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We realize an idea suggested in the literature by Yuzvinsky, giving a construction of logarithmic 1-forms from syzygies of linear functionals.

1. Introduction.

Let V be a vector space of finite dimension l over a field $\mathbf{K}$. An *arrangement* is a finite set $\mathcal{A} = \{X_1, \ldots, X_n\}$ of hyperplanes in V. We assume the arrangement is *central*, i.e., all hyperplanes contain the origin. Let S be the symmetric algebra of V^*, which is canonically isomorphic to the polynomial algebra $\mathbf{K}[x_1, \ldots, x_l]$ for any basis $(x_1, \ldots, x_l)$ of V^*. For each $X \in \mathcal{A}$, fix a linear functional $\alpha_X \in V^*$ with $\ker \alpha_X = X$. Let $Q = \prod_{X \in \mathcal{A}} \alpha_X$.

The module of logarithmic differential forms with poles along a divisor was introduced by Deligne [**De**] for a divisor with normal crossings, and the definition was generalized by Saito [**Sa**] to an arbitrary divisor. Extensive research has been done into the structure of this module when the divisor is a union of hyperplanes, but some important motivating questions, such as [**OT**, Conjecture 4.138], remain unanswered.

Let $\Omega^q[V]$ represent the S-module of differential q-forms on V with coefficients from S and let

$$\Omega^q(\mathcal{A}) = \left\{ \omega \;\middle|\; \begin{array}{l} \omega \text{ is a global rational differential } q\text{-form on } V \\ \text{such that } Q\omega \in \Omega^q[V] \text{ and } Qd\omega \in \Omega^{q+1}[V] \end{array} \right\}.$$

This is the *module of logarithmic differential q-forms corresponding to* $\mathcal{A}$. The S-module $\Omega^1(\mathcal{A})$ and its dual are important algebraic objects studied in arrangement theory, and we herein restrict our attention to the case $q = 1$.

The present work is motivated by several discoveries by Rose, Terao, Yuzvinsky, and Ziegler, appearing in [**RT**], [**Yu**], and [**Zi**], which suggest that there may be a construction of generators for $\Omega^1(\mathcal{A})$ from either combinatoric data or simple algebraic information about the functionals α_X. Some of these results can be elegantly formulated by using a chain complex of relation spaces, here denoted $\bigoplus R_\cdot(\mathcal{A}_X)$, which has appeared in the papers [**BT**] by Brandt and Terao and [**Yu**] by Yuzvinsky. The results in [**RT**] and [**Zi**] imply that if the second relation space, $\bigoplus R_2(\mathcal{A}_X)$, vanishes, then the logarithmic forms $d\alpha_X/\alpha_X$ generate the whole S-module $\Omega^1(\mathcal{A})$. In [**Yu**], Yuzvinsky

gives forms which generate the module $\Omega^1(\mathcal{A})$ whenever the third relation space, $\bigoplus R_3(\mathcal{A}_X)$, vanishes.

In [**Yu**], Yuzvinsky suggests that there may be an interesting construction of logarithmic 1-forms from elements of $\bigoplus R_\cdot(\mathcal{A}_X)$. In this paper we realize Yuzvinsky's idea, giving an explicit construction. The proof that the resulting forms are logarithmic proceeds by the method of deletion and restriction, which was introduced by Zaslavsky in [**Za**] and has become very pervasive in the algebraic theory of arrangements. One family of arrangements for which $\Omega^1(\mathcal{A})$ is generated by these 1-forms is the family associated to the real reflection groups of type A.

In Section 2, we define the chain complex $\bigoplus R_\cdot(\mathcal{A}_X)$. In Sections 3 and 4, we apply Zaslavsky's method of deletion and restriction to $\bigoplus R_\cdot(\mathcal{A}_X)$. In Sections 5 and 6, we recursively construct logarithmic 1-forms ω_μ associated to elements μ of $R_k(\mathcal{A}_X)$ and prove some properties of these forms. In Section 7, we show that $\Omega^1(\mathcal{A})$ is generated by these forms for reflection arrangements of type A.

2. A chain complex of relation spaces.

This section introduces a chain complex, denoted $\bigoplus R_\cdot(\mathcal{A}_X)$, which has appeared in [**BT**] and in [**Yu**].

Let L be the poset of all intersections of the hyperplanes of $\mathcal{A}$, ordered by reverse inclusion, with V itself regarded as the empty intersection. It is well-known that L is a geometric lattice. We let L_k represent the lattice elements of codimension k. For any $Y \in L$, let $\mathcal{A}_Y$ denote the subset of $\mathcal{A}$ consisting of all $X \in \mathcal{A}$ with $Y \subset X$. Note that $\mathcal{A} = L_1 \subset L$.

Let $R_1(\mathcal{A})$ be the vector space with basis $(e_1, \ldots, e_n)$ in one to one correspondence with the hyperplanes of $\mathcal{A}$. Note that there is an obvious isomorphism

$$\bigoplus_{X \in L_1} R_1(\mathcal{A}_X) \longrightarrow R_1(\mathcal{A}),$$

so we may write $\bigoplus_{X \in L_1} R_1(\mathcal{A}_X)$ in place of $R_1(\mathcal{A})$. We will use this notation, since it is more convenient for the recursive definitions in later sections. Let $\delta_1^{\mathcal{A}} : \bigoplus_{X \in L_1} R_1(\mathcal{A}_X) \longrightarrow V^*$ be the linear map defined via $\delta_1^{\mathcal{A}} e_i = \alpha_{X_i}$. Let $R_2(\mathcal{A}) = \ker \delta_1^{\mathcal{A}}$.

For $2 \leq k$, assume that we have vector spaces $R_k(\mathcal{A})$ and $R_{k-1}(\mathcal{A})$, depending on $\mathcal{A}$, such that $R_k(\mathcal{A}) \subset \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X)$. We define a new vector space $R_{k+1}(\mathcal{A})$.

Let $\delta_k^{\mathcal{A}}$ be the map

$$\delta_k^{\mathcal{A}} : \bigoplus_{Y \in L_k} R_k(\mathcal{A}_Y) \longrightarrow \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X)$$

induced by the canonical inclusions

$$R_k(\mathcal{A}_Y) \hookrightarrow \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X).$$

Let $R_{k+1}(\mathcal{A}) = \ker \delta_k^{\mathcal{A}}$.

For $k > l$, note that $R_k(\mathcal{A}) = 0$. For simplicity, we will write $\delta_i^{\mathcal{A}}$ as δ_i for every i. For any $Y \in L$, write $\delta_i^{\mathcal{A}_Y}$ as δ_i^Y for every i. For any $k \geq 2$, $Y \in L_k$ and any $\mu \in \bigoplus_{Y \in L_k} R_k(\mathcal{A}_Y)$, let μ_Y denote the Y coordinate of μ. For any $Y \in L_k$ and $X \in L_{k-1}^Y$, observe that this defines the notation $(\mu_Y)_X$. It is convenient to extend this definition to all $X \in L_{k-1}$ by letting $(\mu_Y)_X = 0$ in case $X \in L_{k-1} \backslash L_{k-1}^Y$. Notice that for $k \geq 2$, $X \in L_{k-1}$, the X coordinate of $\delta_k \mu$ is $\sum_{Y \in L_k} (\mu_Y)_X$.

Observe that the sequence

$$0 \longrightarrow R_l(\mathcal{A}) \xrightarrow{\delta_l} \bigoplus_{X \in L_{l-1}} R_{l-1}(\mathcal{A}_X) \xrightarrow{\delta_{l-1}} \cdots \xrightarrow{\delta_2} \bigoplus_{X \in L_1} R_1(\mathcal{A}_X) \xrightarrow{\delta_1} V^* \longrightarrow 0$$

is a chain complex.

3. Deletion and Restriction.

We now apply Zaslavsky's method of deletion and restriction to the chain complex $\bigoplus R_\cdot(\mathcal{A}_X)$. For convenience in this section and later sections, we put

$$V(p_1, \ldots, p_m) = \cap_{i=1}^m \ker p_i$$

for any $p_1, \ldots, p_m \in V^*$.

Let $\theta \in V^*$. Let $\mathcal{A}' = \mathcal{A} \backslash \{V(\theta)\}$. In particular, if $V(\theta) \notin \mathcal{A}$, then $\mathcal{A}' = \mathcal{A}$. Let $\mathcal{A}'' = \{X \cap V(\theta) \mid X \in \mathcal{A}'\}$. For each $Z \in \mathcal{A}''$, choose some $\eta_Z \in V(\theta)^*$ with $\ker \eta_Z = Z$.

Again, we simplify notation. Let L'' denote $L(\mathcal{A}'')$ and for any $X \in L$, let X'' denote $X \cap V(\theta) \in L''$. For any $\alpha \in V^*$, let α'' denote $\alpha|_{V(\theta)}$. For $i \geq 1$, let δ_i'' denote $\delta_i^{\mathcal{A}''}$.

