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The objective of this paper is to develop criteria that guarantee that a finite group G which acts faithfully on a vector space V possesses "many" orbits of different sizes. This has consequences on (classical and modular) character degrees of finite solvable groups.

Introduction.

The starting point of this investigation was the well-known problem of bounding the derived length $dl(G)$ of a finite solvable group G by the number $|cd(G)|$ of its irreducible complex character degrees. As well-known, in 1985 D. Gluck established the bound

$$dl(G) \ \leq \ 2\ |cd(G)|$$

for arbitrary solvable groups (cf. [2]).

I.M. Isaacs was the first to doubt that this bound asymptotically is the right bound, and there are indications that a much better bound should exist. For example, for the p-Sylow subgroups of some classical groups logarithmic bounds have been established (cf. [6], [12]), and on the other hand for A-groups, i.e. solvable groups all Sylow subgroups of which are abelian, we found a bound that is even stronger than logarithmic (cf. [9]).

Unfortunately, despite the results on some well-known p-groups mentioned above, the abundance of p-groups seems to make it impossible to attack the problem for p-groups with today's methods in general; for instance, no reasonable induction argument is available (cf. [11]). So in order to avoid the problems occuring with p-groups, B. Huppert and I.M. Isaacs suggested to consider the Fitting height instead of the derived length. Note that, as Fitting height and derived length coincide in A–groups, the result in [9] is already a result on the Fitting height. So we aimed at getting a similar bound for more general groups.

Now by standard character theory, in minimal counterexamples one often has a minimal normal subgroup N of G that has a complement H in G, and then character degrees that are induced from the extension of a linear character of N to its inertia group correspond to orbit sizes in the action of H on $\mathrm{Irr}(N)$. This is where the question of the number of different orbit sizes comes into play, a question, that surely is of independent interest and that we are going to study in this paper as generally as possible.

In the first section we prove that any finite group G that possesses a normal subgroup N of a certain type (almost extra-special) possesses (almost) as many distinct orbit sizes in its action on a vector space V as N does, provided that the center of N acts fixed point freely on V (which is a natural and necessary hypothesis); and for N this number is well–known. This result is not far from being best possible and one key step for the proof of our results in Section 2; namely it enables us to find many orbit sizes in quasiprimitive group actions of solvable groups. To handle the imprimitive case, we have to restrict ourselves to groups of odd order (which then are solvable by Feit-Thompson), and we make use of a recent result of A. Seress (cf. [**13**]) stating that the minimal base size of a solvable primitive permutation group of odd order is at most 3. This is why our inductive argument only works for primes greater than 3 whence we have an additional hypthesis in Theorem 2.1, where we examine the relationship between the number of orbit sizes of G on a vector space over a finite field of odd characteristic and the number of distinct orbit sizes of a suitable normal subgroup of G. But still this is strong enough to deal with our original subject, the dependence of the Fitting height $f(G)$ on $|cd(G)|$, at least for groups of odd order, and we find absolute constants C_1, C_2 such that

$$f(G) \ \leq \ C_1 \ \log|cd(G)| \ + \ C_2$$

for any group of odd order (see Corollary 2.4). This improves the former linear bound by Broline-Garrison (cf. [**7**, Corollary (12.21)]) and in fact is the asymptotically best possible bound. We conjecture that such a bound is true for arbitrary solvable groups.

Finally in the third section we discuss a result of I. M. Isaacs on bounding the derived length of a finite solvable group by the number of its irreducible Brauer characters in the light of our results obtained so far.

Results and Proofs.

Notation. All groups considered in this paper are finite. By $f(G)$ we denote the Fitting height (or nilpotent length) of G, and $dl(G)$ is the derived length of G. $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G, and $cd(G) := \{\chi(1) | \chi \in \mathrm{Irr}(G)\}$.

For any group G we denote by $d(G)$ the minimal number of generators of G. If p is a prime and $n \in \mathbb{N}$, then n_p is the p-part of n. For $x \in \mathbb{R}$ we let $\lceil x \rceil = \max\{n \in \mathbb{N} \mid n-1 < x\}$. All other notation is standard.

1. Orbit sizes of quasiprimitive groups.

Definition 1.1. Let G be a group acting on a vector space V. If $0 \neq v \in V$ such that $C_G(v) = 1$, then we call v^G a *regular orbit* of G on V.

If no non-identity element of G fixes any non–zero vector, then we say that G acts *fixed point freely* on V.

For a prime p and any integer $m \geq 0$ we say that a p-group E is *of type* $E(p, m)$ if

either $p > 2$ and E is extra-special of exponent p and order p^{2m+1}

or $p = 2$ and $E = E_1 Z(E)$ (central product) with an extra-special group E_1 of order 2^{2m+1} and $Z(E)$ cyclic of order 2 or 4.

This notion differs from [1] only in that we also allow $m = 0$ (in which case $E = Z(E)$ is cyclic). Note that the Lemmas 2, 3, 4, 5 in [1] remain true for $m = 0$, so that we may apply them also to our groups of type $E(p, m)$.

The following lemma and its proof were brought to my attention by Thomas R. Wolf.

Lemma 1.2. *Let G be a group acting on a vector space V over $K := GF(q)$ for a prime q. Suppose that $E \trianglelefteq G$ is of type $E(p, m)$ for a prime p and an integer $m \geq 0$ such that $Z(E)$ acts fixed point freely on V. If $g \in G$ with $g \notin C_G(E)$, then*

$$\dim_K C_V(g) \;\leq\; \frac{p+1}{2p}\; \dim_K V.$$

Proof. As $Z(E)$ acts fixed point freely on V, E acts faithfully on V and $p \neq q$. So V_E is completely reducible, i.e. there is an $n \in \mathbb{N}$ such that

$$V_E \;=\; V_1 \oplus \cdots \oplus V_n$$

with irreducible and obviously also faithful E–modules V_i. Now if $1 \neq x \in E$, then by [1, Lemma 3] we have $\dim_K C_{V_i}(x) \leq (1/p) \dim_K V_i$ for $i = 1, \dots, n$, and thus we easily obtain

$$\dim_K C_V(x) \;\leq\; \frac{1}{p}\; \dim_K V \;.$$

To prove the lemma, let $g \in G$ with $g \notin C_G(E)$. Then there exists a $1 \neq x \in E$ such that $1 \neq [g, x] \in E$ whence we have $\dim_K C_V([g, x]) \leq (1/p) \dim_K V$. So as clearly $C_V(g^x) \cap C_V(g) \leq C_V([g, x])$, we find that

$$\begin{aligned}
\dim_K V \quad &\geq \quad \dim_K \left(C_V(g^x) \;+\; C_V(g)\right) \\
&= \quad \dim_K C_V(g^x) \;+\; \dim_K C_V(g) \;-\; \dim_K \left(C_V(g^x) \;\cap\; C_V(g)\right) \\
&\geq \quad 2\; \dim_K C_V(g) \;-\; \dim_K C_V([g, x]) \\
&\geq \quad 2\; \dim_K C_V(g) \;-\; \frac{1}{p}\; \dim_K V.
\end{aligned}$$

Hence the assertion follows. □

Next we prove a lemma on the existence of p–regular orbits. We do not try here to figure out the minimal hypothesis we need to establish our claim, but simply give a straightforward argument that suffices to achieve our asymptotic results later on.

