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ON THE DUAL PAIRS $(O(p,q), SL(2,\mathbb{R}))$, $(U(p,q), U(1,1))$ AND $(Sp(p,q), O^*(4))$

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We describe the Howe quotient and theta lift for one-dimensional representations of the dual pairs $(O(p,q), SL(2,\mathbb{R}))$, $(U(p,q), U(1,1))$, $(Sp(p,q), O^*(4))$, by explicitly constructing the Howe quotients (of the representations in correspondence) using the Fock model.

1. Introduction.

Let $Sp(2k,\mathbb{R})$ be the symplectic group on $\mathbb{R}^{2k}$ and $\widetilde{Sp}(2k,\mathbb{R})$ be the metaplectic group. If H is a subgroup of $Sp(2k,\mathbb{R})$, we shall let $\widetilde{H}$ be the pullback of H by the covering map from $\widetilde{Sp}(2k,\mathbb{R})$ to $Sp(2k,\mathbb{R})$. The oscillator representation ω of $\widetilde{Sp}(2k,\mathbb{R})$ may be realized on a space of holomorphic functions on $\mathbb{C}^k$, using the Fock model.

Let (G,G') be a reductive dual pair in $Sp(2k,\mathbb{R})$ (in the sense of [**Ho1**]). The maximal compact subgroup of $\widetilde{Sp}(2k,\mathbb{R})$ is $\widetilde{U}(k)$, the half-determinant cover of $U(k)$. In the Fock model, the space of $\widetilde{U}(k)$-finite vectors of the oscillator representation is $\mathcal{P} = \mathcal{P}(\mathbb{C}^k)$, the set of complex-valued polynomials on $\mathbb{C}^k$. We can also assume that $K = U(k)\cap G$ and $K' = U(k)\cap G'$ are maximal compact subgroups in G and G' respectively. We shall let lower gothic symbols denote Lie algebras of Lie groups, e.g., $\mathfrak{g}$ and $\mathfrak{g}'$ will be the Lie algebras of G and G' respectively.

For a reductive subgroup H (with maximal compact subgroup $K_H = U(k)\cap H$) of $Sp(2k,\mathbb{R})$, we denote by $\mathcal{R}(\mathfrak{h},\widetilde{K}_H,\omega)$ the set of infinitesimal equivalence classes of irreducible $(\mathfrak{h},\widetilde{K}_H)$ modules realizable as quotients of $\mathcal{P}$. Consider the dual pair (G,G'). For $\rho\in\mathcal{R}(\mathfrak{g}',\widetilde{K'},\omega)$, the Howe quotient corresponding to ρ is defined by (see [**Ho2**])

$$\Omega(\rho) = \mathcal{P}/\mathcal{N}_\rho,$$

where $\mathcal{N}_\rho$ is the intersection of all $(\mathfrak{g}',\widetilde{K'})$-invariant subspaces $\mathcal{N}$ of $\mathcal{P}$ such that $\mathcal{P}/\mathcal{N}\cong\rho$ as $(\mathfrak{g}',\widetilde{K'})$ modules. It is known (see [**Ho2**]) that

$$\Omega(\rho)\simeq\rho'\otimes\rho,$$

where ρ' is a $(\mathfrak{g}, \widetilde{K})$ module of finite length, with a unique irreducible quotient $\theta(\rho)$. The correspondence

$$\rho \longmapsto \theta(\rho)$$

is commonly known as the (local) theta correspondence, and $\theta(\rho)$ is often called the theta lift of ρ.

The pullback $\widetilde{H}$ of a Lie subgroup of $Sp(2k, \mathbb{R})$ is a split or non-split extension by $\mathbb{Z}/2\mathbb{Z}$ depending on the dual pair under consideration. The representations which occur in the theta correspondence are genuine, i.e., they do not factor to H. In the case where the cover of H is split, this just means that they are of the form $\pi \otimes sgn$, where π is a representation of H, and sgn is the non-trivial character of $\mathbb{Z}/2\mathbb{Z}$. If $H = O(p, q)$, the non-split cover $\widetilde{H}$ may be realized as $H \times \mathbb{Z}/2\mathbb{Z}$ with group law $(g, \epsilon)(h, \delta) = (gh, \epsilon\delta(\det(g), \det(h))_{\mathbb{R}})$, where $(\cdot, \cdot)_{\mathbb{R}}$ is the Hilbert symbol of $\mathbb{R}$. The character χ of $\widetilde{H}$ given by

$$\chi(g, \epsilon) = \epsilon \cdot \begin{cases} \sqrt{-1} & \text{if } \det(g) = -1, \\ 1 & \text{otherwise} \end{cases}$$

is genuine, and a genuine representation of $\widetilde{H}$ is of the form $\pi \otimes \chi$ for $\pi \in H\widehat{\ }$. In either case we will only refer to π.

It is a central problem in the theory of dual pairs to describe the theta correspondence. There are several techniques used to obtain explicit correspondences, however they are not elementary.

The theta correspondence for the pairs $(O(p, q), SL(2, \mathbb{R}))$, $(U(p, q), U(1, 1))$, $(Sp(p, q), O^*(4))$ is known mostly to experts in the field. Early results dealing with the theta correspondence of $(O(p, q), SL(2, \mathbb{R}))$ could be found in [**Ho4**] which built on results of Rallis and Schiffman [**RS**] and Strichartz [**St**]. Literature on the last two pairs is quite difficult to locate. The objective of this paper is to study the Howe quotient corresponding to a small representation by explicit construction using the Fock model. The representations dealt with here are the one-dimensional representations of $SL(2, \mathbb{R})$, $U(1, 1)$, unitary finite-dimensional representations of $O^*(4) \simeq (SU(2) \times SL(2, \mathbb{R}))/\{\pm I\}$ (see [**Lm**]) as well as the one-dimensional representations of $O(p, q)$, $U(p, q)$ and $Sp(p, q)$. We believe that in the stable range (see [**Ho3**]), the Howe quotient (corresponding to a unitary representation or "small" representation) is irreducible. Evidences in support of this can be found in this paper as well as [**LZ1**], [**LZ2**], [**ZH**] and [**TZ**]. The Howe quotient also features prominently in many applications; see [**KV2**] and [**Zh**] (and the references therein) for applicatons to invariant distributions and [**KR2**] (and the references therein) for applications to the construction of automorphic forms. The setup used to study the duality correspondence enables one to have control on the Howe quotients, and it is our hope to try to extend it to other dual pairs.

The study of Howe quotients was initiated by Kudla and Rallis (see [**KR1**]; for the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$). Their technique is to embed the Howe quotient in an appropriate degenerate principal series of $Sp(2n,\mathbb{R})$ and using the work of Guillemonat (see [**Gu**]) to understand the structure of these representations, thereby extracting the theta lift in some cases. Recently, S.T. Lee and C.B. Zhu used similar ideas to describe the theta lift of a class of one-dimensional representations of $O(p,q)$ (for the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$; see [**LZ2**]) and one-dimensional representations of $U(p,q)$ (for the dual pairs $(U(p,q), U(n,n))$; see [**LZ1**]) using S. T. Lee's work on the degenerate principal series of $Sp(2n,\mathbb{R})$ (see [**Le2**]) and $U(n,n)$ (see [**Le1**]). Their results are more exhaustive. It is also possible to describe the theta lift in the cases that we are considering using the induction principle (see [**Ku**]) since the structure of principal series representations of $\widetilde{SL}(2,\mathbb{R})$, $\widetilde{U}(1,1)$ and $O^*(4)$ are easily understood. However, our approach is elementary - we simply work with concrete objects, i.e., polynomials.

It is easy to see that the theta lift of the trivial representation of $Sp(2n,\mathbb{R})$ exists only if $p+q$ is even, and in that case, it has a multiplicity-free $\widetilde{O}(p)\times\widetilde{O}(q)$ spectrum (see [**ZH**]). The theta lifts of one-dimensional representations of $U(n,n)$ also give irreducible $U(p,q)$ representations with multiplicity-free $U(p)\times U(q)$ spectrum (see [**TZ**]) if $\min(p,q) \geq 2n$. These representations are "small" in the sense that they have small Gelfand-Kirillov dimensions and small rank (in the sense of [**Ho3**]). They "should" arise from appropriate quantization of nilpotent orbits (see [**TZ**] and [**ZH**]) and are generalizations of representations dealt with in [**BZ**], [**Ko1**] and [**Ko2**]. Another objective for the computations in this paper is to provide a basis of K highest weight vectors (where K is a maximal compact subgroup) for the representations treated in [**TZ**] and [**ZH**]. With such a basis, irreduciblity and perhaps unitarity of these representations result from similar considerations as in [**HT2**]. Of course, irreducibility and unitarity would follow from [**Li**]'s results (see [**ZH**]). But our technique has invariant-theoretic flavour and has the advantage of providing a model of the representation space which might be useful to those who would like to make explicit calculations on these representations. Due to the length of the computations involved, we shall discusss these in a separate paper (see [**Ta**]).

2. The Dual Pairs $(O(p,q), SL(2,\mathbb{R}))$.

Let $p \geq 2$ and consider the dual pair $(O(p), SL(2,\mathbb{R}))$ acting on the $\widetilde{U}(p)$-finite vectors of the associated Fock space $\mathbb{C}[x_1, \dots, x_p]$ as follows:

(a) Action of $\mathfrak{o}(p)$: $\qquad x_i\dfrac{\partial}{\partial x_j} - x_j\dfrac{\partial}{\partial x_i}, \qquad 1 \leq i < j \leq p.$

(b) Action of $\mathfrak{sl}(2) = \{H_1, r_1^2, \Delta_1\}$:

$$H_1 = \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} + \frac{p}{2}, \qquad r_1^2 = \sum_{i=1}^{p} x_i^2, \qquad \Delta_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}.$$

It is easy to see that the duality correspondence is as follows:

$$\mathbb{C}[x_1, \dots, x_p]|_{O(p)\times \widetilde{SL}(2,\mathbb{R})} = \sum_{m=0}^{\infty} \mathcal{H}_m^{(p)} \otimes V_{m+\frac{p}{2}},$$

where

$$\mathcal{H}_m^{(p)} = \{f \in \mathbb{C}[x_1, \dots, x_p] \mid \deg f = m, \Delta_1 f = 0\}$$

is the irreducible $O(p)$ module spanned by spherical harmonics in variables $x_1, \dots, x_p$ of degree m and $V_{m+\frac{p}{2}}$ is the $\widetilde{SL}(2,\mathbb{R})$ lowest weight module of lowest weight $m + \frac{p}{2}$ spanned by $\{(r_1^2)^i (x_1 - \sqrt{-1}x_2)^m \mid i = 0, 1, \dots\}$.

The duality correspondence enables us to write

$$\mathbb{C}[x_1, \dots, x_p]|_{O(p)} = \sum_{i,m=0}^{\infty} (r_1^2)^i \mathcal{H}_m^{(p)}, \tag{2.1}$$

where $(r_1^2)^i \mathcal{H}_m^{(p)}$ are $O(p)$ modules isomorphic to $\mathcal{H}_m^{(p)}$.

We note an interesting computation which makes the computablility of this problem even easier:

Lemma 2.1. *Let $\phi \in \mathcal{H}_m^{(p)}$ where $m \geq 1$. If*

$$(x_i\phi)^{\sim} = x_i\phi - \frac{1}{(2m+p-2)} r_1^2 \frac{\partial \phi}{\partial x_i},$$

then

$$x_i\phi = (x_i\phi)^{\sim} + \frac{1}{(2m+p-2)} r_1^2 \frac{\partial \phi}{\partial x_i}$$

gives the projection of $x_i\phi$ into the $O(p)$ modules $\mathcal{H}_{m+1}^{(p)}$ and $r_1^2\mathcal{H}_{m-1}^{(p)}$.

Remark. This is a special case of (2.1).

Proof. Easy. $\square$

For convenience, we shall let

$$c_{p,m} = \frac{1}{2m+p-2}.$$

We note that when $m \geq 1$, $c_{p,m} > 0$.

Likewise, for $q \geq 2$, the dual pair $(O(q), SL(2,\mathbb{R}))$ acting on $\mathbb{C}[y_1, \dots, y_q]$ gives rise to the following decomposition of the Fock space as an $O(q)$ module:

$$\mathbb{C}[y_1, \dots, y_q]|_{O(q)} = \sum_{j,n=0}^{\infty} (r_2^2)^j \mathcal{H}_n^{(q)}, \tag{2.2}$$

where $r_2^2 = \sum_{j=1}^q y_j^2$, $(r_2^2)^j \mathcal{H}_n^{(q)}$ is isomorphic (as $O(q)$ modules) to $\mathcal{H}_n^{(q)}$, the spherical harmonics $\left(\text{i.e., killed by } \Delta_2 = \sum_{j=1}^q \frac{\partial^2}{\partial y_j^2}\right)$ of degree n in the variables $y_1, \dots, y_q$.

