THE ISOMORPHISM PROBLEM FOR INCIDENCE RINGS

Gene Abrams, Jeremy Haefner, and Angel del Río

Let \( P \) and \( P' \) be finite preordered sets, and let \( R \) be a ring for which the number of nonzero summands in a direct decomposition of the regular module \( R_R \) is bounded. We show that if the incidence rings \( I(P, R) \) and \( I(P', R) \) are isomorphic as rings, then \( P \) and \( P' \) are isomorphic as preordered sets. We give a stronger version of this result in case \( P \) and \( P' \) are partially ordered. We show that various natural extensions of these results fail. Specifically, we show that if \( \{ P_j \mid j \in \Omega \} \) is any collection of (locally finite) preordered sets then there exists a ring \( S \) such that the incidence rings \( \{ I(P_j, S) \mid j \in \Omega \} \) are pairwise isomorphic. Additionally, we verify that there exists a finite dimensional algebra \( R \) and locally finite, nonisomorphic partially ordered sets \( P \) and \( P' \) for which \( I(P, R) \cong I(P', R) \).

Throughout this article \( P \) will denote a locally finite preordered set; so \( P \) is a set equipped with a reflexive, transitive relation such that for any two elements \( x, y \in P \), the set \( \{ z \in P \mid x \leq z \leq y \} \) is finite. \( R \) will denote an associative unital ring. The incidence ring of \( P \) with coefficients in \( R \), denoted by \( I(P, R) \) (or simply by \( A \) throughout this article), is the ring of functions \( \{ f : P \times P \to R \mid f(x, y) = 0 \ \forall x \not\leq y \} \); multiplication is given by \( (fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y) \) (and is well-defined due to the local finiteness condition on \( P \)). The ring \( I(P, R) \) may also be viewed as square matrices with entries in \( R \), whose rows and columns are indexed by \( P \), for which the \( (x, y) \) coordinate is 0 whenever \( x \not\leq y \).

Given locally finite preordered sets \( P \) and \( P' \), and rings \( R \) and \( R' \), the Isomorphism Problem for Incidence Rings is posed as follows: If the incidence rings \( I(P, R) \) and \( I(P', R') \) are isomorphic, under what hypotheses may we conclude that \( P \cong P' \) as preordered sets and/or \( R \cong R' \) as rings? Our investigation in this article focuses on a specific version of the Isomorphism Problem; namely, we find various conditions on \( P \), \( P' \), and \( R \) for which the hypothesis \( I(P, R) \cong I(P', R) \) yields the conclusion that \( P \cong P' \) as preordered sets.

A number of authors have investigated this version of the Isomorphism Problem. An outstanding historical sketch and bibliography related to the Problem is presented in [3, Section 7.2]. For our purposes, two articles
in this area deserve explicit mention. In [4, Theorem 4.3], Voss gives an affirmative answer to the Isomorphism Problem in the case where $P, P'$ are locally finite preordered sets, and $R$ is an indecomposable semiperfect ring. In fact, in a parenthetical remark made near the end of [4], Voss actually extends his solution to coefficient rings which are finite direct sums of indecomposable semiperfect rings, each of whose associated preordered sets is pairwise isomorphic. More recently, in [1, Theorem 2.4], Dăscălescu and Van Wyk solve the Isomorphism Problem in the situation where $P, P'$ are finite preordered sets, and $R$ is semiprime noetherian. In both of these articles, various examples are given to show that the hypotheses used cannot be eliminated.

In this article we make two types of contributions to the investigation of the Isomorphism Problem. In Section 1 we extend the aforementioned results in the situation where $P$ and $P'$ are finite. We do so by establishing the following solutions to two versions of the Isomorphism Problem.

**Solution 1** (cf. Theorem 1.12). Let $R$ be a ring with the property that the integer $\max(\{n \mid \text{there exist nonzero right ideals } K_1, K_2, \ldots, K_n \text{ with } R = K_1 \oplus K_2 \oplus \cdots \oplus K_n\})$ exists, and let $P$ and $P'$ be finite preordered sets. If $I(P, R) \simeq I(P', R)$ as rings, then $P \simeq P'$ as preordered sets.

Numerous classes of rings have the property mentioned in Solution 1; we say such rings have *finite summand length*. (It is not hard to show that this property is left/right symmetric.) Such classes include rings with Goldie dimension (e.g. Noetherian rings), and semiperfect rings (e.g. finite dimensional algebras). In particular, Solution 1 extends [1, Theorem 2.4], even more so than is needed to answer in the affirmative the conjecture made at the end of that article. Along the way to developing the tools needed to prove Theorem 1.12, we establish:

**Solution 2** (cf. Theorem 1.6). Let $R$ be a ring with the property that there are only finitely many direct summands of the right regular module $R_R$ which are in addition two-sided ideals of $R$, and let $P$ and $P'$ be finite partially ordered sets. If $I(P, R) \simeq I(P', R)$ as rings, then $P \simeq P'$ as partially ordered sets.

Our method of attack used in establishing these two solutions is somewhat similar in flavor to that used by Voss [4]. As we do not necessarily work over semiperfect coefficient rings, we are not afforded the luxury of working with a canonical set of indecomposable projective modules (which proved so helpful to Voss and others). Instead, we focus on a particular set of projective right ideals $D(T)$ for a general ring $T$, and then show that there is a strong relationship between $D(R)$ and $D(I(P, R))$. This relationship allows us to produce an isomorphism between the direct products of certain preordered (resp. partially ordered) sets, which then puts us in position
to use a ‘cancellation’ theorem of Lovasz [2, Theorem 4.3] to establish the desired result.

Our second contribution surfaces in Section 2, where we show that natural extensions of various solutions to the Isomorphism Problem (our own, as well the solutions of others mentioned above) fail. We show first (Proposition 2.1) that Solution 2 cannot be extended to preordered sets. Next we show (Proposition 2.3) that if \( \{P_j \mid j \in \Omega\} \) is any collection of locally finite preordered sets then there exists a ring \( S \) such that the incidence rings \( \{I(P_j, S) \mid j \in \Omega\} \) are pairwise isomorphic, thereby demonstrating that any solution to the Isomorphism Problem must contain some stipulation on the structure of the coefficient rings. Finally, we answer a question posed at the end of [4] by demonstrating (Proposition 2.4) the existence of a finite dimensional algebra \( R \) and locally finite, nonisomorphic partially ordered sets \( P \) and \( P' \) for which \( I(P, R) \simeq I(P', R) \).

1. Solutions to the Isomorphism Problem.

We begin with some notation and observations. For a ring \( T \) we let \( \text{Id}(T) \) denote the set of two-sided ideals of \( T \), while \( \text{Id}(T_T) \) denotes the set of right ideals of \( T \). We let \( D(T) = \{K \in \text{Id}(T) \mid K \text{ is a direct summand of } T_T\} \)

\[
= \{eT | e = e^2 \in T \text{ and } Te \subseteq eT\}.
\]

In the sequel we will often view \( D(T) \) as a partially ordered set with relation \( \subseteq \). It is not hard to show that \( D(T) \) is closed under finite sums. For two preordered sets \( P \) and \( P' \) with \( x \in P \) we let

\[
\hat{x} = \{y \in P \mid x \leq y \leq x\}
\]

\[
\hat{P} = \{\hat{y} | y \in P\} \quad \text{(the partially ordered set associated to } P\}
\]

\[
\text{AntiHom}(P, P') = \text{ the set of preordered anti-homomorphisms from } P \text{ to } P'.
\]

The set \( \text{AntiHom}(P, P') \) is preordered, where for \( F, G \in \text{AntiHom}(P, P') \) we set \( F \leq G \) in case \( F(x) \leq G(x) \) for all \( x \in P \). The preordered sets \( P \) and \( P' \) are isomorphic in case there exists a set bijection \( H : P \to P' \) such that both \( H \) and \( H^{-1} \) preserve the respective orders on \( P \) and \( P' \). If \( |\hat{x}| = n \), then for any ring \( T \) we have \( I(\hat{x}, T) \simeq M_n(T) \), the ring of \( n \times n \) matrices with coefficients in \( T \). For a preordered set \( P \) and a ring \( R \) we set \( A = I(P, R) \). If \( X \) is a subset of \( P \), \( e_X \) denotes the idempotent of \( A \) given by \( e_X(x, y) = 1 \) if \( x = y \in X \), and \( e_X(x, y) = 0 \) otherwise. For \( x \in P \) we denote \( e_{\{x\}} \) by \( e_x \).
Our first task is to identify the set of two-sided ideals of \( A = I(P,R) \) in case \( P \) is finite. Given a map \( \Psi : P \times P \to \text{Id}(R_R) \), we let

\[
I(\Psi) = \{ \alpha \in A | \alpha(x,y) \in \Psi(x,y) \text{ for all } x \leq y \text{ in } P \}.
\]

**Lemma 1.1.** Let \( P \) be a preordered set, and let \( \Psi : P \times P \to \text{Id}(R_R) \) be a map.

1. \( I(\Psi) \in \text{Id}(A) \) if and only if \( \Psi(x,y) \subseteq \Psi(x,y') \) for every \( x \leq y \leq y' \) in \( P \).
2. \( I(\Psi) \in \text{Id}(A) \) if and only if \( \Psi(x,y) \in \text{Id}(R) \) and \( \Psi(x,y) \subseteq \Psi(x',y') \) for every \( x' \leq x \leq y \) in \( P \).
3. If \( P \) is finite, then every two-sided ideal of \( A \) is of the form \( I(\Psi) \) for some map \( \Psi : P \times P \to \text{Id}(R_R) \) satisfying the conditions of (2).

**Proof.** Statements (1) and (2) are straightforward to verify, using the definition of multiplication in the incidence ring \( A = I(P,R) \).

For statement (3), assume that \( I \) is a two-sided ideal of \( A \). For every \( x \leq y \) in \( P \) let \( \Psi(x,y) = \{ \alpha(x,y) | \alpha \in I \} \). Clearly \( \Psi(x,y) \) is a two-sided ideal of \( R \), and \( I \subseteq I(\Psi) \). Let \( \alpha \in I(\Psi) \). Then for every pair \( x \leq y \) in \( P \) there is \( \alpha_{x,y} \in I \) such that \( \alpha_{x,y}(x,y) = \alpha(x,y) \). By the finiteness of \( P \) we may then write \( \alpha = \sum_{x \leq y} e_x \alpha_{x,y} e_y \), which is an element of \( I \). Therefore \( I = I(\Psi) \) and hence \( \Psi \) satisfies the conditions of (2). \( \square \)

Having associated the two-sided ideals of \( A \) with certain maps from \( P \times P \) to \( \text{Id}(R_R) \), we now determine additional properties of such maps which ensure that the associated ideal of \( A \) is in fact an element of \( D(A) \). Specifically, given any map \( F : \hat{P} \to \text{Id}(R_R) \) we define

\[
\Psi_F : P \times P \to \text{Id}(R_R) \text{ by } \Psi_F(x,y) = F(\hat{x}), \quad \text{and} \quad J(F) = I(\Psi_F).
\]

From the matrix point of view, \( J(F) \) is the collection of matrices in \( I(P,R) \) for which the entry in the \((x,y)\) coordinate is taken from the right ideal \( F(\hat{x}) \). In particular, the entries in a given row of an element of \( J(F) \) belong to the fixed right ideal \( F(\hat{x}) \).