Let $h_0^{\mathcal{A}} : V^* \longrightarrow V(\theta)^*$ be given by $h_0^{\mathcal{A}}(\alpha) = \alpha''$.

For any $X \in \mathcal{A}$, $X \neq V(\theta)$, notice that α_X'' and $\eta_{X''}$ are $\mathbf{K}$-proportional. Thus we may let $\frac{\alpha_X''}{\eta_{X''}}$ denote the scalar $a \in \mathbf{K}$ such that $\alpha_X'' = a\eta_{X''}$. Now define

$$h_1^{\mathcal{A}} : \bigoplus_{X \in L_1} R_1(\mathcal{A}_X) \longrightarrow \bigoplus_{Z \in L_1''} R_1(\mathcal{A}_Z'')$$

via

$$e_X \mapsto \begin{cases} \dfrac{\alpha_X''}{\eta_{X''}} e_{X''} & \text{if } X \neq V(\theta), \text{ and} \\ 0 & \text{if } X = V(\theta). \end{cases} \tag{1}$$

By definition of $h_1^{\mathcal{A}}$, the following diagram commutes.

$$\begin{array}{ccc} \bigoplus_{X\in L_1} R_1(\mathcal{A}_X) & \xrightarrow{\delta_1} & V^* \\ {\scriptstyle h_1^{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle h_0^{\mathcal{A}}} \\ \bigoplus_{Z\in L_1''} R_1(\mathcal{A}_Z'') & \xrightarrow{\delta_1''} & V(\theta)^* \end{array} \tag{2}$$

Lemma 3.1. *For any $Y \in L_2$, $h_1^{\mathcal{A}_Y} R_2(\mathcal{A}_Y) \subset R_2(\mathcal{A}_{Y''}'')$.*

Proof. Let $\mu \in R_2(\mathcal{A}_Y)$. Observe that $(\mathcal{A}_Y)'' \subset \mathcal{A}_{Y''}''$. By (2), $\delta_1^{\mathcal{A}_{Y''}''} h_1^{\mathcal{A}_Y} \mu = h_0^{\mathcal{A}_Y} \delta_1^Y \mu = h_0^{\mathcal{A}_Y} 0 = 0$. □

4. The chain map.

We now recursively generalize the preceding construction and properties, obtaining a chain map. Let $k \geq 1$, and assume that we have a linear map

$$h_k^{\mathcal{A}} : \bigoplus_{X\in L_k} R_k(\mathcal{A}_X) \longrightarrow \bigoplus_{Z\in L_k''} R_k(\mathcal{A}_Z'')$$

so that, for any $Y \in L_k$,

$$h_k^{\mathcal{A}_Y} R_{k+1}(\mathcal{A}_Y) \subset R_{k+1}(\mathcal{A}_{Y''}''). \tag{3}$$

When $k = 1$, the assumption (3) amounts to Lemma 3.1. For the case $k > 1$, the reader is referred to Lemma 4.1. We now define a new linear map

$$h_{k+1}^{\mathcal{A}} : \bigoplus_{Y\in L_{k+1}} R_{k+1}(\mathcal{A}_Y) \longrightarrow \bigoplus_{W\in L_{k+1}''} R_{k+1}(\mathcal{A}_W'').$$

The construction proceeds coordinate by coordinate. For every $Y \in L_{k+1}$, assumption (3) gives rise to a composition

$$R_{k+1}(\mathcal{A}_Y) \xrightarrow{h_k^{\mathcal{A}_Y}} R_{k+1}(\mathcal{A}_{Y''}'') \hookrightarrow \bigoplus_{W\in L_{k+1}''} R_{k+1}(\mathcal{A}_W''). \tag{4}$$

If $\theta \notin Y^\perp$, then the inclusion in the composition (4) is the canonical injection. If $\theta \in Y^\perp$, then

$$R_{k+1}(\mathcal{A}_{Y''}'') = R_{k+1}(\mathcal{A}_Y'') = 0$$

so the inclusion in composition (4) is the trivial inclusion $0 \hookrightarrow \bigoplus_{W\in L_{k+1}''} R_{k+1}(\mathcal{A}_W'')$. The compositions (4) induce the linear map

$$h_{k+1}^{\mathcal{A}} : \bigoplus_{Y\in L_{k+1}} R_{k+1}(\mathcal{A}_Y) \to \bigoplus_{W\in L_{k+1}''} R_{k+1}(\mathcal{A}_W'').$$

Note that we have, for any $W \in L''_{k+1}$,

$$(h^{\mathcal{A}}_{k+1}\mu)_W = \sum_{\substack{Y \in L_{k+1} \\ Y''=W}} h^{\mathcal{A}_Y}_k \mu_Y. \tag{5}$$

By definition of $h^{\mathcal{A}}_{k+1}$, the following diagram commutes.

$$\begin{array}{ccc} \bigoplus_{Y\in L_{k+1}} R_{k+1}(\mathcal{A}_Y) & \xrightarrow{\delta_{k+1}} & \bigoplus_{X\in L_k} R_k(\mathcal{A}_X) \\ \downarrow{\scriptstyle h^{\mathcal{A}}_{k+1}} & & \downarrow{\scriptstyle h^{\mathcal{A}}_k} \\ \bigoplus_{W\in L''_{k+1}} R_{k+1}(\mathcal{A}''_W) & \xrightarrow{\delta''_{k+1}} & \bigoplus_{Z\in L''_k} R_k(\mathcal{A}''_Z) \end{array} .$$

The proof of the following lemma is exactly analogous to Lemma 3.1.

Lemma 4.1. *For any $Y \in L_k$, $h_{k+1}R_{k+1}(\mathcal{A}_Y) \subset R_{k+1}(\mathcal{A}''_{Y''})$.*

For simplicity, we will write $h^{\mathcal{A}}_i$ as h_i for every i. In general, the maps $h^{\mathcal{A}_X}_i$ with $X \in L$ will be written h^X_i. We have now shown that the following diagram commutes.

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \bigoplus_{X\in L_k} R_k(\mathcal{A}_X) & \xrightarrow{\delta_k} & \bigoplus_{X\in L_{k-1}} R_{k-1}(\mathcal{A}_X) & \xrightarrow{\delta_{k-1}} & \\
 & & \downarrow{\scriptstyle h_k} & & \downarrow{\scriptstyle h_{k-1}} & & \\
\cdots & \longrightarrow & \bigoplus_{Z\in L''_k} R_k(\mathcal{A}''_Z) & \xrightarrow{\delta''_k} & \bigoplus_{Z\in L''_{k-1}} R_{k-1}(\mathcal{A}''_Z) & \xrightarrow{\delta''_{k-1}} & \\
\cdots & \xrightarrow{\delta_2} & \bigoplus_{X\in L_1} R_1(\mathcal{A}_X) & \xrightarrow{\delta_1} & V^* & \xrightarrow{\delta_0} & 0 \\
 & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_0} & & \\
\cdots & \xrightarrow{\delta''_2} & \bigoplus_{Z\in L''_1} R_1(\mathcal{A}''_Z) & \xrightarrow{\delta''_1} & V(\theta)^* & \xrightarrow{\delta''_0} & 0\,.
\end{array}$$

The commutativity of the big diagram is summarized in the following proposition.

Proposition 4.2. *Let $k \geq 1$. Let $\mu \in \bigoplus_{X\in L_k} R_k(\mathcal{A}_X)$. Then*

$$\delta''_k h_k \mu = h_{k-1}\delta_k \mu.$$

The following observation is useful in the proof of our main result, Theorem 6.5.

Lemma 4.3. *Let $2 < k \leq l$, $Y \in L_k$, and $\mu \in R_k(\mathcal{A}_Y)$. Assume $\theta \in Y^\perp$. Then*

$$\sum_{\substack{X\in L^Y_{k-1} \\ \theta\notin X^\perp}} h^X_{k-2}\mu_X = 0.$$

Proof. Note that the codomain of $h_k^Y : R_k(\mathcal{A}_Y) \longrightarrow R_k(\,(\mathcal{A}_Y)''\,)$ is 0, and so h_k^Y is the zero map. Thus $\delta_k^{(\mathcal{A}_Y)''} h_k^Y \mu = \delta_k^{(\mathcal{A}_Y)''} 0 = 0$. By Proposition 4.2, this implies $h_{k-1}^Y \delta_k^Y \mu = 0$. The map δ_k^Y is the inclusion $R_k(\mathcal{A}_Y) \hookrightarrow \bigoplus_{X \in L_{k-1}} R_{k-1}(\mathcal{A}_X)$, so $h_{k-1}^Y \mu = h_{k-1} \delta_k^Y \mu = 0$. Now observe $L_{k-1}(\,(\mathcal{A}_Y)''\,) = \{Y\}$. Also observe, for $X \in L_{k-1}^Y$, $X'' = Y$ if and only if $\theta \notin X^\perp$. Therefore, by (5),

$$h_{k-1}^Y \mu = \sum_{\substack{X \in L_{k-1}^Y \\ \theta \notin X^\perp}} h_{k-2}^X \mu_X.$$

This completes the proof. □

5. Construction of differential 1-forms.

In this section we construct differential 1-forms ω_μ in a recursive manner. It will be shown in the next section that these forms are logarithmic.