Consider $\mathcal{P} = \mathbb{C}[x_1, \dots, x_p, y_1, \dots, y_q]$. This is the space of $\widetilde{U}(p+q)$-finite vectors of the associated Fock model for the dual pair $(O(p,q), SL(2,\mathbb{R}))$, and the actions of the complexified Lie algebras of $O(p,q)$ and $\widetilde{SL}(2,\mathbb{R})$ can be described as follows:

(2.3)

(a) Action of $\mathfrak{o}(p,q)_{\mathbb{C}} = \mathfrak{o}(p)_{\mathbb{C}} \oplus \mathfrak{o}(q)_{\mathbb{C}} \oplus \mathfrak{p}$

(i) Action of $\mathfrak{o}(p)_{\mathbb{C}}$: $\quad x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \qquad 1 \le i < j \le p,$

(ii) Action of $\mathfrak{o}(q)_{\mathbb{C}}$: $\quad y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i}, \qquad 1 \le i < j \le q,$

(iii) Action of $\mathfrak{p}$: $\quad x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}, \qquad 1 \le i \le p, \quad 1 \le j \le q.$

(b) Action of $\mathfrak{sl}(2)_{\mathbb{C}} = \text{Span}\,\{H, E, F\}$:

(i) $H = \sum_{i=1}^p x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^q y_j \frac{\partial}{\partial y_j} + \frac{p-q}{2}$,

(ii) $E = \sum_{i=1}^p x_i^2 - \sum_{j=1}^q \frac{\partial^2}{\partial y_j^2} = r_1^2 - \Delta_2$,

(iii) $F = \sum_{j=1}^q y_j^2 - \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} = r_2^2 - \Delta_1$.

We note that $\widetilde{O}(p,q)$ is a non-split extension by $\mathbb{Z}_2$ while $\widetilde{SL}(2,\mathbb{R})$ is split if and only if $p+q$ is even.

Because of the decompositions (2.1) and (2.2), we have the following decomposition of $\mathcal{P}$ as an $O(p) \times O(q)$ module:

$$\mathbb{C}[x_1, \dots, x_p, y_1, \dots, y_q]|_{O(p)\times O(q)} = \sum_{i,j,m,n=0}^{\infty} (r_1^2)^i (r_2^2)^j \mathcal{H}_m^{(p)} \mathcal{H}_n^{(q)}. \tag{2.4}$$

We will take as a "basis" for $\mathcal{P}$ elements of the form

$$[i,j,m,n] = (r_1^2)^i (r_2^2)^j \phi_1 \phi_2 \qquad \text{where } \phi_1 \in \mathcal{H}_m^{(p)} \text{ and } \phi_2 \in \mathcal{H}_n^{(q)}. \tag{2.5}$$

To be strictly correct, we should take ϕ_1 from a basis of $\mathcal{H}_m^{(p)}$ and likewise for ϕ_2. But our computations basically disregard this. In other words, we can disregard the actions of $\tilde{O}(p) \times \tilde{O}(q)$ when we study the $\tilde{O}(p,q)$ structure of modules in $\mathcal{P}$. The reason is simple: The action of $\tilde{O}(p) \times \tilde{O}(q)$ leaves an element as in (2.5) in the same $\tilde{O}(p) \times \tilde{O}(q)$ type, whilst an operator from $\mathfrak{p}$ (the operators transverse to those coming from the Lie algebra of the maximal compact subgroup $\tilde{O}(p) \times \tilde{O}(q)$) moves elements from one $\tilde{O}(p) \times \tilde{O}(q)$ type to another.

Lemma 2.2. *Assume that $p, q \geq 2$. The actions of $\mathfrak{sl}(2)_{\mathbb{C}}$ and $\mathfrak{p} \subset \mathfrak{o}(p,q)_{\mathbb{C}}$ on the basis in* (2.5) *are as follows:*

$$\begin{aligned}
H \cdot [i,j,m,n] &= \left(2i - 2j + m - n + \frac{p-q}{2}\right)[i,j,m,n];\\
E \cdot [i,j,m,n] &= [i+1,j,m,n] - 2j(q+2n+2j-2)[i,j-1,m,n];\\
F \cdot [i,j,m,n] &= [i,j+1,m,n] - 2i(p+2m+2i-2)[i-1,j,m,n];\\
\left(x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}\right) \cdot [i,j,m,n] &= [i,j,m+1,n+1]\\
&\quad + c_{p,m}[i+1,j,m-1,n+1]\\
&\quad + c_{q,n}[i,j+1,m+1,n-1] + c_{p,m}c_{q,n}[i+1,j+1,m-1,n-1]\\
&\quad - 4ij[i-1,j-1,m+1,n+1] - 2i(2jc_{q,n}+1)[i-1,j,m+1,n-1]\\
&\quad - 2j(2ic_{p,m}+1)[i,j-1,m-1,n+1]\\
&\quad - (2ic_{p,m}+1)(2jc_{q,n}+1)[i,j,m-1,n-1].
\end{aligned}$$

Proof. Follows from Lemma 2.1 and the commutation relations

(a) $[\Delta_1, r_1^2] = 4\left(\sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} + \frac{p}{2}\right)$,

(b) $[\Delta_2, r_2^2] = 4\left(\sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j} + \frac{q}{2}\right)$,

(c) $[\Delta_1, (r_1^2)^i] = 2i(p+2i-2)(r_1^2)^{i-1} + 4i(r_1^2)^{i-1}\sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i}$,

(d) $[\Delta_2, (r_2^2)^j] = 2j(q+2j-2)(r_2^2)^{j-1} + 4j(r_2^2)^{j-1}\sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j}$.

□

Proposition 2.3. *Let $(G, G') \subset Sp(2k, \mathbb{R})$ be a reductive dual pair and $\mathcal{P}$ be the space of $\widetilde{U}(k)$-finite vectors of the Fock model. Let χ be a one-dimensional representation of $\widetilde{G'}$ with differential $d\chi$, then*

$$\mathcal{N}_\chi = \operatorname{Span}\{Xf - d\chi(X)f, \omega(k)f - \chi(k)f \mid X \in \mathfrak{g}', k \in \widetilde{K'}, f \in \mathcal{P}\}$$

is $(\mathfrak{g}\times\mathfrak{g}', \widetilde{K}\times\widetilde{K'})$ *invariant and* $\mathcal{P}/\mathcal{N}_\chi$ *is the Howe quotient corresponding to the* $(\mathfrak{g}', \widetilde{K'})$*-module of* χ.

Remark. If G' is connected, one could omit the terms $\omega(k)f - \chi(k)f$. If G' is disconnected, there may be more than one character with the same differential and the term $\omega(k)f - \chi(k)f$ would capture the piece with the correct $\widetilde{K'}$ action.

Proof. Take $\sum(X_i - d\chi(X_i))f_i \in \mathcal{N}_\chi$ where $X_i \in \mathfrak{g}'$ and $f_i \in \mathcal{P}$. Let $(A,B) \in \mathfrak{g}\times\mathfrak{g}'$ and $(a,b)\in\widetilde{K}\times\widetilde{K'}$. Then

$$\begin{aligned}
&(A,B)\sum(X_i - d\chi(X_i))f_i\\
&= \sum AB(X_i - d\chi(X_i))f_i\\
&= \sum(BX_i - d\chi(X_i)B)(Af_i) \quad \text{(since A commutes with B and X_i)}\\
&= \sum(X_iB - [X_i,B] - d\chi(X_i)B)(Af_i)\\
&= \sum(X_i - d\chi(X_i))(BAf_i) - \sum[X_i,B](Af_i).
\end{aligned}$$

Now fix a choice of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}'$. If $B, X_i \in \mathfrak{h}$, $[X_i,B]=0$, so we don't have a problem here. If $B\in\mathfrak{h}$ and X_i is a non-zero root vector (so that $d\chi(X_i)=0$), then $[X_i,B]$ is a multiple of X_i, and we still have $[X_i,B](Af_i)\in\mathcal{N}_\chi$. Likewise we have no problem if X_i and B are both non-zero root vectors. For the general case, extend by linearity to see that $\mathcal{N}_\chi$ is $\mathfrak{g}\times\mathfrak{g}'$ invariant.

For the action of $\widetilde{K}\times\widetilde{K}'$,

$$\begin{aligned}
&(a,b)\sum(X_i - d\chi(X_i))f_i\\
&\quad= \sum ab(X_i - d\chi(X_i))f_i\\
&\quad= \sum(bX_i - d\chi(X_i)b)(af_i) \quad \text{(since a commutes with b and X_i)}\\
&\quad= \sum(bX_ib^{-1} - d\chi(X_i))(baf_i)\\
&\quad= \sum(bX_ib^{-1} - d\chi(bX_ib^{-1}))(baf_i) \quad \text{(since $d\chi(bX_ib^{-1}) = d\chi(X_i)$)}\\
&\quad\in \mathcal{N}_\chi.
\end{aligned}$$

The argument for the term $\omega(k)f - \chi(k)f$ is similar. So $\mathcal{N}_\chi$ is $(\mathfrak{g}\times\mathfrak{g}', \widetilde{K}\times\widetilde{K'})$ invariant.

If $\mathcal{N}\subset\mathcal{P}$ is such that

$$\mathcal{P}/\mathcal{N} \simeq \chi \qquad \text{(as a $(\mathfrak{g}', \widetilde{K'})$ module)},$$

then for $X\in\mathfrak{g}'$ and $\bar{f}\in\mathcal{P}/\mathcal{N}$, we have

$$X\bar{f} = d\chi(X)\bar{f} \Rightarrow (X - d\chi(X))\bar{f} = 0$$

$$\begin{aligned}&\Rightarrow \overline{(X-d\chi(X))f}=0\\&\Rightarrow (X-d\chi(X))f\in\mathcal{N}.\end{aligned}$$

Likewise for $k\in\widetilde{K'}$,

$$\omega(k)\bar{f}=\chi(k)\bar{f}\Rightarrow(\omega(k)-\chi(k))\bar{f}=0.$$

Thus, $\mathcal{N}_\chi\subset\mathcal{N}$ and $\mathcal{P}/\mathcal{N}_\chi$ is the Howe quotient corresponding to the representation χ of $(\mathfrak{g}',\widetilde{K'})$. □

Lemma 2.4. *Assume $p,q\geq 2$. Consider the basis of $\mathcal{P}$ as in (2.5), and take $\chi=\mathbb{1}$, the trivial $(\mathfrak{sl}(2),\widetilde{U}(1))$ module, then $\mathcal{P}/\mathcal{N}_{\mathbb{1}}$ is*

$$\begin{cases}\textit{Span of (images of)}\\ \{[0,0,m,n]\mid m-n+\frac{p-q}{2}=0,m\geq0,n\geq0\} & \textit{if } \frac{p-q}{2}\in\mathbb{Z};\\ \{0\} & \textit{otherwise.}\end{cases}$$

Proof. We note that from Proposition 2.3,

$$\mathcal{N}_{\mathbb{1}}=\text{Span }\{Hf,Ef,Ff\mid f\in\mathcal{P}\}.$$

From Lemma 2.2, we infer that

(a) Action of $H\ \Rightarrow[i,j,m,n]\in\mathcal{N}_{\mathbb{1}}$ if $2i-2j+m-n+\dfrac{p-q}{2}\neq0$;

(b) Action of $E\ \Rightarrow[i,0,m,n]\in\mathcal{N}_{\mathbb{1}}$ if $i>0$, and
$[i,j,m,n]\equiv2j(q+2n+2j-2)[i-1,j-1,m,n]\mod\mathcal{N}_{\mathbb{1}}$;

(c) Action of $F\ \Rightarrow[0,j,m,n]\in\mathcal{N}_{\mathbb{1}}$ if $j>0$, and
$[i,j,m,n]\equiv2i(p+2m+2i-2)[i-1,j-1,m,n]\mod\mathcal{N}_{\mathbb{1}}$.

$$\text{Thus,}\quad\begin{aligned}&[i,j,m,n]\equiv c_1[i-j,0,m,n]\equiv0 &&\mod\mathcal{N}_{\mathbb{1}}\quad\text{if } i>j;\\&[i,j,m,n]\equiv c_2[0,j-i,m,n]\equiv0 &&\mod\mathcal{N}_{\mathbb{1}}\quad\text{if } j>i;\\&[i,j,m,n]\equiv c_3[0,0,m,n] &&\mod\mathcal{N}_{\mathbb{1}}\quad\text{if } i=j>0.\end{aligned}$$

Here c_1,c_2 and c_3 are non-zero constants. The result follows. □

Theorem 2.5. *Assume $p,q\geq2$. The trivial $(\mathfrak{sl}(2),\tilde{U}(1))$ module belongs to $\mathcal{R}(\mathfrak{sl}(2),\tilde{U}(1),\omega)$ if and only if $\frac{p-q}{2}\in\mathbb{Z}$, and if $\frac{p-q}{2}\in\mathbb{Z}$, the theta lift of the trivial representation is the irreducible and unitary ladder representation $L_{p,q}=\mathit{Span}\ \{[0,0,m,n]\mid m-n+\frac{p-q}{2}=0\}$.*

Remark. These representations of $\tilde{O}(p,q)$ are known as ladder representations in the Physics literature (see [**AFR**] and [**BZ**]). They have Gelfand-Kirillov dimension $p+q-3$ (in the sense of [**Vo**]) and correspond to the quantization of certain minimal orbits (see [**BZ**], [**Ko1**] and [**Ko2**]).

Proof. The first part is immediate from Lemma 2.4.