Lemma 1.3. *Let G be a group acting on a vector space V over $K := GF(q)$ for a prime q. Suppose that $E \trianglelefteq G$ is of type $E(p,m)$ for a prime p and an integer $m \geq 0$ such that $C_G(E)$ acts fixed point freely on V.*

If $m \geq 10$, then G has a regular orbit on V. If p is odd, then G has a p–regular orbit on V even for $m \geq 5$.

Proof. Clearly E acts faithfully on V and so $(|E|,|V|) = 1$. Consequently V_E is completely reducible so that for an $n \in \mathbb{N}$ we have

$$V_E \;=\; V_1 \oplus \cdots \oplus V_n$$

with irreducible and obviously also faithful E-modules V_i.

Now put $\delta = 0$ if p is odd, and $\delta = 1$ in case of $p = 2$. Then it is well-known that there is an abelian $A \leq E$ with $A \cap Z(E) = 1$ of order $p^{m-\delta}$ (cf. [**5**, III, §13]). Now with [**1**, Lemma 3] we see that $|C_{V_i}(A)| = |V_i|^{(1/p^{m-\delta})}$ for all $i = 1, \ldots, n$ and thus $|C_V(A)| = |V|^{(1/p^{m-\delta})}$. Moreover $C_G(E)$ normalizes (and so acts fixed point freely on) $C_V(A)$ which implies $|C_G(E)| \mid |C_V(A)|-1$, in particular $|C_G(E)| \leq |C_V(A)|$.

Furthermore, if we put $B := C_G(Z(E))$, by [**10**, Corollary 1.6] $B/EC_B(E)$ $= B/C_B(E/Z)$ is isomorphic to a subgroup of $\mathrm{Sp}(2m,p)$, so that we find the elementary estimation $|B/EC_G(E)| \leq p^{2m^2+m}$ (note that $C_B(E) = C_G(E)$). Finally observe that obviously $|G/B| \leq |\mathrm{Aut}(Z(E))| \leq p$.

Now assume that G has no p–regular orbit on V. Then obviously

$$V \;=\; \bigcup_{g \in G \setminus \{1\}} C_V(g),$$

and thus as $C_G(E)$ acts fixed point freely on V, we obtain with Lemma 1.2 that

$$|V| \;\leq \sum_{g \in G \text{ with } o(g)=p} |C_V(g)| \;\leq\; |G| \;\cdot\; |V|^{\frac{p+1}{2p}}$$

and hence

$$\begin{aligned} |V|^{\frac{p-1}{2p}} \;&\leq\; |G| \;=\; |G/B| \;\cdot\; |B/EC_G(E)| \;\cdot\; |E/Z(E)| \;\cdot\; |C_G(E)| \\ &\leq\; p \;\cdot\; p^{2m^2+m} \;\cdot\; p^{2m} \;\cdot\; |V|^{(1/p^{m-\delta})}. \end{aligned}$$

Now by [**1**, Lemma 2] for $i = 1, \ldots, n$ we have $\dim_K V_i = s\ p^m$, where $s \geq 1$ is the smallest integer such that $|Z(E)|$ divides q^s-1. Hence $p \leq |Z(E)| \leq q^s$

and

$$|V| \geq |V_1| = q^{sp^m} \geq p^{p^m}.$$

So we conclude that

$$p^{2m^2+3m+1} \geq |V|^{\left(\frac{p^{m-\delta}-p^{m-1-\delta}-2}{2p^{m-\delta}}\right)} \geq p^{\left(\frac{p^m-p^{m-1}-2p^\delta}{2}\right)}$$

and further that

$$p^{m-1}(p-1) \leq 4m^2 + 6m + 2 + 2p^\delta.$$

If $p = 2$, then this is only true for $m \leq 9$ contradicting our hypothesis, and if $p \geq 3$, we have $3^{m-1} \leq 2m^2 + 3m + 2$ which only holds for $m \leq 4$, so that again we have a contradiction. Hence the lemma is shown. □

Lemma 1.4. *Let G be a group acting on a vector space V over $K := GF(q)$ for a prime q. Suppose that $E \trianglelefteq G$ is of type $E(p,m)$ for a prime p and an integer $m \geq 0$ such that $Z(E)$ acts fixed point freely on V. Now define*

$$m' = \begin{cases} m & \text{if } p \text{ is odd or } E \text{ is the central product of } m \\ & \text{dihedral groups of order 8 and } Z(E) \\ m-1 & \text{if } m \geq 1 \text{ and } E \text{ is the central product of } m-1 \\ & \text{dihedral groups of order 8, a quaternion group} \\ & \text{of order 8 and } Z(E). \end{cases}$$

Fix $i \in \{0, \dots, m'\}$. Moreover let $Z(E) \leq E_i \trianglelefteq E$ be of type $E(p,i)$. In case of $m' = m-1$ and $i \geq 1$ we also assume that E_i contains a quaternion subgroup. Let $A_i \leq E$ be elementary abelian of order $p^{m'-i}$ such that $E_i \cap A_i = 1$. (Such E_i, A_i exist, as one can see with the proof of [**1**, Lemma 5].) *Hence $[E_i, A_i] = 1$, and (by* [**1**, Lemma 3]) *$C_V(A_i) > 0$.*

Now let

$$\begin{aligned} C_i &= C_G(E_i\ A_i/A_i) \\ &:= \{g \in G \mid (xA_i)^g = x^g A_i^g = xA_i \ \ \text{for all } x \in E_i\ \} \end{aligned}$$

and let $A_i \leq N_i \leq C_i$ be maximal subject to $C_V(N_i) > 0$.

Put $V_i = C_V(N_i)$ and $H_i = N_G(N_i)$, and for any $U \leq H_i$ we write $\overline{U} = UN_i/N_i$.