**Lemma 1.2.** Let \( P \) be a preordered set. Let \( F \) be a map from \( \hat{P} \) to \( \text{Id}(R_R) \). Then:

1. \( J(F) \) is a right ideal of \( A \).
2. \( J(F) \) is a two-sided ideal of \( A \) if and only if \( \text{Im } F \subseteq \text{Id}(R) \) and \( F \) is an anti-homomorphism of partially ordered sets.
3. If \( P \) is finite, then \( D(A) \simeq \{ J(F) | F \in \text{AntiHom}(\hat{P},D(R)) \} \).

**Proof.** (1) and (2) are consequences of Lemma 1.1.
(3) Let $F \in \text{AntiHom}(\hat{P}, D(R))$. For every $x \in \hat{P}$, let $F_1(x) \in \text{Id}(R_R)$ be a complement of $F(x)$, so that $F(x) \oplus F_1(x) = R$. Then it is easy to check that $J(F) \oplus J(F_1) = A$, so that $J(F) \in D(A)$.

Conversely, let $J = fA \in D(A)$; so $f = f^2 \in A$, and $A f \subseteq fA$. For every $X \in \hat{P}$ let $f_X \in M_X(R)$ be given by $f_X(x, y) = f(x, y)$ for every $x, y \in X$. If $x, y \in X$, $f_X^y(x, y) = \sum_{z \in X} f_X(x, z)f_X(z, y) = \sum_{x \leq z \leq y} f(x, z)f(z, y) = f(x, y) = f_X(x, y)$. Thus $f_X$ is an idempotent of $M_X(R)$. Moreover $f_X M_X(R) \in D(M_X(R))$ because if $r \in M_X(R)$, then there is $\alpha \in A$ such that $\alpha f_X = f_X \alpha_X$ (where $\alpha_X$ is defined similarly). Since the ideals of $M_X(R)$ each arise from an ideal of $R$, we have that there is a two-sided ideal $K_X$ of $R$ such that $f_X M_X(R) = M_X(K_X)$. But $M_X(K_X)$ is then also a direct summand of $M_X(R)_{M_X(R)}$, and hence $e_x M_X(K_X)$ is a direct summand of $e_x M_X(R)_{M_X(R)}$ for every $x \in X$. Since the functor given by $e_x M_X(\cdots)$ is precisely the equivalence functor between $\text{Mod} - M_X(R)$ and $\text{Mod} - R$, we conclude that $K_X$ is a direct summand of $R_R$. Thus $K_X = f_X^1 R$ for some idempotent $f_X^1$ of $R$ such that $R f_X^1 \subseteq f_X^1 R$. Consequently we have $f_X M_X(R) = M_X(K_X) = M_X(f_X^1 R) = g_X M_X(R)$, where $g_X$ is the scalar matrix in $M_X(R)$ with $f_X^1$ on the diagonal. In particular, $f_X$ generates the same right (in fact, two-sided) ideal of $M_X(R)$ as the scalar $g_X$.

Now let $F : \hat{P} \to D(R)$ be given by $F(X) = f_X^1 R$. We define $f_1 \in A$ by setting

$$f_1(x, y) = f_X^1 \text{ if } x = y = X, \text{ while } f_1(x, y) = 0 \text{ otherwise.}$$

So $f_1$ is simply the diagonal blocks of $f$. As such, using the finiteness of $P$ we may write $f_1 = \sum_{((x, y) | x = y)} e_x f e_y$, so that $f_1 \in fA$ as $fA$ is two-sided (being an element of $D(A)$). In particular, since $f$ is idempotent we get that $ff_1 = f_1$. Again using the finiteness of $P$ we have that $f - f_1$ is nilpotent (necessarily of degree $\leq |P|$). Expanding the equation $(f - f_1)^t = 0$ and using that $ff_1 = f_1$ yields that $f = f^t \in fA$. Thus we have that $fA = f_1A$.

Now let $f_2$ be the block-diagonal element of $A$ whose entry in the $X - X$ component is $g_X$. By the discussion in the previous paragraph we have that $f_1A = f_2A$, so that $fA = f_2A$. But the diagonal form of $f_2$ immediately yields that $f_2A = J(F)$, so that finally we have shown that $J = fA = f_2A = J(F)$ is of the required form. That $F \in \text{AntiHom}(\hat{P}, D(R))$ follows from part (2).

**Definition 1.3.** Let $P$ be a partially ordered set. We define the partially ordered subset $P_1$ of $P$ by setting

$$P_1 = \{x \in P | \text{there exists } x' < x \text{ with the property that } y \leq x' \text{ for every } y < x\}.$$ In other words, $x$ is in $P_1$ in case there exists a unique maximal element $x'$ in $X$ below $x$. 
If $T$ is any ring then as noted above $D(T)$ is a partially ordered set under inclusion. In the sequel we will focus much attention on the partially ordered subset $D(T)_{1}$ of $D(T)$. We first prove a general result about such subsets.

**Lemma 1.4.** Let $P$ and $Q$ be two partially ordered sets, such that $P$ is finite, and such that $Q$ has a unique minimal element $0$. Let $H : P \times Q \to \text{AntiHom}(P, Q)$ be the map given by setting $H(p, q)(x) = q$ if $x \leq p$ and $H(p, q)(x) = 0$ otherwise. Then $H$ restricts to an isomorphism of partially ordered sets

$$H : P \times Q_{1} \simeq \text{AntiHom}(P, Q)_{1}.$$ 

In particular, for every $(p, q) \in P \times Q_{1}$, the unique maximal element $H(p, q)' < H(p, q)$ is given by setting $H(p, q)'(x) = q$ if $x < p$, $H(p, q)'(p) = q'$, and $H(p, q)'(x) = 0$ otherwise.

**Proof.** First we check that $H = H(p, q) \in \text{AntiHom}(P, Q)$. Indeed, suppose $x \leq y$ in $P$. Then either $y \leq p$, in which case $H(y) = q = H(x)$, or $y \not\leq p$, in which case $H(y) = 0 \leq H(x)$.

It is clear that the restriction of $H$ to $P \times (Q \setminus \{0\})$ is injective and preserves the order. So we must check that $H(P \times Q_{1}) = \text{AntiHom}(P, Q)_{1}$, and that the inverse map preserves the order. To this end, suppose that $(p, q) \in P \times Q_{1}$. Let $H' : P \to Q$ be given by $H'(x) = q$ if $x < p$, $H'(p) = q'$, and $H'(x) = 0$ if $x \not< p$. Then $H' < H = H(p, q)$. We verify that $H'$ satisfies the desired properties. So let $G < H$ in $\text{AntiHom}(P, Q)$; we show $G \leq H'$. Now $G(x) \leq H(x) = 0$ if $x \not< p$ and hence $G(x) = 0$ in this case. Moreover, $G(p) \leq H(p) = q$. Assume that $G(p) = q$. If $x \leq p$, then $q = G(p) \leq G(x) \leq H(p) = q$; but this gives $G = H$, a contradiction. Therefore $G(p) < q$, and hence $G(p) \leq q'$. Finally, if $x < p$, then $G(x) \leq H(x) = q$. Thus $G \leq H'$ and this proves that $H \in \text{AntiHom}(P, Q)_{1}$.

Conversely, let $H \in \text{AntiHom}(P, Q)_{1}$. Let $p$ and $p_{1}$ be two different maximal elements of $\{x \in P : H(x) \neq 0\}$. Let $F_{1}, F_{2} : P \to Q$ be given by

$$F_{1}(x) = F_{2}(x) = H(x) \text{ for } p \neq x \neq p_{1} \text{ and }$$

$$F_{1}(p) = H(p), \quad F_{1}(p_{1}) = 0, \quad F_{2}(p) = 0, \quad F_{2}(p_{1}) = H(p_{1}).$$

Since $F_{1} < H$ and $F_{2} < H$ we have $F_{1} \leq H'$ and $F_{2} \leq H'$. Then $H(x) = F_{1}(x) \leq H'(x) \geq F_{2}(x) = H(x)$ for $p \neq x \neq p_{1}$, $H(p) = F_{1}(p) \leq H'(p) = H(p)$, and $H(p_{1}) = F_{2}(p_{1}) \leq H'(p_{1}) = H(p_{1})$. But this gives $H = H'$, a contradiction. Therefore there is an element $p \in P$, such that $H(p) \neq 0$ and $H(x) = 0$, for every $x \not< p$.

Let $F : P \to Q$ be given by $F(x) = H(x)$ if $x \neq p$ and $F(p) = 0$. Since $F \in \text{AntiHom}(P, Q)$ and $F < H$, then $F \leq H'$. Therefore $H(x) \geq H'(x) \geq F(x) = H(x)$, and hence $H(x) = H'(x)$, for every $x \neq p$. Since $H' < H$, then $H'(p) < H(p)$.

Set $q = H(p)$. If $H \neq H(p, q)$, there is $x < p$ such that $H(x) > q$. Assume that $x$ is maximal with that condition. Let $F \in \text{AntiHom}(P, Q)$ be given
by \( F(y) = H(y) \) if \( y \neq x \), and \( F(x) = q \). Then \( F < H \) and hence \( F \leq H' \). Therefore \( q = F(p) \leq H'(p) \leq H(p) = q \). Thus \( H' = H \) a contradiction. We conclude that \( H = H(p,q) \).

We now show that \( q \in Q_1 \). We prove that \( H'(p) \) has the desired properties; i.e., we prove that \( H'(p) = q' \). If \( x < q \), let \( F \in \text{AntiHom}(P,Q) \) be given by \( F(y) = H(y) \) if \( y \neq p \) and \( F(p) = x \). Since \( F < H \), then \( F \leq H' \) and hence \( x \leq H'(p) \).

Now that we have shown that \( H \) is a set bijection which preserves the order, we need only show that \( H^{-1} \) preserves the order as well. So suppose that \( H(p_1,q_1) \leq H(p_2,q_2) \). Then on applying each of these antihomomorphisms to \( p_1 \in P \) we get \( q_1 = H(p_1,q_1)(p_1) \leq H(p_2,q_2)(p_1) \). But \( q_1 \neq 0 \), so \( H(p_2,q_2)(p_1) \neq 0 \), which by definition of \( H(p_2,q_2) \) gives that \( p_1 \leq p_2 \), and that \( q_1 \leq q_2 \).

### Lemma 1.2 and 1.4 now yield:

**Proposition 1.5.** Let \( P \) be a finite preordered set, and let \( R \) be any ring. Then there is an isomorphism of partially ordered sets

\[
H : \hat{P} \times D(R)_1 \simeq D(A)_1.
\]

Specifically, for \((X,K) \in \hat{P} \times D(R)_1\) we set \(H(X,K) = J(F_{X,K})\), where \(F_{X,K} \in \text{AntiHom}(\hat{P},D(R)_1)\) is defined by setting \(F_{X,K}(W) = K\) if \(W \leq X\), \(F_{X,K}(W) = 0\) otherwise.

In turn, Proposition 1.5 allows us to establish one of the two Solutions to the Isomorphism Problem mentioned in the introduction.

