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SINGULAR LIMIT OF SOLUTIONS OF  
THE EQUATION  $u_t = \Delta \left( \frac{u^m}{m} \right)$  AS  $m \rightarrow 0$

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We will show that for  $n = 1, 2$ , as  $m \rightarrow 0$  the solution  $u^{(m)}$  of the fast diffusion equation  $\partial u / \partial t = \Delta(u^m/m)$ ,  $u > 0$ , in  $R^n \times (0, \infty)$ ,  $u(x, 0) = u_0(x) \geq 0$  in  $R^n$ , where  $u_0 \in L^1(R^n) \cap L^\infty(R^n)$  will converge uniformly on every compact subset of  $R^n \times (0, T)$  to the maximal solution of the equation  $v_t = \Delta \log v$ ,  $v(x, 0) = u_0(x)$ , where  $T = \infty$  for  $n = 1$  and  $T = \int_{R^2} u_0 dx / 4\pi$  for  $n = 2$ .

The degenerate parabolic equation

$$(0.1) \quad \begin{cases} u_t = \Delta \left( \frac{u^m}{m} \right), u > 0, & \text{in } R^n \times (0, \infty) \\ u(x, 0) = u_0(x) \geq 0 & \forall x \in R^n \end{cases}$$

where  $m > 0$  arises in the modelling of many physical phenomenon such as the flow of gases through a porous medium or the flow of viscous fluid on a surface. When  $m > 1$ , the above equation is called the porous medium equation. When  $0 < m < 1$ , it is called the fast diffusion equation and when  $m = 1$ , it becomes the famous heat equation. We refer the reader to the papers by Aronson [A] and Peletier [P] for extensive reference on the above equation.

In this paper we will investigate the convergence of solution  $u^{(m)}$  of the fast diffusion equation (0.1) as  $m \rightarrow 0$ . We will show that for  $n = 1, 2$ , if  $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ , then as  $m \rightarrow 0$  the solution  $u^{(m)}$  of the above fast diffusion equation will converge uniformly on every compact subset of  $R^n \times (0, T)$  to the maximal solution of

$$(0.2) \quad \begin{cases} v_t = \Delta \log v, v > 0, & \text{in } R^n \times (0, T) \\ v(x, 0) = u_0(x) & \forall x \in R^n \end{cases}$$

where

$$(0.3) \quad T = \begin{cases} \infty & \text{if } n = 1 \\ \frac{1}{4\pi} \int_{R^2} u_0 dx & \text{if } n = 2. \end{cases}$$

As proved in [ERV] for  $u_0 \in L^1(R)$  when  $n = 1$  and in [DP], [H] for  $u_0 \in L^1(R^2) \cap L^p(R^2)$  for some  $p > 1$  when  $n = 2$ , there exists infinitely

many solutions of (0.2). However by the results of [ERV] for  $n = 1$  and [DD] for  $n = 2$  there exists only a single maximal solution of (0.2) for  $n = 1, 2$ . So the limit  $\lim_{m \rightarrow 0} u^{(m)}$  is unique.

When  $n = 2$  L.F.Wu [W1], [W2] showed that (0.2) is related to the study of Ricci flow and when  $n = 1$  K.G. Kurtz [K] showed that (0.2) is related to the limiting density of two type of particles moving against each other obeying the Boltzmann equation. Similar type of singular limits for solutions of (0.1) as  $m \rightarrow \infty$  and  $m \rightarrow 1$  are obtained by L.A. Caffarelli and A. Friedman [CF2], P.L. Lions, P.E. Souganidis and J.L. Vazquez [LSV] and for the p-laplacian equation which are related to the sandpile model by L.C. Evans, M. Feldman, etc. [AEW], [EFG], [EG].

For any domain  $\Omega \subset R^n$ ,  $T > 0$ ,  $\phi(u) = u^m/m$  for some  $0 < m < 1$  or  $\phi(u) = \log u$ , we say that  $u$  is a solution (respectively subsolution, supersolution) of

$$(0.4) \quad \frac{\partial u}{\partial t} = \Delta \phi(\tilde{u})$$

in  $\Omega \times (0, T)$  if  $u \in C(\bar{\Omega} \times (0, T)) \cap C^\infty(\Omega \times (0, T))$ ,  $u > 0$  on  $\bar{\Omega} \times (0, T)$ , and satisfies (0.4) (respectively  $\leq, \geq$ ) in  $\Omega \times (0, T)$  in the classical sense and we say that  $u$  has initial value  $u_0$  if the following holds

$$\lim_{t \rightarrow 0} \int_{\Omega} u(x, t) \eta(x) dx = \int_{\Omega} u_0(x) \eta(x) dx \quad \forall \eta \in C_0^\infty(\Omega).$$

We say that  $u$  is a solution (respectively subsolution, supersolution) of

$$\begin{cases} u_t = \Delta \phi(u), u > 0, & \text{in } R^n \times (0, \infty) \\ u(x, 0) = u_0(x) \geq 0 & \forall x \in R^n \end{cases}$$

if  $u$  is a solution (respectively subsolution, supersolution) of (0.4) in  $R^n \times (0, T)$  with initial value  $u_0$ . We say that  $\tilde{v}$  is a maximal solution of (0.2) if  $\tilde{v} \geq v$  for any solution  $v$  of (0.2). For any  $x_0 \in R^n$ ,  $R > 0$ , we let  $B_R(x_0) = \{x \in R^n : |x - x_0| < R\}$  and  $B_R = B_R(0)$ .

The plan of the paper is as follows. In Section 1 we will recall some existence results and construct some subsolutions of (0.1). In Section 2 we will use the subsolutions obtained in Section 1 to obtain lower bound estimates for the solution of (0.1). We will also prove a Harnack type inequality for solution of (0.1) and a convergence result for the case  $n = 1, 2$ , under the assumption that  $u_0 \in L^1(R^n) \cap L^\infty(R^n)$  and  $\varepsilon(1 + |x|^2)^{-\alpha} \leq u_0(x) \leq M$  for some constants  $\varepsilon > 0$ ,  $M \geq 1$ , and  $1 < \alpha < 2$  for all  $x \in R^n$ . The general convergence result for  $u_0 \in L^1(R^n) \cap L^\infty(R^n)$  then follows by an approximation argument. Since we are concerned with the limit  $m \rightarrow 0$ , we will assume  $0 < m < 1$  and let  $u^{(m)}$  be the solution of (0.1) for the rest of the paper.