Then the following holds:

a) *$\overline{H_i}$ acts on V_i, and $E_i \cong \overline{E_i} \trianglelefteq \overline{H_i}$ and $E_iN_i \trianglelefteq H_i$. Furthermore $E \cap H_i = E_i \times A_i \trianglelefteq H_i$, $E \cap N_i = A_i \trianglelefteq H_i$, and $Z(\overline{E_i}) = \overline{Z(E_i)} = \overline{Z(E)}$ acts fixed point freely on V_i; in particular, $\overline{E_i}$ acts faithfully on V_i.*

b) *$C_{\overline{H_i}}(\overline{E_i})$ acts fixed point freely on V_i.*

c) *Assume now that $0 \leq i \leq m'-1$ and that $A_i > A_{i+1}$ and $E_i < E_{i+1}$. If N_{i+1} is given, then we can choose N_i, V_i such that $N_i > N_{i+1}$ and $V_i < V_{i+1}$.*

Proof. a) First observe that $C_i \le N_G(A_i)$ and that $E_i \cap N_i = 1$.

Now of course $\overline{H_i}$ normalizes V_i. As $N_i \le C_i$, for any $x \in N_i$, $y \in E_i$ there is an $a \in A_i$ such that $xyx^{-1} = y^{x^{-1}} = ya$. Hence $x^y = y^{-1}xy = ax \in N_i$. This shows that $E_i \le H_i$. So $Z(E) \le H_i$ acts fixed point freely on V_i, and so does $Z(\overline{E_i}) = \overline{Z(E_i)} \cong Z(E)$. Hence clearly $\overline{E_i} \cong E_i$.

Now we have $A_i \times E_i \le H_i$. If $y \in E \backslash (A_i \times E_i)$, then there is an $x \in A_i$ with $1 \ne [x,y] \in Z(E)$; hence the assumption $y \in H_i$ yields $[x,y] \in N_i$ which contradicts $C_V(N_i) > 0$. This shows $A_i \times E_i = E \cap H_i \trianglelefteq H_i$ and then $\overline{E_i} = \overline{E_i \times A_i} \trianglelefteq \overline{H_i}$ and $E_iN_i \trianglelefteq H_i$. Clearly $\overline{E_i}$ acts faithfully on V_i.

Now finally we have $A_i \le E \cap N_i \le E \cap H_i = A_i \times E_i$, so we have $E \cap N_i = A_i$, because $E_i \cap N_i = 1$. So a) is shown.

b) It suffices to show that all elements of prime order in $C_{\overline{H_i}}(\overline{E_i})$ act fixed point freely on V_i. Let $1 \ne gN_i \in C_{\overline{H_i}}(\overline{E_i})$ with $g \in H_i$. Then we have $1 = [\langle gN_i \rangle, \overline{E_i}] = [\langle g \rangle, E_i]N_i/N_i$, and so a) yields $[\langle g \rangle, E_i] \le N_i \cap E = A_i$. As furthermore g normalizes A_i, it follows that $g \in C_i$.

Let $N_i^* = \langle N_i, g \rangle = N_i \langle g \rangle \ > N_i$. Then we have $A_i \le N_i^* \le C_i$, so the maximal choice of N_i forces $C_V(N_i^*) = 0$. Consequently also $C_{V_i}(N_i^*) = 0$, and as N_i acts trivially on V_i, we find $C_{V_i}(g) = 0$. Hence gN_i acts fixed point freely on V_i, as desired.

c) Let $A_i \cap E_{i+1} = \langle a \rangle$. By a) we have $A_{i+1} \trianglelefteq N_{i+1}$. Now $A_i = A_{i+1} \times \langle a \rangle$, and thus for $h \in N_{i+1}$ we have $a^h = ab$ with a suitable $b \in A_{i+1}$, as $N_{i+1} \le C_{i+1}$. So it follows that $N_{i+1} \le N_G(A_i)$. As $N_{i+1} \le C_{i+1}$, moreover we have that $N_{i+1} \le C_G(E_i \times \langle a \rangle \ A_{i+1}/A_{i+1}) = C_G(E_iA_i/A_{i+1})$. Hence it follows that $N_{i+1} \le C_i$.

Now we build the semidirect product $K_{i+1} := N_{i+1}A_i = N_{i+1}\langle a \rangle \ > N_{i+1}$. Then we have $A_i \le K_{i+1} \le C_i$. Moreover by a) we know that $E_{i+1} \le H_{i+1}$ acts faithfully on V_{i+1}. Hence again with [**1**] we see that V_{i+1} is completely reducible as E_{i+1}–module and then that $C_{V_{i+1}}(a) > 0$. Hence $0 < C_{V_{i+1}}(N_{i+1}\langle a \rangle) = C_{V_{i+1}}(K_{i+1})$ and so $C_V(K_{i+1}) > 0$. Thus K_{i+1} has all the properties that we demand of N_i (except maximality), i.e. we can choose N_i such that $N_i \ge K_{i+1} > N_{i+1}$, as wanted. Then clearly $V_i \le V_{i+1}$, and as E_{i+1} acts faithfully on V_{i+1} whereas a centralizes V_i, it follows that $V_i \le C_{V_{i+1}}(a) < V_{i+1}$, and finally c) is proven. □

Theorem 1.5. *Let G be a group acting on a vector space V over $K := GF(q)$ for a prime q. Suppose that $E \trianglelefteq G$ is of type $E(p,m)$ for a prime p and an integer $m \ge 0$ such that $Z(E)$ acts fixed point freely on V. Then the following holds:*

a) *Suppose that p is odd and that $m \ge 5$. For $i = 5, \dots, m$ take subgroups N_i of G as in Lemma 1.4 such that $N_5 > \cdots > N_m$ (which is possible by Lemma 1.4c)). Then there exist $v_i \in V$ $(i = 5, \dots, m)$ such that $C_G(v_i) =$*

$N_i > 1$. *Hence by Lemma 1.4 the* $|v_i^G|_p$ $(i = 5, \ldots, m)$ *are strictly increasing; in particular, G has at least $m-4$ different nontrivial orbit sizes on V.*

b) *Suppose that $p = 2$, let m' be defined as in Lemma 1.4, and suppose that $m' \geq 10$. For $i = 10, \ldots, m'$ take subgroups N_i of G as in Lemma 1.4 such that $N_{10} > \cdots > N_{m'}$. Then there exist $v_i \in V$ $(i = 10, \ldots, m')$ such that $C_G(v_i) = N_i > 1$, whence the 2-parts of the corresponding orbit sizes are strictly increasing; in particular, G has at least $m' - 9$ different nontrivial orbit sizes on V.*

Proof. a) Fix $i \in \{5, \ldots, m\}$, and let the A_i, E_i, H_i and V_i be defined as in Lemma 1.4. First note that $V_i \leq C_V(A_i)$. By Lemma 1.4a) E_i acts faithfully (and clearly completely reducibly) on V_i, so by [**1**, Lemma 4] it has a regular orbit on V_i. Now for every $v \in V_i$ that lies in a regular orbit of E_i on V_i obviously we have $C_E(v) = A_i$. Consequently $A_i = E \cap C_G(v) \trianglelefteq C_G(v)$, so with $M_i := N_G(A_i)$ we have $C_G(v) \leq M_i$. Hence we have shown that for every $v \in V_i$ that lies in a regular orbit of E_i we have $C_G(v) = C_{M_i}(v) \quad (*)$.