Assume that $\frac{p-q}{2} \in \mathbb{Z}$. We note the action of $\mathfrak{p} \subset o(p,q)_{\mathbb{C}}$ using Lemma 2.2:

$$\begin{aligned}&\left(x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}\right) \cdot [0,0,m,n] \\ &= [0,0,m+1,n+1] - [0,0,m-1,n-1] \qquad \bmod \mathcal{N}_{\mathbb{1}}.\end{aligned}$$

A simple computation on the $O(p)$ and $O(q)$ weights shows that $[0,0,m,n] \neq [0,0,m',n']$ mod $\mathcal{N}_{\mathbb{1}}$ unless both are in $\mathcal{N}_{\mathbb{1}}$. Starting from $[0,0,\frac{q-p}{2},0]$ or $[0,0,0,\frac{p-q}{2}]$ which is clearly not in $\mathcal{N}_{\mathbb{1}}$, the transition formula above shows that $L_{p,q}$ is irreducible as an $(\mathfrak{o}(p,q), \widetilde{O}(p) \times \widetilde{O}(q))$ module. Unitarity follows from [**Li**]'s results. $\square$

Next we compute the Howe quotient corresponding to the trivial representation of $O(p,q)$ for the dual pair $(O(p,q), SL(2,\mathbb{R}))$. We remark that [**LZ1**] has treated these theta lifts in a different way (and for the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$ and have explicit information on the structure of the corresponding Howe quotients.

Assume $p,q \geq 2$. If $\mathcal{P} = \mathbb{C}[x_1,\ldots,x_p,y_1,\ldots,y_q]$ as before, let

$$\mathcal{N}'_{\mathbb{1}} = Span\{Xf | f \in \mathcal{P}, X \in \mathfrak{o}(p,q)_{\mathbb{C}}\}.$$

Then by the remark following Proposition 2.3, the Howe quotient is contained in $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. Recall that the action of $\mathfrak{o}(p,q)_{\mathbb{C}}$ and $\mathfrak{sl}(2)_{\mathbb{C}}$ is given in (2.3).

Lemma 2.6. *Let $\bar{1}, X$, and Y be the elements of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ given by*

$$\begin{aligned}\bar{1} &= 1 + \mathcal{N}'_{\mathbb{1}}, \\ X &= x_1^2 + \mathcal{N}'_{\mathbb{1}}, \\ Y &= y_1^2 + \mathcal{N}'_{\mathbb{1}}.\end{aligned}$$

Then $B = \{\bar{1}, X^k, Y^l | k,l \in \mathbb{Z}_{\geq 1}\}$ is a basis of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$.

Proof. For $\lambda = (\lambda_1,\ldots,\lambda_p)$ and $\mu = (\mu_1,\ldots,\mu_q)$ with $\lambda_i, \mu_j \in \mathbb{Z}_{\geq 0}$, let $x^\lambda y^\mu$ be the monomial $\prod_{i=1}^p x_i^{\lambda_i} \prod_{i=1}^q y_i^{\mu_i}$. For $1 \leq i < j \leq p$ and arbitrary $x^\lambda y^\mu$, we have that

$$\left(x_i \tfrac{\partial}{\partial x_j} - x_j \tfrac{\partial}{\partial x_i}\right) x_i x_j x^\lambda y^\mu = \left((\lambda_j+1)x_i^2 - (\lambda_i+1)x_j^2\right) x^\lambda y^\mu \in \mathcal{N}'_{\mathbb{1}},$$

so that

$$x_i^2 x^\lambda y^\mu \equiv \tfrac{\lambda_i+1}{\lambda_j+1} x_j^2 x^\lambda y^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{2.6a}$$

Similarly, we have for $1 \leq i < j \leq q$,

$$y_i^2 x^\lambda y^\mu \equiv \tfrac{\mu_i+1}{\mu_j+1} y_j^2 x^\lambda y^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{2.6b}$$

If $\lambda_i = 0$, then

$$\left(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i}\right) x_j x^\lambda y^\mu = (\lambda_j + 1)x_i x^\lambda y^\mu.$$

This implies that $x^\lambda y^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\lambda_i = 1$ for some i. Similarly, $x^\lambda y^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\mu_i = 1$ for some i. Applying (2.6) repeatedly if necessary yields

(2.7) λ_i odd for some i or μ_i odd for some $i \Rightarrow x^\lambda y^\mu \in \mathcal{N}'_{\mathbb{1}}$.

For $1 \le i \le p$ and $1 \le j \le q$, applying the operator $x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}$ to the monomial $x_i y_j x^\lambda y^\mu$ yields

$$x_i^2 y_j^2 x^\lambda y^\mu \equiv (\lambda_i + 1)(\mu_j + 1)x^\lambda y^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{2.8}$$

Using (2.6), (2.7), and (2.8), we see that every monomial in $\mathcal{P}$ is either in $\mathcal{N}'_{\mathbb{1}}$ or in one of the cosets listed in B. So we have that B spans $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. To see that the elements of B are linearly independent, we notice that each is a weight vector for $\mathfrak{u}(1)_{\mathbb{C}} \subset \mathfrak{sl}(2)_{\mathbb{C}}$, of the following weights:

$$\begin{aligned} \bar{1} &\quad \text{has weight} \quad \tfrac{p-q}{2}, \\ X^k &\quad \text{has weight} \quad \tfrac{p-q}{2} + 2k, \\ Y^k &\quad \text{has weight} \quad \tfrac{p-q}{2} - 2k. \end{aligned} \tag{2.9}$$

Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. □

Remark. The proof of Lemma 2.6 also shows that all $\widetilde{U}(1)$-types in $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ have multiplicity one.

Theorem 2.7. *Suppose $p, q \ge 2$. The trivial $(\mathfrak{o}(p,q), \tilde{O}(p)\times\tilde{O}(q))$- module $\mathbb{1}$ belongs to $\mathcal{R}(\mathfrak{o}(p,q), \tilde{O}(p)\times\tilde{O}(q), \omega)$ and no other one-dimensional module does, so that $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ is the Howe quotient corresponding to the trivial module.*
(i) *If p and q are both odd then $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ is irreducible and isomorphic to the principal series of the split double cover of $SL(2,\mathbb{R})$ with infinitesimal character $-\frac{p+q}{2} + 1$, and even $U(1)$-types if $\frac{p-q}{2}$ is even, odd $U(1)$-types otherwise.*
(ii) *If p and q are both even then $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ has two irreducible $(\mathfrak{sl}(2), U(1))$-submodules V_1 and V_2 spanned by $\{X^k | k \ge \frac{q}{2}\}$ and $\{Y^k | k \ge \frac{p}{2}\}$ respectively, which are discrete series representations with minimal $U(1)$-types $\frac{p+q}{2}$ and $-\frac{p+q}{2}$ respectively. The theta-lift of $\mathbb{1}$ is the irreducible quotient of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ by $V_1 \oplus V_2$, which is isomorphic to the unique $(\mathfrak{sl}(2), U(1))$- module of dimension $\frac{p+q}{2} - 1$.*
(iii) *If p is even and q is odd then $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ has an irreducible $(\mathfrak{sl}(2), \widetilde{U}(1))$-submodule V spanned by $\{Y^k | k \ge \frac{p}{2}\}$, which is the Harish-Chandra module of the discrete series representation of $\widetilde{SL}(2,\mathbb{R})$ with minimal $\tilde{U}(1)$-type $-\frac{p+q}{2}$.*

The theta-lift of $\mathbb{1}$ *is the irreducible quotient of* $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ *by* V *which is a lowest weight module with lowest weight* $-\frac{p+q}{2}+2$.

(iv) *If* p *is odd and* q *is even then* $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ *has an irreducible* $(\mathfrak{sl}(2), \widetilde{U}(1))$*-submodule* V *spanned by* $\{X^k | k \geq \frac{q}{2}\}$, *which is the Harish-Chandra module of the discrete series representation of* $\widetilde{SL}(2,\mathbb{R})$ *with minimal* $\tilde{U}(1)$*-type* $\frac{p+q}{2}$. *The theta-lift of* $\mathbb{1}$ *is the irreducible quotient of* $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ *by* V *which is a highest weight module with highest weight* $\frac{p+q}{2}-2$.

Further, the theta-lift of $\mathbb{1}$ *is non-unitarizable except in the case* (ii) *with* $p=q=2$.

Proof. First observe that elements of B transform by the trivial $O(p,q)$ character, simply by checking the action of $\widetilde{O}(p) \times \widetilde{O}(q)$. Using the formulas (2.3), (2.6), and (2.8), we compute the action of $\mathfrak{p}' = \mathrm{Span}\{E, F\}$ on the weight vectors in B:

$$\begin{aligned}
&E \cdot \bar{1} = pX;\\
&E \cdot X^k = \frac{2k+p}{2k+1} X^{k+1};\\
&E \cdot Y^k = (2k-1)(p-2k)Y^{k-1};\\
&F \cdot \bar{1} = qY;\\
&F \cdot X^k = (2k-1)(q-2k)X^{k-1};\\
&F \cdot Y^k = \frac{2k+q}{2k+1} Y^{k+1}.
\end{aligned}$$

Notice that E annihilates $Y^{\frac{p}{2}}$ if p is even, and takes all weight vectors of weight $\frac{p-q}{2}+2k$ with $2k \neq p$ to weight vectors of weight $\frac{p-q}{2}+2k+2$. Similarly, F annihilates $X^{\frac{q}{2}}$ if q is even and takes all weight vectors of weight $\frac{p-q}{2}+2k$ with $2k \neq -q$ to weight vectors of weight $\frac{p-q}{2}+2k-2$. The decomposition into submodules and quotients for (i)-(iv) now follows.

The casimir operator of $\mathfrak{sl}(2)$ acts on the Howe quotient by the constant $(\frac{p+q}{2})(\frac{p+q}{2}-2)$. Non-unitarizability then follows from the descriptions of $\mathfrak{sl}(2)$ modules given in Chapter III of [**HT1**]. □

For completeness, we shall provide the results for the cases when either $p=1$ or $q=1$ or $p=q=1$. Suppose $\epsilon_1, \epsilon_2 \in \{+,-\}$ and let $\mathbb{1}_{\epsilon_1,\epsilon_2}$ be the unique character of $O(p,q)$ which restricts to det on $O(p)$ if $\epsilon_1 = -$ and to the trivial character if $\epsilon_1 = +$ and similarly for $O(q)$ and ϵ_2. Recall that all genuine characters of $\widetilde{O}(p,q)$ are obtained by twisting these characters by the character χ defined in the introduction.

Theorem 2.8.

(a) $(O(p,1), SL(2,\mathbb{R}))$, $p \geq 2$:

(a)(i) $\mathbb{1}_{+,+} \in \mathcal{R}(\mathfrak{o}(p,1), \widetilde{O}(p) \times \widetilde{O}(1), \omega)$ *if* p *is even and* $\theta(\mathbb{1}_{+,+})$ *is a lowest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *module of lowest weight* $-\frac{p-3}{2}$ *and it is not unitarizable unless* $p = 2$;

(a)(ii) $\mathbb{1}_{+,-} \in \mathcal{R}(\mathfrak{o}(p,1), \widetilde{O}(p) \times \widetilde{O}(1), \omega)$ *if* p *is even and* $\theta(\mathbb{1}_{+,-})$ *is a highest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *module of highest weight* $\frac{p-3}{2}$ *and it is not unitarizable unless* $p = 2$;

(a)(iii) $\mathbb{1}_{+,+} \in \mathcal{R}(\mathfrak{o}(p,1), \widetilde{O}(p) \times \widetilde{O}(1), \omega)$ *if* p *is odd and* $\theta(\mathbb{1}_{+,+})$ *is the non-unitarizable* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *principal series representation with infinitesimal character* $\frac{p-1}{2}$, *and even* $\widetilde{U}(1)$*-types if* $\frac{p-1}{2}$ *is even, odd* $\widetilde{U}(1)$*-types otherwise;*

(a)(iv) $\mathbb{1}_{+,-} \in \mathcal{R}(\mathfrak{o}(p,1), \widetilde{O}(p) \times \widetilde{O}(1), \omega)$ *if* p *is odd and* $\theta(\mathbb{1}_{+,-})$ *is the finite-dimensional* $(\mathfrak{sl}(2), \widetilde{U}(1))$*-module of dimension* $\frac{p-1}{2}$ *and it is non-unitarizable unless* $p = 3$;

(a)(v) $\mathbb{1} \in \mathcal{R}(\mathfrak{sl}, \widetilde{U}(1), \omega)$ *if* p *is odd and* $\theta(\mathbb{1})$ *is the irreducible and unitarizable* $(\mathfrak{o}(p,1), \widetilde{O}(p) \times \widetilde{O}(1))$ *ladder representation with representatives* $\{(x_1 - \sqrt{-1}x_2)^m y^t \mid t = m + \frac{p-1}{2}\}$.