**Section 1.**

In this section we will recall some existence results and comparison principles for solutions of (0.1) in [AB], [BBC], [DK]. We will also construct some subsolutions of (0.1). We first start with an existence result of [AB], [DK]:

**Theorem 1.1** ([AB], [DK]). *If  $u_0 \in L^1(R^n)$  and  $(n - 2)_+/n < m < 1$ , then (0.1) will have a unique solution  $u^{(m)} \in L^\infty((0, \infty); L^1(R^n)) \cap C^\infty(R^n \times (0, \infty))$ ,  $u^{(m)} > 0$  on  $R^n \times (0, \infty)$  and satisfying*

$$u_t^{(m)} \leq \frac{u^{(m)}}{(1 - m)t} \quad \text{in } R^n \times (0, \infty).$$

*Proof.* Existence of solution to (0.1) is proved in [DK], [AB]. Let  $p = u_t^{(m)}/u^{(m)}$ . Then as in [AB], [CF1], [ERV]  $p$  will satisfy the equation

$$p_t = u^{(m)m-1} \Delta p + 2mu^{(m)m-2} \nabla u^{(m)} \cdot \nabla p - (1 - m)p^2.$$

Since  $1/(1 - m)t$  also satisfies the above equation but with initial value  $\infty$ . By the maximum principle

$$u_t^{(m)} \leq \frac{u^{(m)}}{(1 - m)t}$$

and the theorem follows. □

**Theorem 1.2** ([BBC],[E]). *If  $(n - 2)_+/n < m < 1$  and  $u_1^{(m)}, u_2^{(m)}$ , are subsolution and supersolutions of (0.1) with initial values  $u_{0,1}, u_{0,2} \in L^1(R^n)$  respectively, then*

$$\int_{R^n} \left( u_1^{(m)}(x, t) - u_2^{(m)}(x, t) \right)_+ dx \leq \int_{R^n} (u_{0,1} - u_{0,2})_+ dx \quad \forall t > 0$$

and if  $u_1^{(m)}$  and  $u_2^{(m)}$  are two solutions of (0.1) with initial values  $u_{0,1}, u_{0,2} \in L^1(R^n)$  respectively, then

$$\int_{R^n} \left| u_1^{(m)}(x, t) - u_2^{(m)}(x, t) \right| dx \leq \int_{R^n} |u_{0,1} - u_{0,2}| dx \quad \forall t > 0.$$

In particular if  $u_{0,1} \leq u_{0,2}$ , then  $u_2 \leq u_1$ .

**Lemma 1.3.** *If  $T_1 > 0$ ,  $0 < m < 1$ ,  $\alpha > 1$ ,  $0 < k < 1$  are constants satisfying the condition  $(1 - m)^{-1} < \alpha < 2(1 - m)^{-1}$  and  $0 < k < (4\alpha)^{-1/(\alpha(1-m)-1)}$ , then the function*

$$(1.1) \quad w(x, t) = \frac{[(1 - m)(T_1 - t)_+]^{1/1-m}}{(k + |x|^2)^\alpha}$$

is a subsolution of

$$(1.2) \quad u_t = \Delta(u^m/m)$$

in  $R^n \times (0, T_1)$ ,  $n = 1, 2$ , with initial value  $[(1-m)T_1]^{1/1-m}(k+|x|^2)^{-\alpha}$ .

*Proof.* Write  $w(x, t) = f(t)g(x)$  where  $f(t) = [(1-m)(T_1-t)_+]^{1/1-m}$  and  $g(x) = g(r) = (k+|x|^2)^{-\alpha}$  where  $r = |x|$ . For  $n = 1$  by direct computation,

$$\begin{cases} (g^m)'(r) = -2\alpha mr(k+r^2)^{-\alpha m-1} \\ (g^m)''(r) = 4\alpha m(\alpha m+1)r^2(k+r^2)^{-\alpha m-2} - 2\alpha m(k+r^2)^{-\alpha m-1}. \end{cases}$$

Hence

$$\begin{aligned} & \Delta(g^m) + mg \\ &= m(k+|x|^2)^{-\alpha}(1+4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2 \\ & \quad - 2\alpha(k+|x|^2)^{\alpha(1-m)-1}) \\ &= m(k+|x|^2)^{-\alpha}(1+(2\alpha+4\alpha^2 m)(k+|x|^2)^{\alpha(1-m)-1} \\ & \quad - 4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}k) \\ &\geq m(k+|x|^2)^{-\alpha}(1+(2\alpha+4\alpha^2 m)k^{\alpha(1-m)-1} - 4\alpha(\alpha m+1)k^{\alpha(1-m)-1}) \\ &\geq m(k+|x|^2)^{-\alpha}(1-2\alpha k^{\alpha(1-m)-1}) \geq 0. \end{aligned}$$

Similarly for  $n = 2$ , we have

$$\begin{aligned} & \Delta(g^m) + mg \\ &= m(k+|x|^2)^{-\alpha}(1+4\alpha(\alpha m+1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2 \\ & \quad - 4\alpha(k+|x|^2)^{\alpha(1-m)-1}) \\ &\geq m(k+|x|^2)^{-\alpha}(1-4\alpha k^{\alpha(1-m)-1}) \geq 0. \end{aligned}$$

Hence

$$\Delta\left(\frac{w^m}{m}\right) - w_t = f^m \Delta\left(\frac{g^m}{m}\right) - f_t g = \frac{f^m}{m} \cdot (\Delta(g^m) + mg) \geq 0$$

Thus  $w$  is a subsolution of (0.1) with initial value  $[(1-m)T_1]^{1/1-m}(k+|x|^2)^{-\alpha}$  and the lemma follows.  $\square$

As a consequence of Theorem 1.2 and Lemma 1.3 we have the the following corollary.

**Corollary 1.4.** *If  $u_0 \in L^1(R^n)$  and  $u_0(x) \geq \varepsilon(k+|x|^2)^{-\alpha}$  for some constants  $\varepsilon > 0$ ,  $0 < k < 1$ ,  $0 < m < 1$ , satisfying  $(1-m)^{-1} < \alpha < 2(1-m)^{-1}$  and  $0 < k < (4\alpha)^{-1/(\alpha(1-m)-1)}$ , then the solution  $u^{(m)}$  of (0.1) is bounded below by  $(\varepsilon^{1-m} - (1-m)t)_+^{1/1-m}(k+|x|^2)^{-\alpha}$ .*

**Theorem 1.5.** *If  $u_0 \in L^1(R^1)$  for  $n = 1$  and  $u_0 \in L^1(R^2) \cap L^p(R^2)$  for some  $p > 1$  for  $n = 2$ , then there exists a unique maximal solution  $v$  of (0.2) in  $R^n \times (0, T)$  where  $T$  is given by (0.3).*

*Proof.* For  $n = 1$  the theorem is proved in [ERV]. For  $n = 2$  by Theorem 4.3 of [H] (cf. [DP] Theorem 1.2) there exists a solution  $v$  of (0.2) in  $R^2 \times (0, T)$  satisfying the condition

$$(1.3) \quad \int_{R^2} v(x, t) dx = \int_{R^2} u_0 dx - 4\pi t \quad \forall 0 < t \leq T$$

where  $T$  is given by (0.3). By Prop 3.1 and Prop 4.1 of [DD], there exists a maximal solution  $v_1$  of (0.2) in  $R^2 \times (0, T)$ . By Theorem 1.3 of [DP],

$$\begin{aligned} \int_{R^2} v_1(x, t) dx &\leq \int_{R^2} u_0 dx - 4\pi t \quad \forall 0 < t \leq T \\ \Rightarrow \int_{R^2} v_1(x, t) dx &\leq \int_{R^2} v(x, t) dx \quad \forall 0 < t \leq T \quad \text{by (1.3)} \\ \Rightarrow v(x, t) &= v_1(x, t) \quad \forall x \in R^2, 0 < t \leq T \text{ since } v \leq v_1. \end{aligned}$$