**Theorem 1.6.** Let \( P \) and \( P' \) be finite partially ordered sets, and let \( R \) be a ring for which \( D(R) \) is finite. If the incidence rings \( I(P,R) \) and \( I(P',R) \) are isomorphic, then \( P \) and \( P' \) are isomorphic as partially ordered sets.

**Proof.** As \( P \) and \( P' \) are partially ordered we have \( P = \hat{P} \) and \( P' = \hat{P'} \). Let \( A \) denote \( I(P,R) \) and let \( A' \) denote \( I(P',R) \). It is straightforward to show that a ring isomorphism between \( A \) and \( A' \) yields an isomorphism of the corresponding partially ordered sets \( D(A)_1 \) and \( D(A')_1 \). By Lemma 1.5 this yields an isomorphism of partially ordered sets \( P \times D(R)_1 \simeq P' \times D(R)_1 \). Now an application of [2, Theorem 4.3] yields the desired isomorphism between \( P \) and \( P' \).

As is apparent from the proof of Lemma 1.2, \( D(T) \simeq D(M_n(T)) \) for any ring \( T \) and any matrix ring \( M_n(T) \). Thus, as \( M_n(T) = I(\hat{x},T) \) for \( |\hat{x}| = n \), \( D(-) \) cannot be used to recover preordered sets. Therefore, in order to establish a result similar to that given in Theorem 1.6 in the more general context of preordered sets, we must utilize the more subtle structure \( D^*(-) \) (defined below). The price we pay for weakening the hypothesis on
the ordered sets is the necessity of imposing additional structure on the underlying coefficient ring.

**Definition 1.7.** For any ring $T$ and any right $T$-module $M_T$ we define the **summand length of** $M$, denoted $w(M_T)$, by setting

$$w(M_T) = \max\{n | M = \oplus_{i=1}^{n} M_i, \text{ for some nonzero submodules } M_i \leq M\}.$$ 

We set $w(M_T) = \infty$ if no such maximum exists. We say the ring $T$ has **finite summand length** in case $w(T_T)$ is finite.

We note that the definition of finite summand length for a ring $T$ given above is left/right symmetric. To see this, we observe that any direct decomposition of the right regular module $T_T$ having $n$ nonzero summands gives rise to a complete set of $n$ orthogonal idempotents in $T$. This set of idempotents can then be used to give a direct decomposition of the left regular module $T_T$ having $n$ nonzero direct summands. The desired conclusion follows by symmetry.

**Lemma 1.8.** For any ring $T$, if $T$ has finite summand length, then $D(T)$ is finite.

**Proof.** We denote $w(T_T)$ by $N$. Since any chain of proper inclusions of length $n$ of nonzero elements of $D(T)$ gives rise to a decomposition of $T_T$ containing $n + 1$ summands, we conclude that the partially ordered set $D(T)$ has finite height (in fact, height $\leq N$). Thus to show that $D(T)$ is finite it is enough to show that for every $K \in D(T)$, the set of minimal elements of $\{J \in D(T) | K \subset J\}$ is finite. Indeed, if we denote this set of minimal elements by $\{J_i | i \in \Omega\}$, then we show that this set contains at most $N$ elements, by showing that this set can be used to generate an equal-sized set of nonzero pairwise orthogonal idempotents in $T$. Let $K = gT$, $J_i = e_iT$ for idempotents $g$, $\{e_i\}_{i \in \Omega}$. Since $K \subset J_i$ we have $e_ig = g$ for all $i$. With this it is easy to show that each $e'_i = e_i - ge_i$ is idempotent. Further, it is easy to show that $e_iT \cap e_jT = e_ie_jT \in D(T)$; the minimality property of these ideals then implies that $e_ie_jT = gT$ for $i \neq j$. It follows that $e'_i$ and $e'_j$ are in fact orthogonal for $i \neq j$. Since $e_i \neq g$ we have that $e'_i$ is nonzero for all $i$, and so $\{e'_i\}_{i \in \Omega}$ is a set of nonzero pairwise orthogonal idempotents. But a set of more than $N$ nonzero pairwise orthogonal idempotents in $T$ would yield a decomposition of $T_T$ in violation of the hypothesis that $w(T_T) = N$, so that $| \Omega | \leq N$ as desired. \qed

**Definition 1.9.** Let $T$ be any ring with finite summand length. We define

$$D^*(T) = \{(K,i) | K \in D(T)_1 \text{ and } 1 \leq i \leq w((K/K')T)\}.$$ 

(Here $K'$ is the unique maximal element of $D(T)$ below $K$, as given in Definition 1.3.) We note that $w((K/K')T)$ is indeed an integer, as $K/K'$ is
isomorphic to a direct summand of $T_T$. We define a preorder on $D^*(T)$ by setting

$$(K, i) \preceq (L, j) \iff K \subseteq L.$$  

For any positive integer $s$, let $N_s$ denote the set $\{1, 2, \ldots, s\}$.

**Lemma 1.10.** Let $P$ be a finite preordered set, and let $R$ be a ring with finite summand length. If $K \in D(R)_1$ and $X \in \hat{P}$, then

$$|X| \cdot w(K/K'_R) = w(H(X, K)/H(X, K)'_A).$$

In particular, there exists a bijection

$$\phi_{X,K} : X \times N_{w(K/K')} \to N_{w(H(X,K)/H(X,K'))}.$$  

**Proof.** Let $h(X, K)$ denote the right ideal $e_XH(X, K)$ of $A$. By Lemma 1.4 one can show that $H(X, K) = \sum_{Y \leq X} H(Y, K)$ and that the unique maximal element $H(X, K)'$ below $H(X, K)$ is given by $H(X, K)' = \sum_{Y < X} H(Y, K) \oplus h(X, K')$. Therefore, $H(X, K)/H(X, K)' \cong h(X, K)/h(X, K')$.

Now let $J = I(P, K')$. Since $h(X, K)I \subseteq h(X, K')$, then $h(X, K)/h(X, K')$ is a right $A/I$-module. Moreover, if $B = I(P, R/K')$, then $(h(X, K)/h(X, K'))_{A/J} \cong h(X, K)/h(X, K')_B$ and it follows that

$$w((h(X, K)/h(X, K'))_{A}) = w((h(X, K)/h(X, K'))_{A/J}) = w(h(X, K/K')_B).$$

Consequently, $w(H(X, K)/H(X, K')_A) = w(h(X, K/K')_B)$. Since $K/K'$ is isomorphic to a direct summand of $K$ it is in fact isomorphic to a right ideal of $R$; thus it suffices to prove that if $L$ is a right ideal of $R$, then $w(h(X, L)_A) = |X|w(L_R)$.

If $L = \oplus_{i=1}^n L_i$, then $h(X, L) = \oplus_{x \in X} \oplus_{i=1}^n e_xh(X, L_i)$ since $e_X$ acts like an identity on the left of elements of $h(X, L)$. This shows that $w(h(X, L)) \geq |X|w(L)$. Now suppose $w(h(X, L)) = n$ and so we may write $h(X, L) = \oplus_{i=1}^n L_i$ where each $L_i$ is a nonzero right ideal of $A$. Again, since $e_X$ is an identity on the left of each $L_i$, we have $L_i = \oplus_{x \in X} e_xL_i$ and so $h(X, L) = \oplus_{i=1}^n \oplus_{x \in X} e_xL_i$. Since $n = w(h(X, L))$, it follows that, for every $i = 1, \ldots, n$, there is exactly one $x_i \in X$ such that $L_i = e_{x_i}L_i$. (It may be the case that two such $L_i$ share the same $x$.) For every $i = 1, \ldots, n$, let $J_i = \{\alpha(x_i, x_i) \mid \alpha \in L_i\}$. It is clear that each $J_i$ is a right ideal of $R$ and that, for every $x \in X$, $L = \sum_{x_i = x} J_i$. We claim that this sum is direct. To see this, suppose $\sum_{x_i = x} r_i = 0$, with $r_i \in J_i$, and so there are $\alpha_i \in L_i$ such that $\alpha_i(x, x) = r_i$. Since $\alpha_i \in L_i = e_{x_i}L_i$, it follows that $\sum_{i=1}^n \alpha_i e_x = 0$. Since the original sum $\oplus_{i=1}^n L_i$ is direct, we have each $\alpha_i e_x = 0$ and so each $r_i = 0$ and the claim holds. In particular, the number $m_x$ of the $i$, $i = 1, \ldots, n$ such that $x_i = x$ satisfies $m_x \leq w(L)$. Consequently, $h(X, L) = \oplus_{x \in X} e_xh(X, L) = \oplus_{x \in X} \oplus_{i, x_i = x} e_xh(X, J_i)$. Once we show that $e_xh(X, J_i) = L_i$ for every $i$, the proof will be complete because then we shall have $w(h(X, L)) = \sum_{x_i = x} J_i$. \n
}
\[ \sum_{x \in X} m_x \leq |X| w(L), \] as desired. To show the equality, let \( \alpha \in e_x h(X, J_i) \). Then, for \( y \geq x_i \), there are \( \beta_y \in L_i \) such that \( \alpha(x_i, y) = \beta_y(x_i, x_i) \) and so \( \alpha = \sum_{y \geq x_i} \beta_y e_{x_i, y} \in L_i \) which shows that \( e_x h(X, J_i) \subset L_i \). However, since \( e_x h(X, J_i) \) and \( L_i \) are summands of \( h(X, L) \), we see from the Modular Law that \( L_i = e_x h(X, J_i) \). \( \square \)

**Proposition 1.11.** If \( P \) is any finite preordered set, and \( R \) is a ring with finite summand length, then \( P \times D^*(R) \simeq D^*(A) \).

**Proof.** Let \( H \) denote the isomorphism given in Proposition 1.5. For every \( X \in \hat{P} \) and \( K \in D(R)_1 \), let \( \phi_{X,K} : X \times N_{w(K)} \to N_{w(h(X,K))} \) be the map described in Lemma 1.10. We now define \( \Phi : P \times D^*(R) \to D^*(A) \) by setting

\[
\Phi(x, (K, i)) = (H(\hat{x}, K), \phi_{\hat{x},K}(x, i)).
\]

Using the fact that \( H \) is an isomorphism of partially ordered sets and that \( \phi_{X,K} \) is a bijection, it is straightforward to check that \( \Phi \) is indeed an isomorphism of preordered sets. \( \square \)

We now have all the tools required to establish our second version of the Isomorphism Problem.

**Theorem 1.12.** Let \( P \) and \( P' \) be two finite preordered sets, and let \( R \) be a ring with finite summand length. If \( I(P, R) \simeq I(P', R) \) as rings, then \( P \simeq P' \) as preordered sets.

**Proof.** Let \( A \) denote \( I(P, R) \) and let \( A' \) denote \( I(P', R) \). It is straightforward to show that a ring isomorphism between \( A \) and \( A' \) yields an isomorphism of the corresponding preordered sets \( D^*(A) \) and \( D^*(A') \). By Proposition 1.11, this gives an isomorphism of preordered sets

\[
P \times D^*(R) \simeq P' \times D^*(R).
\]

Now an application of [2, Theorem 4.3] yields the desired isomorphism between \( P \) and \( P' \). \( \square \)

We note that Theorem 1.12 answers the question posed at the end of [1]; specifically, we have shown that the Isomorphism Problem has a positive solution when the preordered sets are finite, and the coefficient ring is right or left noetherian.