Lemma 5.1. *Let $Y \in L_k$, $1 \leq j \leq k$, and $X \in L_{k-j}^Y$. Let $(\beta_1, \ldots, \beta_k)$ be any basis of $Y^\perp$. Let $\langle \beta_1, \ldots, \beta_j \rangle$ denote the span of $\{\beta_1, \ldots, \beta_j\}$ in $Y^\perp$. If $\langle \beta_1, \ldots, \beta_j \rangle \cap X^\perp = 0$, then:*

(i) *There is a unique basis $C^X = (\gamma_{j+1}^X, \ldots, \gamma_k^X)$ of $X^\perp$ so that*

$$C^X|_{V(\beta_1,\ldots,\beta_j)} = (\beta_{j+1}, \ldots, \beta_k)|_{V(\beta_1,\ldots,\beta_j)}.$$

(ii) *Let $j+1 \leq i \leq k$. Let any $\eta_i \in X^\perp$ have the property $\eta_i|_{V(\beta_1,\ldots,\beta_j)} = \beta_i|_{V(\beta_1,\ldots,\beta_j)}$. Then $\eta_i = \gamma_i^X$.*

Proof. Note that $Y^\perp$ has the internal direct sum decomposition

$$Y^\perp = \langle \beta_1, \ldots, \beta_j \rangle \oplus X^\perp.$$

Let $\pi_1 : Y^\perp \longrightarrow \langle \beta_1, \ldots, \beta_j \rangle$ and $\pi_2 : Y^\perp \longrightarrow X^\perp$ be the projections. For each i, $j+1 \leq i \leq k$, let $\gamma_i^X = \pi_2(\beta_i)$, and let $C^X = (\gamma_{j+1}^X, \ldots, \gamma_k^X)$. This proves existence in (i). Part (ii) now follows. The uniqueness of C^X in (i) is a direct consequence of (ii). □

For $Y \in L_k(\mathcal{A})$, $X \in L_{k-1}^Y$, let $(\beta_1, \ldots, \beta_k)$ be a basis of $Y^\perp$ with $\beta_1 \notin X^\perp$. Since $Y^\perp / X^\perp$ is one-dimensional and $\beta_1^X \not\equiv 0 \mod X^\perp$, there is a scalar $a_i^X \in \mathbf{K}$ (possibly zero) so that

$$\beta_i^X \equiv a_i^X \beta_1^X \mod X^\perp.$$

Lemma 5.2. *Let $Y \in L_k(\mathcal{A})$ and $X \in L_{k-1}^Y$. Let $(\beta_1, \ldots, \beta_k)$ be a basis of $Y^\perp$ with $\beta_1 \notin X^\perp$, and $C^X = (\gamma_2^X, \ldots, \gamma_k^X)$ be as in Lemma 5.1. Then for every i, $2 \leq i \leq k$, $\gamma_i^X = \beta_i - a_i^X \beta_1$.*

Proof. Note that $\left(\beta_i - a_i^X \beta_1\right)|_{V(\beta_1)} = \beta_i|_{V(\beta_1)}$, and $\beta_i - a_i^X \beta_1 \in X^\perp$. The result follows by Lemma 5.1(ii). □

Let $1 \leq k \leq l$, $Y \in L_k$, and $\mu \in R_k(A_Y)$ be fixed. Let $B = (\beta_1, \dots, \beta_k)$ be a k-tuple of elements in $Y^\perp$. If B is linearly dependent or $\mu = 0$, then we define $\omega_\mu(B) = 0$. In particular, if $B = (0, \dots, 0)$, then $\omega_\mu(B) = 0$. Otherwise, $\omega_\mu(B)$ is defined recursively as follows.

Suppose $k = 1$. Then $Y \in L_1 = \mathcal{A}$, and, since $\mu \in R_1(\mathcal{A}_Y)$, μ is of the form ce_Y for some $c \in \mathbf{K}$. Furthermore, $B = (\beta_1)$ is a 1-tuple. We let $\omega_\mu(B) = \frac{c}{\beta_1} d\alpha_Y$.

Now suppose $k \geq 2$. Assume for any $X \in L_{k-1}$, any $(k-1)$-tuple C of elements of $X^\perp$, and any $\nu \in R_{k-1}(\mathcal{A}_X)$, we have a global rational 1-form $\omega_\nu(C)$ on V.

Observe that if $X \in L^Y_{k-1}$ and $\beta_1 \notin X^\perp$, then Lemma 5.1 gives us a basis $C^X = (\gamma_2^X, \dots, \gamma_k^X)$ of $X^\perp$ so that $C^X|_{V(\beta_1)} = (\beta_2, \dots, \beta_k)|_{V(\beta_1)}$. We may extend the definition of C^X to any $X \in L^Y_{k-1}$ via

$$C^X = \begin{cases} (\gamma_2^X, \dots, \gamma_k^X) & \text{if } \beta_1 \notin X^\perp \\ (0, \dots, 0) & \text{otherwise.} \end{cases}$$

Observe that $\mu \in R_k(\mathcal{A}_Y) \subset \bigoplus_{X \in L^Y_{k-1}} R_{k-1}(\mathcal{A}_X)$. Recall that, for $X \in L^Y_{k-1}$, the notation μ_X denotes the X coordinate of μ. Now we may define

$$\omega_\mu(B) = \frac{1}{\beta_1} \sum_{X \in L^Y_{k-1}} \omega_{\mu_X}(C^X).$$

If $k = 2$, there is a convenient alternative formula for ω_μ. In this case, $B = (\beta_1, \beta_2)$ for some $\beta_1, \beta_2 \in Y^\perp$. Observe $\mu \in R_2(\mathcal{A}_Y) \subset \bigoplus_{X \in L^Y_1} R_1(\mathcal{A}_X)$. We may write $\mu = \sum_{X \in \mathcal{A}_Y} c_X e_X$, where $c_X \in \mathbf{K}$. For every $X \in \mathcal{A}_Y$, write $\alpha_X = \alpha_{X1}\beta_1 + \alpha_{X2}\beta_2$ with $\alpha_{X1}, \alpha_{X2} \in \mathbf{K}$. Then

$$\omega_\mu(B) = \frac{1}{\beta_1} \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \frac{d\alpha_X}{\alpha_X}. \tag{6}$$

6. Properties.

We prove several properties of the forms constructed in the previous section, including the property that these forms are logarithmic. In Theorem 6.7 it is shown that these forms are, up to scalar multiple, independent of the tuple B.

First we prove a technical lemma. Let $2 < k \leq l$, $Y \in L_k$, and $\mu \in R_k(A_Y)$. Let $B = (\beta_1, \dots, \beta_k)$ be a basis of $Y^\perp$. To simplify notation, for any $X \in L_{k-1}, Z \in L_{k-2}$, write $\mu_X^Z = (\mu_X)_Z$.

For any $Z \in L^Y_{k-2}$, if $Z^\perp \cap \langle \beta_1, \beta_2 \rangle = 0$, then let C^Z be the basis of Z given by Lemma 5.1 so that $C^Z|_{V(\beta_1,\beta_2)} = (\beta_3, \dots, \beta_k)|_{V(\beta_1,\beta_2)}$.