Hence  $v$  is the maximal solution solution of (0.2). □

**Lemma 1.6.** *There exists a constant  $0 < m_1 \leq 1/2$  such that the function*

$$u_1(x, t) = \frac{Am^2 t^{1/1-m}}{|x|^{2/(1-m)} (|x|^m - 1)^2}$$

*is a subsolution of (1.2) in  $R^2 \setminus B_2 \times (0, \infty)$  for all  $0 < m \leq m_1$  where  $0 < A \leq (2/3)^4$  is a constant.*

*Proof.* Let  $b = 2/(1 - m)$  and  $g(x) = g(r) = r^{-mb}(r^m - 1)^{-2m}$  where  $r = |x|$ . Then

$$\begin{aligned} (1.4) \quad \Delta g &= g'' + \frac{1}{r} g' \\ &= (mb)^2 r^{-mb-2} (r^m - 1)^{-2m} + 4m^3 b r^{-mb+m-2} (r^m - 1)^{-2m-1} \\ &\quad + 4m^4 r^{-mb+m-2} (r^m - 1)^{-2m-1} \\ &\quad + 2m^3 (2m + 1) r^{-mb+m-2} (r^m - 1)^{-2m-2} \\ &\geq 2m^3 r^{-mb+m-2} (r^m - 1)^{-2m-2} \quad \forall r \geq 2 \\ &\Rightarrow \Delta \left( \frac{u_1^m}{m} \right) - u_{1,t} \\ &= Am^m 2^{2m-1} t^{m/1-m} \Delta g - Am^2 (1 - m)^{-1} t^{m/1-m} r^{-b} (r^m - 1)^{-2} \\ &\geq 2Am^m 2^{2m+2} t^{m/1-m} r^{-mb+m-2} (r^m - 1)^{-2m-2} \\ &\quad - Am^2 (1 - m)^{-1} t^{m/1-m} r^{-b} (r^m - 1)^{-2} \\ &= m^2 A^m t^{m/1-m} r^{-b} (r^m - 1)^{-2m-2} (2m^{2m} r^m \\ &\quad - (1 - m)^{-1} A^{1-m} (r^m - 1)^{2m}). \end{aligned}$$

Since  $\lim_{m \rightarrow 0} m^m = 1$ , there exists  $0 < m_1 \leq 1/2$  such that  $m^m > 2/3$  for all  $0 < m \leq m_1$ . Thus

$$\begin{aligned} & 2m^{2m}r^m - (1 - m)^{-1}A^{1-m}(r^m - 1)^{2m} \\ & \geq 2(2/3)^2r^m - 2A^{1/2}r^{2m^2} \\ & \geq 2(2/3)^2r^{2m^2}(r^{m(1-2m)} - 1) \geq 0 \quad \forall 0 < A \leq (2/3)^4, 0 < m \leq m_1, r \geq 2. \end{aligned}$$

Hence the right hand side of (1.4) is positive for all  $0 < m \leq m_1, r \geq 2$ , and the lemma follows.  $\square$

By direct computation we also have the follow result:

**Lemma 1.7.** *For any  $0 < A < 1, 0 < m < 1$ , the function*

$$u_2(x, t) = A \left( \frac{t}{|x|^2} \right)^{1/(1-m)}$$

*is a subsolution of (1.2) in  $R \setminus B_2 \times (0, \infty)$ .*

**Theorem 1.8.** *If  $v$  is a solution of (0.2) in  $R^2 \times (0, T')$  and satisfies the condition*

$$v(x, t) \geq \frac{Ct}{(|x| \log |x|)^2} \quad \forall |x| \geq R, 0 < t \leq T'$$

*for some constants  $C > 0$  and  $R > 0$ , then  $v$  is the unique maximal solution of (0.2) in  $R^2 \times (0, T')$ .*

*Proof.* For any  $0 < t_1 < T'$ , since

$$v(x, t) \geq \frac{Ct_1}{(|x| \log |x|)^2} \quad \forall |x| \geq R, t_1 < t \leq T'$$

by Prop 2.1 of [DD]  $v$  is the unique maximal solution of

$$(1.5) \quad \begin{cases} w_t = \Delta \log w, w > 0, & \text{in } R^n \times (t_1, T') \\ w(x, t_1) = v(x, t_1) & \forall x \in R^n \end{cases}$$

with  $n = 2$ . By Theorem 4.3 of [H] or Theorem 1.2 of [DP] there exist a solution  $\bar{v}$  of (1.5) satisfying

$$\int_{R^2} \bar{v}(x, t) dx = \int_{R^2} v(x, t_1) dx - 4\pi(t - t_1) \quad \forall 0 < t_1 \leq t \leq T'.$$

Since  $v$  is the unique maximal solution of (1.5),  $v \geq \bar{v}$ . Thus

$$(1.6) \quad \int_{R^2} v(x, t) dx \geq \int_{R^2} \bar{v}(x, t) dx = \int_{R^2} v(x, t_1) dx - 4\pi(t - t_1) \quad \forall 0 < t_1 \leq t \leq T'.$$



We next let  $R > 0$  and let  $\eta_R \in C_0^\infty(R^2)$  be such that  $0 \leq \eta_R \leq 1$ ,  $\eta_R(x) \equiv 1$  for  $|x| \leq R$ ,  $\eta_R(x) \equiv 0$  for  $|x| \geq 2R$ . Then

$$\begin{aligned} \int_{R^2} v(x, t_1) dx &\geq \int_{R^2} v(x, t_1) \eta_R(x) dx \\ \Rightarrow \liminf_{t_1 \rightarrow 0} \int_{R^2} v(x, t_1) dx &\geq \liminf_{t_1 \rightarrow 0} \int_{R^2} v(x, t_1) \eta_R(x) dx \\ &= \int_{R^2} u_0(x) \eta_R(x) dx \\ \Rightarrow \liminf_{t_1 \rightarrow 0} \int_{R^2} v(x, t_1) dx &\geq \int_{R^2} u_0(x) dx \quad \text{as } R \rightarrow \infty. \end{aligned}$$

Hence letting  $t_1 \rightarrow 0$  in (1.6) we have

$$(1.7) \quad \int_{R^2} v(x, t) dx \geq \int_{R^2} u_0(x) dx - 4\pi t \quad \forall 0 < t \leq T'.$$

By Theorem 4.3 of [H] and the proof of Theorem 1.5 there exists a unique maximal solution  $\tilde{v}$  of (0.2) which satisfies

$$(1.8) \quad \int_{R^2} \tilde{v}(x, t) dx = \int_{R^2} u_0(x) dx - 4\pi t \quad \forall 0 < t \leq T'.$$

Since  $\tilde{v} \geq v$ , by (1.7) and (1.8) we get

$$\begin{aligned} \int_{R^2} v(x, t) dx &= \int_{R^2} \tilde{v}(x, t) dx = \int_{R^2} u_0(x) dx - 4\pi t \quad \forall 0 < t \leq T' \\ \Rightarrow v &\equiv \tilde{v} \quad \text{on } R^2 \times (0, T'). \end{aligned}$$

Hence  $v$  is the unique maximal solution of (0.2). □

**Theorem 1.9.** *If  $v$  is a solution of (0.2) in  $R \times (0, T')$  and satisfies the condition*

$$(1.9) \quad v(x, t) \geq \frac{Ct}{|x|^2} \quad \forall |x| \geq R, 0 < t \leq T'$$