(b) $(O(1,q), SL(2,\mathbb{R}))$, $q \geq 2$:

(b)(i) $\mathbb{1}_{+,+} \in \mathcal{R}(\mathfrak{o}(1,q), \widetilde{O}(1) \times \widetilde{O}(q), \omega)$ *if* q *is even and* $\theta(\mathbb{1}_{+,+})$ *is a highest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *module of highest weight* $\frac{q-3}{2}$ *and it is not unitarizable unless* $q = 2$;

(b)(ii) $\mathbb{1}_{-,+} \in \mathcal{R}(\mathfrak{o}(1,q), \widetilde{O}(1) \times \widetilde{O}(q), \omega)$ *if* q *is even and* $\theta(\mathbb{1}_{-,+})$ *is a lowest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *module of lowest weight* $-\frac{q-3}{2}$ *and it is not unitarizable unless* $q = 2$;

(b)(iii) $\mathbb{1}_{+,+} \in \mathcal{R}(\mathfrak{o}(1,q), \widetilde{O}(1) \times \widetilde{O}(q), \omega)$ *if* q *is odd and* $\theta(\mathbb{1}_{+,+})$ *is the non-unitarizable* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *principal series representation with infinitesimal character* $(\frac{q-1}{2})$, *and even* $\widetilde{U}(1)$*-types if* $\frac{q-1}{2}$ *is even, odd* $\widetilde{U}(1)$*-types otherwise;*

(b)(iv) $\mathbb{1}_{-,+} \in \mathcal{R}(\mathfrak{o}(1,q), \widetilde{O}(1) \times \widetilde{O}(q), \omega)$ *if* q *is odd and* $\theta(\mathbb{1}_{-,+})$ *is the finite-dimensional* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *of dimension* $\frac{q-1}{2}$ *and it is non-unitarizable unless* $q = 3$;

(b)(v) $\mathbb{1} \in \mathcal{R}(\mathfrak{sl}, \widetilde{U}(1), \omega)$ *if* q *is odd and* $\theta(\mathbb{1})$ *is the irreducible and unitarizable* $(\mathfrak{o}(1,q), \widetilde{O}(1) \times \widetilde{O}(q))$ *ladder representation with representatives* $\{x^s(y_1 - \sqrt{-1}y_2)^n \mid s = n + \frac{q-1}{2}\}$.

(c) $(O(1,1), SL(2,\mathbb{R}))$:

(c)(i) $\mathbb{1}_{+,+} \in \mathcal{R}(\mathfrak{o}(1,1), \widetilde{O}(1) \times \widetilde{O}(1), \omega)$ *and* $\theta(\mathbb{1}_{+,+})$ *is the unitarizable* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *principal series representation with infinitesimal character* 0;

(c)(ii) $1\!\!1_{-,+} \in \mathcal{R}(\mathfrak{o}(1,1), \widetilde{O}(1) \times \widetilde{O}(1), \omega)$ *and* $\theta(1\!\!1_{+,-})$ *is the unitarizable lowest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *module of lowest weight* 1 (*i.e., limit of discrete series*);

(c)(iii) $1\!\!1_{+,-} \in \mathcal{R}(\mathfrak{o}(1,1), \widetilde{O}(1) \times \widetilde{O}(1), \omega)$ *and* $\theta(1\!\!1_{+,-})$ *is the unitarizable highest weight* $(\mathfrak{sl}(2), \widetilde{U}(1))$ *of highest weight* -1 (*i.e., limit of discrete series*);

(c)(iv) $1\!\!1 \in \mathcal{R}(\mathfrak{sl}, \widetilde{U}(1), \omega)$ *and* $\theta(1\!\!1)$ *is the non-unitarizable two-dimensional* $(\mathfrak{o}(1,1), \widetilde{O}(1) \times \widetilde{O}(1))$ *module with representatives* $\{1, xy\}$.

Remark. The cases (a)(iv) and (b)(iv) could be interpreted physically, in terms of the Huygens' Principle. Even the cases under Theorem 2.7 could be suitably interpreted in terms of $O(p,q)$ invariant distributions (see [**HT1**]).

Proof. Basically the computations are similar. The results are obtained by tracking the actions of $\widetilde{O}(p)$, $\widetilde{O}(q)$ and $\mathfrak{sl}(2)$ on the representatives in $\mathcal{P}/\mathcal{N}'_{1\!\!1}$. □

3. The Dual Pairs $(U(p,q), U(1,1))$.

Consider the dual pair $(U(p), U(1,1))$ acting on the $\widetilde{U}(2p)$-finite vectors of the associated Fock space $\mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p]$ as follows:

(a) Action of $\mathfrak{u}(p)_{\mathbb{C}}$: $\quad z_i \dfrac{\partial}{\partial z_j} - \bar{z}_j \dfrac{\partial}{\partial \bar{z}_i}, \qquad 1 \le i, j \le p.$

(b) Action of $\mathfrak{u}(1,1)_{\mathbb{C}} = \text{Span}\ \{h_1, h_2, r_1^2, \Delta_1\}$, where

$$h_1 = \sum_{i=1}^p z_i \frac{\partial}{\partial z_i}, \quad h_2 = \sum_{i=1}^p \bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \quad r_1^2 = \sum_{i=1}^p z_i \bar{z}_i, \quad \Delta_1 = \sum_{i=1}^p \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.$$

It is easy to see that the duality correspondence is as follows:

$$\mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p]|_{U(p)\times\widetilde{U}(1,1)} = \sum_{m=0}^{\infty} \mathcal{H}^{(p)}_{\alpha,\beta} \otimes (\det)^{\frac{\alpha-\beta}{2}} V_{p+\alpha+\beta},$$

where $\mathcal{H}^{(p)}_{\alpha,\beta}$ is the irreducible $U(p)$ module characterized as follows:

$$\mathcal{H}^{(p)}_{\alpha,\beta} = \{f \in \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p] \mid h_1 f = \alpha f, h_2 f = \beta f, \Delta_1 f = 0\},$$

and $V_{p+\alpha+\beta}$ is the $SU(1,1)$ lowest weight module of lowest weight $p+\alpha+\beta$ spanned by $\{(r_1^2)^i z_1^\alpha \bar{z}_p^\beta \mid i = 0, 1, \ldots\}$. Note that the representation of $SU(1,1)$ is twisted by the $\frac{\alpha-\beta}{2}$-power of the determinant character det on $\widetilde{U}(1,1)$. Incidentally, $\mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p]$ is bigraded in degrees in the z coordinates and $\bar{z}$ coordinates. The subscript in $\mathcal{H}^{(p)}_{\alpha,\beta}$ indicates that the module lives in the homogeneous component of degree α in the z coordinates and degree β in the $\bar{z}$ coordinates. The $U(p)$ highest weight vector in $\mathcal{H}^{(p)}_{\alpha,\beta}$ is $z_1^{\alpha}\bar{z}_p^{\beta}$.

The duality correspondence enables us to write

$$\mathbb{C}[z_1, \dots, z_p, \bar{z}_1, \dots, \bar{z}_p]|_{U(p)} = \sum_{i,\alpha,\beta=0}^{\infty} (r_1^2)^i \mathcal{H}^{(p)}_{\alpha,\beta}, \tag{3.1}$$

where $(r_1^2)^i\mathcal{H}^{(p)}_{\alpha,\beta}$ are $U(p)$ modules isomorphic to $\mathcal{H}^{(p)}_{\alpha,\beta}$. We note the analogue of Lemma 2.1.

Lemma 3.1. *Let $\phi \in \mathcal{H}^{(p)}_{\alpha,\beta}$ where $\alpha+\beta \geq 1$. If*

$$(z_i\phi)^{\sim} = z_i\phi - \frac{1}{(p+\alpha+\beta-1)} r_1^2 \frac{\partial\phi}{\partial\bar{z}_i}, \qquad \textit{and}$$
$$(\bar{z}_i\phi)^{\sim} = \bar{z}_i\phi - \frac{1}{(p+\alpha+\beta-1)} r_1^2 \frac{\partial\phi}{\partial z_i},$$

then

$$z_i\phi = (z_i\phi)^{\sim} + \frac{1}{(p+\alpha+\beta-1)} r_1^2 \frac{\partial\phi}{\partial\bar{z}_i}$$

gives the projection of $z_i\phi$ into the $U(p)$ modules $\mathcal{H}^{(p)}_{\alpha+1,\beta}$ and $r_1^2\mathcal{H}^{(p)}_{\alpha,\beta-1}$, while

$$\bar{z}_i\phi = (\bar{z}_i\phi)^{\sim} + \frac{1}{(p+\alpha+\beta-1)} r_1^2 \frac{\partial\phi}{\partial z_i}$$

gives the projection of $\bar{z}_i\phi$ into the $U(p)$ modules $\mathcal{H}^{(p)}_{\alpha,\beta+1}$ and $r_1^2\mathcal{H}^{(p)}_{\alpha-1,\beta}$.

Proof. Easy. □

For convenience, we shall let

$$c_{p,\alpha,\beta} = \frac{1}{p+\alpha+\beta-1}.$$

We note that when $\alpha+\beta \geq 1$, $c_{p,\alpha,\beta} > 0$.

Likewise, the dual pair $(U(q), U(1,1))$ acting on $\mathbb{C}[w_1, \dots, w_q, \bar{w}_1, \dots, w_q]$ gives rise to the following decomposition of the Fock space as $U(q)$ modules:

$$\mathbb{C}[w_1, \dots, w_q, \bar{w}_1, \dots, \bar{w}_q]|_{U(q)} = \sum_{j,\gamma,\delta=0}^{\infty} (r_2^2)^j \mathcal{H}^{(q)}_{\gamma,\delta}, \tag{3.2}$$

where $r_2^2 = \sum_{j=1}^q w_j\bar{w}_j$, $(r_2^2)^j\mathcal{H}^{(q)}_{\gamma,\delta}$ is isomorphic to $\mathcal{H}^{(q)}_{\gamma,\delta}$, the spherical harmonics of degree γ, δ (i.e., killed by $\Delta_2 = \sum_{j=1}^q \frac{\partial^2}{\partial w_j \partial\bar{w}_j}$) in the variables $w_1, \dots, w_q, \bar{w}_1, \dots, \bar{w}_q$.

Consider $\mathcal{P} = \mathbb{C}[z_1, \dots, z_p, \bar{z}_1, \dots, \bar{z}_p, w_1, \dots, w_q, \bar{w}_1, \dots, \bar{w}_q]$. This is the space of $\widetilde{U}(2p+2q)$-finite vectors of the associated Fock model for the dual pair $(U(p,q), U(1,1))$, and the actions of the complexified Lie algebras of $U(p,q)$ and $\widetilde{U}(1,1)$ can be described as follows:

(3.3)

(a) Action of $\mathfrak{u}(p,q)_{\mathbb{C}} = \mathfrak{u}(p)_{\mathbb{C}} \oplus \mathfrak{u}(q)_{\mathbb{C}} \oplus \mathfrak{p}$:

(i) Action of $\mathfrak{u}(p)_{\mathbb{C}}$: $\quad z_i \dfrac{\partial}{\partial z_j} - \bar{z}_j \dfrac{\partial}{\partial \bar{z}_i}, \quad 1 \le i, j \le p;$

(ii) Action of $\mathfrak{u}(q)_{\mathbb{C}}$: $\quad w_i \dfrac{\partial}{\partial w_j} - \bar{w}_j \dfrac{\partial}{\partial \bar{w}_i}, \quad 1 \le i, j \le q;$

(iii) Action of $\mathfrak{p}$:

$$S_{ij} = z_i w_j - \frac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_j}, \quad 1 \le i \le p,\ 1 \le j \le q;$$

$$T_{ij} = \bar{z}_i \bar{w}_j - \frac{\partial^2}{\partial z_i \partial w_j}, \quad 1 \le i \le p,\ 1 \le j \le q.$$

(b) Action of $\mathfrak{u}(1,1)_{\mathbb{C}} = \mathrm{Span}\,\{H_1, H_2, E, F\}$:

(i) $H_1 = \displaystyle\sum_{i=1}^{p} z_i \frac{\partial}{\partial z_i} - \sum_{j=1}^{q} w_j \frac{\partial}{\partial w_j} + \frac{p-q}{2},$

(ii) $H_2 = -\displaystyle\sum_{i=1}^{p} \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + \sum_{j=1}^{q} \bar{w}_j \frac{\partial}{\partial \bar{w}_j} - \frac{p-q}{2},$

(iii) $E = r_1^2 - \Delta_2,$

(iv) $F = r_2^2 - \Delta_1.$

We note that $\widetilde{U}(p,q)$ is split while $\widetilde{U}(1,1)$ is split when $p+q$ is even and non-split otherwise.