Extend this definition to all $Z \in L^Y_{k-2}$ via

$$C^Z = \begin{cases} C^Z & \text{(as above) if } \ Z^\perp \cap \langle \beta_1, \beta_2 \rangle = 0 \\ (0, \dots, 0) & \text{otherwise.} \end{cases}$$

Lemma 6.1. *We have the identity*

$$\beta_1\beta_2\omega_\mu(B) = \sum_{\substack{X \in L^Y_{k-1} \\ \beta_1 \notin X^\perp}} \sum_{Z \in L^Y_{k-2}} \frac{\beta_2}{\gamma_2^X} \omega_{\mu_X^Z}(C^Z).$$

Proof. First, for any $X \in L^Y_{k-1}$, $\beta_1 \notin X^\perp$, and $Z \in L^X_{k-2}$ with $\gamma_2^X \notin Z^\perp$, Lemma 5.1 gives a unique basis $C^{X,Z}$ with $C^{X,Z}|_{V(\gamma_2^X)} = (\gamma_3^X, ..., \gamma_k^X)|_{V(\gamma_2^X)}$. Extend this definition to all $X \in L^Y_{k-1}$ with $\beta_1 \notin X^\perp$ and $Z \in L^Y_{k-2}$ via

$$C^{X,Z} = \begin{cases} C^{X,Z} & \text{(as above) if } Z \in L^X_{k-2} \text{ and } \gamma_2^X \notin Z^\perp \\ (0, \dots, 0) & \text{otherwise.} \end{cases}$$

Observe that, for any $X \in L^X_{k-2}$ with $\beta_1 \notin X^\perp$, the definition of $\omega_{\mu_X}(C^X)$ gives us

$$\omega_{\mu_X}(C^X) = \frac{1}{\gamma_1^X} \sum_{Z \in L^Y_{k-2}} \omega_{\mu_X^Z}(C^{X,Z}).$$

By definition of C^Z and $C^{X,Z}$, we have $C^{X,Z} = C^Z$ for every $X \in L^Y_{k-1}$ with $\beta_1 \notin X^\perp$, $Z \in L^X_{k-2}$.

Now we are ready to prove the equation. By definition,

$$\begin{aligned} \beta_1\beta_2\omega_\mu(B) &= \beta_2 \sum_{X \in L^Y_{k-1}} \omega_{\mu_X}(C^X) \\ &= \beta_2 \sum_{\substack{X \in L^Y_{k-1} \\ \beta_1 \notin X^\perp}} \frac{1}{\gamma_2^X} \sum_{Z \in L^X_{k-2}} \omega_{\mu_X^Z}(C^{X,Z}) \\ &= \sum_{\substack{X \in L^Y_{k-1} \\ \beta_1 \notin X^\perp}} \sum_{Z \in L^X_{k-2}} \frac{\beta_2}{\gamma_2^X} \omega_{\mu_X^Z}(C^Z). \end{aligned}$$

If $Z \in L^Y_{k-2} \backslash L^X_{k-2}$, then $\mu_X^Z = 0$ and $\omega_{\mu_X^Z}(C^Z) = 0$, so the last expression is

$$\sum_{\substack{X \in L^Y_{k-1} \\ \beta_1 \notin X^\perp}} \sum_{Z \in L^Y_{k-2}} \frac{\beta_2}{\gamma_2^X} \omega_{\mu_X^Z}(C^Z).$$

□

The following lemma is well-known. It is needed in Lemma 6.3 to establish a necessary and sufficient condition for certain rational differential 1-forms to be logarithmic.

Lemma 6.2. *Let ω be a global rational differential p-form on V such that $Q\omega \in \Omega^p[V]$. Then $\omega \in \Omega^p(\mathcal{A})$ if and only if $Q(d\alpha_X/\alpha_X) \wedge \omega \in \Omega^{p+1}[V]$ for every $X \in \mathcal{A}$.*

Proof. See [**OT**, Proposition 4.78]. □

Lemma 6.3. *Let $\omega \in \Omega^1(\mathcal{A})$ and $X \in \mathcal{A}$. Let ρ denote restriction to X. Assume that α_X divides $Q\omega$ in the S-module $\Omega^1[V]$. Then $\frac{1}{\alpha_X}\omega \in \Omega^1(\mathcal{A})$ if and only if $\rho(\frac{Q}{\alpha_X}\omega) = 0$.*

Proof. Since α_X divides $Q\omega$ in $\Omega^1[V]$, we have $Q\frac{1}{\alpha_X}\omega \in \Omega^1[V]$ and we may apply Lemma 6.2.

Choose a basis $(x_1, x_2, \dots, x_l)$ of V^* with $x_1 = \alpha_1$. For every $i, 1 \le i \le l$, let $\frac{p_i}{q_i}$ represent the dx_i coefficient of ω, where p_i and q_i are chosen to be relatively prime elements of $S = \mathbf{K}[x_1, \dots, x_l]$, with q_i monic.

Assume $\frac{1}{x_1}\omega \in \Omega^1(\mathcal{A})$. Then by Lemma 6.2,

$$Q\frac{dx_1}{x_1} \wedge \frac{1}{x_1}\omega = \sum_{i=2}^{l} Q\frac{p_i}{q_i x_1^2} dx_1 \wedge dx_i \in \Omega^2[V].$$

Let $i, 2 \le i \le l$ be arbitrary. Since $\Omega^2[V]$ is a free S-module with basis $\{dx_j \wedge dx_k | 1 \le j < k \le l\}$ we have $Q\frac{p_i}{q_i x_1^2} dx_1 \wedge dx_i \in \Omega^2[V]$. Therefore $Q\frac{p_i}{q_i x_1^2}$ is polynomial. Thus x_1 divides $Q\frac{p_i}{q_i x_1}$ in the ring S, and so $\rho\left(Q\frac{p_i}{q_i x_1}\right) = 0$. Thus

$$\rho\left(\frac{Q}{x_1}\omega\right) = \rho\left(Q\frac{p_1}{q_1 x_1} dx_1\right) + \rho\left(\sum_{i=2}^{l} Q\frac{p_i}{q_i x_1} dx_i\right) = 0.$$

Conversely, assume $\rho(\frac{Q}{x_1}\omega) = 0$. We have

$$\rho\left(\frac{Q}{x_1}\omega\right) = \sum_{i=2}^{l} \rho\left(Q\frac{p_i}{q_i x_1}\right) d\rho(x_i).$$

Since $\Omega^1[V(x_1)]$ is a free $K[\rho(x_2), \dots, \rho(x_l)]$ module with basis $\{d\rho(x_2), \dots, d\rho(x_l)\}$, $\rho\left(Q\frac{p_i}{q_i x_1}\right) = 0$ for $i = 2, \dots, l$. Therefore, x_1 divides $Q\frac{p_i}{q_i x_1}$ in the ring S. Thus $Q\frac{p_i}{q_i x_1^2} \in S$ and so $Q\frac{p_i}{q_i x_1^2} dx_1 \wedge dx_i \in \Omega[V]$. Therefore $Q\frac{dx_1}{x_1} \wedge \frac{1}{x_1}\omega = \sum_{i=2}^{l} Q\frac{p_i}{q_i x_1^2} dx_1 \wedge dx_i$ is regular. By Lemma 6.2, $\frac{1}{x_1}\omega$ is logarithmic. □

Lemma 6.4. *For $\nu, \upsilon \in R_k(\mathcal{A}_Y)$ and $a, b \in \mathbf{K}$,*

$$a\omega_\nu(B) + b\omega_\upsilon(B) = \omega_{a\nu+b\upsilon}(B).$$

That is, the assignment $\mu \to \omega_\mu$ is $\mathbf{K}$-linear.

Proof. This is obvious for $k = 1$, and follows immediately for arbitrary k by induction. □

The following theorem is our main result. It is a list of properties of $\omega_\mu(B)$, with property (iii) being the most important.

Notation. For Theorem 6.5 and its proof, we make the following notational conventions. The symbol θ will represent an element of V^*, ρ will denote restriction of differential forms to $V(\theta)$, and $\mathcal{A}''$ will denote the restriction of $\mathcal{A}$ to $V(\theta)$. For $1 \leq i \leq l$, let h_i be the linear map defined in Sections 3 and 4. For $X \in L$, $\alpha \in V^*$, and $Z \in \mathcal{A}''$, let X'', α'', and η_Z be as in Section 3.

Theorem 6.5. *Let $k \geq 2$, $Y \in L_k$, and $\mu \in R_k(\mathcal{A}_Y)$. Write $\omega_\mu = \omega_\mu(B)$.*

(i) *For any permutation σ of $(1, \ldots, k)$,*

$$\omega_\mu(\beta_{\sigma(1)}, \ldots, \beta_{\sigma(k)}) = (\operatorname{sgn} \sigma)\omega_\mu.$$

(ii) *Let $\eta = b_1\beta_1 + \ldots + b_k\beta_k$, where each $b_i \in \mathbf{K}$ and $b_1 \neq 0$. Then*

$$b_1\omega_\mu(\eta, \beta_2, \ldots, \beta_k) = \omega_\mu.$$

(iii) $\omega_\mu \in \Omega(\mathcal{A}_Y)$.

(iv) *Assume $\theta \notin Y^\perp$. Then*

$$\rho(\omega_\mu) = \omega_{h_{k-1}^Y \mu}(\beta_1'', \ldots, \beta_k'').$$

Proof. First we prove (i)-(iv) in the case $k = 2$. Write $\mu = \sum_{X \in \mathcal{A}_Y} c_X e_X$, and use Equation (6) to write $\omega_\mu = \frac{1}{\beta_1} \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \frac{d\alpha_X}{\alpha_X}$.