*for some constants  $C > 0$  and  $R > 0$ , then  $v$  is the unique maximal solution of (0.2) in  $R \times (0, T')$ .*

*Proof.* For any  $0 < t_1 < T'$ , since  $v$  satisfies (1.9)

$$\log(1/v) \leq -\log Ct + 2\log|x| \leq o(|x|) \quad \forall |x| \geq R, t_1 < t \leq T'.$$

By the result in Section 3 of [ERV]  $v$  is the unique maximal solution of (1.5) with  $n = 1$ . By an argument similar to the proof of Theorem 1.8 we get that  $v$  is the unique maximal solution of (0.2) and the theorem follows. □

**Section 2.**

In this section we will first prove the convergence of solution  $u^{(m)}$  of (0.1) as  $m \rightarrow 0$  for  $n = 1, 2$ , under the assumption that  $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  and  $\varepsilon(1 + |x|^2)^{-\alpha} \leq u_0(x) \leq M$  for some constants  $\varepsilon > 0$ ,  $M \geq 1$ , and  $1 < \alpha < 2$  for all  $x \in \mathbb{R}^n$ . The general convergence theorem then follows by an approximation argument. We will first start with a Harnack type inequality.

**Lemma 2.1.** *If  $u^{(m)} \in L^\infty((0, \infty); L^1(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times (0, \infty))$  and  $0 \leq u^{(m)} \leq M$  for some constant  $M$ , then for any  $(n - 2)_+/n < m < 1$ ,  $0 < t_1 < t_2$ ,  $t > 0$ ,  $R > 0$ , and  $x_0 \in \mathbb{R}^n$ , the following inequalities hold,*

$$\begin{aligned}
 \text{(i)} \quad & \frac{1}{|B_R|} \int_{B_R(x_0)} \frac{u^{(m)m}(x, t)}{m} dx \leq \frac{u^{(m)m}(x_0, t)}{m} + C \frac{MR^2}{(1 - m)t} \\
 \text{(ii)} \quad & \frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)m}}{m} dx dt \leq I_2 + CMR^2 \\
 \text{(iii)} \quad & I_1 \leq \frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)m}}{m} dx dt + CMR^2
 \end{aligned}$$

where  $C > 0$  is a constant independent of  $m$  and

$$\begin{aligned}
 \text{(2.1)} \quad & I_i = \frac{(t_2 - t_1)}{m} \left\{ 1 - (1 - m)u^{(m)m}(x_0, t_i)t_i^{-m/1-m} \left( \frac{t_2^{1/1-m} - t_1^{1/1-m}}{t_2 - t_1} \right) \right\}, \\
 & i = 1, 2.
 \end{aligned}$$

*Proof.* We will use a modification of the argument of Lemma 6 of [V] and Lemma 4 of [GH] to prove the lemma. Let (cf. [V] p. 509, [DD] p. 653)

$$G_R(x) = \begin{cases} |x - x_0|^{2-n} - R^{2-n} + \frac{n-2}{2}R^{-n}(|x - x_0|^2 - R^2) & \text{if } n \geq 3 \\ \log(R/|x - x_0|) + \frac{1}{2}R^{-2}(|x - x_0|^2 - R^2) & \text{if } n = 2 \\ R - |x - x_0| + \frac{R^{-1}}{2}(|x - x_0|^2 - R^2) & \text{if } n = 1 \end{cases}$$

be the Green's function for  $B_R(x_0)$ . Then  $G_R \geq 0$ ,  $G_R(R) = G'_R(R) = 0$ , and

$$\Delta G_R = \begin{cases} n(n - 2)R^{-n} - (n - 2)|\partial B_1| \delta_{x_0} & \text{if } n \geq 3 \\ 2R^{-2} - 2\pi \delta_{x_0} & \text{if } n = 2 \\ R^{-1} - 2\delta_{x_0} & \text{if } n = 1 \end{cases}$$

where  $\delta_{x_0}$  is the delta mass at  $x_0$ . Then by Theorem 1.1 we have

$$\begin{aligned}
 & a_n \left( \frac{1}{|B_R|} \int_{B_R(x_0)} \frac{u^{(m)m}(x, t)}{m} dx - \frac{u^{(m)m}(x_0, t)}{m} \right) \\
 &= \int_{B_R(x_0)} G_R(x) \Delta \left( \frac{u^{(m)m}}{m} \right) (x, t) dx \\
 (2.2) \quad &= \int_{B_R(x_0)} G_R(x) u_t(x, t) dx \\
 &\leq \frac{1}{(1-m)t} \int_{B_R(x_0)} G_R(x) u(x, t) dx \leq C \frac{MR^2}{(1-m)t}
 \end{aligned}$$

where  $a_1 = 2$ ,  $a_2 = 2\pi$ , and  $a_n = (n - 2)|\partial B_1|$  for  $n \geq 3$ . Thus (i) follows. Integrating (2.2) over  $(t_1, t_2)$  we get

$$\begin{aligned}
 & \frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)m}}{m} dx dt \\
 &= \int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt - \frac{1}{a_n} \int_{B_R(x_0)} G_R(x) u(x, t) dx \Big|_{t_1}^{t_2}.
 \end{aligned}$$

By Theorem 1.1,

$$\frac{u(x, t_2)}{t_2^{1/1-m}} \leq \frac{u(x, t)}{t^{1/1-m}} \leq \frac{u(x, t_1)}{t_1^{1/1-m}} \quad \forall t_1 \leq t \leq t_2.$$

Hence

$$\int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt \leq \int_{t_1}^{t_2} \frac{1 - u^{(m)m}(x_0, t_2)(t/t_2)^{m/1-m}}{m} dt \leq I_2$$

where  $I_2$  is given by (2.1). Since  $G_R \geq 0$  and

$$\int_{B_R(x_0)} G_R(x) u(x, t) dx \leq CMR^2 \quad \forall t > 0$$

(ii) follows. Similarly

$$\int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt \geq I_1$$

where  $I_1$  is given by (2.1) and (iii) follows. □

**Lemma 2.2.** *Let  $u^{(m)} \in L^\infty((0, \infty); L^1(R^n)) \cap L^\infty(R^n \times (0, \infty))$  be the solution of (0.1). If there exists a constant  $M \geq 1$  such that  $0 \leq u^{(m)} \leq M$  for all  $0 < m < 1$  and*

$$(2.3) \quad \liminf_{m \rightarrow 0} u^{(m)}(x_0, t_0) > 0$$

for some  $x_0 \in R^n$ ,  $n = 1, 2$ ,  $0 < t_0 < T$ ,  $T > 0$ . Then for any  $R > 0$ ,  $0 < t_1 < t_2 < t_0$ , there exists  $C > 0$  and  $0 < m_0 < 1$  depending on  $t_1, t_2, t_0 - t_2$ , and  $R > 0$  such that

$$(2.4) \quad u^{(m)}(x, t) \geq C \quad \forall 0 < m \leq m_0, x \in B_R, t_1 \leq t \leq t_2.$$