Now clearly $E_iA_i = E_i \times A_i = E \cap M_i \trianglelefteq M_i$ and $C_i \leq M_i$ and thus $C_i = C_{M_i}(E_iA_i/A_i) \trianglelefteq M_i$, and so we have $N_i \leq C_i \trianglelefteq M_i$.

Next let $v \in V_i$ be an element out of a regular orbit of E_i on V_i and put $S_i = C_{M_i}(v) \cap C_i \trianglelefteq C_{M_i}(v)$. We claim that $S_i = N_i$. Obviously $N_i \leq S_i$. On the other hand we have $A_i \leq S_i \leq C_i$ and $v \in C_V(S_i)$, in particular $C_V(S_i) > 0$. Hence our maximal choice of N_i forces $S_i = N_i$, establishing our claim.

So now by $(*)$ we have $N_i = S_i \trianglelefteq C_{M_i}(v) = C_G(v)$, whence $C_G(v) \leq N_G(N_i) = H_i$, and as v was chosen arbitrarily, now we know that for every $v \in V_i$ that lies in a regular orbit of E_i on V_i we have $C_G(v) = C_{H_i}(v) \quad (**)$.

Now by Lemma 1.4a), b) the action of $\overline{H_i} := H_i/N_i$ on V_i fulfills the hypotheses of Lemma 1.3, so that Lemma 1.3 yields a $v_i \in V_i$ that lies in a regular orbit of $\overline{H_i}$ on V_i, i.e. $|C_{\overline{H_i}}(v_i)| = 1$. In particular, as $E_i \cong \overline{E_i}$, v_i lies in a regular orbit of E_i on V_i, so that with $(**)$ we obtain $C_G(v_i) = C_{H_i}(v_i) = C_{N_i}(v_i) = N_i$, as desired.

Finally as $N_i \geq N_{i+1}$ and as by Lemma 1.4a) we have $A_i \leq N_i$ and $A_i \not\leq N_{i+1}$, it readily follows that $|N_i|_p > |N_{i+1}|_p$, and the proof of a) is complete.

b) is proved analogously with the help of the results in the Lemmata 1.3 and 1.4 for the case $p = 2$. □

Remember that if G is a group and V a G–module such that V_N is homogeneous for every $N \trianglelefteq G$, then V is called a quasiprimitive G–module. In this case the structure of G is well–known (cf. [**10**, §1]).

As a special case of Theorem 1.5 we get a result on quasiprimitive actions.

Corollary 1.6. *Let G be a solvable group acting faithfully and quasiprimitively on a vector space V over a finite field. Then there is an $E \trianglelefteq G$ of type $E(p, m)$ for a prime p and an integer $m \geq 0$, and the following hold:*

a) *If p is odd and $m \geq 5$, then G has at least $m-4$ different nontrivial orbit sizes on V.*

b) *Let $p=2$ and m' be defined as in Lemma 1.4. If $m' \geq 10$, then G has at least $m'-9$ different nontrivial orbit sizes on V.*

Proof. The existence of E is clear from [**10**, Corollary 1.10]. Moreover as $Z(E) \trianglelefteq G$, $V_{Z(E)}$ is homogeneous, and so $Z(E)$ acts fixed point freely on V. Thus by Theorem 1.5 the assertion follows. $\square$

2. The main results for groups of odd order.

Theorem 2.1. *Let G be a (solvable) group of odd order that acts faithfully on a vector space V over a finite field K of odd characteristic q. Suppose that $N \trianglelefteq G$ is a p-group of class 1 or 2 and of exponent p for a prime $p \geq 5$ with $p \neq q$. Let $A \leq N$ be an abelian subgroup of N, and put $s = d(A)$. Then G has on V at least $m := \lceil \frac{s}{6} \rceil$ orbits, whose sizes have mutually distinct nontrivial p-parts. In particular, G has at least m different nontrivial orbit sizes on V.*

Proof. We prove the result by induction on $|G|+|V|$.

Note that A is elementary abelian. If $s \leq 5$, the assertion is trivial, so let $s \geq 6$.

First suppose that V_N is homogeneous. Then V is a multiple of a faithful and irreducible N-module. Hence (by [**3**, Theorem 3.2.2]) $Z(N)$ is cyclic, and as $N' \leq Z(N)$, we see that either $N'=1$ and N is cyclic of order p or $|N'| = |Z(N)| = p$. In the first case our assertion is trivially fulfilled, so we only have to consider the latter. As $\exp(N) = p$, we also have $N' = \phi(N)$ (which is the Frattini subgroup of N). Consequently N is extra–special and thus of type $E(p,n)$ for an integer n. Thus $6 \leq s \leq n+1$, and by Theorem 1.5 G has on V at least $n-4 \geq s-5 \geq \lceil \frac{s}{6} \rceil = m$ orbits, whose sizes have mutually distinct nontrivial p-parts. So we are done in this case.

It remains to consider the case that V_N is inhomogeneous.

First assume that V is not irreducible, i.e. $V = V_1 \oplus V_2$ for nontrivial G-modules V_i $(i=1,2)$. Let $C = C_G(V_1)$, $s_1 = d(AC/C)$ and $m_1 = \lceil \frac{s_1}{6} \rceil$. Applying induction to the action of G/C on V_1 yields $v_1, \ldots, v_{m_1} \in V_1$ such that $|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{m_1})|_p$. Now let $H = C_G(v_{m_1})$, $D = C_H(V_2)$, $s_2 = d((A \cap H)D/D)$ and $m_2 = \lceil \frac{s_2}{6} \rceil$. This time induction, applied to the action of H/D on V_2, yields $w_1, \ldots, w_{m_2} \in V_2$ such that $|H|_p > |C_H(w_1)|_p > \cdots > |C_H(w_{m_2})|_p$. Now put $v_{m_1+i} = v_{m_1} + w_i$ for $i = 1, \ldots, m_2$. Then $C_G(v_{m_1+i}) = C_G(v_{m_1}) \cap C_G(w_i) = C_H(w_i)$ for $i = 1, \ldots, m_2$. Thus we obtain $|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{m_1+m_2})|_p$. Now as $C_A(V_1) \leq H$ surely acts faithfully on V_2, we see that $s_1 + s_2 \geq s$ and consequently $m_1 + m_2 \geq \lceil \frac{s_1}{6} + \frac{s_2}{6} \rceil \geq \lceil \frac{s}{6} \rceil$ which shows that we have established our assertion.