**2. Non-extendability of Solutions.**

In the second section of this article we present three constructions which demonstrate the failure of possible extensions of various solutions of the Isomorphism Problem.

Our first non-extendability result shows that the hypotheses of Theorem 1.12 cannot be replaced by the weaker hypotheses of Theorem 1.6.
Proposition 2.1. There exists a ring $S$ having finite $D(S)$ and there exist nonisomorphic preordered sets $P$ and $P'$ such that $I(P, S) \simeq I(P', S)$.

Proof. Let $k$ be any unital ring, and let $S$ denote the ring $RFM(k)$ of countably infinite row-finite matrices over $k$. It is well-known that $S$ has the property that $S \simeq M_2(S)$, which yields that $I(P, S) \simeq I(P', S)$ where $|P| = 1$ and $P' = \hat{x}$ where $|\hat{x}| = 2$. Furthermore, since $S$ has only one nontrivial two-sided ideal, $D(S)$ is finite. \qed

Prior to presenting our second non-extendability result (Proposition 2.3 below), we review some of the ideas introduced in [4]. Let $X$ be any locally finite preordered set, and let $a \in X$. We define $\hat{X}^a = \left\{ (x_n) \in \prod_{n \in \mathbb{N}} X \mid \exists N \text{ with } x_n = a \ \forall n \geq N \right\}$. In [4, Proposition 1] Voss shows that $\hat{X}^a$ is a locally finite preordered set having the property that $X \times \hat{X}^a \simeq \hat{X}^a$ as preordered sets. Indeed, the isomorphism $\phi$ is given by setting $\phi((x, (x_n))) = (y_n)$, where $y_1 = x$ and $y_n = x_{n-1}$ for all $n \geq 2$.

We now let $\{X_i\}_{i \in \Omega}$ be any collection of preordered sets. For each $i \in \Omega$ let $a_i$ denote a fixed element of $X_i$, and let $\bar{a}_i$ denote the element $(a_i, a_i, ...)$ of $\hat{X}^{a_i}$. Finally, let $B$ denote the set

$$B = \left\{ (b_i) \in \prod_{i \in \Omega} \hat{X}^{a_i} \mid b_i = \bar{a}_i \text{ for all but finitely many } i \right\}.$$ 

Lemma 2.2. The preordered set $B$ defined above is locally finite provided that each $X_i$ is locally finite. Moreover, for each $i \in \Omega$ we have $X_i \times B \cong B$ as preordered sets.

Proof. The set $B$ inherits the preorder from the product $\prod_{i \in \Omega} \hat{X}^{a_i}$. Now choose $(b_i) \leq (c_i) \leq (d_i)$ in $B$. Then for all but finitely many $i \in \Omega$ we have that $b_i = c_i = d_i = \bar{a}_i$. The first statement now follows as each $\hat{X}^{a_i}$ is locally finite.

Define $\phi_i : X_i \times B \to B$ by setting $\phi_i((x_i, (b_j))) = (c_j)$, where $c_j = b_j$ in $\hat{X}^{a_j}$ for $j \neq i$, and $c_i = \phi((x_i, b_i))$ (using the map $\phi$ described above). Then it is easy to check that $\phi_i$ is an isomorphism of the desired type. \qed

Our second non-extendability result shows that the Recovery Question must always fail without some stipulation on the structure of the underlying coefficient rings.

Proposition 2.3. Let $\{X_i\}_{i \in \Omega}$ be any collection of locally finite preordered sets. Then there exists a ring $S$ with the property that the incidence rings
I(X_i, S) and I(X_j, S) are isomorphic for all i, j ∈ Ω. Indeed, each of these incidence rings is isomorphic to S itself.

Proof. Let B be the locally finite preordered set defined above, let T be any ring, and let S denote the incidence ring I(B, T). Then for each i ∈ Ω,

\[ I(X_i, S) \cong I(X_i, I(B, T)) \cong I(X_i \times B, T) \cong I(B, T) \cong S. \]

□

We mention that the above result generalizes an example from [1]. In [1, Example 1.1] Dăscălescu and Van Wyk construct a ring R for which the following five ‘structural matrix rings over R’ are isomorphic:

\[ R \oplus R \quad M_2(R) \quad \begin{pmatrix} R & R \\ 0 & R \end{pmatrix} \quad \begin{pmatrix} R & 0 \\ R & R \end{pmatrix}. \]

These structural matrix rings are precisely the incidence rings which arise from the five preordered sets (respectively)

\{a\} {a, b} {a, b | a ≤ b, b ≤ a} {a, b | a ≤ b} {a, b | b ≤ a}.

The final non-extendability result (Proposition 2.5) addresses the necessity of the finiteness condition on the preordered sets. The key ingredient in this result is an example of non-cancellability for certain preordered sets. In [4, Theorem 4] and a subsequent remark, Voss proves that if X, Y and Z are preordered sets for which X and Y are locally finite, and Z is finite and has the property that the connected components of Z are pairwise isomorphic, then \( X \times Z \cong Y \times Z \) implies \( X \cong Y \). We now show that the condition on the connected components of Z cannot be dropped from the hypotheses of Voss’ result. We recall some notation: For \( x, y \) in a preordered set X we say that \( y \) covers \( x \) in case \( x < y \), and there does not exist \( z \) in X with \( x < z < y \).

**Proposition 2.4.** There exists a finite (necessarily non-connected) partially ordered set Z and locally finite (necessarily infinite) partially ordered sets X and Y such that \( X \times Z \cong Y \times Z \) but \( X \ncong Y \).

**Proof.** Let U be any preordered set in which there exists a unique minimal element \( a \in U \). Let \( \tilde{U}^a \) be the preordered set constructed as above, and let \( \tilde{a} \) denote the constant tuple \( \tilde{a} = (a, a, \ldots) \). Set

\[ \tilde{U}^a_0 = \{(a_n) \in \tilde{U}^a | \max\{n | a_n \neq a\} \text{ is even}\} \cup \{\tilde{a}\} \quad \text{and} \quad \tilde{U}^a_1 = \{(a_n) \in \tilde{U}^a | \max\{n | a_n \neq a\} \text{ is odd}\} \cup \{\tilde{a}\}. \]

We first show that \( U \times \tilde{U}^a_0 \cong \tilde{U}^a_1 \) and \( U \times \tilde{U}^a_1 \cong \tilde{U}^a_0 \) as preordered sets. To verify this, it suffices to prove \( U \times \tilde{U}^a_0 \cong \tilde{U}^a_1 \), since the other isomorphism is
shown symmetrically. Define \( \phi : U \times \hat{U}_0^a \to \hat{U}_1^a \) by setting \( \phi(x, f)(1) = x \) and \( \phi(x, f)(n) = f(n - 1) \) for \( n \geq 2 \). That \( \phi \) is a set bijection which preserves the underlying preordering is clear.

We next show, perhaps counter to one’s intuition, that the preordered sets \( \hat{U}_0^a \) and \( \hat{U}_1^a \) are not isomorphic, as long as there exists a unique cover \( \alpha \) of \( a \) in \( U \).

Clearly \( \bar{a} \) is the unique minimal element of both \( \hat{U}_0^a \) and \( \hat{U}_1^a \). Let \( C_0 \) denote the set of covers of \( \bar{a} \) in \( \hat{U}_0^a \); so the elements of \( C_0 \) are of the form \( \alpha_{2n} \), the tuple whose \( 2n \)-th coordinate is \( \alpha \), and all other coordinates equal \( a \). Similarly, the set \( C_1 \) of covers of \( \bar{a} \) in \( \hat{U}_1^a \) are of the form \( \alpha_{2n+1} \).

It is easy to see that every element of \( C_0 \) is covered by an element which does not cover any other element of \( C_0 \); for instance, \( \alpha_{2n} \) is covered in such a way by \( \bar{a} \), the tuple which is \( \alpha \) in coordinates \( 1 \) and \( 2n \), and is \( a \) elsewhere. However, the element \( \alpha_1 \) of \( C_1 \) does not have this property; that is, every cover of \( \alpha_1 \) necessarily covers another element of \( C_1 \).

Since \( \bar{a} \) serves as the unique minimal element for both \( \hat{U}_0^a \) and \( \hat{U}_1^a \), and since covers are preserved by preordered set isomorphism, the above discussion yields that \( \hat{U}_0^a \not\cong \hat{U}_1^a \) as claimed.

Now we construct the example in which cancellation fails. Let \( U \) and \( V \) be two preordered sets, with elements \( a \in U \) and \( b \in V \). Set \( Z = U \uplus V \) (the disjoint union), \( X = \hat{U}_1^a \times \hat{V}_0^b \), and \( Y = \hat{U}_0^a \times \hat{V}_1^b \). Then

\[
X \times Z \simeq (U \uplus V) \times (\hat{U}_1^a \times \hat{V}_0^b) \simeq (U \times \hat{U}_1^a \times V_0^b) \uplus (V \times \hat{U}_1^a \times V_0^b) \\
\simeq (U_0^a \times V_1^b) \cup (U_1^a \times V_0^b) \\
\simeq (U \uplus V) \times (\hat{U}_0^a \times \hat{V}_1^b) = Y \times Z.
\]

Consequently, to find an example of the desired type, it suffices to find \( U \) and \( V \) such that \( X = \hat{U}_0^a \times V_1^b \not\cong \hat{U}_1^a \times V_0^b = Y \). But if \( U = \{a, \alpha\} \) such that \( a < \alpha \) and \( V = \{b\} \), then \( \hat{V}_0^b = \{b\} = V_1^b \), while \( \hat{U}_0^a \not\cong \hat{U}_1^a \) as shown above. Then \( X \simeq \hat{U}_0^a \not\cong \hat{U}_1^a \simeq Y \), and we are done. \( \Box \)

Our final non-extendability result shows that if we allow the preordered sets to be infinite, then we need not be able to recover the preordered sets, even from incidence rings with coefficients taken from a finite dimensional algebra.

**Proposition 2.5.** There exist locally finite, non-isomorphic partially ordered sets \( P \) and \( P' \) and a finite dimensional algebra \( R \) such that the incidence rings \( I(P, R) \) and \( I(P', R) \) are isomorphic.

**Proof.** Let \( X, Y \) and \( Z \) be as in Proposition 2.4 and let \( k \) be any field. Let \( P = X \), let \( P' = Y \), and let \( R \) denote the incidence ring \( I(Z, k) \). In particular, \( R \) is a 4-dimensional algebra over \( k \). By hypothesis we have...
\( P \not\equiv P' \). But we do have \( P \times Z \equiv P' \times Z \), and so
\[
I(P, R) = I(P, I(Z, k)) \equiv I(P \times Z, k) \equiv I(P', Z, k)
\equiv I(P', I(Z, k)) = I(P', R).
\]

Since the ring \( R \) given in Proposition 2.5 has finite \( w(R) \) (and therefore also finite \( D(R) \)), we cannot extend either Theorem 1.6 or Theorem 1.12 to locally finite partially ordered sets.

