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MAXIMALITY OF THE MICROSTATES FREE ENTROPY  
FOR  $R$ -DIAGONAL ELEMENTS

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## MAXIMALITY OF THE MICROSTATES FREE ENTROPY FOR $R$ -DIAGONAL ELEMENTS

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A non-commutative non-self adjoint random variable  $z$  is called  $R$ -diagonal, if its  $*$ -distribution is invariant under multiplication by free unitaries: if a unitary  $w$  is  $*$ -free from  $z$ , then the  $*$ -distribution of  $z$  is the same as that of  $wz$ . Using Voiculescu's microstates definition of free entropy, we show that the  $R$ -diagonal elements are characterized as having the largest free entropy among all variables  $y$  with a fixed distribution of  $y^*y$ . More generally, let  $Z$  be a  $d \times d$  matrix whose entries are non-commutative random variables  $X_{ij}$ ,  $1 \leq i, j \leq d$ . Then the free entropy of the family  $\{X_{ij}\}_{ij}$  of the entries of  $Z$  is maximal among all  $Z$  with a fixed distribution of  $Z^*Z$ , if and only if  $Z$  is  $R$ -diagonal and is  $*$ -free from the algebra of scalar  $d \times d$  matrices. The results of this paper are analogous to the results of our paper [3], where we considered the same problems in the framework of the non-microstates definition of entropy.

### 1. Introduction.

Let  $(M, \tau)$  be a tracial non-commutative  $W^*$ -probability space. A (non-self-adjoint) element  $z \in M$  is called  $R$ -diagonal if its  $*$ -distribution is invariant under multiplication by free unitaries; i.e., if  $w$  is a unitary,  $*$ -free from  $z$ , the  $*$ -distributions of  $wz$  and  $z$  coincide. The concept of  $R$ -diagonality was introduced in [4], where it was shown to be equivalent to several conditions; we mention that if  $z^*z$  has a (possibly unbounded) inverse (in particular, if the distribution of  $z^*z$  is non-atomic), then  $z$  is  $R$ -diagonal if and only if in its polar decomposition  $z = u(z^*z)^{1/2}$ ,  $u$  is  $*$ -free from  $(z^*z)^{1/2}$  and satisfies  $\tau(u^k) = 0$  for  $k \in \mathbb{Z} \setminus \{0\}$ .

In our recent paper [3]  $R$ -diagonal elements appeared in connection with certain maximization problems in free entropy. Free entropy was introduced by Voiculescu in [8]; later, a different definition was given by him in [10]. The first definition involves approximating the given  $n$ -tuple of variables using finite-dimensional matrices (so-called microstates); the normalized limit of the logarithms of volumes of all such possible microstates is then the free entropy. On the other hand, Voiculescu's definition in [10] does not involve

microstates, but uses free Fisher information measure and non-commutative Hilbert transform. At present it is not known whether the two definitions of free entropy always give the same quantity. Our approach in [3] used the second definition of Voiculescu.

In this paper we prove two theorems for the microstates free entropy, which are analogous to our results in [3] for the second (non-microstates) definition of entropy. One of our results can be interpreted as saying that  $R$ -diagonal elements  $z$  are characterized by the statement that the free entropy  $\chi(z)$  is maximal among all possible  $\chi(y)$ , so that the distributions of  $y^*y$  and  $z^*z$  are the same.

When this paper was almost finished we received a preprint of Hiai and Petz [1], where the same kind of problems were considered.

If  $Y_1, \dots, Y_n \in M$  (not necessarily self-adjoint), we denote by  $\chi(Y_1, \dots, Y_n)$  the free entropy of  $Y_1, \dots, Y_n$  as defined by Voiculescu in [11]. We denote by  $\chi^{\text{sa}}(X_1, \dots, X_n)$  for  $X_i \in M$  self-adjoint the free entropy of a self-adjoint  $n$ -tuple as defined in [8]; we give a brief review of these quantities below in §2.3. A unitary  $u$  in a non-commutative probability space  $(M, \tau)$  is called a Haar unitary if  $\tau(u^k) = 0$  for all  $k \in \mathbb{Z} \setminus \{0\}$ .

**Theorem 1.** *Let  $y \in M$ , and let  $u \in M$  be a Haar unitary which is  $*$ -free from  $b = (y^*y)^{1/2}$ . Let  $x$  be an element such that  $\tau(x^{2k}) = \tau(b^{2k})$  and  $\tau(x^{2k+1}) = 0$ , for all  $k \in \mathbb{N}$  (i.e.,  $x$  is symmetric). Then:*

- (a)  $\chi(y) \leq \chi(ub)$ .
- (b)  $\chi(ub) = \chi^{\text{sa}}(b^2/2) + 3/4 + 1/2 \log 2\pi = 2\chi^{\text{sa}}(2^{-\frac{1}{2}}x)$ .
- (c) *If  $\chi(y) = \chi(ub) > -\infty$ , then  $y$  is  $R$ -diagonal, i.e., in the polar decomposition  $y = vb$  we have:  $v$  is a Haar unitary and is  $*$ -free from  $b$ .*

Let  $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$  be a free ultrafilter; i.e., a homomorphism from the algebra  $C(\mathbb{N})$  of all bounded (continuous) functions on  $\mathbb{N}$  to  $\mathbb{C}$ , which is not given by the evaluation at a point in  $\mathbb{N}$ . For  $d \in \mathbb{N}$  we write  $d\omega$  for the free ultrafilter corresponding to the functional  $f \mapsto \lim_{n \rightarrow \omega} f(dn)$ . Given  $\omega$ , one can construct (see [11] and see also a brief review below) free entropy quantities  $\chi^{\text{sa}\omega}$  and  $\chi^\omega$ , which have properties similar to those of  $\chi^{\text{sa}}$  and  $\chi$ ; it is in fact not known whether these quantities are different. It is known that in the one-variable case,  $\chi^{\text{sa}}(X) = \chi^{\text{sa}\omega}(X)$ .