Because of the decompositions (3.1) and (3.2), we have the following decomposition of $\mathcal{P}$ as a $U(p) \times U(q)$ module:

$$\mathcal{P}|_{U(p)\times U(q)} = \sum_{i,j,\alpha,\beta,\gamma,\delta=0}^{\infty} (r_1^2)^i (r_2^2)^j \mathcal{H}^{(p)}_{\alpha,\beta} \mathcal{H}^{(q)}_{\gamma,\delta}. \tag{3.4}$$

Again, we will take as a "basis" for $\mathcal{P}$, elements of the form

$$[i,j,\alpha,\beta,\gamma,\delta] = (r_1^2)^i (r_2^2)^j \phi_1 \phi_2, \quad \text{where } \phi_1 \in \mathcal{H}^{(p)}_{\alpha,\beta} \text{ and } \phi_2 \in \mathcal{H}^{(q)}_{\gamma,\delta}. \tag{3.5}$$

Lemma 3.2. *The actions of* $\mathfrak{u}(1,1)_{\mathbb{C}}$ *and* $\mathfrak{p} \subset \mathfrak{u}(p,q)_{\mathbb{C}}$ *on the basis in* (3.5) *are as follows:*

$$H_1 \cdot [i,j,\alpha,\beta,\gamma,\delta] = \left(i - j + \alpha - \gamma + \frac{p-q}{2}\right)[i,j,\alpha,\beta,\gamma,\delta];$$

$$H_2 \cdot [i,j,\alpha,\beta,\gamma,\delta] = \left(j - i + \delta - \beta - \frac{p-q}{2}\right)[i,j,\alpha,\beta,\gamma,\delta];$$

$$\begin{aligned}
E\cdot[i,j,\alpha,\beta,\gamma,\delta] &= [i+1,j,\alpha,\beta,\gamma,\delta]\\
&\quad - j(q+\gamma+\delta+j-1)[i,j-1,\alpha,\beta,\gamma,\delta];\\
F\cdot[i,j,\alpha,\beta,\gamma,\delta] &= [i,j+1,\alpha,\beta,\gamma,\delta]\\
&\quad - i(p+\alpha+\beta+i-1)[i-1,j,\alpha,\beta,\gamma,\delta]\\
S_{kl}\cdot[i,j,\alpha,\beta,\gamma,\delta] &= [i,j,\alpha+1,\beta,\gamma+1,\delta]\\
&\quad + c_{p,\alpha,\beta}[i+1,j,\alpha,\beta-1,\gamma+1,\delta]\\
&\quad + c_{q,\gamma,\delta}[i,j+1,\alpha+1,\beta,\gamma,\delta-1]\\
&\quad + c_{p,\alpha,\beta}c_{q,\gamma,\delta}[i+1,j+1,\alpha,\beta-1,\gamma,\delta-1]\\
&\quad - ij[i-1,j-1,\alpha+1,\beta,\gamma+1,\delta]\\
&\quad - i(jc_{q,\gamma,\delta}+1)[i-1,j,\alpha+1,\beta,\gamma,\delta-1]\\
&\quad - j(ic_{p,\alpha,\beta}+1)[i,j-1,\alpha,\beta-1,\gamma+1,\delta]\\
&\quad - (ic_{p,\alpha,\beta}+1)(jc_{q,\gamma,\delta}+1)[i,j,\alpha,\beta-1,\gamma,\delta-1];\\
T_{kl}\cdot[i,j,\alpha,\beta,\gamma,\delta] &= [i,j,\alpha,\beta+1,\gamma,\delta+1]\\
&\quad + c_{p,\alpha,\beta}[i+1,j,\alpha-1,\beta,\gamma,\delta+1]\\
&\quad + c_{q,\gamma,\delta}[i,j+1,\alpha,\beta+1,\gamma-1,\delta]\\
&\quad + c_{p,\alpha,\beta}c_{q,\gamma,\delta}[i+1,j+1,\alpha-1,\beta,\gamma-1,\delta]\\
&\quad - ij[i-1,j-1,\alpha,\beta+1,\gamma,\delta+1]\\
&\quad - i(jc_{q,\gamma,\delta}+1)[i-1,j,\alpha,\beta+1,\gamma-1,\delta]\\
&\quad - j(ic_{p,\alpha,\beta}+1)[i,j-1,\alpha-1,\beta,\gamma,\delta+1]\\
&\quad - (ic_{p,\alpha,\beta}+1)(jc_{q,\gamma,\delta}+1)[i,j,\alpha-1,\beta,\gamma-1,\delta].
\end{aligned}$$

Proof. Similar to Lemma 2.2. □

For $\nu \in \frac{1}{2}\mathbb{Z}$, let $\det^\nu$ be the ν-power of the determinant character on $\widetilde{U}(1,1)$. Let

$$\mathcal{N}_\nu = \text{Span}\ \{H_1 f - \nu f, H_2 f - \nu f, Ef, Ff \mid f \in \mathcal{P}\}.$$

Lemma 3.3. *For $\nu \in \frac{1}{2}\mathbb{Z}$, $\det^\nu \in \mathcal{R}(\mathfrak{u}(1,1), \tilde{U}(1)\times\tilde{U}(1), \omega)$ if and only if $\frac{p-q}{2} - \nu \in \mathbb{Z}$. Consider the basis of $\mathcal{P}$ as in* (3.5). *If $\frac{p-q}{2} - \nu \in \mathbb{Z}, \mathcal{P}/\mathcal{N}_\nu =$*

Span of (images of)

$$\left\{[0,0,\alpha,\beta,\gamma,\delta] \,\middle|\, \alpha-\gamma+\frac{p-q}{2}-\nu = \delta-\beta-\frac{p-q}{2}-\nu = 0\right\}.$$

Proof. From Lemma 3.2, we infer that

(a) Action of H_1 and $H_2 \Rightarrow$ $[i,j,\alpha,\beta,\gamma,\delta] \in \mathcal{N}_\nu$ if $i-j+\alpha-\gamma+\frac{p-q}{2}-\nu \neq 0$, or if $j-i+\delta-\beta-\frac{p-q}{2}-\nu \neq 0$;

(b) Action of $E \Rightarrow$ $[i,0,\alpha,\beta,\gamma,\delta] \in \mathcal{N}_\nu$ if $i>0$, and

$$[i,j,\alpha,\beta,\gamma,\delta] \equiv j(q+\gamma+\delta+j-1)[i-1,j-1,\alpha,\beta,\gamma,\delta] \mod \mathcal{N}_\nu;$$

(c) Action of $F \Rightarrow$ $[0,j,\alpha,\beta,\gamma,\delta] \in \mathcal{N}_\nu$ if $j>0$, and

$$[i,j,\alpha,\beta,\gamma,\delta] \equiv i(p+\alpha+\beta+i-1)[i-1,j-1,\alpha,\beta,\gamma,\delta] \mod \mathcal{N}_\nu.$$

Thus,

$$\begin{aligned}
&[i,j,\alpha,\beta,\gamma,\delta] \equiv c_1[i-j,0,\alpha,\beta,\gamma,\delta] \equiv 0 && \mod \mathcal{N}_\nu \quad \text{if } i>j;\\
&[i,j,\alpha,\beta,\gamma,\delta] \equiv c_2[0,j-i,\alpha,\beta,\gamma,\delta] \equiv 0 && \mod \mathcal{N}_\nu \quad \text{if } j>i;\\
&[i,j,\alpha,\beta,\gamma,\delta] \equiv c_3[0,0,\alpha,\beta,\gamma,\delta] && \mod \mathcal{N}_\nu \quad \text{if } i=j>0,
\end{aligned}$$

where c_1, c_2 and c_3 are non-zero constants. The result follows. □

Theorem 3.4. *Assume that* $\frac{p-q}{2}-\nu \in \mathbb{Z}$. *The theta lift of the representation* $\det^\nu$ *of* $\tilde{U}(1,1)$ *is the irreducible and unitarizable* $(\mathfrak{u}(p,q), U(p)\times U(q))$ *module*

$$H_{p,q,\nu} = \text{Span (\textit{of the images})}\Big\{[0,0,\alpha,\beta,\gamma,\delta] \mid \alpha-\gamma+\frac{p-q}{2}-\nu=0,\\ \delta-\beta-\frac{p-q}{2}-\nu=0\Big\}.$$

Remark. The representations are also known as ladder representations (see [**AFR**]), even though their K-spectrum has two parameters. They are restrictions of the ladder representation $L_{2p,2q}$ of $O(2p,2q)$:

$$L_{2p,2q}|_{U(p,q)} = \sum_{\nu\in\frac{1}{2}\mathbb{Z}} H_{p,q,\nu}$$

and have Gelfand-Kirillov dimensions $2p+2q-4$ (compare with $2p+2q-3$ of $L_{2p,2q}$).

Proof. We note the action of $\mathfrak{p} \subset u(p,q)_\mathbb{C}$ using Lemma 3.2:

$$\begin{aligned}
&S_{kl}\cdot[0,0,\alpha,\beta,\gamma,\delta]\\
&= [0,0,\alpha+1,\beta,\gamma+1,\delta] - [0,0,\alpha,\beta-1,\gamma,\delta-1] \mod \mathcal{N}_\nu;\\
&T_{kl}\cdot[0,0,\alpha,\beta,\gamma,\delta]\\
&= [0,0,\alpha,\beta+1,\gamma,\delta+1] - [0,0,\alpha-1,\beta,\gamma-1,\delta] \mod \mathcal{N}_\nu.
\end{aligned}$$

This shows that $H_{p,q,\nu}$ is irreducible as a $(\mathfrak{u}(p,q), \widetilde{U}(p)\times\widetilde{U}(q))$ module. Unitarity follows from the observation that they are restrictions of the unitarizable $(\mathfrak{o}(2p,2q), \widetilde{O}(2p)\times\widetilde{O}(2q))$ module $L_{2p,2q}$. □

Now we compute the Howe quotient corresponding to the trivial representation of $U(p,q)$ for the dual pair $(U(p,q),U(1,1))$. Assume that $p,q \geq 1$. If

$$\mathcal{P} = \mathbb{C}[z_1,\dots,z_p,\bar{z}_1,\dots,\bar{z}_p,w_1,\dots,w_q,\bar{w}_1,\dots,\bar{w}_q]$$

as before, let

$$\mathcal{N}'_{\mathbb{1}} = \operatorname{Span}\{Xf | f \in \mathcal{P}, X \in \mathfrak{u}(p,q)_{\mathbb{C}}\}.$$

Then by Proposition 2.3, the Howe quotient is $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. Recall that the action of $\mathfrak{u}(p,q)_{\mathbb{C}}$ and $\mathfrak{u}(1,1)_{\mathbb{C}}$ is given in (3.3).

Lemma 3.5. *Let $\bar{1}, Z$, and W be the elements of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ given by*

$$\begin{aligned} \bar{1} &= 1 + \mathcal{N}'_{\mathbb{1}}, \\ Z &= z_1\bar{z}_1 + \mathcal{N}'_{\mathbb{1}}, \\ W &= w_1\bar{w}_1 + \mathcal{N}'_{\mathbb{1}}. \end{aligned}$$

Then $B = \{\bar{1}, Z^k, W^l | k,l \in \mathbb{Z}_{\geq 1}\}$ is a basis of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$.

Proof. For $\lambda = (\lambda_1,\dots,\lambda_p,\bar{\lambda}_1,\dots,\bar{\lambda}_p)$ and $\mu = (\mu_1,\dots,\mu_q,\bar{\mu}_1,\dots,\bar{\mu}_q)$ with $\lambda_i,\bar{\lambda}_i,\mu_j,\bar{\mu}_j \in \mathbb{Z}_{\geq 0}$, let $z^\lambda w^\mu$ be the monomial $\prod_{i=1}^p z_i^{\lambda_i}\bar{z}_i^{\bar{\lambda}_i}\prod_{i=1}^q w_i^{\mu_i}\bar{w}_i^{\bar{\mu}_i}$. For $1 \leq i,j \leq p$, and arbitrary $z^\lambda w^\mu$, we have that

$$\left(z_i\tfrac{\partial}{\partial z_j} - \bar{z}_j\tfrac{\partial}{\partial \bar{z}_i}\right)\bar{z}_i z_j z^\lambda w^\mu = \left((\lambda_j+1)z_i\bar{z}_i - (\bar{\lambda}_i+1)z_j\bar{z}_j\right)z^\lambda w^\mu \ \in \mathcal{N}'_{\mathbb{1}},$$

so that

$$z_i\bar{z}_i z^\lambda w^\mu \equiv \tfrac{\bar{\lambda}_i+1}{\lambda_j+1} z_j\bar{z}_j z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{3.6a}$$

Similarly, we have for $1 \leq i,j \leq q$,

$$w_i\bar{w}_i z^\lambda w^\mu \equiv \tfrac{\bar{\mu}_i+1}{\mu_j+1} w_j\bar{w}_j z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{3.6b}$$

If $\bar{\lambda}_i = 0$, then

$$\left(z_i\tfrac{\partial}{\partial z_j} - \bar{z}_j\tfrac{\partial}{\partial \bar{z}_i}\right) z_j z^\lambda w^\mu = (\lambda_j+1)z_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}.$$

This implies that $z_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\bar{\lambda}_i = 0$. Similarly, $\bar{z}_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\lambda_i = 0$, $w_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\bar{\mu}_i = 0$, and $\bar{w}_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\mu_i = 0$. Using (3.6) with $i = j$, we see that if $\lambda_i \neq \bar{\lambda}_i$ for some i, then $z_i\bar{z}_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ (and similarly if $\mu_i \neq \bar{\mu}_i$), so that we have

$$\lambda_i \neq \bar{\lambda}_i \text{ for some } i \text{ or } \mu_i \neq \bar{\mu}_i \text{ for some } i \Rightarrow z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}. \tag{3.7}$$

For $1 \leq i \leq p$ and $1 \leq j \leq q$, applying the operator $z_i w_j - \frac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_j}$ to the monomial $\bar{z}_i\bar{w}_j z^\lambda w^\mu$ yields

$$z_i\bar{z}_i w_j\bar{w}_j z^\lambda w^\mu \equiv (\bar{\lambda}_i+1)(\bar{\mu}_j+1)z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{3.8}$$

Using (3.6), (3.7), and (3.8), we see that every monomial in $\mathcal{P}$ is either in $\mathcal{N}'_{\mathbb{1}}$ or in one of the cosets listed in B. So we have that B spans $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. To see that the elements of B are linearly independent, we notice that each is a weight vector for $\big(\mathfrak{u}(1)\oplus\mathfrak{u}(1)\big)_{\mathbb{C}} \subset \mathfrak{u}(1,1)_{\mathbb{C}}$, of the following weights:

$$\begin{aligned} \bar{1} &\quad \text{has weight} \quad (\tfrac{p-q}{2}, -\tfrac{p-q}{2}),\\ Z^k &\quad \text{has weight} \quad (\tfrac{p-q}{2}+k, -\tfrac{p-q}{2}-k),\\ W^k &\quad \text{has weight} \quad (\tfrac{p-q}{2}-k, -\tfrac{p-q}{2}+k). \end{aligned} \tag{3.9}$$

Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. □

Remark. The proof of Lemma 3.5 also shows that all $\widetilde{U}(1)\times\widetilde{U}(1)$-types in $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ have multiplicity one.