At the end of [4], Voss asks whether or not his solution to the Isomorphism Problem (where the coefficient rings are assumed to be finite direct sums of indecomposable semiperfect rings each of whose associated preordered sets is pairwise isomorphic) extends to all artinian rings. Proposition 2.5 provides a negative answer to this question. In fact, the coefficient ring used in the Proposition is in some sense a minimal counterexample, as any algebra of dimension three or less over any field can be shown to satisfy Voss' hypotheses.

References


Received September 12, 1997. This work was completed while the third author was in residence at the University of Colorado at Colorado Springs, to which he wishes to express his thanks for its hospitality. The second author was supported by NSA grant MDA904-96-0-0030, while the third author was supported by grants DGICYT (PB93-0515-C02-02) and DGICYES (PR95-469).

University of Colorado
Colorado Springs, CO 80933
E-mail address: abrams@math.uccs.edu

University of Colorado
Colorado Springs, CO 80933
E-mail address: haefner@math.uccs.edu

Universidad de Murcia
30071 Murcia
Spain
E-mail address: adelrio@fcu.um.es
FINITE GROUPS OF LIE TYPE OF SMALL RANK

Frauke M. Bleher

In this paper, it is proven that a conjecture of Zassenhaus is valid for all finite simple groups of Lie type of rank 1 and of rank 2 which are not of type \( 2A_3 \) or \( 2A_4 \). In particular, this conjecture holds for all finite simple groups with abelian Sylow 2–subgroups.

1. Introduction.

One of the key problems in the representation theory of finite groups is to represent a given finite group \( G \) in a suitable way. To do so, invariants which classify \( G \) up to isomorphisms are determined. One of the questions derived in this context is the so-called Isomorphism Problem (IP), which asks whether the existence of an isomorphism between the integral group algebras \( ZG \) and \( ZH \) implies that \( G \) and \( H \) are isomorphic. In the seventies, Zassenhaus conjectured a stronger version of (IP) describing the structure of the group of units \( V(ZG) \) of \( ZG \) with augmentation 1:

\((ZC)\) Any two subgroups \( X, Y \) of \( V(ZG) \) which have the same order as \( G \) are conjugate by a unit of \( QG \).

The conjecture (ZC) is not true in general, as was shown by Roggenkamp and Scott who constructed a metabelian counter-example \[38, IX \S1\]. Note that since this counter-example is a metabelian group, (IP) is still valid for this group. Recently, Hertweck also found a counter-example to the general validity of (IP) which is a soluble group \[18\].

Nevertheless, it still remains an interesting question for which classes of finite groups (ZC) is valid. Roggenkamp and Scott showed that (ZC) is valid for groups whose generalized Fitting subgroup is a \( p \)-group \[37\]. Moreover, Weiss was able to show that for nilpotent groups each finite subgroup of \( V(ZG) \) is conjugate to a subgroup of \( G \) by a unit of \( QG \) \[45, 46\]. In case \( G \) is soluble, Čech style cohomology sets can be used to get obstructions for (ZC) to be true \[30\].

For finite simple groups however, new methods have to be developed to examine (ZC). In \[4\] it is proven that (ZC) is valid for all minimal simple groups and for all simple Zassenhaus groups. But it is still an open question whether the conjecture (ZC) is valid for all finite simple groups.
If $G$ is finite simple, then (IP) has always a positive solution [29, Thm. 2.3]. Furthermore, from [5, Thm. 1] it follows that (IP) is valid for all finite groups of Lie type $G^F$, where $G$ is a simply connected simple algebraic group over an algebraically closed field of finite characteristic and $F$ is a Frobenius map in the sense of [8, §1.17]. Note that these groups $G^F$ are certain central extensions of the finite simple groups of Lie type. In general, if (IP) holds for $G$, (ZC) is equivalent to the following description of the group $\text{Aut}_{n}(ZG)$ of augmentation preserving ring automorphisms of $ZG$:

(ZCAut). Every $\sigma \in \text{Aut}_{n}(ZG)$ can be written as $\sigma = \tau \circ \alpha$, where $\alpha$ is the $Z$–linear extension of a group automorphism of $G$ and $\tau \in \text{Aut}_{n}(ZG)$ fixes the class sums of $G$.

Thus, if (IP) is valid for $G$, the structure of certain ring automorphisms of $ZG$ can be used to study the group of units $V(ZG)$. Since (IP) has a positive solution for all groups considered in this paper, we will always examine (ZCAut) to verify (ZC).

The main results of this paper are the following two theorems:

**Theorem 1.** The conjecture (ZC) is valid for all finite simple groups with abelian Sylow $2$-subgroups.

**Theorem 2.** The conjecture (ZC) is valid for all finite simple groups of Lie type of rank $1$ and of rank $2$, which are not isomorphic to the unitary groups $\text{PSU}(4,q^2)$ and $\text{PSU}(5,q^2)$.

Note that because of the close relationship of the ordinary and modular representation theory of linear and unitary groups (see [26] and [43]), $\text{PSU}(4,q^2)$ and $\text{PSU}(5,q^2)$ should be examined together with the groups $\text{PSL}(4,q)$ and $\text{PSL}(5,q)$.

To prove these statements we use ordinary and modular representation theory. In particular, modular representations in the defining characteristic as described by Steinberg’s tensor product theorem [43] play an important role. Here the knowledge of the action of $\sigma \in \text{Aut}_{n}(ZG)$ on tensor products as given in [4, Prop. 2.1] is essential. Furthermore, we need to examine the generic ordinary character tables of the considered groups. Except for the groups $^2F_4(q^2)$, these character tables can be found in [9], [13], [14], [15], [16], [34], [41], [42], or [44]. All character tables are also given in the computer algebra program CHEVIE [17] which provides generic ordinary character tables for several series of finite groups of Lie type of small rank. Furthermore, in CHEVIE known character tables have been verified, mistakes have been corrected and the data has been completed.

The article is organized as follows: In Section 2, we state basic results and techniques which are needed to prove the theorems. In particular, we show that if $G$ is a finite simple group of Lie type with defining characteristic $p$, or a certain central extension, then the action of $\sigma \in \text{Aut}_{n}(ZG)$ on the
p-modular character table commutes with the operation induced by field automorphisms. In Section 3, the validity of (ZC) for the finite Ree groups of type $F_4$ and $G_2$ is proved. This establishes, together with the results of [4], Theorem 1. Section 4 is then dedicated to the proof of Theorem 2. Note that in [4] it is already shown that (ZC) is valid for the groups $\text{SL}(2, q)$, $\text{PSL}(2, q)$ and $^2B_2(q^2)$ for all possible prime powers $q$.

The notations used are mainly standard, see for example [11]. The set of the conjugacy classes of $G$ is denoted by $\text{Cl}(G)$, $\text{Cl}(g)$ is the conjugacy class of $g \in G$, and $\text{Cl}(G_{p'})$ denotes the set of the conjugacy classes of $p$-regular elements of $G$. Suppose $(K,R,k)$ is a $p$-modular system with $K$ sufficiently large relative to $G$ and $\text{char}(K) = 0$. Thus both $K$ and $k$ are splitting fields for $G$. Then the set of the ordinary irreducible characters $\text{Irr}(G)$ is identified with the set of the characters afforded by simple $KG$-modules. The irreducible Brauer characters $\text{IBr}(G)$ are the irreducible Brauer characters with respect to $(K,R,k)$.

2. Preliminaries.

In this section, we want to provide basic results and techniques needed to prove the Theorems 1 and 2.

Suppose $G$ is an arbitrary finite group and $(K,R,k)$ is a $p$-modular system with $K$ sufficiently large relative to $G$. Given $\sigma \in \text{Aut}_n(\mathbb{Z}G)$, $\sigma$ induces augmentation preserving algebra automorphisms $\sigma \in \text{Aut}_n(KG)$, respectively $\sigma \in \text{Aut}_n(kG)$, because there are ring homomorphisms from $\mathbb{Z}$ to $K$, respectively $k$. Then $\sigma$ defines an autoequivalence of the category of finitely generated $KG$-modules, respectively $kG$-modules, by mapping a module $M$ to $M^\sigma$. The twisted module $M^\sigma$ is defined to be equal to $M$ as vector space, but $g \in G$ operates on $M^\sigma$ as $\sigma(g)$. Thus $\sigma$ induces an operation on the ordinary characters, respectively Brauer characters, of $G$. Because $\sigma$ also induces a class sum correspondence [38, IV], i.e. $\sigma$ maps class sums to class sums, $\sigma$ defines also an action on the conjugacy classes of $G$. These two actions are compatible in the sense that $\sigma$ induces a character table automorphism of the ordinary, respectively $p$-modular, character table of $G$. This means that for all $\chi \in \text{Irr}(G)$, $\varphi \in \text{IBr}(G)$, $C \in \text{Cl}(G)$ and $C' \in \text{Cl}(G_{p'})$

$$\chi^\sigma(C) = \chi(C^\sigma)$$
$$\varphi^\sigma(C') = \varphi(C'^\sigma).$$

Furthermore, the operation of $\sigma$ on characters commutes with tensor products:

**Proposition 2.1** ([4, Prop. 2.1]). Let $\xi$ and $\zeta$ be two ordinary characters, respectively Brauer characters, then

$$(\xi \otimes \zeta)^\sigma = \xi^\sigma \otimes \zeta^\sigma.$$
By [28, Thm. V.1(c)], it follows that \( \sigma \) preserves also the power map in the sense that

\[
(C^n)^\sigma = (C^\sigma)^n
\]

for all positive integers \( n \). Note that if \( C = \text{Cl}(g) \), then \( C^n = \text{Cl}(g^n) \).

The operation on the ordinary character table induced by \( \sigma \in \text{Aut}_n(ZG) \) yields a statement which is equivalent to \( (ZCAut) \):

**Lemma 2.2.** The conjecture \( (ZCAut) \) is valid for \( G \) if and only if for every \( \sigma \in \text{Aut}_n(ZG) \) there exists \( \alpha \in \text{Aut}(G) \) such that \( \sigma \) and \( \alpha \) induce the same character table automorphism on the ordinary character table of \( G \).

Thus Lemma 2.2 provides a necessary and sufficient criterion for the validity of \( (ZCAut) \) which uses only the ordinary character table and its automorphisms.

For the remainder of the paper, we want to concentrate on finite groups which are associated to Chevalley groups of universal type over an algebraically closed field.

Let \( G \) be a simple Lie algebra over \( \mathbb{C} \) with Cartan-subalgebra \( \mathcal{H}_\mathbb{C} \). Let \( \Phi \) be the root system of \( G \) with respect to \( \mathcal{H}_\mathbb{C} \), and let \( \Delta = \{ e_1, \ldots, e_l \} \) be the corresponding system of fundamental roots.

Suppose \( k \) is an algebraically closed field of characteristic \( p > 0 \). The simply connected simple algebraic group, i.e. the Chevalley group of universal type, \( G \) of type \( \Phi \) over \( k \) is generated by \( \{ x_r(t) \mid r \in \Phi, \; t \in k \} \) with relations as in [7, Thm. 12.1.1]. If \( h_s(t) = x_s(t)x_{-s}(-t^{-1})x_s(t)x_s(-1)x_{-s}(1)x_s(-1) \), then \( \mathbb{T} = \{ h_s(t) \mid s \in \Delta, \; t \in k \} \) is a maximal torus of \( G \).