**Theorem 2.** *Let  $X_{ij}, 1 \leq i, j \leq d$  be a family of non-commutative random variables in a tracial non-commutative probability space  $(M, \hat{\tau})$ . Let  $Z \in M \otimes M_d$  be given by*

$$Z = \sum_{i,j=1}^d X_{ij} \otimes e_{ij},$$

where  $e_{ij}$  are matrix units in the algebra of  $d \times d$  matrices. We denote by  $\tau$  the normalized trace on  $M \otimes M_d$ . Let  $\omega$  be a free ultrafilter. Let  $X$

be a self-adjoint variable with  $\tau(X^{2n+1}) = 0$  for all  $n \in \mathbb{N}$ , and such that  $\tau(X^{2n}) = \tau((Z^*Z)^n)$ ,  $\forall n \in \mathbb{N}$ . Then we have:

- (a)  $\chi^\omega(\{X_{ij}\}_{1 \leq i, j \leq d}) \leq d^2 \chi^{d\omega}(Z) - d^2 \log d \leq 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$ .
- (b) If  $Z$  is  $R$ -diagonal and  $*$ -free from the algebra  $1 \otimes M_d$ , then  $\chi^{d\omega}(\{X_{ij}\}_{1 \leq i, j \leq d}) = d^2 \chi^\omega(Z) - d^2 \log d = 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$ .
- (c) If  $\chi^\omega(\{X_{ij}\}_{1 \leq i, j \leq d}) = 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$  and  $\chi^{\text{sa}}(2^{-\frac{1}{2}}X) \neq -\infty$ , then  $Z$  is  $R$ -diagonal and is  $*$ -free from the algebra  $1 \otimes M_d$ .

The proof of the first theorem is quite different in nature than our proof in [3] (the microstates-free proof relied on the notion of free entropy with respect to a completely-positive map introduced in [6]). On the other hand, the proof of the second theorem is analogous to the one we gave in [3], and relies on the microstates analog [5] of the relative entropy [10] that we used in the microstates-free approach.

**2. Maximality of microstates free entropy for  $R$ -diagonal pairs.**

Let  $(M, \tau)$  be a tracial  $W^*$ -probability space, and  $b \in M$  be a fixed positive element. Let  $u \in M$  be a Haar unitary which is  $*$ -free from  $b$ . Lastly, let  $x \in M$  be such that for all  $k \in \mathbb{N}$ ,  $\tau(x^{2k+1}) = 0$  and  $\tau(x^{2k}) = \tau(b^{2k})$ . The main result of the section is:

**Theorem 2.1.** *Let  $u, b$  and  $x$  be as above. Assume that  $y \in M$  satisfies  $(y^*y)^{1/2} = b$ . Then:*

- (a)  $\chi(y) \leq \chi(ub)$ .
- (b)  $\chi(ub) = \chi^{\text{sa}}(b^2/2) + 3/4 + 1/2 \log 2\pi = 2\chi^{\text{sa}}(2^{-\frac{1}{2}}x)$ .
- (c) *If  $\chi(y) = \chi(ub) > -\infty$ , then  $y$  is  $R$ -diagonal, i.e., in the polar decomposition  $y = vb$ , we have:  $v$  is a Haar unitary and is  $*$ -free from  $b$ .*

*The same conclusions hold for  $\chi^\omega$  in place of  $\chi$ .*

Before starting the proof of the theorem, we fix some notation and definitions.

**Notation 2.2.** We use the following notation:

- $U(k)$  is the unitary group of  $k \times k$  unitary matrices.
- $M_k$  is the set of all  $k \times k$  matrices;  $M_k^{\text{sa}}$  is the set of all self-adjoint matrices in  $M_k$ .
- $M_k^+ \subset M_k$  is the set of all positive  $k \times k$  matrices.
- $\mu_k$  is the normalized Haar measure on  $U(k)$ ; thus  $\mu_k(U(k)) = 1$ .
- $\lambda_k$  is the measure on  $M_k$ , coming from its Euclidean structure  $\langle a, b \rangle = \text{Re Tr}(ab^*)$ , where  $\text{Tr}$  is the usual matrix trace,  $\text{Tr}(I) = k$ ;  $\lambda_k^{\text{sa}}$  is the Lebesgue measure on  $M_k^{\text{sa}}$  coming from its Euclidean structure  $\langle a, b \rangle = \text{Re Tr}(ab^*)$ .

- $\lambda_k^+$  is the measure on  $M_k^+$  coming from its structure of a cone in the Euclidean space of  $k \times k$  matrices.
- $P : U(k) \times M_k^+ \rightarrow M_k$  is given by  $(v, p) \mapsto vp$ .
- $\Omega_k$  is the canonical volume form on  $M_k$  giving rise to Lebesgue measure.
- $\Omega_k^u \wedge \Omega_k^+$  is the canonical volume form on  $U(k) \times M_k^+$ , giving rise to the product measure  $\mu_k \times \lambda_k^+$ .
- $\mathfrak{u}(k)$  is the Lie algebra of  $U(k)$ .
- $C_k$  is the volume of  $U(k)$  with respect to the bi-invariant volume form arising from the Euclidean structure on  $\mathfrak{u}(k)$  coming from the Killing form  $\langle a, b \rangle = \text{Re Tr}(ab)$ .

