}}$.

In applying this remark to the set $E$, we obtain
$$\hat{\phi}_{\rho_{s} \rho_{(t)} n_{-1}}^{s} = 0$$
for all $s, t$ in $\mathbb{R}$. This relation tells us that a set $A$ contained in $\Lambda_{t_{+}} \cup \Lambda_{t_{-}}$ has positive Plancherel measure if and only if the set $\cup_{\rho_{t} \in A T_{\ell}}$ has positive Plancherel measure in $\Lambda_{t_{1}}$.

In applying this remark to the set $E$, we obtain
$$\hat{\phi}_{\rho_{s} \rho_{(t)} n_{-1}}^{s} = 0$$
for all $\rho_{t} n_{-1}$ in $E' \subset \hat{G}_{n-1}$ with positive Plancherel measure.

By the induction hypothesis $\phi^{s} = 0$ almost everywhere on $G_{n-1}$, which implies that $\phi = 0$ almost everywhere on $G$ by using Fubini’s theorem.

**Second case:** $g^{\ell} \not\subset g_{n-1}$ for all $\ell \in W_{\epsilon}$. Let $\rho \in C_c^{\infty}(G)$. By hypothesis, for all $\rho_{t}$, such that $\ell \in E$ we have $\hat{\phi}_{\rho_{t}} = 0$; let us show that $\hat{\phi}_{\rho_{t}} = 0$ for all $\ell \in W_{\epsilon}$.

Let $\ell \in E$. For all $\alpha \in \mathbb{R}$ we have
$$\langle \hat{\phi}_{\rho_{t_{\alpha}}} \xi, \eta \rangle = \int_{\mathbb{R}} e^{i\alpha s} \rho_{\ell}(\exp(sX)) \hat{\phi}_{\rho_{t_{n_{-1}}}}^{s} \xi, \eta \rangle ds;$$
hence
$$\hat{\phi}_{\rho_{t_{\alpha}}} = \int_{\mathbb{R}} e^{i\alpha s} \rho_{\ell}(\exp(sX)) \hat{\phi}_{\rho_{t_{n_{-1}}}}^{s} ds.$$
Set
$$\Psi(s) = \rho_{\ell}(\exp(sX)) \hat{\phi}_{\rho_{t_{n_{-1}}}}^{s}.$$
Hence
$$\hat{\phi}_{\rho_{t_{\alpha}}} = \int_{\mathbb{R}} \Psi(s) e^{i\alpha s} ds$$
$$= \hat{\Psi}(\alpha).$$
By hypothesis, for all $\ell \in E$ we have $\hat{\phi}_{\rho_{t}} = 0$. The above calculation tells us that there exists a set $E' \subset E$ with positive Plancherel measure such that $\hat{\Psi}(\alpha) = 0$ for $\alpha$ belonging to a set of reals with positive Lebesgue measure and $\ell \in E'$. Hence $\Psi = 0$ almost everywhere; consequently we have $\Psi(s) = 0$ for almost every $s \in \mathbb{R}$. Hence
$$0 = \hat{\phi}_{\rho_{t_{\alpha}}} = \int_{\mathbb{R}} e^{i\alpha s} \rho_{\ell}(\exp(sX)) \hat{\phi}_{\rho_{t_{n_{-1}}}}^{s} ds$$
for all $\alpha$ in $\mathbb{R}$, which implies that $\hat{\phi}_{\rho_{t_{n_{-1}}}}^{s} = 0$ for all $\ell_{n_{-1}}$ in $E_{1}$ (path of $E$ on $g_{n_{-1}}^{s}$) with positive Plancherel measure on $\hat{G}_{n-1}$. By using the induction hypothesis $\hat{\phi}_{\rho_{t_{n_{-1}}}}^{s} = 0$ for almost all $\ell_{n_{-1}} \in W_{\epsilon}'$ (path of $W_{\epsilon}$ on $g_{n_{-1}}^{s}$). Hence, $0 = \hat{\phi}_{\rho_{t}}$ for almost all $\ell \in W_{\epsilon}$ (from the above calculation of $\hat{\phi}_{\rho_{t_{\alpha}}}$).
Hence \( \hat{\phi}_{\rho} = 0 \) for almost all \( \rho \) relating with the Plancherel measure. By the Plancherel formula for completely solvable Lie groups, we have

\[
\phi(e) = \sum_{\epsilon} \int_{\Lambda_{\epsilon}} \text{Tr}(A_{\psi,\lambda,\epsilon}^{-1/2} \rho_{\lambda,\epsilon}(\phi) A_{\psi,\lambda,\epsilon}^{-1/2}) |r_{\psi,\epsilon}(\lambda)| d\lambda
\]

which implies that \( \phi = 0 \).

Now, we consider \( \phi \in L_c^\infty(G) \). Let \( \{f_n\}_n \) be an approximate identity in \( C_c^\infty(G) \). For all integers \( n \), \( f_n * \phi \in C_c^\infty(G) \). Let \( \rho \in E \). If \( \hat{\phi}_{\rho} \) vanishes, then \( (f_n * \phi)_\rho \) also vanishes. Hence by what precedes, \( f_n * \phi = 0 \) (for all integers \( n \)). But, \( (f_n * \phi)_n \in \mathbb{N} \) converges to \( \phi \) in \( L^1(G) \), which implies that \( \phi = 0 \) almost everywhere on \( G \).