(i) Since $k = 2$, σ is either the identity or the transposition $(1, 2)$. If $\sigma =$ identity, the claim is obvious. If $\sigma = (1, 2)$, the claim is $\omega_\mu = -\omega_\mu(\beta_2, \beta_1)$. By definition of $R_2(\mathcal{A}_Y)$, $\delta_1^Y \mu = 0$. However, $\delta_1^Y(\mu) = \sum_{X \in \mathcal{A}_Y} c_X \alpha_X$, so

$$\sum_{X \in \mathcal{A}_Y} c_X \alpha_X = 0.$$

Therefore

$$\begin{aligned} \beta_1\beta_2(\omega_\mu(\beta_2, \beta_1) + \omega_\mu) &= \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X1} \beta_1 \frac{d\alpha_X}{\alpha_X} + \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \beta_2 \frac{d\alpha_X}{\alpha_X} \\ &= \sum_{X \in \mathcal{A}_Y} c_X d\alpha_X \\ &= 0. \end{aligned}$$

There are no zero divisors in the S-module of global rational 1-forms on V. Since $\beta_1\beta_2 \neq 0$, we have $\omega_\mu(\beta_2, \beta_1) + \omega_\mu = 0$, proving the claim.

(ii) Since $b_1 \neq 0$, (η, β_2) is a basis of $Y^\perp$. For any $\alpha_X \in \mathcal{A}_Y$, write

$$\alpha_X = \alpha'_{X1}\eta + \alpha'_{X2}\beta_2.$$

Then $\alpha_{X1} = b_1\alpha'_{X1}$. Therefore

$$\begin{aligned} b_1\omega_\mu(\eta, \beta_2) &= -b_1\omega_\mu(\beta_2, \eta) \\ &= -\frac{1}{\beta_2}\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X1}\frac{d\alpha_X}{\alpha_X} \\ &= -\omega_\mu(\beta_2, \beta_1) \\ &= \omega_\mu(\beta_1, \beta_2) \end{aligned}$$

which proves property (ii).

(iii) Without loss of generality we may assume that $V(\beta_1) \in \mathcal{A}_Y$. Indeed, choose any $X \in \mathcal{A}_Y$ so that $X \neq V(\beta_2)$. Then, writing $\alpha_X = \alpha_{X1}\beta_1 + \alpha_{X2}\beta_2$, we have $\alpha_{X1} \neq 0$. By (ii), $\alpha_{X1}\omega(\alpha_X, \beta_2) = \omega(\beta_1, \beta_2)$. Since $\Omega^1(\mathcal{A})$ is closed under scalar multiplication, it suffices to show $\omega(\alpha_X, \beta_2) \in \Omega^1(\mathcal{A})$. Thus we may and shall assume $V(\beta_1) \in \mathcal{A}_Y$.

Since for every $X \in \mathcal{A}_Y$, $\frac{d\alpha_X}{\alpha_X} \in \Omega(\mathcal{A}_Y)$, the sum $\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X}$ is in the module $\Omega(\mathcal{A}_Y)$. We now check that β_1 divides $Q_Y\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X}$ in the S-module $\Omega^1[V]$ so that we may apply Lemma 6.3. Since $V(\beta_1) \in \mathcal{A}_Y$, β_1 divides Q_Y. For any $\alpha_X \in \mathcal{A}_Y$ with $\alpha_X \neq \beta_1$, α_X is nonproportional to β_1, so α_X and β_1 are not associates in the unique factorization domain S. Thus β_1 divides $\frac{Q_Y}{\alpha_X}$ in S, so β_1 divides $Q_Y c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X}$ in $\Omega^1[V]$. Therefore, β_1 divides each summand of

$$\begin{aligned} Q_Y\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X} &= \sum_{X\in\mathcal{A}_Y} Q_Y c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X} \\ &= \sum_{\substack{X\in\mathcal{A}_Y \\ \alpha_X\neq\beta_1}} Q_Y c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X} \end{aligned}$$

in $\Omega^1[V]$, and so β_1 divides $Q_Y\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X}$.

Let ρ_1 denote restriction to $V(\beta_1)$. By Lemma 6.3, it suffices to show

$$\begin{aligned} \rho_1(Q_Y\omega_\mu) &= \rho_1\left(\frac{Q_Y}{\beta_1}\sum_{X\in\mathcal{A}_Y} c_X\alpha_{X2}\frac{d\alpha_X}{\alpha_X}\right) \\ &= 0. \end{aligned}$$

By definition, $\mu \in \ker \delta_1^Y$. But

$$\begin{aligned}\delta_1^Y \mu &= \sum_{X \in \mathcal{A}_Y} c_X \alpha_X \\ &= \sum_{X \in \mathcal{A}_Y} c_X (\alpha_{X1}\beta_1 + \alpha_{X2}\beta_2) \\ &= \left(\sum_{X \in \mathcal{A}_Y} c_X \alpha_{X1} \right) \beta_1 + \left(\sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \right) \beta_2.\end{aligned}$$

Since (β_1, β_2) is a basis of $Y^\perp$, $\sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} = 0$.

Thus

$$\begin{aligned}\rho_1 \left(\frac{Q_Y}{\beta_1} \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \frac{d\alpha_X}{\alpha_X} \right) &= \rho_1 \left(\frac{Q_Y}{\beta_1} \right) \left(\sum_{\substack{X \in \mathcal{A}_Y \\ X \neq V(\beta_1)}} c_X \alpha_{X2} \right) \frac{d\rho_1(\beta_2)}{\rho_1(\beta_2)} \\ &= 0.\end{aligned}$$

Therefore, by Lemma 6.3, $\omega_\mu = \frac{1}{\beta_1} \sum_{X \in \mathcal{A}} c_X \alpha_{X2} \frac{d\alpha_X}{\alpha_X} \in \Omega(\mathcal{A}_Y)$.

(iv) For any $Z \in (\mathcal{A}_Y)''$, write

$$\eta_Z = \eta_{Z1}\beta_1'' + \eta_{Z2}\beta_2''.$$

For $X \in \mathcal{A}_Y$, write $\alpha_X'' = \alpha_{X1}\beta_1'' + \alpha_{X2}\beta_2''$, and note that

$$\alpha_{X2}\eta_{X''} = \eta_{X''2}\alpha_X''$$

i.e.

$$\frac{\alpha_X''}{\eta_{X''}} \eta_{X''2} = \alpha_{X2}.$$

Now recall that $h_1^Y \mu = \sum_{X \in \mathcal{A}_Y} c_X \frac{\alpha_X''}{\eta_{X''}} e_{X''}$, so

$$\begin{aligned}\omega_{h_1^Y \mu}(\beta_1'', \ldots, \beta_k'') &= \frac{1}{\beta_1''} \sum_{X \in \mathcal{A}_Y} c_X \frac{\alpha_X''}{\eta_{X''}} \eta_{X''2} \frac{d\eta_{X''}}{\eta_{X''}} \\ &= \frac{1}{\beta_1''} \sum_{X \in \mathcal{A}_Y} c_X \alpha_{X2} \frac{d\alpha_X''}{\alpha_X''} \\ &= \rho(\omega_\mu).\end{aligned}$$

This proves property (iv).

Now we prove properties (i)-(iv) in the case when $k > 2$.

(i) *Step* 1. We claim $\omega_\mu(\beta_2, \beta_1, \beta_3, \ldots, \beta_k) = -\omega_\mu$, i.e., the property holds if σ is the transposition $(1, 2)$.

For $X \in L_{k-1}^Y$, $\beta_1 \notin X^\perp$, let a_i^X be as in Lemma 5.2. Then $C^X = (\gamma_2^X, \ldots, \gamma_k^X)$ has $\gamma_i = \beta_i - a_i^X \beta_1$ for each i. Now consider the basis $(\beta_2, \beta_1, \beta_3, \beta_4, \ldots, \beta_k)$ of $Y^\perp$. For any $X \in L_{k-1}^Y$ with $\beta_2 \notin X^\perp$, let D^X be

the unique basis of $X^\perp$ given by Lemma 5.1 with $D^X|_{V(\beta_2)} = (\beta_1, \beta_3, \beta_4, \dots, \beta_k)|_{V(\beta_2)}$. Then for each i, $i = 1, 3, 4, \dots, k$, let b_i^X be the scalar with $b_i^X \beta_2 \equiv \beta_i \pmod{X^\perp}$. Lemma 5.2 applied in this situation shows

$$D^X = (\beta_1 - b_1^X \beta_2\ ,\ \beta_3 - b_3^X \beta_2,\ \beta_4 - b_4^X \beta_2\ , \dots\ ,\ \beta_k - b_k^X \beta_2).$$

Now apply Lemma 6.1, obtaining

$$\beta_1\beta_2\omega_\mu(B) = \sum_{\substack{X \in L_{k-1}^Y \\ \beta_1 \notin X^\perp}} \sum_{Z \in L_{k-2}^Y} \frac{\beta_2}{\beta_2 - a_2^X \beta_1} \omega_{\mu_X^Z}(C^Z).$$