So now we may assume that V is an irreducible imprimitive G–module. By [**10**, Proposition 0.2(ii)] there exists a $N \leq C \trianglelefteq G$ such that $V_C =$

$V_1 \oplus \cdots \oplus V_n$ (with $n > 1$) with C-invariant V_i ($i = 1, \ldots, n$) that are faithfully and primitively permuted by G/C.

If for any $M \subseteq \{V_1, \ldots, V_n\}$ we put $\mathrm{stab}_{G/C}(M) = \{hC \in G/C \mid M^{hC} = M\}$, then by [13, Theorem 1.3] we may assume that

$$(*) \qquad \bigcap_{j=1}^{3} \mathrm{stab}_{G/C}(\{V_j\}) \ = \ 1, \text{ i.e. } \quad \bigcap_{j=1}^{3} N_G(V_i) \ = C.$$

Now $H = N_G(V_1)$ is known to be a maximal subgroup of G, and V is induced from the H-module V_1, i.e. if $G = \bigcup_{j=1}^{|G:H|} Hg_j$ with suitable $g_j \in G$ (and $g_1 := 1$) is the decomposition of G into right cosets of H, then $V = V_1 \otimes_{KH} KG$, and thus (by identifying V_1 and $V_1 \otimes 1$) we have $V_i = V_1 \otimes g_i$ for $i = 1, \ldots, n$.

Next observe that (by [10, Lemma 4.1]) $v^H \neq (-v)^H$ for all $v \in V_1$. Hence there is an $X \subseteq V_1$ such that with $Y := -X$ we have $V_1 \backslash \{0\} = X \cup Y$ and $X \cap Y = \emptyset$. Now we define $B \subseteq V$ as follows: If $v \in V$ and $v = \sum_{i=1}^{n} x_i \otimes g_i$ with $x_i \in V_1$ for all i, then $v \in B$ if and only if

(1) $x_i \in X$ for $i = 1, \ldots, \min\{3, n\}$ and $x_i \in Y$ for $i = 4, \ldots, n$, and

(2) whenever $x_i = 0$ for an $i < n$, then $x_{i+1} = 0$.

Now let $v = \sum_{i=1}^{n} x_i \otimes g_i \in B$, and define $S(v) := \bigcap_{i=1}^{n} C_G(x_i \otimes g_i)$. Then we claim that $S(v) \trianglelefteq C_G(v)$ such that

$$(**) \qquad |C_G(v)/S(v)| \ \in \ \{1, 3\}.$$

To see this, first let $j \in \{1, \ldots, n\}$ be maximal subject to $x_j \neq 0$. If $j = 1$, clearly $C_G(v) = S(v)$, and this is also true for $j = 2$, as $|G|$ is odd. So let $n \geq j \geq 3$. Now note that for any $g \in G$ we have $v^g \ \in \ \bigoplus_{i=1}^{j} x_i^H \ \otimes \ g_{\sigma(i)}$, where $\sigma \in S_n$ is the permutation defined by the permution action of gC on the V_i, i.e. $V_i^{gC} = V_{\sigma(i)}$ for $i = 1, \ldots, n$. So as X and Y are H-invariant, by $(*)$ and our definition of B we find that if $g \in C_G(v)$, then $gC \in T/C := \mathrm{stab}_{G/C}(\{V_1, V_2, V_3\})$. Now by $(*)$ we have that T/C is isomorphic to a subgroup of S_3, so as $|G|$ is odd, either $T = C$ or $|T/C| = 3$. Furthermore as $j \geq 3$ and as naturally $C_G(x_i \otimes g_i) \leq H^{g_i} = N_G(V_i)$ for all i, $(*)$ also implies that $S(v) \leq C$. So we conclude that $S(v) = C \cap C_G(v) \trianglelefteq C_G(v)$ and further that $|C_G(v)/S(v)| = |C_G(v)C/C| \mid |T/C| \in \{1, 3\}$ whence $(**)$ is shown.

Now we can construct representatives of the orbits that we are looking for. We will do this by an inductive process, namely we put $v_0 = 0 \in V$, $t_0 = 0$ and for all $i = 1, \ldots, n$ proceed as follows:

($\ast\!\ast\!\ast$) Suppose that we have already $t_{i-1} \in \mathbb{N}$ and $v_0, \ldots, v_{t_{i-1}} \in B$ with $|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_{i-1}})|_p$, then let $H_i = C_G(v_{t_{i-1}})$ and $C_i = C_{H_i}(V_i)$. Furthermore let $s_i = d((A \cap H_i)C_i/C_i)$ and $m_i = \lceil \frac{s_i}{6} \rceil$. First suppose that $m_i \geq 1$. Apply induction to the action of $N_{H_i}(V_i)/C_i$ on V_i. As obviously $C_{H_i}(w) \leq N_{H_i}(V_i)$ for all $w \in V_i$, this yields elements $w_{i,1}, \ldots, w_{i,m_i} \in V_i$ such that $|H_i|_p > |C_{H_i}(w_{i,1})|_p > \cdots > |C_{H_i}(w_{i,m_i})|_p$. If $i \leq 3$, we choose the $w_{i,j}$ $(j = 1, \ldots, m_i)$ to be out of $X \otimes g_i$, if $i \geq 4$, we take them out of $Y \otimes g_i$ (which is obviously possible). Then put $v_{t_{i-1}+j} = v_{t_{i-1}} + w_{i,j} \in B$ $(j = 1, \ldots, m_i)$ and put $t_i := t_{i-1} + m_i$. Hence as $p \geq 5$ and because of ($\ast\ast$) we conclude $|C_G(v_{t_{i-1}+j})|_p = |S(v_{t_{i-1}+j})|_p = |S(v_{t_{i-1}}) \cap C_G(w_{i,j})|_p = |C_G(v_{t_{i-1}}) \cap C_G(w_{i,j})|_p = |C_{H_i}(w_{i,j})|_p < |H_i|_p$ for $j = 1, \ldots, m_i$. Thus we have $|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_i})|_p$. Now suppose that $m_i = 0$. If $i \leq 3$, take an arbitrary $v \in X \otimes g_i$, if $i \geq 4$, then take an arbitrary $v \in Y \otimes g_i$. Put $t_i = t_{i-1}$ and simply replace v_{t_i} by $v_{t_i} + v$. Then still $v_{t_i} \in B$ and $|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_i})|_p$.