Since \( G \) is simply connected, the group \( X = X(\mathbb{T}) \) of rational characters of \( \mathbb{T} \) is the full lattice of weights, with a basis consisting of the fundamental weights \( \{ \lambda_1, \ldots, \lambda_l \} \). Denote by \( X^+ \) the set \( \sum Z_{\mathbb{N}}^+ \lambda_i \) of all dominant weights. There exists a natural partial ordering of \( X \) given by \( \mu \leq \lambda \) if \( \lambda - \mu \) is a sum of positive roots. The group \( X/pX \) of restricted weights with respect to \( p \) is identified with \( X_p = \{ \sum c_i \lambda_i \mid 0 \leq c_i \leq p-1 \} \). Given \( \lambda \in X^+ \), \( \lambda \) can be written uniquely as \( \lambda = \mu_0 + p\mu_1 + \cdots + p^{m-1}\mu_{m-1} \) for a suitable \( m \geq 1 \) with \( \mu_i \in X_p \). If \( X_q = \{ \sum c_i \lambda_i \mid 0 \leq c_i \leq q-1 \} \), \( q = p^m \), then it follows that \( \lambda \in X_q \).

Since \( G \) has a Chevalley basis \( \{ e_r, h_s \mid r \in \Phi, \; s \in \Delta \} \) as \( \mathbb{C} \)-basis whose structure constants all lie in \( \mathbb{Z} \), we can define \( G_{\mathbb{Z}} \) to be the \( \mathbb{Z} \)-lattice spanned by the Chevalley basis, and \( G_k = k \otimes_{\mathbb{Z}} G_{\mathbb{Z}} \). Then \( G_k \) is a Lie algebra over \( k \), and we denote \( 1 \otimes e_r \) and \( 1 \otimes h_s \) again by \( e_r \) and \( h_s \). A \( G_k \)-module \( M \) is called restricted, if \( e_r^p \cdot m = 0 \) and \( h_s^\rho \cdot m = h_s \cdot m \) for all \( r \in \Phi, \; s \in \Delta \) and \( m \in M \).

The non-isomorphic irreducible restricted \( G_k \)-modules, which were described by Curtis (see [43, (2.7)]), correspond bijectively to the restricted weights in \( X_p \). Let \( \mathcal{M} \) be a full set of representatives of these modules, and
let $M_\lambda$ denote the irreducible restricted $G_k$-module in $\mathcal{M}$ which corresponds to $\lambda \in X_p$.

According to [43, §4] the $G_k$-module $M_\lambda$ can be lifted to a simple $kG$-module $M_\lambda$ which is generated by a maximal vector $v^+$ of highest weight $\lambda$. This means that if we look at the weight spaces $M_\lambda(\mu) = \{ m \in M_\lambda \mid x \cdot m = \mu(x) \cdot m \text{ for all } x \in \mathbb{T} \}$, $\mu \in X$, then $v^+ \in M_\lambda(\lambda)$ and $x_r(t) \cdot v^+ = v^+$ for all $r \in \Phi^+$. For all $\mu \in X$ with $M_\lambda(\mu) \neq 0$, it follows that $\mu \leq \lambda$.

Note that each field automorphism $\alpha$ of $\bar{k}$ induces a group automorphism $\alpha$ of $G$ by defining $\alpha(x_r(t)) = x_r(\alpha(t))$, $r \in \Phi$, $t \in \bar{k}$.

The isomorphism classes of simple $kG$-modules can now be described as follows:

**Theorem 2.3** (Chevalley, Kostant, Steinberg, see [23, §2.1]). Let $\alpha_i$ be the field automorphism of $\bar{k}$ which is defined by $\alpha_i(t) = t^{q^i}$. Let $\lambda \in X^+$, such that $\lambda \in X_q$ for a suitable $q = p^n$, $\lambda = \mu_0 + p\mu_1 + \cdots + p^{m-1}\mu_{m-1}$ with $\mu_i \in X_p$. Then

$$M_\lambda = M_{\mu_0}^{\alpha_0} \otimes_k M_{\mu_1}^{\alpha_1} \otimes_k \cdots \otimes_k M_{\mu_{m-1}}^{\alpha_{m-1}}$$

is a simple $kG$-module. All $M_\lambda$ ($\lambda \in X^+$) are pairwise non-isomorphic and exhaust the isomorphism classes of simple $kG$-modules.

Let $q = p^n$ and let $F = GF(q)$ be the finite subfield of $\bar{k}$ with $q$ elements. Then the finite Chevalley group $G = G(F)$ over $F$ is generated by $\{ x_r(t) \mid r \in \Phi, t \in F \}$ with the same relations as $\mathbb{G}$. Note: $\bar{G} = G/\text{center}(G)$ is a simple group except for the cases $\text{SL}(2,2)$, $\text{PSL}(2,3)$, $\text{Sp}(4,2)$ and $G_2(2)$ [7, Thm. 11.1.2].

The twisted groups are certain subgroups of finite Chevalley groups:

Suppose $G$ is a finite Chevalley group of type $A_l$, $l \geq 2$, $D_l$, $l \geq 4$, $E_6$, or $D_4$, respectively, over a finite field $F$. Let $\omega$ be a nontrivial symmetry of order $e$ of the Dynkin diagram and let $\gamma$ be the corresponding graph automorphism. Let $\alpha$ be a nontrivial field automorphism such that $\Omega = \gamma\alpha$ satisfies $\Omega^e = 1$, and let $F_0$ be the subfield of $F$ fixed by $\alpha$. Note that $F$ is of the form $F = GF(p^{ne})$ and $F_0 = GF(p^n)$. If $G^1$ is defined to be the group of all elements of $G$ fixed by $\Omega$, then $G^1$ is a twisted group of type $^2A_l$, $l \geq 2$, $^2D_l$, $l \geq 4$, $^2E_6$, or $^3D_4$, respectively. $\bar{G}^1 = G^1/\text{center}(G^1)$ is simple except for $\text{PSU}(3,2^2)$ [7, Thm. 14.4.1].

The definition of the finite Suzuki and Ree groups is slightly different:

Suppose $G$ is a finite Chevalley group of type $B_2$, $G_2$, or $F_4$, respectively, over a finite field $F$ which has characteristic 2 for $B_2$ and $F_4$, and characteristic 3 for $G_2$. Let $\omega$ be the nontrivial symmetry of the Dynkin diagram with corresponding graph automorphism $\gamma$. Let $\alpha$ be a nontrivial field automorphism such that $\Omega^2 = 1$ for $\Omega = \gamma\alpha$. Note that $F$ is of the form $F = GF(p^{2m+1})$, $p = 2$ or 3. If $G^1$ is the group of all elements of $G$ fixed by
Ω, then $G^1$ is a Suzuki group if $G$ has type $B_2$, and a Ree group if $G$ has type $G_2$ or $F_4$. $G^1$ is simple for all positive $m$ [7, Thm. 14.4.1].

Since the $\bar{k}G$-modules $M_\lambda$, $\lambda \in X_q$, define by restriction $\bar{k}G$-modules, respectively $\bar{k}G^1$-modules, we get the following theorem:

**Theorem 2.4** ([43, Thm. 7.4, Thm. 9.3 and Thm. 12.2]). Let $G$ be a finite Chevalley group over $GF(p^n)$, or a twisted group of type $^2A_l$, $l \geq 2$, $^2D_l$, $l \geq 4$, $^2E_6$, or $^3D_4$ over $GF(p^{ne})$ where $e$ is the order of the symmetry of the corresponding Dynkin diagram. Then every simple $\bar{k}G$-module can be expressed uniquely as a tensor product

$$M_0^\alpha \otimes_{\bar{k}} M_1^{\alpha_1} \otimes_{\bar{k}} \cdots \otimes_{\bar{k}} M_{n-1}^{\alpha_{n-1}}$$

with $M_i \in \mathcal{M}$.

In case that $G$ is a Suzuki or Ree group of type $^2B_2$, $^2G_2$, or $^2F_4$ over $GF(p^n)$, $p = 2$ or $3$, set $X'_p = \{ \lambda = \sum c_i \lambda_i \in X_p \mid c_i = 0 \text{ for all long roots } s_i \in \Delta \}$ and $\mathcal{M}' = \{ M_\lambda \mid \lambda \in X'_p \}$. Then every simple $\bar{k}G$-module can be expressed uniquely as a tensor product

$$M_0^\alpha \otimes_{\bar{k}} M_1^{\alpha_1} \otimes_{\bar{k}} \cdots \otimes_{\bar{k}} M_{n-1}^{\alpha_{n-1}}$$

with $M_i \in \mathcal{M}'$.

**Remark 2.5.** Let $G$ be a finite Chevalley group, a twisted group or a finite Suzuki or Ree group.

(i) Since the simple $\bar{k}G$-modules are given as restrictions of certain simple modules of the corresponding Chevalley group $\mathbb{G}$ of universal type, we can attach weights $\lambda \in X^+$ to the simple $\bar{k}G$-modules. If we choose $\lambda \in X_{p^n}, n$ as in Theorem 2.4, then these weights determine the simple $\bar{k}G$-modules uniquely up to isomorphisms. In this sense we will write the simple $\bar{k}G$-modules also as $M_\lambda$ for suitable weights $\lambda \in X_{p^n}$. Furthermore, the Brauer character corresponding to $M_\lambda$ will be denoted by $\beta_\lambda$.

(ii) We will also use $M_\lambda$ to denote the $\bar{k}G$-module obtained by restriction of the $k\mathbb{G}$-module $M_\lambda$ in case $\lambda \in X^+ \setminus X_q$. Note that then this restriction is not necessarily a simple $\bar{k}G$-module. Again the corresponding Brauer character is denoted by $\beta_\lambda$.

(iii) The simple modules for the groups $\bar{G} = G/\text{center}(G)$ are exactly those simple $\bar{k}G$-modules on which the center acts trivially.

(iv) For the finite groups the algebraically closed field $\bar{k}$ can be replaced by any splitting field of the same characteristic.

We want now to describe some general properties of the considered finite groups of Lie type which are also important for the proofs of the Theorems 1 and 2.

For the remainder of the paper, we fix the following notation:

Let $G$ be a finite Chevalley group over $GF(p^n)$, a twisted group over $GF(p^{ne})$, or a finite Suzuki or Ree group over $GF(p^n)$ as in Theorem 2.4, and let $q = p^n$. Let $Z = \text{center}(G)$ and let $\bar{G} = G/Z$. The group automorphism
of $G$ induced by the field automorphism $t \mapsto t^p$ will be denoted by $\alpha_i$. Then $\tilde{\alpha}_i$ with $\tilde{\alpha}_i(gZ) = \alpha_i(g)Z$ is the corresponding group automorphism of $\tilde{G}$. Let $\sigma \in \text{Aut}_n(\mathbb{Z}G)$ and $\tilde{\sigma} \in \text{Aut}_n(\mathbb{Z}\tilde{G})$ be arbitrary elements.

Let $(K, R, k)$ be a $p$-modular system with $K$ sufficiently large relative to $G$ such that $\tilde{k}$ is an algebraic closure of $k$. Since $k$ is a splitting field for $G$, the isomorphism classes of simple $kG$-modules can be identified with those of simple $\tilde{kG}$-modules. Let $G$ be the Chevalley group of universal type over $k$ corresponding to $G$.