Another application of the same lemma yields

$$\beta_1\beta_2\omega_\mu(\beta_2, \beta_1, \beta_3, \dots, \beta_k) = \sum_{\substack{X \in L_{k-1}^Y \\ \beta_2 \notin X^\perp}} \sum_{Z \in L_{k-2}} \frac{\beta_1}{\beta_1 - b_1^X \beta_2} \omega_{\mu_X^Z}(C^Z).$$

Now notice that for any $X \in L_{k-1}^Y$ with $\beta_1 \notin X^\perp$, $\beta_2 \notin X^\perp$, we have $b_1^X = \frac{1}{a_2^X}$ by definition of these scalars. Also note that, for X with $\beta_1 \notin X^\perp$, $a_2^X = 0$ if and only if $\beta_2 \in X^\perp$ and for X with $\beta_2 \notin X^\perp$, $b_1^X = 0$ if and only if $\beta_1 \in X^\perp$. Together, these observations imply that

$$\beta_1\beta_2(\omega_\mu(\beta_2, \beta_1, \beta_3, \dots, \beta_k) + \omega_\mu) = \sum_{Z \in L_{k-2}^Y} \sum_{X \in L_{k-1}^Y} \omega_{\mu_X^Z}(C^Z).$$

By Lemma 6.4, the last expression is

$$\sum_{Z \in L_{k-2}^Y} \omega_\diamond(C^Z),$$

where $\diamond = \sum_{X \in L_{k-1}^Y} \mu_X^Z$. Now, by definition of δ_{k-1}^Y, $\diamond = (\delta_{k-1}^Y \mu)_Z$. However, $\delta_{k-1}^Y \mu = 0$. So by definition, each $\omega_\diamond(C^Z) = 0$. Therefore

$$\beta_1\beta_2(\omega_\mu(\beta_2, \beta_1, \beta_3, \dots, \beta_k) + \omega_\mu) = 0.$$

Since $\beta_1\beta_2$ is nonzero,

$$\omega_\mu(\beta_2, \beta_1, \beta_3, \dots, \beta_k) + \omega_\mu = 0$$

i.e., $\omega_\mu(\beta_2, \beta_1, \beta_3, \dots, \beta_k) = -\omega_\mu$. This proves the claim in Step 1.

Step 2. We now show, for any $q > 1$,

$$\omega_\mu(\beta_1, \beta_2, \dots, \beta_{q-1}, \beta_{q+1}, \beta_q, \beta_{q+2} \dots, \beta_k) = -\omega_\mu,$$

that is, the claim in (i) holds when σ is the transposition $(q, q+1)$.

Proof of Step 2: Let $C^X = (\gamma_2^X, \dots, \gamma_k^X)$ be as in Lemma 5.1. Then by the inductive hypothesis,

$$\begin{aligned}\omega_\mu &= \frac{1}{\beta_1}\sum \omega_{\mu_X}(C^X)\\ &= -\frac{1}{\beta_1}\sum \omega_{\mu_X}(\gamma_2^X, \dots, \gamma_{q+1}^X, \gamma_q^X, \dots, \gamma_k^X)\\ &= -\omega_\mu(\beta_1, \beta_2, \dots, \beta_{q+1}, \beta_q, \dots, \beta_k).\end{aligned}$$

Step 3. We now show $\omega_\mu(\beta_{\sigma(1)}, \dots, \beta_{\sigma(k)}) = \operatorname{sgn}\sigma\omega_\mu$.

Write σ as a product $\sigma_N \circ \sigma_{N-1} \circ \cdots \circ \sigma_1$ of transpositions σ_i with the following property.

Let $\tau_i = \sigma_{i-1} \circ \cdots \circ \sigma_1$ for $i > 1$ and τ_1 be the identity. Then for every i, $1 \le i \le N$, σ_i switches the positions of two adjacent entries in the tuple $(\tau_i(1), \dots, \tau_i(k))$.

By Steps 1 and 2,

$$\begin{aligned}\omega_\mu(\beta_{\sigma(1)}, \dots, \beta_{\sigma(k)}) &= \omega_\mu(\beta_{\tau_{N+1}(1)}, \dots, \beta_{\tau_{N+1}(k)})\\ &= -\omega_\mu(\beta_{\tau_N(1)}, \dots, \beta_{\tau_N(k)})\\ &= \cdots\\ &= (-1)^N\omega_\mu\\ &= \operatorname{sgn}\sigma\omega_\mu.\end{aligned}$$

(ii) By property (i), Lemma 5.2, and induction we have, for some $s_\eta^X, s_1^X, s_3^X, \cdots s_k^X \in \mathbf{K}$,

$$\begin{aligned}b_1\omega_\mu(\eta, \beta_2, \dots, \beta_k) &= -b_1\omega_\mu(\beta_2, \eta, \beta_3 \dots, \beta_k)\\ &= -\frac{b_1}{\beta_2}\sum_{X\in L_{k-1}} \omega_\mu(\eta - s_\eta^X\beta_2, \beta_3 - s_3^X\beta_2, \dots, \beta_k - s_k^X\beta_2)\\ &= -\frac{1}{\beta_2}\sum_{X\in L_{k-1}} \omega_\mu(\beta_1 - s_1^X\beta_2, \beta_3 - s_3^X\beta_2 \dots, \beta_k - s_k^X\beta_2)\\ &= \omega_\mu.\end{aligned}$$

(iii) We may assume without loss of generality that $V(\beta_1) \in \mathcal{A}_Y$. Indeed, take any $X \in \mathcal{A}_Y$ with $\alpha_{X1} \neq 0$, where $\alpha_X = \alpha_{X1}\beta_1 + \alpha_{X2}\beta_2 + \cdots + \alpha_{Xk}\beta_k$. Then by (ii), $\omega_\mu(\alpha_X, \beta_2, \dots, \beta_k) \doteq \omega_\mu$. Thus it suffices to show $\omega_\mu(\alpha_X, \beta_2, \dots, \beta_k) \in \Omega(\mathcal{A})$. Thus we may and shall assume that $V(\beta_1) \in \mathcal{A}_Y$.

By the inductive hypothesis (iii), $Q_X\omega_{\mu_X}(C^X) \in \Omega^1[V]$ for each $X \in L_{k-1}^Y$. For any $X \in L_{k-1}^Y$ with $\beta_1 \notin X^\perp$, we have $\beta_1 \notin \mathcal{A}_X$ so β_1 does not divide Q_X in S. However, β_1 divides Q_Y so β_1 divides $\frac{Q_Y}{Q_X}$. Thus β_1 divides

$\frac{Q_Y}{Q_X}Q_X\omega_{\mu_X}(C^X) = Q_Y\omega_{\mu_X}(C^X)$ in $\Omega^1[V]$. Any X with $\beta_1 \in X^\perp$ has $\mu_X = 0$ so β_1 divides

$$Q_Y \sum_{X\in L^Y_{k-1}} \omega_{\mu_X}(C^X) = Q_Y \sum_{\substack{X\in L^Y_{k-1}\\ \beta_1\notin X^\perp}} \omega_{\mu_X}(C^X) = \sum_{\substack{X\in L^Y_{k-1}\\ \beta_1\notin X^\perp}} Q_Y\omega_{\mu_X}(C^X)$$

in $\Omega^1[V]$. Thus by Lemma 6.2 it suffices to check that the restriction of $Q_Y\omega_\mu = \frac{Q_Y}{\beta_1}\sum_{X\in L^Y_{k-1}} \omega_{\mu_X}(C^X)$ to $V(\beta_1)$ is 0.

To see this, we apply the inductive hypothesis in the case $\theta = \beta_1$. Note that for $X \in L^Y_{k-1}$ with $\beta_1 \notin X^\perp$, and $i \geq 2$, $(\gamma_i^X)'' = \beta_i''$ by definition. Also note that for $X \in L^Y_{k-1}$ with $\beta_1 \notin X^\perp$, $X'' = Y$.

By Lemma 6.4 we have

$$\begin{aligned}
\rho_1\left(\frac{Q_Y}{\beta_1}\sum_{X\in L^Y_{k-1}} \omega_{\mu_X}(C_X)\right) &= \rho_1\left(\frac{Q_Y}{\beta_1}\sum_{\substack{X\in L^Y_{k-1}\\ \beta\notin X^\perp}} \omega_{\mu_X}(C_X)\right)\\
&= \rho_1\left(\frac{Q_Y}{\beta_1}\right)\sum_{\substack{X\in L^Y_{k-1}\\ \beta\notin X^\perp}} \omega_{h^X_{k-2}\mu_X}(\beta_2'',\dots,\beta_k'')\\
&= \rho_1\left(\frac{Q_Y}{\beta_1}\right)\omega_\diamond(\beta_2'',\dots,\beta_2'').
\end{aligned}$$

Where $\diamond = \sum_{\substack{X\in L^Y_{k-1}\\ \beta_1\notin X^\perp}} h^X_{k-2}\mu_X$. This is 0 by Lemma 4.3.