If $i = n$, then we are done, else we repeat the procedure ($\ast\!\ast\!\ast$).

So this process yields t_n orbits on V the sizes of which have mutually distinct p–parts > 1. It remains to show that $t_n \geq m$. If we show that $t := \sum_{i=1}^{n} s_i \geq s$, then we obtain $t_n = \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \lceil \frac{s_i}{6} \rceil \geq \lceil \sum_{i=1}^{n} \frac{s_i}{6} \rceil \geq \lceil \frac{s}{6} \rceil = m$, as wanted.

So we have to prove $t \geq s$. For this let $1 \neq a \in A$, and surely it suffices to show that there is an $i \in \{1, \ldots, n\}$ such that $a \in H_i$ and $a \notin C_i$, i.e. $1 \neq aC_i \in (A \cap H_i)C_i/C_i$. To see this note first that if $a \notin C_1$, then we are done with $i = 1$, as $a \in A \leq N \leq C \leq H = H_1$. So let $a \in C_1$, and let $j \in \{1, \ldots, n\}$ be maximal such that $a \in C_l$ for $l = 1, \ldots, j$. Surely $j \leq n-1$ because else we would get the contradiction $1 \neq a \in \bigcap_{k=1}^{n} C_G(V_k) = 1$. Then clearly $a \in H_{j+1}$ and $a \notin C_{j+1}$. So we choose $i := j + 1$. This completes the proof of the theorem. □

Notice that Theorem 2.1 is also true for $p = 3$ if K is not too small, more precisely, if besides -1 there is another $a \in K^{\times}$ whose order does not divide $|G|$. Namely then in an obvious way we can define B in the proof of Theorem 2.1 even such that $|C_G(v)/S(v)| = 1$ for all $v \in B$. Details are left to the reader.

Notice furthermore that obviously with the proof of Theorem 2.1 one even can get elements $w_1, \ldots, w_m \in V$ whose centralizers have orders with increasing $3'$–parts (but this is not relevant for our purposes).

In the following for any solvable group G we shall make use of subgroups $K_i(G)$ $(i \in \mathbb{N} \cup \{0\})$ which are defined as follows:

$K_0(G) = G$, and for $i \geq 1$ and given $K_{i-1}(G)$ let $K_i(G)$ be the smallest normal subgroup of $K_{i-1}(G)$ such that $K_{i-1}(G)/K_i(G)$ is nilpotent.

Then by [5, III, Satz 4.6] we know that the $K_i(G)$ are characteristic subgroups of G and that if n is the minimal number such that $K_n(G) = 1$, then $n = f(G)$. Furthermore, by the definition of the K_i in [5] it is clear that for any $N \trianglelefteq G$ and for all $i \in \mathbb{N}$ we have

$$K_i(G/N) \;=\; K_i(G)N/N.$$

Hence for any $N \trianglelefteq G$ with $K_{n-1}(G)N/N > 1$ we have $f(G/N) = f(G)$.

Lemma 2.2. *Let G be a solvable group, p a prime and H a Hall-p'-subgroup of G. Put $n = f(G)$ and $K = K_{n-1}(G)$. Then*

$$dl(H) \;\geq\; \begin{cases} \frac{f(G)}{2} & \text{if } K \text{ is not a } p\text{-group} \\ \frac{f(G)-1}{2} & \text{if } K \text{ is a } p\text{-group.} \end{cases}$$

Proof. We prove the result by induction on $|G|$. It is obviously true if G is nilpotent. So let $f(G) \geq 2$.

First suppose that K is a p-group. Then, as $K_{n-2}(G)$ is not nilpotent, $K_{n-2}(G/K) = K_{n-2}(G)/K$ is not a p-group, and thus induction yields $dl(H) = dl(HK/K) \geq \frac{f(G/K)}{2} = \frac{f(G)-1}{2}$, as wanted.

Hence it remains that K is not a p-group. If $p \mid |K|$, then let P be a Sylow-p-subgroup of K. As K is nilpotent, we have $P \trianglelefteq G$, and as $K/P > 1$, we have $f(G/P) = n = f(G)$ and $K_{n-1}(G/P) = K/P$ is not a p-group, whence induction yields $dl(H) = dl(HP/P) \geq \frac{f(G/P)}{2} = \frac{f(G)}{2}$. Hence we may assume that K is a p'-group.

Now if there is an $N \trianglelefteq G$ with $1 < N < K$, then induction yields $dl(H) \geq dl(H/N) \geq \frac{f(G/N)}{2} = \frac{f(G)}{2}$. Thus we may assume that K is a minimal normal subgroup of G, and as such it is an elementary abelian q-group for a prime $q \neq p$.

Let $k = dl(H)$ and $T = H^{(k-1)}$ the $(k-1)$-th commutator subgroup of H, then $T > 1$ and T is abelian. Put $C := C_G(K)$. Clearly $K_{n-2}(G) \not\leq C$ and thus $K_{n-2}(G/C) = K_{n-2}(G)C/C > 1$ and so $f(G/C) = n - 1 = f(G) - 1$. Thus if $T \leq C$, by induction we conclude $dl(H) \geq dl(HC/C) + 1 \geq \frac{f(G/C)-1}{2} + 1 = \frac{f(G)-2}{2} + 1 = \frac{f(G)}{2}$, as desired.

Hence we may assume that $\overline{T} := TC/C > 1$. We will lead this to a contradiction. Now as K may be regarded as a faithful and irreducible G/C-module over $GF(q)$, we have that $\overline{T}$ is a q'-group of automorphisms of K (cf. [5, V, Satz 5.17]). Hence by [3, Theorem 5.3.6] for $1 < R := [\overline{T}, K] \leq K$ we have $[\overline{T}, R] = R$. Going back to G, this obviously means that $R = [T, K]$ and $[T, R] = R$ $(*)$.