**Lemma 2.6.** The action of $\sigma$ on the simple $kG$-modules $S$ commutes with the action of $\alpha_i$ for all $i$, i.e. $S^{\sigma\alpha_i} \cong S^{\alpha_i\sigma}$ for all $i$. The action of $\alpha_i$ on the $p$-regular conjugacy classes of $G$ is given by $\text{Cl}(g) \mapsto \text{Cl}(\tilde{g}^p)$.

**Proof.** Because of Proposition 2.1 and Theorem 2.4, it suffices to show this for the simple modules $M \in \mathcal{M}$ or $\mathcal{M}'$, respectively. Consider $M$ as a $kG$-module. Thus we have to show $M^{\alpha_i\sigma} \cong M^{\sigma\alpha_i}$ as $kG$-modules.

Suppose $\rho : G \to \text{GL}(m, \tilde{k})$ is the representation corresponding to $M$ as a $\tilde{k}G$-module. Let $\nu_i$ be the automorphism of $\text{GL}(m, \tilde{k})$, which maps a matrix $(a_{rs})$ to $(a_{rs}^{-1})$. Then, if $M^{(i)}$ denotes the module corresponding to the representation $\nu_i \circ \rho$, we get $M^{\alpha_i} \cong M^{(i)}$ as $\tilde{k}G$-modules (see for example [31, Prop. 5.4.2.(i)]). A simple matrix calculation shows that, if $\xi_1, \ldots, \xi_m$ are the eigenvalues of $\rho(g)$, then $\xi_1^p, \ldots, \xi_m^p$ are the eigenvalues of $\nu_i(\rho(g))$ and of $\rho(\tilde{g}^p)$.

If $\varphi$ is the Brauer character corresponding to $\rho$, and $\varphi^{(i)}$ is the Brauer character corresponding to $\nu_i \circ \rho$, then it follows that

$$\varphi^{\alpha_i}(g) = \varphi^{(i)}(g) = \varphi(\tilde{g}^p)$$

for all $p$-regular elements $g \in G$.

Since Equation (2.1) is valid for all Brauer characters of $G$ by Proposition 2.1, and since $g \mapsto g^p$ is a permutation of the $p$-regular elements, the permutation of the $p$-regular conjugacy classes of $G$ corresponding to $\alpha_i$ is given by $\text{Cl}(g) \mapsto \text{Cl}(\tilde{g}^p)$.

Since $\sigma$ preserves the power map, i.e. $(C^\sigma)^\sigma = (C^\sigma)^\alpha$ for all conjugacy classes $C = \text{Cl}(g)$ and $C^\alpha = \text{Cl}(\tilde{g})$, it follows that $\varphi^{\sigma\alpha_i} = \varphi^{\alpha_i\sigma}$ and therefore, $M^{\sigma\alpha_i} \cong M^{\alpha_i\sigma}$ for all $M \in \mathcal{M}$ or $\mathcal{M}'$, respectively.

**Corollary 2.7.** The operation of $\tilde{\alpha}_i$ on the $p$-regular conjugacy classes of $\tilde{G}$ is given by $\text{Cl}(\tilde{g}) \mapsto \text{Cl}(\tilde{g}^p)$. In particular, it follows that $S^{\sigma\tilde{\alpha}_i} \cong S^{\tilde{\alpha}_i\sigma}$ for all simple $kG$-modules $S$.

**Proof.** By Lemma 2.6, $F(\alpha_i(g)) = \text{Cl}(\tilde{g})$, i.e. $\alpha_i(g)$ is conjugate to $\tilde{g}^p$ in $G$ for all $p$-regular elements $g \in G$. Therefore, $\tilde{\alpha}_i(\tilde{g}Z)$ is conjugate to $(\tilde{g}Z)^p$ in $\tilde{G}$, i.e. $\alpha_i(g)$ is conjugate to $\tilde{g}^p$ for all $p$-regular elements $\tilde{g} \in \tilde{G}$. Thus we have that $\tilde{\alpha}_i(\text{Cl}(\tilde{g})) = \text{Cl}(\tilde{g}^p)$. Because $\tilde{g} \mapsto \tilde{g}^p$ is a permutation
of the $p$-regular elements of $\bar{G}$ and because $\sigma$ preserves the power map, the corollary follows. \hfill \Box

With respect to the weights of tensor products we have the following result.

**Lemma 2.8.** Let $\lambda, \mu \in X^+$, and let $M_\lambda$ and $M_\mu$ be the two (not necessarily simple) $kG$-modules which are restrictions of the corresponding simple $\bar{k}G$-modules. Then $M_{\lambda+\mu}$ is a factor module of $M_\lambda \otimes M_\mu$ as $kG$-modules. Furthermore, as $\bar{k}G$-modules, $\lambda + \mu$ is the highest weight occurring in $M_\lambda \otimes M_\mu$.

Suppose now that $M_\lambda$ and $M_\mu$ are simple $kG$-modules with $\lambda, \mu \in X_q$. If $M_\lambda \otimes M_\mu$ is also a simple $kG$-module then $M_\lambda \otimes M_\mu \cong M_{\lambda+\mu}$ as $kG$-modules. Since $\lambda + \mu \in X_{pq}$, it follows that $\lambda + \mu = \nu_0 + q\nu_1$ for appropriate $\nu_0 \in X_q$ and $\nu_1 \in X_p$. Thus

$$M_{\lambda+\mu} \cong M_{\nu_0} \otimes M_{\nu_1}^\alpha$$

as $kG$-modules. Note that $M_{\nu_0}^\alpha \cong M_{\nu_1}$ if $G$ is a finite Chevalley group or a Suzuki or Ree group. Otherwise $M_{\nu_1}^\alpha \cong M_{\omega^{-1}(\nu_1)}$, where $\omega$ is the symmetry of the Dynkin diagram in the definition of the twisted groups.

**Proof.** Let $\lambda, \mu \in X^+$. Then there exist maximal vectors $v$, respectively $w$, of highest weight $\lambda$, respectively $\mu$, in the $\bar{k}G$-module $M_\lambda$, respectively $M_\mu$. By [22, §31.4], $v \otimes w$ is then a maximal vector of the $\bar{k}G$-module $M_\lambda \otimes M_\mu$ of weight $\lambda + \mu$, which is the highest weight occurring in $M_\lambda \otimes M_\mu$. So $M_{\lambda+\mu}$ is a factor module of $M_\lambda \otimes M_\mu$ as $\bar{k}G$-modules and thus as $kG$-modules.

Let now $\lambda, \mu \in X_q$. If $M_\lambda \otimes M_\mu$ is simple as a $kG$-module then $M_\lambda \otimes M_\mu$ is also simple as a $kG$-module. Thus $M_\lambda \otimes M_\mu$ is simple with weight $\lambda + \mu$. Since $\lambda, \mu \in X_q$ and $2(q - 1) \leq pq - 1$, it follows that $\lambda + \mu = \nu_0 + q\nu_1$ for appropriate $\nu_0 \in X_q$ and $\nu_1 \in X_p$. Thus $M_{\lambda+\mu} \cong M_{\nu_0} \otimes_k M_{\nu_1}^\alpha$ as $kG$-modules. \hfill \Box

We want now to outline the strategy to prove the conjecture (ZC) for certain $G$ and $\bar{G}$.

**Definition 2.9.** Denote by $\bar{X}^+$ all dominant weights $\lambda \in X^+$ such that $M_\lambda$ is a $kG$-module. In case that $G$ is a Suzuki or a Ree group, let $X^{+'} = \{ \lambda = \sum c_i \lambda_i \in X^+ \mid c_i = 0 \text{ for all long roots } s_i \in \Delta \}$. Let $Y^+$ be either $X^+$, $X^{+'}$, or $\bar{X}^+$. 

(i) A relation $\prec$ on $Y^+$ is called a weak order if $\prec$ is reflexive and transitive.

(ii) An allowable system on $Y^+$ is a subset $R \subseteq Y^+$ such that all $\lambda \in Y^+$ can be written as

$$\lambda = \sum_{\rho \in R} d_\rho \rho$$

for a unique choice of integers $d_\rho \geq 0$. 

•
(iii) Let $R$ be an allowable system and $\prec$ a weak order on $Y^+$. Then $\prec$ is called an allowable weak order if the following properties are fulfilled:

- (a) If $\lambda \in \{0\} \cup R$ and $\mu \in Y^+$ with $\mu \prec \lambda$, then $\mu \in \{0\} \cup R$.
- (b) If $\mu_1 \leq \mu_2$ and $\mu_1 \neq \mu_2$, it follows that $\mu_1 \prec \mu_2$ and $\mu_2 \prec \mu_1$.
- (c) If $\lambda = \sum d_{\rho} \rho$ is a dominant weight with $\lambda \notin \{0\} \cup R$, then there exists $\rho_0 \in R$ with $\rho_0 \prec \lambda$ and $\lambda - \rho_0 \prec \lambda$, but $\lambda \not\prec \lambda - \rho_0$.

**Remark 2.10.** (i) If $\prec$ is a weak order on $Y^+$ then we can define an equivalence relation $\sim$ on $Y^+$ by

$$\lambda \sim \mu \iff \lambda \prec \mu \text{ and } \mu \prec \lambda.$$ 

Then $\prec$ defines a partial ordering of $Y^+/\sim$, the equivalence classes of $Y^+$ with respect to $\sim$.

(ii) For $Y^+ = X^+$ (respectively $X^{\prime,+}$) we can and will always choose the fundamental weights $\lambda_i \in X_\rho$ (respectively $X^{\prime}_\rho$) as allowable system.

(iii) Definition 2.9 is written in this generality since we want to prove (ZC) for groups $G$ and $\bar{G}$.

(iv) The order relation $\leq$ on $X^+$ does not always fulfill property (c). For example, for $\text{SL}(3,q)$, $\lambda_1 \not\leq \lambda_1 + \lambda_2$ and $\lambda_2 \not\leq \lambda_1 + \lambda_2$.

The following proposition plays an important role in the examination of (ZC).

**Proposition 2.11.** Let $(H, Y^+)$ be either $(G, X^+)$, $(G, X^{\prime,+})$, or $(\bar{G}, \bar{X}^+)$, let $\tau$ be either $\sigma$ or $\bar{\sigma}$, and let $R$ be an allowable system of $Y^+$. Suppose

- (i) there exists a group automorphism $\alpha \in \text{Aut}(H)$ with $M^\tau_\rho \cong M^\sigma_\rho$ for all $\rho \in R$, and
- (ii) there exists an allowable weak order $\prec$ on $Y^+$.

Then $\tau$ and $\alpha$ induce the same operation on the $p$-modular character table of $H$.

**Proof.** We have to show that $\beta^\tau_\lambda = \beta^\sigma_\lambda$ for all $\lambda \in Y^+ \cap X_q$.

By (i) there exists a group automorphism $\alpha$ such that $\beta^\tau_\rho = \beta^\sigma_\rho$ for all $\rho \in R$. Thus, by property (a) of $\prec$, the statement is valid for the smallest weights with respect to $\prec$.