Thus we have shown

$$\rho_1(Q_Y\omega_\mu) = \omega_0(\beta_2'',\dots,\beta_2'') = 0.$$

This proves (iii).

(iv) To simplify notation, we write $\gamma_i^{X''}$ to stand for $(\gamma_i^X)''$. For any $Z \in L''$, let $Z^\perp_{V(\theta)}$ represent the orthogonal complement of Z in $V(\theta)^*$. Observe that since $\theta \notin Y^\perp$, the codimension of Y'' in $V(\theta)$ is k, and $\beta_1'',\dots,\beta_k''$ form a basis for $(Y'')^\perp_{V(\theta)}$.

Recall that $(\mathcal{A}_Y)'' \subset \mathcal{A}''_{Y''}$, and in particular, $L(\,(\mathcal{A}_Y)''\,) \subset L(\mathcal{A}''_{Y''})$. Thus for $Z \in L_{k-1}(\,(\mathcal{A}_Y)''\,)$ with $\beta_1'' \notin Z^\perp_{V(\theta)}$, Lemma 5.1 gives us a basis E^Z of elements in $Z^\perp_{V(\theta)}$ with $E^Z|_{\ker\beta_1''} = (\beta_2'',\dots,\beta_k'')|_{\ker\beta_1''}$. For any $X \in L_{k-1}$ with $X'' = Z$, the tuple $(\gamma_2^{X''},\dots,\gamma_k^{X''})$ consists of elements of $Z^\perp_{V(\theta)}$. Furthermore, for any i, $i = 2,\dots,k$, $\gamma_i^{X''}|_{\ker\beta_1''} = \beta_i''|_{\ker\beta_1''}$. Thus by part (ii) of Lemma 5.1, $E^Z = (\gamma_2^{X''},\dots,\gamma_k^{X''})$.

We have

$$\begin{aligned}\rho(\omega_\mu(\beta_1,\dots,\beta_k)) &= \rho\left(\frac{1}{\beta_1}\sum_{X\in L_{k-1}^Y}\omega_{\mu_X}(\gamma_2^X,\dots,\gamma_k^X)\right)\\ &= \frac{1}{\beta_1''}\sum_{X\in L_{k-1}^Y}\rho(\omega_{\mu_X}(\gamma_2^X,\dots,\gamma_k^X))\\ &= \frac{1}{\beta_1''}\sum_{X\in L_{k-1}^Y}\omega_{h_{k-2}^X\mu_X}(\gamma_2^{X''},\dots,\gamma_k^{X''})\\ &= \frac{1}{\beta_1''}\sum_{X\in L_{k-1}^Y}\omega_{h_{k-2}^X\mu_X}(E^{X''}).\end{aligned}$$

By Lemma 6.4, we obtain

$$\rho(\omega_\mu(\beta_1,\dots,\beta_k)) = \frac{1}{\beta_1''}\sum_{Z\in L_{k-1}((\mathcal{A}_Y)'')}\omega_\diamond(E^Z).$$

Where $\diamond = \sum_{\substack{X\in L_{k-1}^Y\\ X''=Z}} h_{k-2}^X\mu_X = (h_{k-1}^Y\mu)_Z$. This proves (iv). □

To prove that the definition of $\omega_\mu(B)$ is, up to scalar multiple, independent of the tuple B, we need the following well-known result.

Lemma 6.6. (Chio's Pivotal Condensation Method.) *Let* $\Delta = \left|(\eta_{ij})_{1\le i,j\le k}\right|$ *be a determinant. Let* $1 \le m \le k$.

Let

$$\Delta' = \begin{vmatrix} \chi_{12} & \cdots & \chi_{1k}\\ \cdots & \cdots & \cdots\\ \chi_{m-12} & \cdots & \chi_{m-1k}\\ \chi_{m+12} & \cdots & \chi_{m+1k}\\ \cdots & \cdots & \cdots\\ \chi_{k2} & \cdots & \chi_{kk}\end{vmatrix} \qquad \textit{where} \qquad \chi_{ij} = \begin{vmatrix}\eta_{m1} & \eta_{mj}\\ \eta_{i1} & \eta_{ij}\end{vmatrix}.$$

Then $\Delta' = (-1)^{m-1}\eta_{m1}^{k-2}\Delta$.

Proof. This is an easy generalization of [**Ev**, Theorem 3.6.1]. □

The next theorem shows that the definition of $\omega_\mu(B)$ is, up to scalar multiple, independent of the tuple B.

Theorem 6.7. *Let* $Y \in L_k$, *and let* $B = (\beta_1,\dots,\beta_k)$ *and* $E = (\eta_1,\dots,\eta_k)$ *be bases of* $Y^\perp$. *Let* $\mu \in R_k(\mathcal{A}_Y)$. *For each* i, $1 \le i \le k$, *let* $\eta_i = \sum_{j=1}^k \eta_{ij}\beta_j$. *Let* Δ *denote the determinant* $\Delta = \left|(\eta_{ij})_{1\le i,j\le k}\right|$. *Then* $\omega_\mu(B) = \Delta\omega_\mu(E)$.

Proof. The proof proceeds by induction on k.

For $k = 2$, we give the argument in case $\eta_{11} \neq 0$. The proof for the only other case, $\eta_{11} = 0$ and $\eta_{21} \neq 0$, is similar.

First observe

$$\eta_2 = \frac{\eta_{21}}{\eta_{11}}\eta_1 + \left(\eta_{22} - \frac{\eta_{21}\eta_{12}}{\eta_{11}}\right)\beta_2.$$

Now

$$\begin{aligned}\omega(\beta_1, \beta_2) &= \eta_{11}\omega(\eta_1, \beta_2) \\ &= -\eta_{11}\omega(\beta_2, \eta_1) \\ &= -\eta_{11}\left(\eta_{22} - \frac{\eta_{21}\eta_{12}}{\eta_{11}}\right)\omega(\eta_2, \eta_1) \\ &= (\eta_{11}\eta_{22} - \eta_{12}\eta_{21})\omega(\eta_1, \eta_2).\end{aligned}$$

This completes the proof for $k = 2$.

For $k > 2$, assume that the result holds for every k', $2 \leq k' < k$. Choose $m, 1 \leq m \leq k$ so that $\eta_{m1} \neq 0$. By Theorem 6.5

$$\begin{aligned}\omega_\mu(\beta_1, \beta_2, \ldots, \beta_k) &= \eta_{m1}\omega(\eta_m, \beta_2, \ldots, \beta_k) \\ &= \frac{\eta_{m1}}{\eta_m}\sum_{X \in L_{k-1}} \omega_{\mu_X}(\xi_2^X, \ldots, \xi_k^X).\end{aligned}$$

Where $(\xi_2^X, \ldots, \xi_k^X)$ is a basis of $X^\perp$ with $(\xi_2^X, \ldots, \xi_k^X)|_{V(\eta_m)} = (\beta_2, \ldots, \beta_k)|_{V(\eta_m)}$.

Note that for $i \neq m$,

$$\eta_i = \frac{\eta_{i1}}{\eta_{m1}}\eta_m + \sum_{j=2}^{k}\left(\eta_{ij} - \frac{\eta_{i1}}{\eta_{m1}}\eta_{mj}\right)\beta_j. \tag{7}$$

Fix an $i, 1 \leq i \leq k$ with $i \neq m$, and consider $\sum_{j=2}^{k}\left(\eta_{ij} - \frac{\eta_{i1}}{\eta_{m1}}\eta_{mj}\right)\xi_j^X$. By (7) and Lemma 5.2, this is a linear functional of the form $\eta_i - a\eta_m$ for some $a \in \mathbf{K}$. Thus this functional's restriction to $V(\eta_m)$ is exactly the functional $\eta_i|_{V(\eta_m)}$. It is in $X^\perp$, since each ξ_j^X is in $X^\perp$. For $1 \leq i \leq m$, $i \neq m$, let

$$\zeta_i^X = \sum_{j=2}^{k}\left(\eta_{ij} - \frac{\eta_{i1}}{\eta_{m1}}\eta_{mj}\right)\xi_j^X.$$