Next we claim that $R \leq H^{(i)}$ for $i = 0, \ldots, k-1$. For $i = 0$ this is trivial, and if we already have $R \leq H^{(i)}$ for an $i \in \{0, \ldots, k-2\}$, then, as also

$T \leq H^{(i)}$, by ($*$) we have $H^{(i+1)} \geq [T, R] = R$, which proves our claim. Hence we have $R \leq H^{(k-1)} = T$ whence (by ($*$)) we get the contradiction $1 = [T, T] \geq [T, R] = R > 1$. We are done. $\square$

Theorem 2.3. *Let G be a (solvable) group of odd order that acts faithfully and irreducibly on a vector space V over a finite field of odd characteristic q. Put $n = f(G)$ and let m be the number of different orbit sizes of G on V. Assume further that $K := K_{n-1}(G)$ is not a 3-group. Then*

$$f(G) \; \leq \; 8 \; \log_2 m \; + \; 30.$$

Proof. We may assume that $n \geq 2$. Let $p \geq 5$ be a prime dividing $|K|$. Then $p \neq q$ (by [**5**, V, Satz 5.17]). Let $P \in Syl_p(K)$ and put $C = C_G(P)$. Then $P \trianglelefteq G$ and $K_{n-2}(G) \not\leq C$, because else $K_{n-2}(G)/O_{p'}(K)$ would be nilpotent contradicting the definition of K. Hence we have $f(G/C) \geq f(G) - 1$. Now let H be a Hall-p'-subgroup of G/C. Then H acts faithfully on P with $(|H|, |P|) = 1$. Hence by [**3**, Theorem 5.3.13] there is a characteristic subgroup N of P of class at most 2 and of exponent p such that H acts faithfully on N. Note that $N \trianglelefteq G$. By [**3**, Theorem 5.1.4] H acts also faithfully on $N/\phi(N)$ (which is of order $p^{d(N)}$) whence by [**10**, Corollary 3.12(b)] we have $dl(H) \leq 2\log_2 d(N) + 2$. Now by Lemma 2.2 we know that $dl(H) \geq \frac{f(G/C)-1}{2} \geq \frac{f(G)}{2} - 1$, so that we obtain $f(G) \leq 2dl(H) + 2 \leq 4\log_2 d(N) + 6$. As p is odd, by [**5**, III, Satz 12.3] there is an abelian $A \trianglelefteq N$ with $d(N) \leq \frac{d(A)(d(A)+1)}{2} \leq d(A)^2$, so that $f(G) \leq 8\log_2 d(A) + 6$. By Theorem 2.1 we have $m \geq \frac{d(A)}{6}$, whence we finally conclude $f(G) \leq 8\log_2(6m) + 6 \leq 8\log_2 m + 30$, as desired. $\square$

Actually we have not tried to optimize the constants in Theorem 2.3.

Corollary 2.4. *Let G be a group of odd order. Then*

$$f(G) \; \leq \; 8 \; \log_2 |cd(G)| \; + \; 32.$$

Proof. Put $n = f(G)$, $K = K_{n-1}(G)$ and (in case of $n \geq 2$) $L = K_{n-2}(G)$ and $F = F(L)$, the Fitting subgroup of L. First we show:

($*$) If $n \leq 3$ or if $n \geq 4$ and L/F is not a 3-group, then $f(G) \leq 8\log_2 |cd(G)| + 31$.

To prove ($*$), we argue by induction on $|G|$. For $n \leq 3$ there is nothing to prove. Let $n \geq 4$ and let $p \geq 5$ be a prime dividing $|L/F|$, and let $P \in Syl_p(L)$. Surely $PF/F > 1$, and there is a prime $q \neq p$ such that for $Q_0 \in Syl_q(K)$ we have $[P, Q_0] > 1$ because else P would centralize $O_{p'}(K)$ and thus $P \times O_{p'}(K) = PK \leq F$ against our choice of P. Clearly $Q_0 \trianglelefteq G$, and with [**3**, Theorem 5.3.6] we conclude that for $Q := [P, Q_0] \leq Q_0$ we have $[P, Q] = Q > 1$.

Now if there is a $1 < N \trianglelefteq G$ such that $QN/N > 1$, then surely we have $f(G/N) = f(G)$ and $QN/N \le Q_0N/N \le F(LN/N) = F(K_{n-2}(G/N))$, and as $[PN/N, QN/N] = [P,Q]N/N = QN/N > 1$, it follows that $1 < PN/N \le LN/N = K_{n-2}(G/N)$ and $PN/N \not\le F(K_{n-2}(G/N))$. Consequently we see that $K_{n-2}(G/N)/F(K_{n-2}(G/N))$ is not a 3–group whence induction yields $f(G) = f(G/N) \le 8\log_2|cd(G/N)| + 31 \le 8\log_2|cd(G)| + 31$, as desired. Hence we may assume that no such N exists (+); in particular, $O_{q'}(G) = 1$, thus $K = Q_0$ and $F(G)$ is a q-group. Now as $PK/K \le F(G/K)$ and $F(G)/K \le F(G/K)$, we see that $PF(G)/K$ is nilpotent and hence $1 = [PK/K, F(G)/K] = [PK, F(G)]K/K$ and therefore $[P, F(G)] \le K$. Thus with [**3**, Theorem 5.3.6] we see that $Q \le [P, F(G)] = [P, [P, F(G)]] \le [P, K] = Q$ and hence $1 < Q = [P, F(G)] \not\le \phi(F(G))$ (by [**3**, Theorem 5.3.5]). Thus $Q\phi(F(G))/\phi(F(G)) > 1$, and by (+) we conclude that $\phi(F(G)) = 1$, i.e., $F(G)$ is elementary abelian, and now with (+) we easily see that $F(G) = K = Q$ is a minimal normal subgroup of G.

Hence by [**5**, III, Hilfssatz 4.4] let H be a complement of K in G; then K is an irreducible and faithful H-module. Now let $V = \mathrm{Irr}(K)$. By [**10**, Proposition 12.1] V is a faithful and irreducible H–module. As by [**7**, Problem (6.18)] every $\lambda \in V$ can be extended to its inertia group in G, we conclude by standard character theory that every orbit size in the action of H on V is an irreducible complex character degree of $G = HK$. So if m is the number of different orbit sizes of H on V, then we have $m \le |cd(G)|$. Now as $K_{n-2}(H) \cong K_{n-2}(G)/K = K_{n-2}(G)/F(K_{n-2}(G))$ is not a 3–group and $f(H) = n-1$, by Theorem 2.3 we have $f(H) \le 8\log_2 m + 30 \le 8\log_2|cd(G)| + 30$, and we get $f(G) \le 8\log_2|cd(G)| + 31$. So ($*$) is shown.

To prove the corollary, note first that by ($*$) we may assume that $n \ge 4$ and that L/F is a 3–group. Let $\overline{G} = G/F$. Then $f(\overline{G}) = n-1$, and surely $K_{n-3}(\overline{G})/F(K_{n-3}(\overline{G})) \cong (K_{n-3}(G)/F)/F(K_{n-3}(G)/F)$ is not a 3-group, because else $K_{n-3}(\overline{G})/O_{3'}(K_{n-3}(\overline{G}))$ would be a 3-group, hence $K_{n-3}(\overline{G})$ would be isomorphic to a subgroup of $K_{n-3}(\overline{G})/O_{3'}(K_{n-3}(\overline{G})) \times K_{n-3}(\overline{G})/K_{n-2}(\overline{G})$ and as such be nilpotent against $f(\overline{G}) = n-1$. Hence application of ($*$) to G/F yields $f(G) = f(G/F) + 1 \le 8\log_2|cd(G/F)| + 31 + 1 \le 8\log_2|cd(G)| + 32$, and we are done. □

The result in Corollary 2.4 is asymptotically best possible, as the following trivial example shows (which also can be used to see that Theorem 2.3 is asymptotically best possible, too).