Theorem 3.6. *Suppose $p, q \geq 1$. The trivial $(\mathfrak{u}(p,q), U(p)\times U(q))$-module $\mathbb{1}$ belongs to $\mathcal{R}(\mathfrak{u}(p,q), U(p)\times U(q), \omega)$. The Howe quotient $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ has two irreducible $(\mathfrak{u}(1,1), \tilde{U}(1)\times\tilde{U}(1))$-submodules V_1 and V_2 spanned by $\{Z^k | k \geq q\}$ and $\{W^k | k \geq p\}$ respectively, which are discrete series representations with minimal $\widetilde{U}(1)\times\widetilde{U}(1)$-types $(\frac{p+q}{2}, -\frac{p+q}{2})$ and $(-\frac{p+q}{2}, \frac{p+q}{2})$ respectively. The theta-lift of $\mathbb{1}$ is the irreducible quotient of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ by $V_1\oplus V_2$, the $(\mathfrak{u}(1,1), \widetilde{U}(1)\times\widetilde{U}(1))$-module of dimension $p+q-1$, with $\widetilde{U}(1)\times\widetilde{U}(1)$-types $\{(k+\frac{p-q}{2}, -k-\frac{p-q}{2}) | -q+1 \leq k \leq p-1\}$. The theta-lift of $\mathbb{1}$ is unitarizable only in the case when $p=q=1$.*

Remark. The situation in this case is very much controlled by the situation for the dual pair $(O(2p,2q), SL(2,\mathbb{R}))$ (see Theorem 2.7(ii)). Again, Lee and Zhu [**LZ2**] has treated these representations of $U(n,n)$ for the dual pairs $(U(p,q), U(n,n))$ and have explicit information on the structure of the corresponding Howe quotients.

Proof. Using the formulas (3.3), (3.6), and (3.8), we compute the action of $\mathfrak{p}' = \mathrm{Span}\{E, F\}$ on the weight vectors in B:

$$\begin{aligned} E\cdot\bar{1} &= pZ;\\ E\cdot Z^k &= \frac{k+p}{k+1}Z^{k+1};\\ E\cdot W^k &= k(p-k)W^{k-1};\\ F\cdot\bar{1} &= qW;\\ F\cdot Z^k &= k(q-k)Z^{k-1};\\ F\cdot W^k &= \frac{k+q}{k+1}W^{k+1}. \end{aligned}$$

Notice that E annihilates W^p, and takes all weight vectors of weight $(k+\frac{p-q}{2}, -k-\frac{p-q}{2})$ to weight vectors of weight $(k+1+\frac{p-q}{2}, -k-1-\frac{p-q}{2})$. Similarly, F annihilates Z^q and takes all weight vectors of weight $(k+\frac{p-q}{2}, -k-\frac{p-q}{2})$ to weight vectors of weight $(k-1+\frac{p-q}{2}, -k+1-\frac{p-q}{2})$. The result follows since we know that only the trivial finite-dimensional module is unitarizable. ☐

4. The Dual Pairs $(Sp(p,q), O^*(4))$.

Consider the dual pair $(Sp(p), O^*(4))$ acting on the $\widetilde{U}(4p)$-finite vectors of the associated Fock space $\mathbb{C}[z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p}]$ as follows:

(4.1)

(a) Action of $\mathfrak{sp}(p)_{\mathbb{C}}$:

(i) $z_i \dfrac{\partial}{\partial z_j} - \bar{z}_j \dfrac{\partial}{\partial \bar{z}_i} - z_{p+j} \dfrac{\partial}{\partial z_{p+i}} + \bar{z}_{p+i} \dfrac{\partial}{\partial \bar{z}_{p+j}}, \quad 1 \le i, j \le p;$

(ii) $z_i \dfrac{\partial}{\partial z_{p+j}} - \bar{z}_{p+j} \dfrac{\partial}{\partial \bar{z}_i} + z_j \dfrac{\partial}{\partial z_{p+i}} - \bar{z}_{p+i} \dfrac{\partial}{\partial \bar{z}_j}, \quad 1 \le i \le j \le p;$

(iii) $z_{p+i} \dfrac{\partial}{\partial z_j} - \bar{z}_j \dfrac{\partial}{\partial \bar{z}_{p+i}} + z_{p+j} \dfrac{\partial}{\partial z_i} - \bar{z}_i \dfrac{\partial}{\partial \bar{z}_{p+j}}, \quad 1 \le i \le j \le p.$

(b) Action of $\mathfrak{o}^*(4)_{\mathbb{C}} = \mathrm{Span}\{E_{11}^z + p, E_{22}^z + p, E_{12}^z, E_{21}^z, r_1^2, \Delta_1\}$:

$$E_{11}^z = \sum_{i=1}^{2p} z_i \frac{\partial}{\partial z_i}, \quad E_{22}^z = \sum_{i=1}^{2p} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \quad E_{12}^z = \sum_{i=1}^{p} \left(z_{p+i} \frac{\partial}{\partial \bar{z}_i} - z_i \frac{\partial}{\partial \bar{z}_{p+i}} \right),$$

$$E_{21}^z = \sum_{i=1}^{p} \left(\bar{z}_i \frac{\partial}{\partial z_{p+i}} - \bar{z}_{p+i} \frac{\partial}{\partial z_i} \right), \quad r_1^2 = \sum_{i=1}^{2p} z_i \bar{z}_i, \quad \Delta_1 = \sum_{i=1}^{2p} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.$$

Observe that

$$\begin{aligned} \mathfrak{o}^*(4)_{\mathbb{C}} &\simeq \mathrm{Span}\{E_{11}^z - E_{22}^z, E_{12}^z, E_{21}^z\} \oplus \mathrm{Span}\{E_{11}^z + E_{22}^z + 2p, r_1^2, \Delta_1\} \\ &\simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{sl}(2)_{\mathbb{C}} \Longrightarrow O^*(4) \simeq (SU(2) \times SL(2,\mathbb{R}))/\{\pm I\}. \end{aligned}$$

Define the spherical harmonics as in the last section:

$$\mathcal{H}(\mathbb{C}^{2p}) = \{ f \in \mathbb{C}[z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p}] \mid \Delta_1 f = 0 \}.$$

We have the following decomposition (see [**HT2**]):

$$\mathcal{H}(\mathbb{C}^{2p})\big|_{Sp(p)\times SU(2)} = \sum_{\xi_1,\xi_2 \ge 0} \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1},$$

where $SU(2)$ is the group with (complexified) Lie algebra $\{E_{11}^z - E_{22}^z, E_{12}^z, E_{21}^z\}$, V_1^m is the irreducible unitary representation of $SU(2)$ of dimension $m+1$, and $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)}$ is the $Sp(p)$ module with highest weight $(\xi_1+\xi_2, \xi_2, 0, ..., 0)$ with respect to the Cartan subalgebra spanned by

$$\left\{ z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - z_{p+i} \frac{\partial}{\partial z_{p+i}} + \bar{z}_{p+i} \frac{\partial}{\partial \bar{z}_{p+i}} \,\middle|\, i = 1, \dots, p \right\}.$$

The joint $Sp(p) \times SU(2)$ highest weight vector of $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1}$ is given by

$$\gamma_{(\xi_1,\xi_2)} = z_1^{\xi_1} \begin{vmatrix} z_1 & \bar{z}_{p+1} \\ z_2 & \bar{z}_{p+2} \end{vmatrix}^{\xi_2}. \tag{4.2}$$

More precisely, there are $\xi_1 + 1$ copies of the $Sp(p)$ representations in $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1}$, and the $Sp(p)$ highest weight vectors are

$$\begin{aligned} u_{(p,\xi_1,\xi_2,j)} &= \bar{z}_{p+1}^{j} v_{(p,\xi_1+\xi_2-j,\xi_2)} \\ &= \bar{z}_{p+1}^{j} z_1^{\xi_1-j} \begin{vmatrix} z_1 & \bar{z}_{p+1} \\ z_2 & \bar{z}_{p+2} \end{vmatrix}^{\xi_2}, \qquad j = 0, 1, \dots, \xi_1. \end{aligned}$$

We shall denote the $Sp(p)$ module with highest weight vector $u_{(p,\xi_1,\xi_2,j)}$ by $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2),j}$. Thus

$$\mathcal{H}(\mathbb{C}^{2p})\big|_{Sp(p)} = \sum_{\xi_1 \geq 0, \xi_2 \geq 0, j=0,\dots,\xi_1} \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2),j}.$$

We also note that $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2),j} \subset \mathcal{H}^{(2p)}_{\xi_1+\xi_2-j,\xi_2+j}$. In particular, we can extract the decomposition of $\mathcal{H}^{(2p)}_{\alpha,\beta}$ into $Sp(p)$ modules:

$$\begin{aligned} \mathcal{H}^{(2p)}_{\alpha,\beta}\Big|_{Sp(p)} &= \mathcal{K}^{(p)}_{(\alpha+\beta,0),\beta} \oplus \mathcal{K}^{(p)}_{(\alpha+\beta-1,1),\beta-1} \oplus \mathcal{K}^{(p)}_{(\alpha+\beta-2,2),\beta-2} \\ &\qquad \oplus \dots \oplus \mathcal{K}^{(p)}_{(\max(\alpha,\beta),\min(\alpha,\beta)),\beta-\min(\alpha,\beta)}. \end{aligned}$$

The duality correspondence is as follows:

$$\mathbb{C}[z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p}]\big|_{Sp(p)\times O^*(4)} = \sum_{\xi_1,\xi_2=0}^{\infty} \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1} \otimes V_{p+\xi_1+2\xi_2},$$

where $V_{p+\xi_1+2\xi_2}$ is the $SL(2,\mathbb{R})$ lowest weight module of lowest weight $p + \xi_1 + 2\xi_2$ spanned by $\{(r_1^2)^i \gamma_{(\xi_1,\xi_2)} \mid i = 0, 1, \dots\}$. The duality correspondence enables us to write

$$\mathbb{C}[z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p}]\big|_{Sp(p)\times SU(2)} = \sum_{i,\xi_1,\xi_2=0}^{\infty} (r_1^2)^i \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1}, \tag{4.3}$$

where $(r_1^2)^i \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1}$ are $Sp(p) \times SU(2)$ modules isomorphic to $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes V_1^{\xi_1}$.

Likewise, the dual pair $(Sp(q), O^*(4))$ acting on $\mathbb{C}[w_1, ..., w_{2q}, \bar{w}_1, ..., w_{2q}]$ gives rise to the following decomposition as $Sp(q) \times SU(2)$ modules:
(4.4)

$$\mathbb{C}[w_1, \dots, w_{2q}, \bar{w}_1, \dots, \bar{w}_{2q}]|_{Sp(q)\times SU(2)} = \sum_{i,\eta_1,\eta_2=0}^{\infty} (r_2^2)^j \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)} \otimes V_1^{\eta_1},$$

where $(r_2^2)^i \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)} \otimes V_1^{\eta_1}$ are $Sp(q) \times SU(2)$ modules isomorphic to $\mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)} \otimes V_1^{\eta_1}$, which are analogously defined as in the $(Sp(p), O^*(4))$ case.

Consider $\mathcal{P} = \mathbb{C}[z_1, \dots, z_{2p}, \bar{z}_1, \dots, \bar{z}_{2p}, w_1, \dots, w_{2q}, \bar{w}_1, \dots, \bar{w}_{2q}]$. This is the space of $\widetilde{U}(4p+4q)$-finite vectors of the associated Fock model for the dual pair $(Sp(p,q), O^*(4))$, and the actions of the complexified Lie algebras of $Sp(p,q)$ and $O^*(4)$ can be described as follows:

Lemma 4.1.