**2.3. Definitions of free entropy.** Let  $X_1, \dots, X_n \in M$  be self-adjoint, and  $Y_1, \dots, Y_n \in M$  be not necessarily self-adjoint. Let  $\epsilon > 0, R > 0$  be real numbers and  $k > 0, m > 0$  be integers. Then define the sets (cf. [8, 11])

$$\Gamma_R^{\text{sa}}(X_1, \dots, X_n; m, k, \epsilon) = \left\{ (x_1, \dots, x_n) \in (M_k^{\text{sa}})^n : \right. \\ \left. \left| \frac{1}{k} \text{Tr}(x_{i_1} \dots x_{i_p}) - \tau(X_{i_1} \dots X_{i_p}) \right| < \epsilon \right. \\ \left. \text{for all } p \leq m, 1 \leq i_j \leq n, 1 \leq j \leq p \right\};$$

$$\Gamma_R(Y_1, \dots, Y_n; m, k, \epsilon) = \left\{ (y_1, \dots, y_n) \in (M_k)^n : \right. \\ \left. \left| \frac{1}{k} \text{Tr}(y_{i_1}^{g_1} \dots y_{i_p}^{g_p}) - \tau(Y_{i_1}^{g_1} \dots Y_{i_p}^{g_p}) \right| < \epsilon \right. \\ \left. \text{for all } p \leq m, 1 \leq i_j \leq n, g_j \in \{*, \cdot\}, 1 \leq j \leq p \right\}.$$

Define next

$$\chi^{\text{sa}}(X_1, \dots, X_n; m, \epsilon) \\ = \limsup_{k \rightarrow \infty} \left[ \frac{1}{k^2} \log \lambda_k \Gamma_R^{\text{sa}}(X_1, \dots, X_n; m, k, \epsilon) + \frac{n}{2} \log k \right]$$

and similarly

$$\chi(Y_1, \dots, Y_n; m, \epsilon) \\ = \limsup_{k \rightarrow \infty} \left[ \frac{1}{k^2} \log \lambda_k \Gamma_R(Y_1, \dots, Y_n; m, k, \epsilon) + n \log k \right].$$

For  $\omega$  a free ultrafilter on  $\mathbb{N}$ , the quantities  $\chi^\omega(Y_1, \dots, Y_n; m, \epsilon)$  and  $\chi^{\text{sa}\omega}(X_1, \dots, X_n; m, \epsilon)$  are defined in exactly the same way, except that

$\limsup_{k \rightarrow \infty}$  is replaced by  $\lim_{k \rightarrow \omega}$ . Next, the free entropy is defined by

$$\chi^{\text{sa}}(X_1, \dots, X_n) = \sup_R \inf_{m, \epsilon} \chi^{\text{sa}}(X_1, \dots, X_n; m, \epsilon);$$

the quantities  $\chi^{\text{sa}\omega}$ ,  $\chi$ ,  $\chi^\omega$  are defined in exactly the same way, using in the place of  $\chi^{\text{sa}}(\dots; m, \epsilon)$  the quantities  $\chi^{\text{sa}\omega}(\dots; m, \epsilon)$ ,  $\chi(\dots; m, \epsilon)$ , and  $\chi^\omega(\dots; m, \epsilon)$ , respectively.

**Definition 2.4.** Let  $(X_R(k, m, \epsilon), \mu_{R,k,m,\epsilon}^X)$  and  $(Y_R(k, m, \epsilon), \mu_{R,k,m,\epsilon}^Y)$  be two sequences of measure spaces depending on  $k, m \in \mathbb{N}$  and  $R, \epsilon \in (0, +\infty)$ . We shall say that  $X$  is asymptotically included in  $Y$ , if for all  $m, \epsilon, R$ , there is  $k_0, m' \geq m, \epsilon' \leq \epsilon, R' > R$ , such that for all  $k > k_0$ , there is a map

$$\phi = \phi_{R',k,m',\epsilon'} : X_{R'}(k, m', \epsilon') \rightarrow Y_R(k, m, \epsilon),$$

which is measure preserving. We say that  $X$  and  $Y$  are asymptotically equal, if both  $X$  is asymptotically included in  $Y$  and  $Y$  is asymptotically included in  $X$ .

**Remark 2.5.** Note that if  $X$  is asymptotically included into  $Y$ , we obtain that

$$\begin{aligned} & \sup_R \inf_{m, \epsilon} \limsup_k \alpha_k \log \mu_{R,k,m,\epsilon}^X(X_R(k, m, \epsilon)) + a_k \\ & \leq \sup_R \inf_{m, \epsilon} \limsup_k \alpha_k \log \mu_{R,k,m,\epsilon}^Y(Y_R(k, m, \epsilon)) + a_k, \end{aligned}$$

for all sequences  $a_k, \alpha_k$ .

It is not hard to see that the sets

$$\Gamma_R(Y_1, \dots, Y_n; k, m, \epsilon)$$

and

$$\Gamma_R^{\text{sa}}(\text{Re}(Y_1), \text{Im}(Y_1), \dots, \text{Re}(Y_n), \text{Im}(Y_n); k, m, \epsilon)$$

are asymptotically equal; the relevant maps  $\phi$  send the  $n$ -tuple  $(y_1, \dots, y_n)$  of non-self-adjoint matrices to the  $2n$ -tuples of self-adjoint matrices  $(\text{Re}(y_1), \text{Im}(y_1), \dots, \text{Re}(y_n), \text{Im}(y_n))$ . This implies (using the Remark 2.5) that

$$\chi(Y_1, \dots, Y_n) = \chi^{\text{sa}}(\text{Re}(Y_1), \text{Im}(Y_1), \dots, \text{Re}(Y_n), \text{Im}(Y_n)).$$

We proceed to prove several lemmas that will be used in the proof of the main theorem.

**Lemma 2.6.** Let  $\Gamma \subset M_k^+$  and  $U_k \subset U(k)$  be measurable sets. Let

$$U_k \Gamma = \{vp : v \in U_k, p \in \Gamma\} \quad \text{and} \quad S(\Gamma) = \left\{ \frac{p^2}{2} : p \in \Gamma \right\}.$$

Then

$$\lambda_k(U_k \Gamma) = C_k \mu_k(U_k) \lambda_k^+(S(\Gamma)).$$

In other words, the map  $Q : (v, p) \mapsto v\sqrt{2p}$  from  $U(k) \times M_k^+$ , endowed with the measure  $\mu_k \times C_k\lambda_k^+$ , to  $M_k$ , endowed with the measure  $\lambda_k$ , is measure preserving.