Example 2.5. Let A, B be groups of order 3 and 5, respectively. For $i \in \mathbb{N}$ we let $Z_i = A$ if i is even, and $Z_i = B$ if i is odd, and for $n \in \mathbb{N}$ we define $G_n = Z_1 \wr \cdots \wr Z_n$ to be iterated (regular) wreath products of the Z_i. Then obviously $f(G_n) = n$, and if t is the number of divisors of $|G_n|$, then clearly

$|cd(G_n)| \leq t$. Now if $B_n = B \wr \cdots \wr B$ (with n factors), then it is well–known that $|B_n| = 5^s$ with $s = \sum_{i=0}^{n-1} 5^i = \frac{1}{4}(5^n - 1) \leq 5^n - 1$.

Now $|G_n| \leq |B_n| \leq 5^{5^n - 1}$ and thus obviously $|G_n| \mid (3 \cdot 5)^{5^n - 1}$. This implies that $t \leq 5^{2n}$. Thus we have

$$f(G_n) \; = n \; \geq \; \frac{1}{2} \; \log_5 t \; \geq \; \frac{1}{2} \; \log_5 |cd(G_n)|,$$

whence asymptotically Corollary 2.4 is the right bound.

3. An application to modular character theory.

In this section we want to discuss briefly a modification of results obtained by Isaacs in [**8**] on orbit sizes and modular character degrees. Because of the hypotheses in Theorem 2.1, we have to restrict ourselves to groups of odd order and to primes greater than 3. In this case, for groups whose p-Sylow subgroups have bounded exponent, we can asymptotically improve the bounds obtained in [**8**] considerably. The reason why we need that bounded exponent is that with Theorem 2.1 we only get many different orbit sizes of a p-group P acting on a vector space, if $d(A)$ is large for some abelian subgroup of P. But, as well-known, there are p-groups of arbitrarily large derived length all of whose abelian normal subgroups are generated by only three elements (cf. [**4**]).

Theorem 3.1. *Let G be a (solvable) group of odd order and $p \geq 5$ be a prime. Suppose that G acts on an abelian p'-group V of odd order, and let b be the number of different p-parts among the orbit sizes of G on V. Let $P = O_p(G)$ and $\exp(P) = p^e$. Then*

$$dl(P/C_P(V)) \;\leq\; 2 \;\log_2 b \;+\; 2 \;\log_2 e \;+\; 8.$$

Proof. As in the proof of [**8**, Theorem (2.1)] we see that we may assume that V is elementary abelian and a faithful and irreducible G–module. Now let $k = dl(P)$. If P is abelian, there is nothing to prove, hence let $k \geq 2$.

Now let $P^{(k-1)} \leq D$ be a maximal characteristic abelian subgroup of $P^{(k-2)}$. Then we define subgroups C, A, M of G as follows:

If D is also a maximal normal abelian subgroup of $P^{(k-2)}$, then let $C = D = M$ and $A = \Omega_1(M) \leq C$; in particular $d(A) = d(M)$.

If not, then let $D < M \leq P^{(k-2)}$ be a maximal normal abelian subgroup of $P^{(k-2)}$. Thus $M \leq C_{P^{(k-2)}}(D) =: H$. By the choice of D clearly $H/D > 1$ is abelian, so let $D \leq C \leq P^{(k-2)}$ such that $C/D = H/D \cap \Omega_1(P^{(k-2)}/D) = \Omega_1(H/D)$. Then $\Omega_1(M/D) \leq C/D$ and hence obviously $A := \Omega_1(M) \leq C$ and $d(A) = d(M)$.

This shows that in any case there is a subgroup C of $P^{(k-2)}$ such that $\Omega_1(C)$ contains an abelian subgroup A with $d(A) = d(M)$ for some maximal abelian normal subgroup M of $P^{(k-2)}$. Now observe that by our construction of C and the proof of [**3**, Theorem 5.3.12] we see that C is a critical subgroup of $P^{(k-2)}$ in the sense of [**3**]; so by the proof of [**3**, Theorem 5.3.13] we know that $\Omega_1(C) \trianglelefteq G$ is a p-group of class 1 or 2 and of exponent p which contains the abelian subgroup A. Hence we may apply 2.1 to the action of G on V which yields that

$$b \;\geq\; \frac{1}{6}\, d(A).$$

Now by the proof of [**5**, III, Satz 7.11] we have $|P^{(k-2)}| = p^n$ with $n \geq 2^{k-2}+2$, so that with [**5**, III, Satz 7.3b)] we have $|M| = p^a$ with $2a^2 \geq a(a+1) \geq 2n \geq 2^{k-1}+4$ whence we conclude that $a \geq 2^{(k-2)/2}$ and consequently $d(A) \geq a/e \geq e^{-1}\, 2^{(k-2)/2}$. Thus altogether we have

$$b \;\geq\; \frac{1}{6e}\, 2^{\frac{k-2}{2}}$$

from which we easily deduce the assertion of the theorem. □

The following corollary is the counterpart of Theorem B in [**8**].

Corollary 3.2. *Let $p \geq 5$ be a prime and P be a p-group that acts faithfully on an abelian p'-group V of odd order. Let $\exp(P) = p^e$ and b be the number of different orbit sizes in the action of P on V. Then*

$$dl(P) \;\leq\; 2\, \log_2 b \;+\; 2\, \log_2 e \;+\; 8.$$

Proof. This follows from Theorem 3.1 with $G = P$. □

Now we can also derive a counterpart of Theorem A of [**8**].

Theorem 3.3. *Let G be a (solvable) group of odd order and $p \geq 5$ be a prime. Let $cd_p(G)$ denote the set of the degrees of the irreducible Brauer characters at the prime p. Put $c = |cd_p(G)|$ and let p^e be the exponent of a p–Sylow subgroup of G. Then*

$$dl(G/O_p(G)) \;\leq\; 3\,(c-1)\,(2\,\log_2 c \;+\; 2\,\log_2 e \;+\; 8) \;+\; 1.$$

Proof. The proof runs completely the same as the proof of Theorem A in [**8**], i.e. first one proves a counterpart of [**8**, Theorem (3.2)] by using Theorem 3.1 instead of [**8**, Theorem (2.1)], and then one completes the proof with the help of [**8**, Theorem (3.3)]. □

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