\( \square \)

5. Example: The \( ax + b \) Group.

Consider the group

\[
G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a > 0, b \in \mathbb{R} \right\}.
\]

We use the notation

\[
(a, b) = \begin{pmatrix} a \\ b \\ 0 \\ 1 \end{pmatrix}.
\]

The Matrix multiplication gives:

\[
(a_1, b_1)(a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)
\]

and the inverse

\[
(a, b)^{-1} = (a^{-1}, -ba^{-1}).
\]

Let \( H = (1, b) \) be the derived group of \( G \) which is identified with \( \mathbb{R} \). Let \( y \in \mathbb{R} \), \( \chi_y \) the character of \( H \) defined by \( \chi_y((1, b)) = e^{iby} \).

Remark that \( (a, b) = (1, b)(a, 0) \). Let \( \rho_y = \text{Ind}_H^G \chi_y \) be the induced representation of \( G \). This representation is realized in the space \( L^2(\mathbb{R}) \). Recall that for all \( y > 0 \), \( \rho_y \) is equivalent to \( \rho_1 \) and we denote by \( \rho_+ \) the class of the representation \( \rho_1 \). If \( y < 0 \), \( \rho_y \) is equivalent to \( \rho_{-1} \); we denote by \( \rho_- \) the equivalence class of this representation.

The Lie algebra \( g \) of \( G \) is the set of matrices

\[
g = \left\{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}, (x, y) \in \mathbb{R}^2 \right\}.
\]

In the basis

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]

we have \([X, Y] = Y\). With the basis \( X \) and \( Y \) we have

\[
\text{Ad}(a, b) = \begin{pmatrix} 1 & 0 \\ -b & a \end{pmatrix}.
\]
Also in the dual basis $\{X^*, Y^*\}$

$$Ad^*(a, b) = \begin{pmatrix} 1 & ba^{-1} \\ 0 & a^{-1} \end{pmatrix}.$$  

For $\ell = \alpha X + \beta Y^* \in \mathfrak{g}^*$ the orbits of $G$ in $\mathfrak{g}^*$ are the upper half plane $\beta > 0$, the lower half plane $\beta < 0$ and the points $(\alpha, 0)$.

Let $\mathfrak{g} = \{X, Y\}$ be the basis of $\mathfrak{g}$ defined above, and $\mathfrak{g}^* = \{X^*, Y^*\}$ the dual basis of $\mathfrak{g}^*$. There exists a set $J = \{j_1, j_2\} \subseteq \{1, 2\}$ and $M = \{j_2\}$ a subset of $J$, so that $V \subset \mathbb{R}^2$, $V = ]0, \infty[ \times \mathbb{R}$. We have $W_D = \emptyset$ and $W_M$ is spanned by the vector $\{X^*_j \mid j_2 \in M\}$.

The Zariski open sets $U_+$ and $U_-$ are the half planes of $\mathfrak{g}^*$ defined above and $U = U_+ \cup U_-$. Here, $a = 1$ and $\epsilon \in \{1, -1\}$.

Since there are only two orbits, the set

$$W = \{\ell \in W_M \cap U \mid |q_{j_2}(\ell)| = 1, j_2 \in M\}$$

is a union of two points in $\mathfrak{g}^*$. We have $W_+ = W \cap U_+$ and $W_- = W \cap U_-$. Let $\epsilon \in \{1, -1\}$. In this case the Zariski open set is $\Lambda_\epsilon = \Lambda_\epsilon$ or $\Lambda_\epsilon = \Lambda_\epsilon$ of $W_D$, which reduces to a point.

In this particular case we can prove weak Paley–Wiener property by direct calculations.

Let $\phi \in \mathcal{C}_c^\infty(G)$, $f \in L^2(\mathbb{R}_+^*)$ and $(t, 0) \in \mathbb{R}_+^*$: then

$$(\hat{\phi}_{pt} f)(t) = \int_G \phi((a, b))\rho_t((a, b))f(t)a^{-2}dadb$$

$$= \int_G \phi((a, b))f((a, b)^{-1}(t, 0))a^{-2}dadb$$

$$= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \phi((a, b))f((a^{-1}t, -ba^{-1}))a^{-2}dadb$$

$$= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \phi((a, b))f((a^{-1}t, 0)(1, -bt^{-1}))a^{-2}dadb$$

$$= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \phi((a, b))\chi_y((1, -bt^{-1}))f((a^{-1}t, 0))a^{-2}da$$

$$= \int_{\mathbb{R}_+^*} \int_{\mathbb{R}} \phi^a(b)e^{-ibt^{-1}}f((a^{-1}t, 0))a^{-2}da$$

$$= \int_{\mathbb{R}_+^*} \phi^a_{\chi_yt^{-1}}f((a^{-1}t, 0))a^{-2}da,$$

where $\phi^a(b) = \phi((a, b))$.

Remark that $\phi^a \in \mathcal{C}_c^\infty(\mathbb{R})$. By hypothesis we have $\hat{\phi}_{pt} = 0$ for all $\ell \in E$. The above calculation implies that for all $a > 0$ we have $\hat{\phi}^a_{\chi_yt^{-1}} = 0$ for almost all $t > 0$ and for fixed $y$. 

As \( \phi^a \in C_\infty^\infty(\mathbb{R}) \), \( \hat{\phi}^a_{\lambda,\mu^{-1}} \) extends as an entire function over \( \mathbb{C} \). \( \hat{\phi}^a_{\lambda,\mu^{-1}} \) vanishes on a set in which the Plancherel measure \( d\mu_1 \) is positive hence by the classical Paley–Wiener theorem, we can conclude that \( \phi^a = 0 \), and then \( \phi = 0 \) almost everywhere on \( G \).

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References


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