Then

$$(\zeta_1^X, \ldots, \zeta_{m-1}^X, \zeta_{m+1}^X, \ldots, \zeta_k^X)|_{V(\eta_m)} = (\eta_1, \ldots, \eta_{m-1}, \eta_{m+1}, \ldots, \eta_k)|_{V(\eta_m)}$$

so, by definition,

$$\omega_\mu(\eta_m, \eta_1, \eta_2, \ldots, \hat{\eta}_m, \ldots, \eta_k) = \frac{1}{\eta_m}\sum_{X \in L_{k-1}^Y} \omega_{\mu_X}(\zeta_1^X, \ldots, \hat{\zeta}_m^X, \ldots, \zeta_k^X).$$

By the inductive hypothesis and the definition of ζ_i^X,

$$\begin{aligned}\omega(\beta_1,\beta_2,\dots,\beta_k) &= \frac{\eta_{m1}}{\eta_m}\sum_{X\in L_{k-1}^Y}\omega_{\mu_X}(\xi_2^X,\dots,\xi_k^X)\\ &= \frac{\eta_{m1}}{\eta_m}\sum_{X\in L_{k-1}^Y}\Delta_1\omega_{\mu_X}(\zeta_1^X,\dots,\zeta_{m-1}^X,\zeta_{m+1}^X,\dots,\zeta_k^X).\end{aligned}$$

Where

$$\Delta_1 = \begin{vmatrix} A_{12} & \cdots & A_{1k}\\ \vdots & & \vdots\\ A_{m-12} & \cdots & A_{m-1k}\\ A_{m+12} & \cdots & A_{m+1k}\\ \vdots & & \vdots\\ A_{k2} & \cdots & A_{kk}\end{vmatrix}$$

and $A_{ij} = \frac{1}{\eta_{m1}}\begin{vmatrix}\eta_{m1} & \eta_{mj}\\ \eta_{i1} & \eta_{ij}\end{vmatrix}$.

Let

$$\Delta_2 = \begin{vmatrix} \hat{A}_{12} & \cdots & \hat{A}_{1k}\\ \vdots & & \vdots\\ \hat{A}_{m-12} & \cdots & \hat{A}_{m-1k}\\ \hat{A}_{m+12} & \cdots & \hat{A}_{m+1k}\\ \vdots & & \vdots\\ \hat{A}_{k2} & \cdots & \hat{A}_{kk}\end{vmatrix}$$

where $\hat{A}_{ij} = \begin{vmatrix}\eta_{m1} & \eta_{mj}\\ \eta_{i1} & \eta_{ij}\end{vmatrix}$.

Then by elementary row operations, $\Delta_1 = \left(\frac{1}{\eta_{m1}}\right)^{k-1}\Delta_2$. Thus

$$\omega(\beta_1,\beta_2,\dots,\beta_k) = \frac{\left(\frac{1}{\eta_{m1}}\right)^{k-2}\Delta_2}{\eta_m}\sum_{X\in L_{k-1}^Y}\omega_{\mu_X}(\zeta_1^X,\dots,\zeta_{m-1}^X,\zeta_{m+1}^X,\dots,\zeta_k^X).$$

By applying Lemma 6.6,

$$\left(\frac{1}{\eta_{m1}}\right)^{k-2}\Delta_2 = \frac{\eta_{m1}^{k-2}}{\eta_{m1}^{k-2}}(-1)^{m-1}\Delta = (-1)^{m-1}\Delta.$$

So

$$\begin{aligned}\omega(\beta_1,\beta_2,\dots,\beta_k) &= \frac{\eta_{m1}}{\eta_m}\sum_{X\in L_{k-1}^Y}\Delta_1\omega_{\mu_X}(\zeta_1^X,\dots,\zeta_{m-1}^X,\zeta_{m+1}^X,\dots,\zeta_k^X)\\ &= (-1)^{m-1}\frac{\Delta}{\eta_m}\sum_{Z\in L_{k-1}}\omega_{\mu_Z}(\zeta_1,\dots,\zeta_{m-1},\zeta_{m+1},\dots,\zeta_k)\\ &= (-1)^{m-1}\Delta\omega_\mu(\eta_m,\eta_1,\dots,\eta_{m-1},\eta_{m+1},\dots,\eta_k)\\ &= (-1)^{m-1}(-1)^{m-1}\Delta\omega_\mu(\eta_1,\dots,\eta_k).\end{aligned}$$

This proves the theorem. □

7. Type A arrangements.

Consider a vector space of dimension $l+1$ with basis $u_1,\dots,u_{l+1}$. The *Coxeter arrangement of type* A_l consists of the planes perpendicular to u_i-u_j for $1\le i<j\le l+1$. This arrangement is the product of an empty 1-arrangement with an irreducible arrangement. We consider here the irreducible factor $\mathcal{A}$ of the Coxeter arrangement of type A_l. We compute certain elements of the relation spaces associated to $\mathcal{A}$, compute their corresponding logarithmic 1-forms, and show that they generate the module $\Omega^1(\mathcal{A})$. It is an easy consequence that the logarithmic 1-forms described in Section 5 generate all logarithmic 1-forms for Coxeter arrangements of type A_l. The argument here is field-independent.

For the example in this section, we will need the following well-known property.

Proposition 7.1. (Saito's criterion.) *Let* $\omega_1,\dots,\omega_l\in\Omega_1(\mathcal{A})$. *Then* $\Omega^1(\mathcal{A})$ *is a free* S*-module with basis* $\omega_1,\dots,\omega_l$ *if and only if* $\omega_1\wedge\cdots\wedge\omega_l$ *is a* $\mathbf{K}$*-multiple of* $Q^{-1}dx_1\wedge\cdots\wedge dx_l$.

Proof. See [**OT**, Proposition 4.80]. □

Let $x_1,\dots,x_l$ be a basis of V^*. For $1\le p\le l$, let $X_p=V(x_p)$, and for $1\le p<q\le l$, let $Y_{pq}=V(x_p-x_q)$. Let $\mathcal{A}$ be the arrangement consisting of all the hyperplanes X_p and Y_{pq}. We choose linear functionals associated to elements of $\mathcal{A}$ as follows. For $1\le p\le l$, let $\alpha_{X_p}=x_p$, and for $1\le p<q\le l$, let $\alpha_{Y_{pq}}=x_p-x_q$.

For $1\le p\le l$, denote the basis element of $\bigoplus_{X\in L_1}R_1(\mathcal{A}_X)$ corresponding to X_p by e_p, and for $1\le p<q\le l$, denote the basis element corresponding to Y_{pq} by e_{pq}.

We now recursively compute certain elements of the spaces $R_k(\mathcal{A}_X)$ for $k\ge 2$. For $k\ge 2$ and $1\le i_1<i_2<\cdots<i_k\le l$, let $X_{i_1\cdots i_k}=V(x_{i_1},\dots,x_{i_k})\in L_k$, and let $Y_{i_1\cdots i_k}=V(x_{i_1}-x_{i_2},x_{i_1}-x_{i_3},\dots,x_{i_1}-x_{i_k})\in L_{k-1}$. Observe that this is consistent with the definition of Y_{pq} above.

For every p and q with $1 \leq p < q \leq l$ we have a relation $\mu_{pq} = e_p - e_q - e_{pq} \in R_2(\mathcal{A}_{X_{pq}})$, and for every p, q, and r with $1 \leq p < q < r \leq l$, we have a relation $\nu_{pqr} = -e_{pq} + e_{pr} - e_{qr} \in R_2(\mathcal{A}_{Y_{pqr}})$.

These relations generalize recursively as follows. For $k \geq 3$ and $1 \leq i_1 < \cdots < i_k \leq l$, let

$$\mu_{i_1 \cdots i_k} = \sum_{j=1}^{k} (-1)^j \mu_{i_1 \cdots \hat{i}_j \cdots i_k} + \nu_{i_1 \cdots i_k} \in R_k(\mathcal{A}_{X_{i_1 \cdots i_k}}),$$

and let

$$\nu_{i_1 \cdots i_k} = \sum_{j=1}^{k} (-1)^{j+1} \nu_{i_1 \cdots \hat{i}_j \cdots i_k} \in R_{k-1}(\mathcal{A}_{Y_{i_1 \cdots i_k}}).$$

It now follows by direct computation and recursion that, for any $k \geq 2$ and $1 \leq i_1 < \cdots < i_k \leq l$, we have

$$\omega_{\mu_{i_1 \cdots i_k}}(x_{i_1}, \ldots, x_{i_k}) = \sum_{j=1}^{k} (-1)^{j+1} \frac{dx_{i_j}}{x_{i_j} \prod_{1 \leq p < j} (x_{i_p} - x_{i_j}) \prod_{j < p \leq k} (x_{i_j} - x_{i_p})}.$$

Proposition 7.1 now implies that the forms

$$\omega_{i_1}(x_1), \omega_{i_1 i_2}(x_1, x_2), \ldots, \omega_{i_1 i_2 \cdots i_l}(x_1, x_2, \ldots, x_l)$$

constitute a basis for the free S-module $\Omega^1(\mathcal{A})$.

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