*Proof.* Since invertible matrices are a set of comeasure zero in  $M_k$ , we see by existence of polar decomposition that  $P : (v, p) \mapsto vp$  is invertible as a map of measure spaces. We start by computing the pull-back of Lebesgue measure on  $M_k$  to  $U(k) \times M_k^+$ . Note that since  $P$  is equivariant with respect to the actions of  $U(k)$  by left multiplication, and Lebesgue measure is invariant under this action (since the Euclidean structure is), the resulting measure on  $U(k) \times M_k^+$  is the product of Haar measure on  $U(k)$  and some measure  $\nu_k$  on  $M_k^+$ , hence  $\lambda_k(U_k\Gamma) = \mu_k(U_k)\nu_k(\Gamma)$ . It remains to identify  $\nu_k$ .

We have the equation

$$(1) \quad d\mu_k(v)d\nu_k(p) = (P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+)d\mu_k(v)d\lambda_k^+(p),$$

where  $P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+$  is the ratio of the two volume forms. Furthermore, in view of the mentioned invariance under an action of  $U(k)$ , it is sufficient to compute  $(P^*(\Omega_k) : \Omega_k^u \wedge \Omega_k^+)$  in (1) at the point  $(1, p) \in U(k) \times M_k^+$ .

Note that the tangent space  $T_{1,p}(U(k) \times M_k^+)$  is isomorphic to the direct sum  $\mathfrak{u}(k) \times M_k^{\text{sa}}$ , where  $\mathfrak{u}(k) = iM_k^{\text{sa}}$  is the Lie algebra of  $U(k)$ . Identify  $T_{(1,p)}(U(k) \times M_k^+) = iM_k^{\text{sa}} \oplus M_k^{\text{sa}}$  with  $M_k = T_p(M_k)$ . Then the inner product given by the trace  $\langle a, b \rangle = \text{Re Tr}(ab^*)$  defines on  $T_{1,p}$  a Euclidean structure, for which the subspaces  $M_k^{\text{sa}}$  and  $iM_k^{\text{sa}}$  are perpendicular. Since the restriction of this inner product to  $\mathfrak{u}(k)$  is the Killing form on this Lie algebra, and the restriction to  $T_pM_k^+$  is the inner product we chose before on this space,  $\Omega_k$  (which via the above identification is a volume form on  $U(k) \times M_k^+$ ) has the form  $C_k\Omega_k^u \wedge \Omega_k^+$ . Further,  $C_k$  is the ratio of the volume form on  $U(k)$  arising from the Euclidean structure on  $\mathfrak{u}(k)$  coming from the Killing form and the volume form corresponding to the normalized Haar measure. Hence  $C_k$  is just the volume of  $U(k)$  with respect to the volume form arising from the Euclidean structure on  $\mathfrak{u}(k)$  coming from the Killing form.

Thus from (1) we get that

$$d\nu_k(p)d\mu_k(v) = C_kd\mu_k(v) \det(DP)(p)d\lambda_k^+(p).$$

It remains to compute  $DP$ . We note that  $P$  is the identity map restricted to  $M_k^+$ . Choose a basis in which  $p$  is diagonal with eigenvalues  $l_1, \dots, l_k$ , and let  $e_{ij} \in M_k$  be the matrix all of whose entries are zero, except that the  $i, j$ -th entry is 1. Consider the orthonormal basis  $\xi_{\alpha\beta}$  for  $iM_k^{\text{sa}}$ , given by:

$$\xi_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{2}}(e_{\alpha\beta} - e_{\beta\alpha}) & \text{if } \alpha < \beta \\ ie_{\alpha\alpha} & \text{if } \alpha = \beta \\ i\frac{1}{\sqrt{2}}(e_{\alpha\beta} + e_{\beta\alpha}) & \text{if } \alpha > \beta. \end{cases}$$

Then

$$DP(\xi_{\alpha\beta})p = \xi_{\alpha\beta}p = \frac{1}{2}(l_\alpha + l_\beta)\xi_{\alpha\beta} + \frac{1}{2}(l_\alpha - l_\beta)\eta_{\alpha\beta}, \quad \eta_{\alpha\beta} \in M_k^{\text{sa}}.$$

It follows that

$$\det(DP)(p) = \frac{1}{2^{k^2}} \prod_{\alpha,\beta=1}^k (l_\alpha + l_\beta).$$

Hence we record the final answer:

$$d\nu_k(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^k (l_\alpha + l_\beta) d\lambda_k^+(p)$$

where  $l_i$  are the eigenvalues of  $p$ .

Consider the map  $S : p \mapsto \frac{p^2}{2}$  from  $M_k^+$  to itself. This map is a.e. invertible; moreover, its Jacobian  $\det(DS)$  at  $p$  is given by  $\det(\frac{1}{2}(1 \otimes p + p \otimes 1))$ , where  $1 \otimes p$  and  $p \otimes 1$  are viewed as elements of  $M_k \otimes M_k \cong M_{k^2}$  (see e.g. [8]). To compute this determinant, let  $\zeta_i, i = 1, \dots, k$  be orthonormal eigenvectors of  $p$ , such that  $p\zeta_i = l_i\zeta_i$ . Then  $\zeta_i \otimes \zeta_j$  is an orthonormal basis for  $\mathbb{C}^{k^2}$ , on which  $M_{k^2} = M_k \otimes M_k$  acts naturally. Moreover,  $\frac{1}{2}(1 \otimes p + p \otimes 1)(\zeta_i \otimes \zeta_j) = \frac{1}{2}(l_i + l_j)\zeta_i \otimes \zeta_j$ . So the determinant is  $2^{-k^2} \prod_{\alpha,\beta=1}^k (l_\alpha + l_\beta)$ . Hence the push-forward of  $\nu_k$  by  $S$  is given by

$$d(S_*\nu_k)(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^k (l_\alpha + l_\beta) d\lambda_k^+(p) \cdot \det(DS)^{-1}(p) = C_k d\lambda_k^+(p).$$

Thus we have

$$S_*\nu_k = C_k \lambda_k^+,$$

which is our assertion. □

We have the following standard lemma (see [8]).