*Proof.* Without loss of generality we may assume that  $R > |x_0|$ . By (2.3) there exists  $0 < m_1 < 1$  and  $\delta > 0$  such that

$$u^{(m)}(x_0, t_0) \geq \delta \quad \forall 0 < m \leq m_1.$$

Since  $B_1(y) \subset B_{2R+1}(x_0)$  for any  $y \in B_R$ , by Lemma 2.1 we have for any  $y \in B_R, 0 < t_1 \leq t \leq t_2 < t_0$ ,

$$(2.5) \quad \begin{aligned} & \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)m}(y, t)t^{-m/1-m} \left( \frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t} \right) \right\} \\ & \leq \frac{1}{|B_1|} \int_t^{t_0} \int_{B_1(y)} \frac{1 - u^{(m)m}}{m} dx ds + CMR^2 \\ & \leq \frac{1}{|B_1|} \int_t^{t_0} \int_{B_1(y) \cap \{u^{(m)} \leq 1\}} \frac{1 - u^{(m)m}}{m} dx ds + CMR^2 \\ & \leq \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} \leq 1\}} \frac{1 - u^{(m)m}}{m} dx ds + CMR^2 \\ & \leq \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0)} \frac{1 - u^{(m)m}}{m} dx ds \\ & \quad + \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} > 1\}} \frac{u^{(m)m} - 1}{m} dx ds + CMR^2 \\ & \leq (2R + 1)^n \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)m}(x_0, t_0)t_0^{-m/1-m} \right. \\ & \quad \cdot \left. \left( \frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t} \right) \right\} \\ & \quad + \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} > 1\}} \frac{u^{(m)m} - 1}{m} dx ds + CMR^2 \end{aligned}$$

where  $C > 0$  is a constant. Since

$$\frac{z^m - 1}{m} \leq z \log z \leq M \log M \quad \forall 1 \leq z \leq M.$$

The second term on the right hand side above is bounded above by

$$(2.6) \quad (t_0 - t)(2R + 1)^n M \log M.$$

By the mean value theorem there exists  $t_3 \in (t, t_0)$  and  $0 < \theta_1 < 1$  such that

$$\begin{aligned}
 (2.7) \quad & \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)m}(x_0, t_0)t_0^{-m/1-m} \left( \frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t} \right) \right\} \\
 & = \frac{(t_0 - t)}{m} \left\{ 1 - u^{(m)m}(x_0, t_0)(t_3/t_0)^{m/1-m} \right\} \\
 & = (t_0 - t)(u^{(m)}(x_0, t_0)(t_3/t_0)^{1/1-m})^{\theta_1 m_1} \log \frac{(t_0/t_3)^{1/1-m}}{u^{(m)}(x_0, t_0)} \\
 & \leq TM \log \frac{(T/t_1)^{1/1-m_1}}{\delta} \quad \forall 0 < m \leq m_1.
 \end{aligned}$$

Similarly there exists  $t_4 \in (t, t_0)$  such that

$$\begin{aligned}
 (2.8) \quad & \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)m}(y, t)t_0^{-m/1-m} \left( \frac{t_0^{1/1-m} - t^{1/1-m}}{t_0 - t} \right) \right\} \\
 & = \frac{(t_0 - t)}{m} \left\{ 1 - u^{(m)m}(y, t)(t_4/t)^{m/1-m} \right\} \\
 & \geq \frac{(t_0 - t_2)}{m} \left\{ 1 - u^{(m)m}(y, t)(T/t_1)^{m/1-m} \right\}.
 \end{aligned}$$

By (2.5), (2.6), (2.7), (2.8), there exists a constant  $c_1 > 0$  such that for all  $0 < m \leq m_1$  we have

$$\begin{aligned}
 (2.9) \quad & \frac{(t_0 - t_2)}{m} \left\{ 1 - u^{(m)m}(y, t)(T/t_1)^{m/1-m} \right\} \leq c_1 \\
 \Rightarrow \quad & u^{(m)}(y, t) \geq \left( 1 - \frac{mc_1}{t_0 - t_2} \right)^{1/m} (t_1/T)^{1/1-m} \quad \forall 0 < m \leq m_1.
 \end{aligned}$$

Since the right hand side of (2.9) tends to some positive constant independent of  $y \in B_R$ ,  $t_1 \leq t \leq t_2$ , there exists constants  $0 < m_0 \leq m_1$  and  $C > 0$  such that

$$u^{(m)}(y, t) \geq C > 0 \quad \forall y \in B_R, t_1 \leq t \leq t_2, 0 < m \leq m_0$$

and the lemma follows. □

**Theorem 2.3.** For  $n = 1, 2$ , if  $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$  and  $\varepsilon(k + |x|^2)^{-\alpha} \leq u_0(x) \leq M$  for some constants  $\varepsilon > 0$ ,  $0 < k < 1$ ,  $1 < \alpha < 2$ ,  $M \geq 1$ , satisfying  $\alpha > (1 - m_2)^{-1}$  and  $0 < k < (4\alpha)^{-1/(\alpha(1 - m_2) - 1)}$  for some constant  $0 < m_2 < 1/2$ , then as  $m \rightarrow 0$  the solution  $u^{(m)}$  of (0.1) will converge uniformly on every compact subset of  $\mathbb{R}^n \times (0, T)$  to the unique maximal solution  $v$  of (0.2) where  $T$  is given by (0.3).

*Proof.* We first consider the case  $n = 2$ . For any  $0 < m \leq m_2$ , we have  $(1 - m)^{-1} < \alpha < 2(1 - m)^{-1}$  and  $\alpha(1 - m) - 1 \geq \alpha(1 - m_2) - 1$ . Thus  $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)} \leq (4\alpha)^{-1/(\alpha(1-m)-1)}$  for all  $0 < m \leq m_2$ . By Theorem 1.2 and Corollary 1.4,

$$\begin{aligned} (\varepsilon^{1-m} - (1 - m)t)_+^{1/1-m} (k + |x|^2)^{-\alpha} \\ \leq u^{(m)}(x, t) \leq M \quad \forall x \in R^2, t > 0, 0 < m \leq m_2. \end{aligned}$$

Since  $(\varepsilon^{1-m} - (1 - m)t)_+^{1/1-m} \geq \varepsilon/2$  for all  $0 < t \leq \varepsilon/2$  and  $0 < m \leq 1/2$ ,

$$M \geq u^{(m)}(x, t) \geq C_K > 0 \quad (x, t) \in K$$

for any compact subset  $K$  of  $R^2 \times (0, \varepsilon/2)$ . Let

$$T_0 = \max \left\{ s > 0 : \liminf_{m \rightarrow 0} u^{(m)}(x_0, s) > 0 \text{ for some } x_0 \in R^n \right\}.$$

Then  $T_0 \geq \varepsilon/2 > 0$ . For any  $0 < t_1 < t_2 < T_0$ , let  $t_0 = (t_2 + T_0)/2$ . Then there exists  $x_0 \in R^n$  such that

$$(2.10) \quad \liminf_{m \rightarrow 0} u^{(m)}(x_0, t_0) > 0.$$

By Lemma 2.2 for any  $R > 0$  there exists  $0 < m_0 < m_2$  such that (2.4) holds. Then there exists constants  $C_2 > 0, C_3 > 0$  independent of  $0 < m < m_0$  such that

$$C_2 \leq u^{(m)m-1}(x, t) \leq C_3 \quad \forall x \in B_R, t_1 \leq t \leq t_2.$$