(a) *Action of* $\mathfrak{sp}(p,q)_{\mathbb{C}} = \mathfrak{sp}(p)_{\mathbb{C}} \oplus \mathfrak{sp}(q)_{\mathbb{C}} \oplus \mathfrak{p}$:

(i) *Action of* $\mathfrak{sp}(p)_{\mathbb{C}}$*: as in* (4.1);

(ii) *Action of* $\mathfrak{sp}(q)_{\mathbb{C}}$*: similar to* (4.1);

(iii) *Action of* $\mathfrak{p}$*:*

$$P_{ij} = z_i w_j - \frac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_j} + \bar{z}_{p+i}\bar{w}_{q+j} - \frac{\partial^2}{\partial z_{p+i}\partial w_{q+j}}, \quad 1 \le i \le p,\ 1 \le j \le q;$$

$$Q_{ij} = z_i w_{q+j} - \frac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_{q+j}} - \bar{z}_{p+i}\bar{w}_j + \frac{\partial^2}{\partial z_{p+i}\partial w_j}, \quad 1 \le i \le p,\ 1 \le j \le q;$$

$$R_{ij} = z_{p+i} w_j - \frac{\partial^2}{\partial \bar{z}_{p+i} \partial \bar{w}_j} - \bar{z}_i\bar{w}_{q+j} + \frac{\partial^2}{\partial z_i\partial w_{q+j}}, \quad 1 \le i \le p,\ 1 \le j \le q;$$

$$S_{ij} = z_{p+i} w_{q+j} - \frac{\partial^2}{\partial \bar{z}_{p+i} \partial \bar{w}_{q+j}} + \bar{z}_i\bar{w}_j - \frac{\partial^2}{\partial z_i\partial w_j}, \quad 1 \le i \le p,\ 1 \le j \le q.$$

(b) *Action of* $\mathfrak{o}^*(4)_{\mathbb{C}} = \mathrm{Span}\{E_{11}+p-q, E_{22}+p-q, E_{12}, E_{21}, E, F\}$:

$$E_{11} = E^z_{11} - E^w_{11}, \quad E_{22} = E^z_{22} - E^w_{22}, \quad E_{12} = E^z_{12} - E^w_{21},$$
$$E_{21} = E^z_{21} - E^w_{12}, \quad E = r_1^2 - \Delta_2, \quad F = r_2^2 - \Delta_1.$$

Proof. Omitted. □

We note that $\widetilde{Sp}(p,q)$ and $\widetilde{O}^*(4)$ are both split extensions. Because of the decompositions (4.3) and (4.4), we have the following decomposition of $\mathcal{P}$ as $Sp(p) \times Sp(q) \times SU(2)$ modules:

$$\text{(4.5)} \qquad \mathcal{P}|_{Sp(p)\times Sp(q)\times SU(2)} = \sum (r_1^2)^i (r_2^2)^j \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)} \otimes V_1^{\mu},$$

where the sum is over the 7-tuples $(i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu) \in \mathbb{Z}_{\geq 0}$ such that

$$|\xi_1 - \eta_1| \leq \mu \leq \xi_1 + \eta_1,$$
$$\mu \equiv \xi_1 + \eta_1 \mod 2.$$

This comes about by a direct application of the Clebsh-Gordan formula for the decomposition of a tensor product of two $SU(2)$ modules:

$$V_1^{\xi_1} \otimes V_1^{\eta_1} = V_1^{\xi_1+\eta_1} \oplus V_1^{\xi_1+\eta_1-2} \oplus V_1^{\xi_1+\eta_1-4} \oplus \ldots \oplus V_1^{|\xi_1-\eta_1|}. \tag{4.6}$$

Observe that $Sp(q)$ acts in a contragredient fashion, so to obtain a set of $Sp(p) \times Sp(q) \times SU(2)$ highest weight vectors in $\mathcal{P}$, we need a little adjustment. Recall that

$$\gamma_{(\xi_1,\xi_2)} = z_1^{\xi_1} \begin{vmatrix} z_1 & \bar{z}_{p+1} \\ z_2 & \bar{z}_{p+2} \end{vmatrix}^{\xi_2}$$

is a joint $Sp(p) \times SU(2)$ highest weight vector. In a similar way,

$$\theta_{(\eta_1,\eta_2)} = \bar{w}_{q+1}^{\eta_1} \begin{vmatrix} w_1 & \bar{w}_{q+1} \\ w_2 & \bar{w}_{q+2} \end{vmatrix}^{\eta_2} \tag{4.7}$$

is a joint $Sp(q) \times SU(2)$ highest weight vector (relative to another choice of positive system for $SU(2)$) satisfying

$$\begin{aligned} &- E_{11}^w \theta_{(\eta_1,\eta_2)} = -\eta_2 \theta_{(\eta_1,\eta_2)}, \\ &- E_{22}^w \theta_{(\eta_1,\eta_2)} = -(\eta_1 + \eta_2) \theta_{(\eta_1,\eta_2)}, \\ &- E_{21}^w \theta_{(\eta_1,\eta_2)} = 0. \end{aligned}$$

We will take as a “basis” for $\mathcal{P}$ elements of the form (see (4.2) and (4.7) for the definitions of γ_{ξ_1,ξ_2} and θ_{η_1,η_2})

$$[i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (r_1^2)^i (r_2^2)^j \gamma_{(\xi_1-\nu,\xi_2)} \theta_{(\eta_1-\nu,\eta_2)} (z_1 w_1 + \bar{z}_{p+1} \bar{w}_{q+1})^\nu, \tag{4.8}$$

where $\mu = \xi_1 + \eta_1 - 2\nu$ and $|\xi_1 - \eta_1| \leq \mu \leq \xi_1 + \eta_1$.

Note that if we set $i = j = 0$, then

$$\begin{aligned} E_{11}[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu] &= (\xi_1 + \xi_2 - \eta_2 - \nu)[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu], \\ E_{22}[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu] &= (\xi_2 - \eta_1 - \eta_2 + \nu)[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu], \\ E_{12}[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu] &= 0. \end{aligned}$$

In other words, $[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu]$ is the set of joint $Sp(p) \times Sp(q) \times SU(2)$ pluriharmonics (up to multiples, of course).

Lemma 4.2. *The actions of $\mathfrak{sl}(2)_{\mathbb{C}} \subset \mathfrak{o}^*(4)$ on the basis in* (4.8) *are as follows:*

(a) $E_{11} \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (2i + \xi_1 + \xi_2 - \eta_2)\, [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$;

(b) $E_{22} \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (-2j + \xi_2 - \eta_1 - \eta_2)\, [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$;

(c) $E \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = [i + 1, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$

$$- j(2q + \eta_1 + 2\eta_2 + j - 1)[i, j - 1, \xi_1, \xi_2, \eta_1, \eta_2, \mu];$$

(d) $F \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = [i, j + 1, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$
$$- i(2p + \xi_1 + 2\xi_2 + i - 1)[i - 1, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu].$$

Proof. Similar to Lemma 2.2. □

Proposition 4.3. *Consider the basis of* $\mathcal{P}$ *as in* (4.8). *Then* $V_1^\mu \otimes \mathbb{1} \in \mathcal{R}(\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\}, \omega)$ *if and only if* μ *is even. If* μ *is even, the Howe quotient corresponding to* $V_1^\mu \otimes \mathbb{1}$ *(i.e., the trivial representation of* $SL(2, \mathbb{R})$ *twisted by a unitary representation of* $SU(2)$*) of* $O^*(4)$ *is*

$$\mathcal{P}/\mathcal{N}_\mu = \text{Span of (images of) } \{[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] \mid |\xi_1 - \eta_1| \le \mu \le \xi_1 + \eta_1, \\ \mu \equiv \xi_1 + \eta_1 \mod 2,\ \xi_1 + 2\xi_2 + 2p - 2q = \eta_1 + 2\eta_2\}.$$

In particular,

$$\mathcal{P}/\mathcal{N}_\mu \simeq L_\mu \otimes (V_1^\mu \otimes \mathbb{1})$$

is an irreducible $(\mathfrak{sp}(p, q), Sp(p) \times Sp(q)) \times (\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\})$ *module, so the theta lift of* $V_1^\mu \otimes \mathbb{1}$ *is the irreducible and unitarizable representation* L_μ.

Remark. This is not surprising; in fact, the restriction of the ladder representation $L_{4p,4q}$ of $O(4p, 4q)$ to $Sp(p, q)$ decomposes as follows:

$$L_{4p,4q}|_{Sp(p,q) \times SU(2)} = \sum_{\mu=0, \mu \text{ even}}^{\infty} L_\mu \otimes V_1^\mu.$$

Here $SU(2)$ is not embedded in $O(4p, 4q)$. It arises from the exponentiated action of the Lie algebra $\mathfrak{su}(2) = \text{Span}\,\{E_{11} - E_{22}, E_{12}, E_{21}\}$ (see Lemma 4.1). We shall call the L_μ ladder representations. They have Gelfand-Kirillov dimensions $4p + 4q - 6$ (compare with $4p + 4q - 3$ of $L_{4p,4q}$).

Proof. Let $\mathcal{P}/\mathcal{N}_\mu$ be the Howe quotient corresponding to the representation $V_1^\mu \otimes \mathbb{1}$. As an $SU(2)$ module, $\mathcal{P}$ is completely reducible, i.e., $\mathcal{P} = \sum_{\mu \in \mathbb{Z}} \mathcal{P}_\mu$ where $\mathcal{P}_\mu$ denotes the V_1^μ-isotypic component of $\mathcal{P}$. Thus as an $SU(2)$ module,

$$\mathcal{P}/\mathcal{N}_\mu \simeq \mathcal{P}_\mu/(\mathcal{N}_\mu \cap \mathcal{P}_\mu).$$

Let $\mathcal{N}_{\mathbb{1}} = \{Xf \mid X \in \mathfrak{sl}(2), f \in \mathcal{P}\}$. Since $\mathfrak{sl}(2)$ acts trivially, we have $\mathcal{N}_{\mathbb{1}} \cap \mathcal{P}_\mu \subset \mathcal{N}_\mu \cap \mathcal{P}_\mu$. But $\mathcal{N}_\mu = \cap \mathcal{N}$ where $\mathcal{N} \subset \mathcal{P}$ is such that $\mathcal{P}/\mathcal{N} \simeq \mathcal{P}_\mu/(\mathcal{N} \cap \mathcal{P}_\mu) \simeq V_1^\mu \otimes \mathbb{1}$ as $SU(2)$ modules. Thus $\mathcal{N}_\mu \cap \mathcal{P}_\mu = \cap(\mathcal{N} \cap P_\mu)$ is the smallest subspace in $\mathcal{P}_\mu$ such that $\mathcal{P}_\mu/(\mathcal{N}_\mu \cap \mathcal{P}_\mu) \simeq V_1^\mu \otimes \mathbb{1}$. Hence $\mathcal{N}_{\mathbb{1}} \cap \mathcal{P}_\mu = \mathcal{N}_\mu \cap \mathcal{P}_\mu$ and thus

$$\mathcal{P}/\mathcal{N}_\mu \simeq \mathcal{P}_\mu/(\mathcal{N}_{\mathbb{1}} \cap \mathcal{P}_\mu)$$

as $(\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\})$ modules.

From the above and Lemma 4.2, the $(\mathfrak{sp}(p,q), Sp(p)\times Sp(q))$ module L_μ has a multiplicity free $Sp(p)\times Sp(q)$ spectrum:

$$L_\mu|_{Sp(p)\times Sp(q)} = \sum \mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)},$$

where the sum runs through non-negative integer tuples $(\xi_1,\xi_2,\eta_1,\eta_2)$ satisfying

$$\begin{aligned}
&|\xi_1-\eta_1| \le \mu \le \xi_1+\eta_1,\\
&\mu \equiv \xi_1+\eta_1 \mod 2,\\
&\xi_1+2\xi_2+2p-2q = \eta_1+2\eta_2.
\end{aligned}$$

The above relations show that μ must be even.

There are several ways to show irreducibility. In the spirit of this paper, we note the transitions from $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)}$ to neighbouring $Sp(p)\times Sp(q)$ types as follows. Let the operators P_{11}, P_{12} and P_{21} from $\mathfrak{p}$ be given as in Lemma 4.1 and

$$\begin{aligned}
X_{21} &= z_2\frac{\partial}{\partial z_1} - \bar{z}_1\frac{\partial}{\partial \bar{z}_2} - z_{p+1}\frac{\partial}{\partial z_{p+2}} + \bar{z}_{p+2}\frac{\partial}{\partial \bar{z}_{p+1}},\\
Y_{21} &= w_2\frac{\partial}{\partial w_1} - \bar{w}_1\frac{\partial}{\partial \bar{w}_2} - w_{p+1}\frac{\partial}{\partial w_{p+2}} + \bar{w}_{p+2}\frac{\partial}{\partial \bar{w}_{p+1}}
\end{aligned}$$

be operators coming from $\mathfrak{sp}(p)_{\mathbb{C}}$ and $\mathfrak{sp}(q)_{\mathbb{C}}$ (see (4.1)). Then the formulae

(a) $P_{11}[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu] = [0,0,\xi_1+1,\xi_2,\eta_1+1,\eta_2,\mu]$,

(b) $(P_{11}X_{21}-\xi_1P_{21})[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu]$
$$= \left(\frac{\eta_1-\xi_1-\mu}{2}\right)[0,0,\xi_1-1,\xi_2+1,\eta_1+1,\eta_2,\mu],$$

(c) $(P_{11}Y_{21}-\eta_1P_{12})[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu]$
$$= \left(\frac{\eta_1+\mu-\xi_1}{2}\right)[0,0,\xi_1+1,\xi_2,\eta_1-1,\eta_2+1,\mu],$$