**Lemma 2.7.** *Let  $p$  be a positive element in  $M$ . Then the sequences of sets  $\Gamma_R^{\text{sa}}(p, m, k, \epsilon)$  and  $\Gamma_R^{\text{sa}}(p, m, k, \epsilon) \cap M_k^+$ , each taken with the measure  $\lambda_k$ , are asymptotically equal.*

**Lemma 2.8.**  $\lim_k \frac{1}{k^2} \log(C_k) + \frac{1}{2} \log k = \frac{3}{4} + \frac{1}{2} \log 2\pi.$

In this exact form this lemma can be found, for example, in [2] (the reader is cautioned that the cited paper uses a slightly different normalization of the Killing form, different from ours by a factor).

**Lemma 2.9.** *Let  $y \in (M, \tau)$  be a (not necessarily self-adjoint) random variable. Then*

$$\chi(y) \leq \chi^{\text{sa}}\left(\frac{y^*y}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$

*Proof.* Denote by  $S : M_k \rightarrow M_k^+$  the map

$$y \mapsto \frac{y^*y}{2}.$$

Note that

$$S(\Gamma_R(y; m, k, \epsilon)) \subset \Gamma_{R^2}^{\text{sa}}\left(\frac{y^*y}{2}; m/2, k, \epsilon\right),$$

hence the former is asymptotically included in the latter. Note that

$$\Gamma_R(y; m, k, \epsilon) \subset U(k)\Gamma_R(y; m, k, \epsilon).$$

We therefore get

$$\begin{aligned} \lambda_k(\Gamma_R(y; m, k, \epsilon)) &\leq \lambda_k(U(k)\Gamma_R(y; m, k, \epsilon)) \\ &\leq \lambda_k(U(k)\{(a^*a)^{1/2} : a \in \Gamma_R(y; m, k, \epsilon)\}) \\ &\leq C_k \lambda_k(S(\Gamma_R(y; m, k, \epsilon))) \\ &\leq C_k \lambda_k\left(\Gamma_{R^2}^{\text{sa}}\left(\frac{y^*y}{2}; \frac{m}{2}, k, \epsilon\right)\right). \end{aligned}$$

Taking the logarithm and passing to the limits gives the result. □

**Lemma 2.10.** *Let  $u, b \in (M, \tau)$  be such that  $u$  is a Haar unitary  $*$ -free from the positive element  $b$ . Let  $z = ub$ . Given  $\delta > 0$ , there exists  $k_0$ , such that for all  $k > k_0$ , there is a subset  $X_k \subset U(k) \times (\Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+)$ ,*

$$\log \frac{\mu_k \times \lambda_k^+(X_k)}{\mu_k \times \lambda_k^+(U_k \times \Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon))} \geq -\delta,$$

such the map

$$(2) \quad Q : (v, p) \mapsto v\sqrt{2p}$$

is an asymptotic inclusion of  $X_k$ , endowed with the measure  $\mu_k \times C_k \lambda_k^+$ , into  $\Gamma_R(z; m, k, \epsilon)$ , endowed with the measure  $\lambda_k$ .

*Proof.* Note that by Lemma 2.6, the map defined in Equation (2) is measure preserving.

Let  $R > 0$ ,  $\epsilon > 0$  and  $\delta > 0$  be fixed. For  $x \in M_k^+$ , let  $U_k(x, \epsilon) \subset U(k)$  be the maximal set of unitaries, for which  $U_k(x, \epsilon) \cdot x \in \Gamma_R(wx; m, k, \epsilon)$ , where  $w$  is a Haar unitary  $*$ -free from  $x$  (in other words, “elements of  $U_k(x, \epsilon)$  and  $x$  are  $*$ -free to order  $m, \epsilon$ ”). Note that  $U_k(x, \epsilon)$  is open. By Corollary 2.12 of [11], there exists  $k_0$ , such that for all  $k > k_0$ , and any  $x \in M_k^+$ ,  $\|x\| < R$ ,  $\log \mu_k(U_k(x, \epsilon/2)) > -\delta$ . Let

$$\hat{X}_k = \bigcup_{x \in \Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+} U_k(x, \epsilon) \times \{x\}.$$

Since whenever  $x \in \Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$ ,  $U_k(x) \cdot \sqrt{2x} \subset \Gamma_R(z; m, k, \epsilon)$ ,  $Q(\hat{X}_k)$  lies in  $\Gamma_R(z; m, k, \epsilon)$ .

We claim that there exists a measurable subset  $X_k \subset \hat{X}_k$  of measure at least  $\exp(-\delta)$  times that of  $\Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon)$ . First, it is sufficient to show (because of Lemma 2.7) that the measure of  $X_k$  is at least  $\exp(-\delta)$  times that of  $\Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$ . Next, let  $x \in \Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$ , and let  $V(x)$  be an open neighborhood of  $x$  for the norm topology. Then if  $V(x)$  is sufficiently small, for all  $x' \in V$ ,  $U_k(x, \epsilon/2) \subset U_k(x', \epsilon)$ . Hence

$$O(x) = U_k(x, \epsilon/2) \times V(x) \subset \hat{X}_k.$$

Moreover, the volume of  $O(x)$  is at least  $\exp(-\delta)$  times the volume of  $V(x)$ . Let

$$X_k = \bigcup_x O(x).$$

Then  $X_k$  is open, and its volume is at least  $\exp(-\delta)$  times that of  $\Gamma_R^{\text{sa}}(\frac{z^*z}{2}; m, k, \epsilon) \cap M_k^+$ . □

*Proof of 2.1(a) and 2.1(b) in Theorem 2.1.* Assume that  $x, u$  and  $b$  are as in the statement of Theorem 2.1(b) and let  $z = ub$ ; note that  $z$  is  $R$ -diagonal. By Lemma 2.10 and Lemma 2.8, we have that

$$\chi^{\text{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \leq \chi(z).$$

Since, by Lemma 2.9, we always have the other inequality, we obtain

$$(3) \quad \chi(z) = \chi^{\text{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$

This can be expressed in terms of the free entropy of the symmetric variable  $x$  as follows (by using the explicit formula for  $\chi^{\text{sa}}$  of one variable given by Voiculescu in [8]):

$$\begin{aligned} \chi(z) &= \chi^{\text{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \\ &= 2\left(\frac{3}{4} + \frac{1}{2} \log 2\pi\right) + \iint \log |s - t| d\mu_{\frac{z^*z}{2}}(s) d\mu_{\frac{z^*z}{2}}(t) \\ &= 2\left(\frac{3}{4} + \frac{1}{2} \log 2\pi\right) + 2 \iint \log |s - t| d\mu_{2^{-1/2}x}(s) d\mu_{2^{-1/2}x}(t) \\ &= 2\chi^{\text{sa}}(2^{-1/2}x). \end{aligned}$$

This proves 2.1(b).