Hence (0.1) is uniformly parabolic for  $\{u^{(m)}\}_{0 < m < m_0}$  on any compact subset of  $R^2 \times (0, T_0)$ . By standard parabolic theory [LSU],  $\{u^{(m)}\}_{0 < m \leq m_0}$  is uniformly equi-Holder continuous on any compact subset of  $R^2 \times (0, T_0)$ . Let  $\{u^{(m_i)}\}, m_i \rightarrow 0$  as  $i \rightarrow \infty$ , be a sequence of  $u^{(m)}$ . By the Ascoli Theorem and a diagonalization argument  $u^{(m_i)}$  will have a subsequence  $u^{(m'_i)}$  converging uniformly on every compact subset  $K$  of  $R^2 \times (0, T_0)$  to a continuous function  $v$  satisfying  $M \geq v \geq C_K > 0$  on  $K$  for some constant  $C_K > 0$ . Without loss of generality we may assume  $u^{(m_i)}$  converges uniformly on every compact subset  $K$  of  $R^2 \times (0, T_0)$  to  $v$  as  $i \rightarrow \infty$ . Since  $u^{(m_i)}$  satisfies (0.1), for any  $\eta \in C_0^\infty(R^2 \times (0, T_0))$  we have

$$\begin{aligned} (2.11) \quad & \int_{t'_1}^{t'_2} \int_{R^2} \left( u^{(m_i)} \eta_t + \frac{u^{(m_i)m_i} - 1}{m_i} \Delta \eta \right) dx ds \\ & = \int_{R^2} u^{(m_i)} \eta dx \Big|_{t'_1}^{t'_2} \quad \forall 0 < t'_1 < t'_2 < T_0. \end{aligned}$$

Now by the mean value theorem, for each  $(x, t) \in K$ , there exists  $\theta = \theta(x, t)$  such that

$$\begin{aligned} & \left| \frac{u^{(m_i)m_i}(x, t) - 1}{m_i} - \log v \right| \\ & \leq \left| \frac{u^{(m_i)m_i}(x, t) - 1}{m_i} - \log u^{(m_i)} \right| + \left| \log u^{(m_i)} - \log v \right| \\ & \leq \left| \frac{e^{m_i \log u^{(m_i)}} - 1}{m_i} - \log u^{(m_i)} \right| + \left| \log u^{(m_i)} - \log v \right| \\ & = \left| u^{(m_i)\theta m_i} \log u^{(m_i)} - \log u^{(m_i)} \right| + \left| \log u^{(m_i)} - \log v \right| \\ & \leq \left| u^{(m_i)\theta m_i} - 1 \right| \left| \log u^{(m_i)} \right| + \left| \log u^{(m_i)} - \log v \right| \\ & \leq \max(M^{m_i} - 1, |C_K^{m_i} - 1|) \cdot \max(\log M, |\log C_K|) + \left| \log u^{(m_i)} - \log v \right| \\ & \rightarrow 0 \quad \text{as } i \rightarrow \infty \end{aligned}$$

uniformly on  $K$ . Hence letting  $i \rightarrow \infty$  in (2.11), for any  $\eta \in C_0^\infty(R^2 \times (0, T_0))$  we have

$$\int_{t'_1}^{t'_2} \int_{R^2} \left( v \eta_t + \log v \Delta \eta \right) dx ds = \int_{R^2} v \eta dx \Big|_{t'_1}^{t'_2} \quad \forall 0 < t'_1 < t'_2 < T_0.$$

By standard parabolic theory [LSU]  $v \in C^\infty(R^2 \times (0, T_0))$ . Thus  $v$  is a solution of (0.3) in  $R^2 \times (0, T_0)$  with  $\phi(v) = \log v$ . We next claim that  $v$  has initial value  $u_0$ . To prove the claim we let  $\eta \in C_0^\infty(R^2)$  be such that  $\text{supp } \eta \subset B_R$  for some constant  $R > 2$  and fix a  $0 < t_2 < T_0$ . Then by the definition of  $T_0$ , if  $t_0 = (t_2 + T_0)/2$ , then there exists  $x_0 \in R^2$  such that (2.10) holds. By Lemma 2.2 there exists  $\delta > 0$  and  $0 < M_0 < \min(m_1, m_2)$  such that

$$u^{(m)}(x, t_2) \geq \delta \quad \forall |x| \leq R, 0 < m < M_0$$

where  $m_1$  is as in Lemma 1.6. By Theorem 1.1, we have for any  $0 < m < M_0$ ,

$$\begin{aligned} (2.12) \quad u^{(m)}(x, t) & \geq \frac{u^{(m)}(x, t_2)}{t_2^{1/1-m}} t^{1/1-m} \\ & \geq \frac{\delta}{t_2^{1/1-m}} t^{1/1-m} \geq c_1 t^{1/1-m} \quad \forall |x| \leq R, 0 < t \leq t_2 \end{aligned}$$

where  $c_1 = \delta/(1 + T_0)^2$ . Hence

$$\begin{aligned}
& \left| \int_{R^2} u^{(m)}(x, t) \eta(x) dx - \int_{R^2} u_0 \eta(x) dx \right| \\
&= \left| \int_0^t \int_{R^2} u_t^{(m)} \eta dx ds \right| \\
&= \left| \int_0^t \int_{R^2} \Delta \left( \frac{u^{(m)m} - 1}{m} \right) \eta dx ds \right| \\
&= \left| \int_0^t \int_{B_R} \left( \frac{u^{(m)m} - 1}{m} \right) \Delta \eta dx ds \right| \\
&\leq \|\Delta \eta\|_{L^\infty} \left( \int_0^t \int_{B_R \cap \{u^{(m)} \geq 1\}} \frac{u^{(m)m} - 1}{m} dx ds \right. \\
&\quad \left. + \int_0^t \int_{B_R \cap \{u^{(m)} < 1\}} \frac{1 - u^{(m)m}}{m} dx ds \right) \\
&\leq \|\Delta \eta\|_{L^\infty} \left( \int_0^t \int_{B_R \cap \{u^{(m)} \geq 1\}} \frac{M^m - 1}{m} dx ds \right. \\
&\quad \left. + \int_0^t \int_{B_R \cap \{u^{(m)} < 1\}} \frac{1 - (c_1 s^{1/1-m})^m}{m} dx ds \right) \\
&\leq \|\Delta \eta\|_{L^\infty} |B_R| \left( tM \log M - \int_0^t (c_1 s^{1/1-m})^{\theta m} \log(c_1 s^{1/1-m}) ds \right) \\
&\leq C(t + t \log t) \quad \forall 0 < t < t_2
\end{aligned}$$

for some constants  $C > 0$  and  $0 < \theta < 1$  by the mean value theorem. Letting  $m = m_i \rightarrow 0$ , we get

$$\left| \int_{R^2} v(x, t) \eta(x) dx - \int_{R^2} u_0 \eta(x) dx \right| \leq C(t + t \log t) \rightarrow 0 \text{ as } t \rightarrow 0.$$