(d) $(P_{11}X_{21}Y_{21}-\eta_1P_{12}Y_{21}-\xi_1P_{12}X_{21}+\xi_1\eta_1P_{12}P_{21})[0,0,\xi_1,\xi_2,\eta_1,\eta_2,\mu]$
$$\begin{aligned}
&= \left(\frac{\xi_1+\eta_1-\mu}{2} - \frac{(\eta_1-\xi_1-\mu)(\xi_1-\eta_1-\mu)}{4}\right)\\
&\quad \cdot [0,0,\xi_1-1,\xi_2+1,\eta_1-1,\eta_2+1,\mu],
\end{aligned}$$

describes the transitions (of $Sp(p)\times Sp(q)$-types) $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2)}$ to $\mathcal{K}^{(p)}_{(\xi_1+\xi_2+1,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2+1,\eta_2)}$, to $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2+1)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2+1,\eta_2)}$, to $\mathcal{K}^{(p)}_{(\xi_1+\xi_2+1,\xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2+1)}$, and to $\mathcal{K}^{(p)}_{(\xi_1+\xi_2,\xi_2+1)} \otimes \mathcal{K}^{(q)}_{(\eta_1+\eta_2,\eta_2+1)}$ respectively. Noting that the lowest joint harmonic (see [**Ho2**]) has trivial $Sp(p)\times Sp(q)$ type, these transitions are enough to show that the $(\mathfrak{sp}(p,q), Sp(p)\times Sp(q))$ module L_μ

is irreducible. Unitarity follows from the unitarizability of the $(\mathfrak{o}(4p,4q), \widetilde{O}(4p)\times\widetilde{O}(4q))$ module $L_{4p,4q}$. $\square$

Now we compute the Howe quotient corresponding to the trivial representation of $Sp(p,q)$ for the dual pair $(Sp(p,q), O^*(4))$. Assume that $p,q \geq 1$. If

$$\mathcal{P} = \mathbb{C}[z_1,\ldots,z_{2p},\bar{z}_1,\ldots,\bar{z}_{2p},w_1,\ldots,w_{2q},\bar{w}_1,\ldots,\bar{w}_{2q}]$$

as before, let

$$\mathcal{N}'_{\mathbb{1}} = \mathrm{Span}\{Xf | f\in\mathcal{P}, X\in\mathfrak{sp}(p,q)_{\mathbb{C}}\}.$$

Then by Proposition 2.3, the Howe quotient is $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. Recall that the action of $\mathfrak{sp}(p,q)_{\mathbb{C}}$ and $\mathfrak{o}^*(4)_{\mathbb{C}} \cong (\mathfrak{su}(2)\oplus\mathfrak{sl}(2))_{\mathbb{C}}$ is given in Lemma 4.1.

Lemma 4.4. *Let $\bar{1}, Z$, and W be the elements of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ given by*

$$\begin{aligned}\bar{1} &= 1+\mathcal{N}'_{\mathbb{1}},\\ Z &= z_1\bar{z}_1+\mathcal{N}'_{\mathbb{1}},\\ W &= w_1\bar{w}_1+\mathcal{N}'_{\mathbb{1}}.\end{aligned}$$

Then $B = \{\bar{1}, Z^k, W^l | k,l\in\mathbb{Z}_{\geq 1}\}$ is a basis of $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$.

Proof. For $\lambda = (\lambda_1,\ldots,\lambda_{2p},\bar{\lambda}_1,\ldots,\bar{\lambda}_{2p})$ and $\mu = (\mu_1,\ldots,\mu_{2q},\bar{\mu}_1,\ldots,\bar{\mu}_{2q})$ with $\lambda_i,\bar{\lambda}_i,\mu_j,\bar{\mu}_j\in\mathbb{Z}_{\geq 0}$, let $z^\lambda w^\mu$ be the monomial $\prod_{i=1}^{2p} z_i^{\lambda_i}\bar{z}_i^{\bar{\lambda}_i}\prod_{i=1}^{2q} w_i^{\mu_i}\bar{w}_i^{\bar{\mu}_i}$. For $1\leq i\leq p$, and arbitrary $z^\lambda w^\mu$, and using (4.1)(a)(ii) with $i=j$, we have that

$$\begin{aligned}&\left(z_i\tfrac{\partial}{\partial z_{p+i}} - \bar{z}_{p+i}\tfrac{\partial}{\partial\bar{z}_i}\right)\bar{z}_i z_{p+i} z^\lambda w^\mu\\ &= \left((\lambda_{p+i}+1)z_i\bar{z}_i - (\bar{\lambda}_i+1)z_{p+i}\bar{z}_{p+i}\right) z^\lambda w^\mu \ \in \mathcal{N}'_{\mathbb{1}},\end{aligned}$$

so that

$$z_i\bar{z}_i z^\lambda w^\mu \equiv \tfrac{\bar{\lambda}_i+1}{\lambda_{p+i}+1} z_{p+i}\bar{z}_{p+i}z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.9 a}$$

Similarly, we have for $1\leq i\leq q$,

$$w_i\bar{w}_i z^\lambda w^\mu \equiv \tfrac{\bar{\mu}_i+1}{\mu_{q+i}+1} w_{q+i}\bar{w}_{q+i}z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.9 b}$$

If $\lambda_i = 0$, then using (4.1)(a)(iii) with $i=j$ we get

$$\left(z_{p+i}\tfrac{\partial}{\partial z_i} - \bar{z}_i\tfrac{\partial}{\partial\bar{z}_{p+i}}\right)\bar{z}_{p+i}z^\lambda w^\mu = -(\bar{\lambda}_{p+i}+1)\bar{z}_i z^\lambda w^\mu \ \in \mathcal{N}'_{\mathbb{1}}.$$

This implies that $\bar{z}_i z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$ whenever $\lambda_i = 0$. Analogous statements hold for the cases $\bar{\lambda}_i = 0$, $\lambda_{p+i}=0$, $\bar{\lambda}_{p+i}=0$, and for $1\leq i\leq 2q$, $\mu_i=0$, and $\bar{\mu}_i = 0$. Applying (4.9) repeatedly if necessary yields

$$\left.\begin{array}{c}\lambda_i\neq\bar{\lambda}_i \text{ for some } i\leq 2p\\ \text{or}\\ \mu_i\neq\bar{\mu}_i \text{ for some } i\leq 2q\end{array}\right\} \Rightarrow z^\lambda w^\mu\in\mathcal{N}'_{\mathbb{1}}. \tag{4.10}$$

Now suppose $z^\lambda w^\mu$ satisfies

$$\lambda_i = \bar{\lambda}_i \quad \text{for} \quad 1 \le i \le 2p \quad \text{and} \quad \mu_i = \bar{\mu}_i \quad \text{for} \quad 1 \le i \le 2q. \tag{4.11}$$

Let $1 \le i, j \le p$. Then

$$\begin{aligned}&\left(z_i \tfrac{\partial}{\partial z_j} - \bar{z}_j \tfrac{\partial}{\partial \bar{z}_i} - z_{p+j} \tfrac{\partial}{\partial z_{p+i}} + \bar{z}_{p+i} \tfrac{\partial}{\partial \bar{z}_{p+j}}\right) \bar{z}_i z_j z^\lambda w^\mu \\ &= (\lambda_j + 1) z_i \bar{z}_i z^\lambda w^\mu - (\bar{\lambda}_i + 1) z_j \bar{z}_j z^\lambda w^\mu - M_1 + M_2,\end{aligned}$$

where M_1 and M_2 are in $\mathcal{N}'_{\mathbb{1}}$ by (4.10), so we have that

$$z_i \bar{z}_i z^\lambda w^\mu \equiv \tfrac{\bar{\lambda}_i + 1}{\lambda_j + 1} z_j \bar{z}_j z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.12 a}$$

Similarly,

$$z_{p+i} \bar{z}_{p+i} z^\lambda w^\mu \equiv \tfrac{\bar{\lambda}_{p+i} + 1}{\lambda_{p+j} + 1} z_{p+j} \bar{z}_{p+j} z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}, \tag{4.12 b}$$

and for $1 \le i, j \le q$,

$$w_i \bar{w}_i z^\lambda w^\mu \equiv \tfrac{\bar{\mu}_i + 1}{\mu_j + 1} w_j \bar{w}_j z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}, \tag{4.12 c}$$

and

$$w_{q+i} \bar{w}_{q+i} z^\lambda w^\mu \equiv \tfrac{\bar{\mu}_{q+i} + 1}{\mu_{q+j} + 1} w_{q+j} \bar{w}_{q+j} z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.12 d}$$

Now suppose again that $z^\lambda w^\mu$ satisfies (4.11), and that $1 \le i \le p$ and $1 \le j \le q$. Then

$$\begin{aligned}&\left(z_i w_j - \tfrac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_j} + \bar{z}_{p+i} \bar{w}_{q+j} - \tfrac{\partial^2}{\partial z_{p+i} \partial w_{q+j}}\right) \bar{z}_i \bar{w}_j z^\lambda w^\mu \\ &= z_i \bar{z}_i w_j \bar{w}_j z^\lambda w^\mu - (\lambda_i + 1)(\mu_j + 1) z^\lambda w^\mu + M_3 - M_4,\end{aligned}$$

where M_3 and M_4 are in $\mathcal{N}'_{\mathbb{1}}$ by (4.10). Consequently,

$$z_i \bar{z}_i w_j \bar{w}_j z^\lambda w^\mu \equiv (\lambda_i + 1)(\mu_j + 1) z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.13a}$$

Similarly,

$$z_{p+i} \bar{z}_{p+i} w_{q+j} \bar{w}_{q+j} z^\lambda w^\mu \equiv (\lambda_{p+i} + 1)(\mu_{q+j} + 1) z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{1}}. \tag{4.13b}$$

Notice that (4.12) and (4.13) also hold if $z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{1}}$.

Using (4.9), (4.10), (4.12), and (4.13), we see that every monomial in $\mathcal{P}$ is either in $\mathcal{N}'_{\mathbb{1}}$ or in one of the cosets listed in B. So we have that B spans $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. To see that the elements of B are linearly independent, we notice that each is a weight vector for $\mathfrak{u}(1)_{\mathbb{C}} \subset \mathfrak{sl}(2)_{\mathbb{C}}$, of the following weights:

$$\begin{aligned} \bar{1} &\quad \text{has weight} \quad 2(p-q), \\ Z^k &\quad \text{has weight} \quad 2(p-q) + 2k, \\ W^k &\quad \text{has weight} \quad 2(p-q) - 2k. \end{aligned} \tag{4.14}$$

Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. $\square$

Remark. The proof of Lemma 3.5 also shows that all $U(1)$-types in $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ have multiplicity one.

Theorem 4.5. *Suppose $p, q \geq 1$. The trivial $(\mathfrak{sp}(p,q), Sp(p) \times Sp(q))$-module $\mathbb{1}$ belongs to $\mathcal{R}(\mathfrak{sp}(p,q), Sp(p) \times Sp(q), \omega)$. Since $O^*(4)$ is a quotient of $SU(2) \times SL(2,\mathbb{R})$, we may regard the Howe quotient $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$ as an $(\mathfrak{su}(2) \oplus \mathfrak{sl}(2), SU(2) \times U(1))$-module. This module is of the form $\mathbb{1} \otimes V$. The $(\mathfrak{sl}(2), U(1))$-module V has two irreducible submodules V_1 and V_2 spanned by $\{Z^k | k \geq 2q\}$ and $\{W^k | k \geq 2p\}$ respectively, which are discrete series representations with minimal $U(1)$-types $2p$ and $-2q$ respectively. The quotient of V by $V_1 \oplus V_2$ is irreducible and of dimension $2(p+q)-1$. If $\sigma_{p,q}$ is the unique irreducible $(\mathfrak{sl}(2), U(1))$-module of dimension $2(p+q)-1$, then the theta-lift of $\mathbb{1}$ is $\mathbb{1} \otimes \sigma_{p,q}$, and it is not unitarizable.*

Remark. The situation in this case is again controlled by the situation for the dual pair $(U(2p,2q), U(1,1))$ (see Theorem 3.6) which is in turn controlled by the situation in $(O(4p,4q), SL(2,\mathbb{R}))$ (see Theorem 2.7(ii)).

Proof. Using the formulas of Lemma 4.1, it is easy to confirm that $\mathfrak{su}(2)$ (and hence $SU(2)$) acts trivially on $\mathcal{P}/\mathcal{N}'_{\mathbb{1}}$. Using (4.9), (4.12), and (4.13), we compute the action of $\mathfrak{p}' = \text{Span}\,\{E, F\} \subset \mathfrak{sl}(2)_{\mathbb{C}}$ (see Lemma 4.1) on the weight vectors in B:

$$\begin{aligned}
E \cdot \bar{1} &= 2pZ;\\
E \cdot Z^k &= \frac{k+2p}{k+1} Z^{k+1};\\
E \cdot W^k &= k(2p-k) W^{k-1};\\
F \cdot \bar{1} &= 2qW;\\
F \cdot Z^k &= k(2q-k) Z^{k-1};\\
F \cdot W^k &= \frac{k+2q}{k+1} W^{k+1}.
\end{aligned}$$

Notice that E annihilates W^{2p}, and takes all weight vectors of $U(1)$-weight $2(p-q)+2k$ to weight vectors of $U(1)$-weight $2(p-q)+2k+2$. Similarly, F annihilates Z^{2q} and takes all weight vectors of $U(1)$-weight $2(p-q)+2k$ to weight vectors of $U(1)$-weight $2(p-q)+2k-2$. The result follows. $\square$

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National University of Singapore
Kent Ridge, Singapore 119260
Singapore
E-mail address: mattanec@nus.edu.sg