Combining the above with Lemma 2.9 we get 2.1(a):

$$\begin{aligned} \chi(y) &\leq \chi^{\text{sa}}\left(\frac{y^*y}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \\ &= \chi^{\text{sa}}\left(\frac{z^*z}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \\ &= \chi(z). \end{aligned}$$

□

**Proposition 2.11** (A change of variables formula for polar decomposition).

Let  $y_1, \dots, y_n$  be elements of a  $W^*$ -probability space  $(M, \tau)$ , and let  $y_i = v_i(y_i^*y_i)^{1/2}$  be their polar decompositions. Assume that  $f_i : [0, +\infty) \rightarrow [0, +\infty)$  are  $C^1$ -diffeomorphisms, and let  $z_i = v_i[2f(y_i^*y_i/2)]^{1/2}$ . Then

$$(4) \quad \chi(z_1, \dots, z_n) = \chi(y_1, \dots, y_n) + \sum_{j=1}^n \iint \log \left| \frac{f(s) - f(t)}{s - t} \right| d\mu_i(s) d\mu_i(t),$$

where  $\mu_i$  is the distribution of  $y_i^*y_i/2$  for  $i = 1, \dots, n$ . The same statement holds for  $\chi^\omega$  in the place of  $\chi$ .

*Proof.* If for some  $i$  the distribution of  $y_i^*y_i$  contains atoms, then so does the distribution of  $z_i^*z_i$ . In this case we have

$$\chi(y_1, \dots, y_n) \leq \sum_j \chi(y_j) = -\infty,$$

since by Lemma 2.9,  $\chi(y_i) \leq \chi^{\text{sa}}(y_i^*y_i/2) + \text{const} = -\infty$ . Similarly,  $\chi(z_1, \dots, z_n) = -\infty$ , and there is nothing to prove. Hence we may assume that the distributions of  $y_i^*y_i$ , and thus the distributions of  $z_i^*z_i$ , are non-atomic for all  $i$ ; in particular, that  $v_i$  are unitaries.

We may also assume that  $f_i$  for  $i \neq 1$  are the identity diffeomorphisms; moreover, by replacing  $f_i$  with  $f_i^{-1}$ , we only need to prove that the left-hand side of the statement of Equation (4) is greater than or equal to the right hand side. We write  $f = f_1$ .

Consider the mappings

$$T : M_k \ni x \mapsto v[2f(x^*x/2)]^{1/2} \in M_k,$$

where  $x = v(x^*x)^{1/2}$  is the polar decomposition of  $x$ , and

$$\hat{T} : M_k^n \ni (x_1, \dots, x_n) \mapsto (T(x_1), x_2, \dots, x_n) \in M_k^n.$$

Note that the set  $\hat{T}(\Gamma_R(y_1, \dots, y_n; m, k, \epsilon))$ , taken with the measure  $\lambda_k \times \dots \times \lambda_k$  is asymptotically included into the set  $\Gamma_R(z_1, \dots, z_n; m, k, \epsilon)$ , taken with the same measure. Moreover, the infimum of the Jacobian of  $\hat{T}$  on the set  $\Gamma_R(y_1, \dots, y_n; m, k, \epsilon)$  is not less than the infimum of the Jacobian of  $T$  on the set  $\Gamma_R(y_1; m, k, \epsilon)$ . View  $T$  as a map from  $U(k) \times M_k^+$  to itself, using

the identification of measure spaces  $U(k) \times M_k^+ \cong M_k$ ,  $(v, p) \mapsto vp$ . Then  $T$  acts trivially on the unitary component. Recall that the measure on  $M_k^+$ , arising from the identification of  $M_k$  with  $U(k) \times M_k^+$ , is the push-forward of Lebesgue measure on  $M_k^+$  to  $M_k^+$  by the map  $p \mapsto p^2/2$ . Hence the infimum of the Jacobian of  $T$  is equal to the infimum of the Jacobian of the map  $p \mapsto [2f(p^2/2)]^{1/2}$  viewed as a map from  $M_k^+$  endowed with Lebesgue measure to itself, on the set  $\Gamma_R(y_1^*y_1/2; m, k, \epsilon)$ . The rest of the computation is exactly as in the proof of Proposition 3.1 of [9].  $\square$

**Remark 2.12.** Let  $B \subset M$  be a subalgebra of  $M$ . The proof of the proposition above also works if we replace  $\chi(\cdot)$  with the relative entropy  $\chi(\cdot|B)$  introduced in [5]; we leave the details to the reader.

*Proof of 2.1(c) of Theorem 2.1.* Assume that  $\chi(y) = \chi(ub) > -\infty$ . Because of part 2.1(b), we conclude that  $\chi(b) > -\infty$ ; in particular, the distribution of  $b$  is non-atomic (see [8]). Since  $(y^*y)^{1/2} = b$ , this implies that in the polar decomposition of  $y = v(y^*y)^{1/2}$ ,  $v$  is a unitary.