Hence  $v$  have initial value  $u_0$  and is thus a solution of (0.2) in  $R^2 \times (0, T_0)$ . We next claim that  $v$  is the unique maximal solution of (0.2). To prove the claim we observe that by (2.12)

$$\begin{aligned}
(2.13) \quad u^{(m)}(x, t) &\geq \frac{2^{2/1-m} \delta}{(1 + T_0)^2} \left( \frac{2^m - 1}{m} \right)^2 \frac{m^2 t^{1/1-m}}{|x|^{2/(1-m)} (|x|^m - 1)^2} \\
&\geq \frac{Am^2 t^{1/1-m}}{|x|^{2/(1-m)} (|x|^m - 1)^2} \quad \forall |x| = 2, 0 < t \leq t_2, 0 < m \leq M_0
\end{aligned}$$

where

$$A = \min \left( (2/3)^4, \frac{(\log 2)^2 \delta}{16^8 (1 + T_0)^2} \right)$$



since  $(2^m - 1)/m = 2^{\theta m} \log 2 \geq \log 2$  for some constant  $0 < \theta < 1$ . By Lemma 1.6, (2.12), and the maximum principle (Lemma 3.4 of [HP]), for any  $0 < m \leq M_0$  we have

(2.14)

$$u^{(m)}(x, t) \geq Am^2 t^{1/1-m} |x|^{-2/(1-m)} (|x|^m - 1)^{-2} \quad \forall 0 < t \leq t_2, |x| \geq 2$$

(2.15)  $\Rightarrow v(x, t) = \lim_{i \rightarrow \infty} u^{(m_i)}(x, t) \geq \frac{At}{(|x| \log |x|)^2} \quad \forall 0 < t \leq t_2, |x| \geq 2.$

Then by Theorem 1.8  $v$  is the unique maximal solution of (0.2) in  $R^2 \times (0, t_2)$  for all  $0 < t_2 < T_0$ . Hence  $v$  is the unique maximal solution of (0.2) in  $R^2 \times (0, T_0)$  and  $u^{(m)}$  converges uniformly to  $v$  on any compact subset of  $R^2 \times (0, T_0)$  as  $m \rightarrow 0$ . Suppose  $T_0 < T$  where  $T = \int_{R^2} u_0 dx / 4\pi$ . By Theorem 1.5  $v$  can be extended to the unique maximal solution of (0.2) in  $R^2 \times (0, T)$ . Since  $v > 0$  in  $R^2 \times (0, T)$  and  $v \in C^\infty(R^2 \times (0, T))$ , there exists a constant  $\delta_1 > 0$  such that

$$v(x, t) \geq \delta_1 > 0 \quad \forall |x| \leq 2, T_0/2 \leq t \leq (T_0 + T)/2.$$

Let

$$A_1 = \min \left( (2/3)^4, \frac{(\log 2)^2 (\delta_1/2)}{16^8 (1 + T_0)^2} \right)$$

and choose  $t_3 > 0$  such that  $T_0/(1 + A_1) < t_3 < T_0$ . Since  $u^{(m)} \rightarrow v$  as  $m \rightarrow 0$  uniformly on  $B_2 \times \{t_3\}$ . There exists  $0 < M_1 \leq m_1$  such that

$$u^{(m)}(x, t_3) \geq \delta_1/2 \quad \forall |x| \leq 2, 0 < m \leq M_1.$$

By the same argument as the proof of (2.14) we get

$$u^{(m)}(x, t) \geq \frac{A_1 m^2 t^{1/1-m}}{|x|^{2/(1-m)} (|x|^m - 1)^2} \quad \forall |x| \geq 2, 0 < t \leq t_3, 0 < m \leq M_1.$$

Let  $\alpha_m = (1 - m)^{-1} + 1 + m$ ,  $k_1 = (2A_1(1 + T_0)^2/\delta_1)^{1/\alpha_m}$ , and let  $w$  be as in Lemma 1.3 with  $T_1 = (1 + A_1)t_3$  and  $k = k_1$ . Then  $1/(1 - m) < \alpha_m < 2/(1 - m) < 4$  for  $0 < m < 1/2$  and

$$\begin{aligned} \alpha_m(1 - m) - 1 &= (1 + m)(1 - m) = 1 - m^2 > \frac{1}{2} \quad \forall 0 < m < 1/2 \\ \Rightarrow \frac{1}{\alpha_m(1 - m) - 1} &< 2 < \frac{8}{\alpha_m} \\ \Rightarrow 0 < k_1 &= (2A_1(1 + T_0)^2/\delta_1)^{1/\alpha_m} \leq 16^{-8/\alpha_m} \leq (4\alpha_m)^{-1/\alpha_m(1-m)-1}. \end{aligned}$$

Since by the mean value theorem there exists  $0 < \theta < 1$  such that

$$\frac{x^m - 1}{m} = |x|^{\theta m} \log |x| \leq |x|^m \log |x| \leq |x|^{1+m} \quad \forall |x| \geq 2$$

and

(2.16)

$$T_1 = (1 + A_1)t_3 \Rightarrow ((1 - m)(T_1 - t_3)_+)^{1/1-m} \leq (A_1 t_3)^{1/1-m} \leq A_1 t_3^{1/1-m}$$

hence for all  $|x| \geq 2$ ,  $0 < m < M_1$ , we have

$$\begin{aligned} \frac{A_1 m^2 t_3^{1/1-m}}{|x|^{2/(1-m)}(|x|^m - 1)^2} &\geq \frac{A_1 t_3^{1/1-m}}{|x|^{2+2m+2/(1-m)}} \geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} \\ \Rightarrow u^{(m)}(x, t_3) &\geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3) \\ &\quad \forall |x| \geq 2, 0 < m \leq M_1. \end{aligned}$$

Since  $k_1^{\alpha_m} = 2A_1(1 + T_0)^2/\delta_1$ , by (2.16) for any  $|x| \leq 2$ ,  $0 < m < M_1$ , we have

$$\begin{aligned} u^{(m)}(x, t_3) &\geq \frac{\delta_1}{2} = \frac{A_1(1 + T_0)^2}{k_1^{\alpha_m}} \geq \frac{A_1 t_3^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} \\ &\geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1-m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3). \end{aligned}$$

Hence

$$u^{(m)}(x, t_3) \geq w(x, t_3) \quad \forall x \in R^2.$$

By the maximum principle,

$$u^{(m)}(x, t) \geq w(x, t) \quad \forall x \in R^2, t_3 \leq t < T_1, 0 < m \leq M_1.$$

Since  $T_1 = (1 + A_1)t_3 > T_0$ , we have  $(T_0 + T_1)/2 > T_0$  and

$$u^{(m)}(x, (T_0 + T_1)/2) \geq w(x, (T_0 + T_1)/2) > 0 \quad \forall x \in R^2.$$

This contradicts the maximality of  $T_0$ . Hence  $T_0 = T$  where  $T$  is given by (0.3) and the theorem for the case  $n = 2$  follows. For  $n = 1$  by an argument similar to the case  $n = 2$  but with the subsolution  $u_2$  of Lemma 1.7 in place of  $u_1$  of Lemma 1.6 in the argument we get that for  $n = 1$  as  $m \rightarrow 0$  the solution  $u^{(m)}$  also converges uniformly on every compact subset of  $R \times (0, \infty)$  to the maximal solution  $v$  of (0.2) in  $R \times (0, \infty)$  and the theorem follows.  $\square$