Arguing as in Lemma 4.2 of [9], we may assume that there exists a family  $f_j$  of  $C^1$  diffeomorphisms on  $[0, +\infty)$ , and a continuous function  $f : [0, +\infty) \rightarrow [0, +\infty)$ , such that  $f(\frac{y^*y}{2})$  is the square of a  $(0, 1)$ -semicircular random variable,  $\|f_j(y^*y) - f(y^*y)\| \rightarrow 0$  as  $j \rightarrow \infty$ ,  $W^*(y^*y) = W^*(f(y^*y))$ , and  $\lim_j \chi^{\text{sa}}(f_j(y^*y)) = \chi^{\text{sa}}(f(y^*y))$ . Let  $y = v(y^*y)^{1/2}$  be the polar decomposition of  $y$ ; let  $z = v[2f(y^*y/2)]^{1/2}$ , and similarly  $z_j = v[2f_j(y^*y/2)]^{1/2}$ . Then by Proposition 2.11 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [8]), we get for all  $j$ ,

$$\chi(z_j) = \chi(y) + \chi^{\text{sa}}\left(f_j\left(\frac{y^*y}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{y^*y}{2}\right).$$

Applying Proposition 2.6 of [8], we get that

$$\begin{aligned} \chi(z) &\geq \limsup_j \chi(z_j) \\ &= \limsup_j \left[ \chi(y) + \chi^{\text{sa}}\left(f_j\left(\frac{y^*y}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{y^*y}{2}\right) \right] \\ &= \chi(y) + \chi^{\text{sa}}\left(f\left(\frac{y^*y}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{y^*y}{2}\right). \end{aligned}$$

Since  $\chi(y) = \chi(ub)$  by assumption, and  $\chi(ub) = \chi^{\text{sa}}(\frac{y^*y}{2}) + \frac{3}{4} + \frac{1}{2} \log 2\pi$  by Theorem 2.1(b) we get that

$$\chi(z) \geq \chi^{\text{sa}}\left(f\left(\frac{y^*y}{2}\right)\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$

By assumption, the distribution of  $(z^*z)^{1/2}$  is quarter-circular (i.e., it is the absolute value of a  $(0, 2)$ -semicircular). Let  $c$  be a circular variable (i.e., its

real and imaginary parts are free  $(0, 1)$ -semicircular variables). Then, since  $c$  is  $R$ -diagonal (see [4]), we have by 2.1(b), that

$$\begin{aligned} \chi(c) &= \chi^{\text{sa}}\left(\frac{c^*c}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \\ &= \chi^{\text{sa}}\left(f\left(\frac{y^*y}{2}\right)\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi, \end{aligned}$$

since  $c^*c$  has the same distribution as  $z^*z = 2f(y^*y/2)$ . Hence  $\chi(z) \geq \chi(c)$ .

On the other hand,  $c$  is  $R$ -diagonal, with the same distribution of the positive part as  $z$ , so by 2.1(a), we have  $\chi(z) \leq \chi(c)$ . So  $\chi(z) = \chi(c)$ .

We claim that  $z$  is circular. This will prove the proposition, since then the polar and positive parts of  $z$  are  $*$ -free (see [7] or [4]), and thus the polar and positive parts of  $y$  are  $*$ -free, since the polar part of  $y$  is the same as the polar part of  $z$ , and the positive part of  $y$  is some function of the positive part of  $z$ .

Now, for the claim that  $z$  is circular, let  $\gamma$  be a complex number of modulus one; then  $\chi(\gamma z) = \chi(z)$ . Let

$$X_\gamma = \frac{1}{2}(\gamma z + \bar{\gamma} z^*), \quad Y_\gamma = \frac{1}{2i}(\gamma z - \bar{\gamma} z^*).$$

Then

$$\tau(X_\gamma^2) = \frac{1}{4} \left[ 2\tau(z z^*) + \gamma^2 \tau(z^2) + \bar{\gamma}^2 \cdot \overline{\tau(z^2)} \right].$$

Similarly,

$$\tau(Y_\gamma^2) = \frac{1}{4} \left[ 2\tau(z z^*) - \gamma^2 \tau(z^2) - \bar{\gamma}^2 \cdot \overline{\tau(z^2)} \right].$$

We choose  $\gamma$  such that  $\gamma^2 \tau(z^2)$  is purely imaginary. Since  $\tau(z^*z) = 2$ , we have then  $\tau(X_\gamma^2) = \tau(Y_\gamma^2) = 1$ . But  $\chi(z) = \chi(c) = \chi^{\text{sa}}(x_1, x_2)$ , where  $x_i$  are free  $(0, 1)$  semicircular variables. Hence we have

$$\chi(z) = \chi^{\text{sa}}(X_\gamma, Y_\gamma) = \chi(\gamma z) = \chi^{\text{sa}}(x_1, x_2),$$

where  $X_\gamma$  and  $Y_\gamma$  are some self-adjoint random variables of covariance 1. But then by Voiculescu’s Proposition 2.4 of [9],  $X_\gamma$  and  $Y_\gamma$  are both semicircular and free, so that  $\gamma z$  is circular, so  $z$  is circular. □

### 3. Maximization of free entropy for matrices.

**Theorem 3.1.** *Let  $X_{ij}$ ,  $1 \leq i, j \leq d$  be non-commutative random variables in a tracial non-commutative probability space  $(M, \hat{\tau})$ . Let  $Z \in M \otimes M_d$  be given by*

$$Z = \sum_{i,j=1}^d X_{ij} \otimes e_{ij},$$

where  $e_{ij}$  are matrix units in the algebra of  $d \times d$  matrices. We denote by  $\tau$  the normalized trace on  $M \otimes M_d$ . Let  $\omega$  be a free ultrafilter. Let  $X$

be a self-adjoint variable with  $\tau(X^{2n+1}) = 0$  for all  $n \in \mathbb{N}$ , and such that  $\tau(X^{2n}) = \tau((Z^*Z)^n)$ ,  $\forall n \in \mathbb{N}$ . Then we have:

- (a)  $\chi^{d\omega}(\{X_{ij}\}_{1 \leq i, j \leq d}) \leq d^2 \chi^\omega(Z) - d^2 \log d \leq 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$ .
- (b) If  $Z$  is  $R$ -diagonal and  $*$ -free from the algebra  $1 \otimes M_d$ , then  $\chi^{d\omega}(\{X_{ij}\}_{1 \leq i, j \leq d}) = d^2 \chi^\omega(Z) - d^2 \log d = 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$ .
- (c) If  $\chi^{d\omega}(\{X_{ij}\}_{1 \leq i, j \leq d}) = 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X) - d^2 \log d$ , and  $\chi^{\text{sa}}(2^{-\frac{1}{2}}X) \neq -\infty$ , then  $Z$  is  $R$ -diagonal and is  $*$ -free from the algebra  $1 \otimes M_d$ .

*Proof.* Let  $B = 1 \otimes M_d$ . We have by [5] that

$$\chi^{d\omega}(\{X_{ij}\}) = d^2 \chi^\omega(Z|B) - d^2 \log d \leq d^2 \chi^\omega(Z) - d^2 \log d.$$

(We have the summand  $-d^2 \log d$  rather than  $-\frac{d^2}{2} \log d$  appearing above because we are dealing with  $\chi$ , not  $\chi^{\text{sa}}$ .) Moreover,  $d^2 \chi^\omega(Z) \leq 2d^2 \chi^{\text{sa}}(2^{-\frac{1}{2}}X)$  by Theorem 2.1, hence 3.1(a).

If  $Z$  is  $*$ -free from  $B$ , then, by [5], we have  $\chi^\omega(Z|B) = \chi^\omega(Z)$ . Moreover, if  $Z$  is  $R$ -diagonal, we have, by Theorem 2.1, that  $\chi^\omega(Z) = 2\chi^{\text{sa}}(X/\sqrt{2})$ , which proves 3.1(b).

Assuming the conditions in 3.1(c) are satisfied, we get that  $\chi^\omega(Z) = 2\chi^{\text{sa}}(2^{-\frac{1}{2}}X) > -\infty$ , so  $Z$  is  $R$ -diagonal by Theorem 2.1(c), i.e,  $Z$  has polar decomposition  $Z = v(Z^*Z)^{1/2}$ , where  $v$  is a Haar unitary, which is  $*$ -free from  $Z^*Z$ . Note also that we are given that  $\chi^\omega(Z|B) = \chi^\omega(Z)$ . We may assume, as in the proof of statement 2.1(c) of Theorem 2.1 that there exists a family  $f_i$  of  $C^1$  diffeomorphisms on  $[0, +\infty)$ , and a continuous function  $f : [0, +\infty) \rightarrow [0, +\infty)$ , such that  $f(\frac{Z^*Z}{2})$  is the square of a  $(0, 1)$ -semicircular random variable,  $\|f_j(Z^*Z) - f(Z^*Z)\| \rightarrow 0$  as  $j \rightarrow \infty$ ,  $W^*(Z^*Z) = W^*(f(Z^*Z))$ , and  $\lim_j \chi^{\text{sa}}(f_j(Z^*Z)) = \chi^{\text{sa}}(f(Z^*Z))$ . Given the polar decomposition  $Z = v(Z^*Z)^{1/2}$ , let  $z = v[2f(Z^*Z/2)]^{1/2}$ , and similarly  $z_j = v[2f_j(Z^*Z/2)]^{1/2}$ . Notice that  $z$  is circular; moreover, since  $W^*(Z^*Z) = W^*(f(Z^*Z)) = W^*(z^*z)$ , we have that  $Z \in W^*(z)$ . Hence it will suffice to prove that  $z$  is  $*$ -free from  $B$ , as then also  $Z$  is  $*$ -free from  $B$ .

By Remark 2.12 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [8]), we get for all  $j$ ,

$$\chi^\omega(z_j|B) = \chi^\omega(Z|B) + \chi^{\text{sa}}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{Z^*Z}{2}\right).$$

We get

$$\begin{aligned} \chi^\omega(z|B) &\geq \limsup_j \chi^\omega(z_j|B) \\ &= \limsup_j \left[ \chi^\omega(Z|B) + \chi^{\text{sa}}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{Z^*Z}{2}\right) \right] \\ &= \chi^\omega(Z|B) + \chi^{\text{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{Z^*Z}{2}\right). \end{aligned}$$

By assumption, we have that  $\chi^\omega(Z|B) = \chi^\omega(Z)$ ; moreover, by  $R$ -diagonality of  $Z$  we get by Theorem 2.1(b) that  $\chi(Z) = \chi^{\text{sa}}(Z^*Z/2) + 3/4 + (1/2) \log 2\pi$ . Therefore, we get that

$$\begin{aligned} \chi^\omega(z|B) &\geq \chi^{\text{sa}}\left(\frac{Z^*Z}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi \\ &\quad + \chi^{\text{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right) - \chi^{\text{sa}}\left(\frac{Z^*Z}{2}\right) \\ &= \frac{3}{4} + \frac{1}{2} \log 2\pi + \chi^{\text{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right). \end{aligned}$$

But  $z$  is circular, in particular  $R$ -diagonal; moreover,  $z^*z/2 = f(Z^*Z/2)$ . So from the formula in 2.1(b), we get that

$$\chi^\omega(z) = \frac{3}{4} + \frac{1}{2} \log 2\pi + \chi^{\text{sa}}\left(f\left(\frac{Z^*Z}{2}\right)\right).$$

Thus  $\chi^\omega(z|B) \geq \chi^\omega(z)$ . Since  $\chi^\omega(z|B) \leq \chi^\omega(z)$  in general, we get that  $\chi^\omega(z|B) = \chi^\omega(z)$ .

Now let  $S_1, S_2$  be the real and imaginary parts of  $z$ . Then we have that  $\chi^{\text{sa}}(S_1, S_2|B) = \chi^{\text{sa}}(S_1, S_2)$ . Since  $S_1$  and  $S_2$  are two free semicircular variables, it follows by Theorem 4.5 from [5] that  $W^*(S_1, S_2)$  is free from  $B$ . Hence  $z$  is  $*$ -free from  $B$ ; hence  $Z$  is  $*$ -free from  $B$ .  $\square$

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