**Theorem 2.4.** *If  $n = 1, 2$ , and  $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ , then as  $m \rightarrow 0$ , the solution  $u^{(m)}$  of (0.1) will converge uniformly on every compact subset of  $R^n \times (0, T)$  to the unique maximal solution  $v$  of (0.2) where  $T$  is given by (0.3).*

*Proof.* Choose  $1 < \alpha < 2$ ,  $0 < m_2 < 1/2$ ,  $0 < k < 1$ , satisfying the conditions  $\alpha > (1 - m_2)^{-1}$  and  $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)}$  and let  $u_j^{(m)}$  be

the solution of (0.1) with initial value

$$u_{0j}(x) = u_0(x) + \frac{1}{j}(k + |x|^2)^{-\alpha} \quad j = 1, 2, \dots$$

Then by Theorem 2.3 for each  $j = 1, 2, \dots$ ,  $u_j^{(m)}$  will converge uniformly on every compact subset of  $R^n \times (0, T_j)$  to the unique maximal solution  $v_j$  of (0.2) with initial value  $u_{0j}$  as  $m \rightarrow 0$  where

$$T_j = \begin{cases} \infty & \text{if } n = 1 \\ \frac{1}{4\pi} \int_{R^2} u_{0,j}(x) dx & \text{if } n = 2. \end{cases}$$

Suppose the theorem is false. Then there exists  $\varepsilon > 0$ ,  $R > 0$ ,  $0 < t_1 < t_2 < T$  and a sequence  $u^{(m_i)}$ ,  $m_i \rightarrow 0$  as  $i \rightarrow \infty$ , of  $u^{(m)}$ , such that

$$(2.17) \quad \|u^{(m_i)} - v\|_{L^\infty(B_R \times (t_1, t_2))} \geq \varepsilon \quad \forall i = 1, 2, \dots$$

where  $v$  is the unique maximal solution of (0.2) with initial value  $u_0$ . Since  $\|u^{(m)}\|_{L^\infty} \leq \|u_0\|_{L^\infty}$ ,  $u^{(m_i)}$  will have a subsequence  $u^{(m'_i)}$  converging weakly in  $L^\infty(R^n \times (0, T))$  and a.e.  $(x, t) \in R^n \times (0, T)$  to some function  $\tilde{v}$  as  $i \rightarrow \infty$ . By the uniqueness of maximal solution and comparison principle for maximal solution of (0.2) (Lemma 4.2 of [H] or Lemma 4.1 of [DP] for  $n = 2$  and [ERV] for  $n = 1$ ) we have  $v_1 \geq v_2 \geq \dots \geq v > 0$  in  $R^n \times (0, T)$ . Hence  $v_j$  are uniformly bounded below by some positive constant on any compact subset of  $R^n \times (0, T)$ . Thus (0.4) with  $\phi(u) = \log u$  is uniformly parabolic for  $v_j$  on any compact subset of  $R^n \times (0, T)$ . Hence  $v_j$  are equi-Holder continuous on any compact subset of  $R^n \times (0, T)$  and  $v_j$  will converge uniformly on any compact subset of  $R^n \times (0, T)$  to some function  $\bar{v} \geq v$  as  $j \rightarrow \infty$ .

By Theorem 1.2 and Fatou's lemma, for any  $0 < t < T$  we have

$$\begin{aligned} & \int_{R^n} |u_j^{(m)}(x, t) - u^{(m)}(x, t)| dx \leq \int_{R^n} |u_{0j} - u_0| dx \\ & \leq \frac{1}{j} \int_{R^n} (k + |x|^2)^{-\alpha} dx \leq \frac{C}{j} \\ \Rightarrow & \int_0^T \int_{R^n} |u_j^{(m)}(x, t) - u^{(m)}(x, t)| dx dt \leq \frac{CT}{j} \\ \Rightarrow & \int_0^T \int_{R^n} |v_j(x, t) - \tilde{v}(x, t)| dx dt \leq \frac{CT}{j} \quad \text{as } m = m'_i \rightarrow 0 \\ \Rightarrow & \int_0^T \int_{R^n} |\bar{v}(x, t) - \tilde{v}(x, t)| dx dt = 0 \quad \text{as } j \rightarrow \infty \\ \Rightarrow & \bar{v}(x, t) = \tilde{v}(x, t) \quad \text{a.e. } (x, t) \in R^n \times (0, T). \end{aligned}$$

Since  $\bar{v} \geq v > 0$  in  $R^n \times (0, T)$ ,  $\tilde{v}(x, t) > 0$  for a.e.  $(x, t) \in R^n \times (0, T)$ . Thus there exists a set  $E \subset R^n \times (0, T)$  of measure zero such that

$$\lim_{i \rightarrow \infty} u^{(m'_i)}(x, t) = \tilde{v}(x, t) = \bar{v}(x, t) \geq v(x, t) > 0 \quad \forall (x, t) \in (R^n \times (0, T)) \setminus E.$$

Hence for any  $R' > 0$ ,  $0 < t'_1 < t'_2 < T$ , there exist  $x_0 \in R^n$ ,  $t'_2 < t_0 < T$ , such that

$$\lim_{i \rightarrow \infty} u^{(m'_i)}(x_0, t_0) > 0.$$

By Lemma 2.2 there exists a constant  $C > 0$  such that

$$\|u_0\|_{L^\infty} \geq u^{(m'_i)}(x, t) \geq C > 0 \quad \forall x \in B_{R'}, t'_1 \leq t \leq t'_2, i = 1, 2, \dots$$

Then there exists constants  $C_1 > 0$ ,  $C_2 > 0$ , independent of  $m'_i$  such that

$$C_1 \leq u^{(m'_i)m'_i-1}(x, t) \leq C_2 \quad \forall x \in B_{R'}, t'_1 \leq t \leq t'_2, i = 1, 2, \dots$$

Hence (1.2) is uniformly parabolic for  $u^{(m'_i)}$ . By standard parabolic theory [LSU],  $\{u^{(m'_i)}\}$  is uniformly equi-Holder continuous on any compact subset of  $R^2 \times (0, T)$ . By the Ascoli Theorem and a diagonalization argument similar to the proof of Theorem 2.3  $u^{(m'_i)}$  will have a subsequence converging uniformly on every compact subset of  $R^2 \times (0, T)$  to the maximal solution  $v$  of (0.2). This contradicts (2.17). Hence the theorem must be true and  $u^{(m)}$  converges uniformly on every compact subset of  $R^2 \times (0, T)$  to the maximal solution  $v$  of (0.2) as  $m \rightarrow 0$  and we are done.  $\square$

By the proof of Theorems 2.3 and 2.4 we have the following corollary:

**Corollary 2.5.** *If  $v$  is the maximal solution of (0.2) in  $R^n \times (0, T)$ ,  $n = 1, 2$ , where  $T$  is given by (0.3), then for any  $0 < T_0 < T$  there exists a constant  $C > 0$  such that*

$$v(x, t) \geq \begin{cases} \frac{Ct}{|x|^2} & \forall |x| \geq 2, 0 < t \leq T_0 \quad \text{if } n = 1 \\ \frac{Ct}{(|x| \log |x|)^2} & \forall |x| \geq 2, 0 < t \leq T_0 \quad \text{if } n = 2. \end{cases}$$

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