Let now $\lambda \in Y^+$, $\lambda \notin \{0\} \cup R$. Suppose $\beta^\tau_\mu = \beta^\sigma_\mu$ for all $\mu \in Y^+$ with $\mu \prec \lambda$ and $\mu \not\prec \mu$, which means, by property (b) of $\prec$, in particular for all $\mu \in Y^+$ with $\mu \leq \lambda$, $\mu \neq \lambda$. By property (c), there exists $\rho_0 \in R$ such that $\mu_1 = \rho_0 \prec \lambda$, and $\mu_2 = \lambda - \rho_0 \prec \lambda$, but $\lambda \not\prec \mu_2$. Thus $\beta^\tau_{\mu_s} = \beta^\sigma_{\mu_s}$ for $s \in \{1,2\}$. By Lemma 2.8, $\beta_{\mu_1} \otimes \beta_{\mu_2}$ can be written as

$$\beta_{\mu_1} \otimes \beta_{\mu_2} = \sum_{\nu \in Y^+} a_\nu \beta_\nu$$

with $a_\nu \geq 0$

such that $a_\lambda \neq 0$ and $\nu \leq \lambda$ for all other $\nu \in Y^+$ with $a_\nu \neq 0$. Using Proposition 2.1, it follows that $\beta^\tau_\lambda = \beta^\sigma_\lambda$. 

$\square$
Remark 2.12. (i) Proposition 2.11 implies that $\tau$ operates as $\alpha$ on all ordinary irreducible characters which are not $p$-exceptional. Note that a character $\chi \in \text{Irr}(H)$ is called $p$-exceptional, if there exists another character $\chi' \in \text{Irr}(H)$ such that $\chi$ and $\chi'$ have the same values on the $p$-regular classes of $H$.

(ii) All group automorphisms of $G$, respectively $\bar{G}$, are given as the composition of an inner, a diagonal, a graph and a field automorphism [7, Thm. 12.5.1]. Since the inner and diagonal automorphisms operate trivially on the $p$-regular classes, $\alpha$ can be chosen as the composition of a graph and a field automorphism.

Let $H$, $\tau$ and $\alpha$ be as in Proposition 2.11. Then $\tau \alpha^{-1}$ operates trivially on all ordinary irreducible characters which are not $p$-exceptional. We will use ad-hoc techniques to refine $\alpha$ to a group automorphism $\beta$ of $H$ such that $\tau \beta^{-1}$ fixes also all $p$-exceptional characters. Using Lemma 2.2 this then implies that (ZC) is valid for $H$.

In this context, $l$-blocks of $H$ which have cyclic defect groups and their corresponding Brauer trees are important. Note that usually the prime $l$ is different from $p$.

There are two interpretations of the Brauer tree corresponding to a block $B$ with cyclic defect groups. One approach is to define the Brauer tree by only using the category of finitely generated $B$-modules. This is described in [1, Chapter V]. The other possibility is to define the Brauer tree using the decomposition map. This follows from Brauer’s and Dade’s theory of blocks with cyclic defect groups (see [12]). By [12, Thm. 1] it follows that these two definitions coincide. As a reference for both interpretations of the Brauer tree of $B$ use for example [6].

With respect to the operation of $\tau$ on the Brauer trees corresponding to $l$-blocks with cyclic defect groups, we need the following two results. Note that $\tau$ permutes the $l$-blocks of $H$.

**Lemma 2.13.** Let $B$ be an $l$-block of $H$ with cyclic defect groups such that $\tau$ fixes $B$. Then $\tau$ induces a graph automorphism of the Brauer tree of $B$ which is either trivial or a rotation.

**Proof.** This follows from [6, Prop. 3.7] and the proof of [6, Cor. 3.9].

**Lemma 2.14 ([6, Cor. 3.10]).** Let $H$ have cyclic Sylow $l$-subgroups and let $B$ be the principal $l$-block. If $\chi \in \text{Irr}(H)$ belongs to $B$ and if $\chi$ is not $l$-exceptional, then $\tau$ fixes $\chi$.

We use these two lemmas in the following way. To refine $\alpha$ of Proposition 2.11, we want to show that $\tau \alpha^{-1}$ operates on the $p$-exceptional characters as the power of a diagonal automorphism. In case all diagonal automorphisms of $H$ are inner, we use Lemma 2.14 for certain $l$ to show that $\tau \alpha^{-1}$ fixes all $p$-exceptional characters. In the case of the Ree groups of type $F_4$, Lemma
2.13 is used to eliminate operations on characters which are not induced by group automorphisms.

In case that $H$ has a nontrivial diagonal automorphism $\delta$, the characters permuted by $\delta$ have to be examined separately. Here we use, if necessary, Lemma 2.13 to study all possible operations on Brauer trees which contain such characters.

**Remark 2.15.**

(i) To determine the $p$-exceptional characters we use mostly the generic character tables provided in CHEVIE [17]. As outlined in the introduction there are literature references for most of the considered tables, which are either verified or corrected in CHEVIE.

(ii) In the examination of the $p$-exceptional characters, these characters are decomposed into a union of pairwise disjoint subsets such that all characters which have the same values on the $p$-regular classes are in one such subset.

(iii) When Brauer trees are given, a black circle • denotes the exceptional vertex.

According to the above described strategy, the proofs of the validity of (ZC) for certain $G$ (respectively $\bar{G}$) will always be divided into two parts:

(A) Operation of $\sigma$ (respectively $\bar{\sigma}$) on the $p$-modular character table.
   Here we only have to check the hypotheses of Proposition 2.11.

(B) Operation of $\sigma$ (respectively $\bar{\sigma}$) on the $p$-exceptional characters.
   Here we use ad-hoc arguments together with Lemmas 2.13 and 2.14.

The proofs involve the following Dynkin diagrams: $A_2, B_2, G_2, D_4, F_4$. 
A weight $\lambda = \sum c_i\lambda_i$ will be denoted by $\lambda = c_1 \cdots c_l$.

3. The Ree groups $^2G_2(q^2)$ and $^2F_4(q^2)$.

In this section, we want to prove that (ZC) is valid for the finite Ree groups of type $G_2$ and $F_4$, and thus establish Theorem 1.

The Ree groups $^2G_2(q^2)$ and $^2F_4(q^2)$ are simple except for $^2G_2(3)$ and $^2F_4(2)$; their automorphisms are given by inner and field automorphisms. The generic ordinary character table of $^2G_2(3^{2m+1})$ was mostly determined by Ward in [44]. Shinoda described the conjugacy classes of $^2F_4(2^{2m+1})$ in [40]. The complete ordinary character table of $^2G_2(3^{2m+1})$ can be found in CHEVIE; most of the character table of $^2F_4(2^{2m+1})$ and all Green functions are also listed there.

Proposition 3.1. (i) The conjecture (ZC) is valid for $^2G_2(q^2)$ for all $q^2 = 3^{2m+1}$, $m \geq 1$.

(ii) The conjecture (ZC) is valid for $^2F_4(q^2)$ for all $q^2 = 2^{2m+1}$, $m \geq 1$.

Proof. Let $G = ^2G_2(q^2)$ or $G = ^2F_4(q^2)$, respectively.
Part (A).

(i) \( X'_p = \{00, 01, 02\} \) with \( \dim(M_{01}) = 7 \) and \( \dim(M_{02}) = 27 \). Because of dimensions, there exists \( 0 \leq i \leq 2m \) with \( M_{01}^i \cong M_{01}^0 \). Since \( \leq \) defines an allowable weak order \( \prec \) on \( X^+ \), the hypotheses of Proposition 2.11 are satisfied.

(ii) \( X'_p = \{0000, 0010, 0001, 0011\} \) with \( \dim(M_{0001}) = 26 \), \( \dim(M_{0010}) = 246 \) and \( \dim(M_{0011}) = 4096 \). Because of dimensions, there exist \( 0 \leq i,j \leq 2m \) with \( M_{0001}^i \sim M_{0010}^j \). We have to show that \( i = j \). If \( i \neq j \) then \( \sigma \) maps the simple module \( M_{0001}^i \otimes M_{0010}^j \) to \( M_{0001} \otimes M_{0010} \). This implies that \( M_{0001} \otimes M_{0010} \) must be simple, thus by Lemma 2.8 of weight 0011. A comparison of the dimensions shows that this is impossible. Thus it follows that \( i = j \) which yields hypothesis (i) of Proposition 2.11.

Part (B).

(i) \( G \) does not have any nontrivial diagonal automorphisms, thus we have to show that \( \sigma \alpha^{-1}_i, \alpha_i \) from Part (A), fixes all ordinary characters which are \( p \)-exceptional. The generic character table given in CHEVIE shows that there are exactly 6 unipotent characters which are \( p \)-exceptional: Using the notation of [20], these characters correspond to \( \{\xi_3, \xi_4\}, \{\xi_5, \xi_6\} \) and \( \{\xi_7, \xi_8\} \). If \( l > 3 \) is a rational prime which divides \( |G| \), then \( G \) has cyclic Sylow \( l \)-subgroups [25, XI Thm. 13.2]. By [20, Thm. 4.2], \( \xi_5, \xi_6 \) and \( \xi_7, \xi_8 \) lie in the principal \( l \)-block for \( l((q^2 + 1) \) and are not \( l \)-exceptional. By [20, Thm. 4.3], \( \xi_3, \xi_4 \) and \( \xi_5, \xi_6 \) lie in the principal \( l \)-block for \( l((q^2 + \sqrt{3}q + 1) \) and are not \( l \)-exceptional. Thus, by Lemma 2.14, \( \sigma \alpha^{-1}_i \) fixes all ordinary characters, which proves Proposition 3.1(i).

(ii) Since the generic character table of \( G \) given in CHEVIE is not complete, we have to look at this case more carefully. Again we have to show that \( \sigma \alpha^{-1}_i \) operates trivially on the characters which are \( p \)-exceptional.

Let \( G \) be the Chevalley group of universal type of type \( F_4 \) over \( \bar{k} \), and let \( F \) be the Frobenius map (notation used as in [8, §1.17]) with \( G^F = G \). Using the Jordan decomposition for characters, the ordinary irreducible characters of \( G \) can be described as follows (cf. [8, §12.9]): Since the Ree groups are self-dual, the characters of \( G \) are parametrized by \( G \)-conjugacy classes of pairs \( (s, \lambda) \) where \( s \) is a semisimple element of \( G \) and \( \lambda \) is a unipotent character of \( C_G(s) \). This character is then denoted by \( \chi_{s, \lambda} \). Since \( G \) is self-dual, the \( G \)-conjugacy classes of semisimple elements of \( G \) correspond to geometric conjugacy classes.
We use the notation in [40] for the 11 conjugacy classes of the maximal tori and the representatives of the semisimple conjugacy classes, i.e. of the semisimple class types. Note that the $G$-conjugacy classes of the maximal tori $T$ of $G$ are uniquely determined by the isomorphism type of $T^F$.

In CHEVIE, all ordinary irreducible characters of $G$ can be found except for those corresponding to the semisimple representatives $t_7$, $t_9$, $t_{12}$ and $t_{13}$. From the part of the character table given in CHEVIE we can read off the following:

Exactly 10 of the unipotent characters are $p$-exceptional. These are, using the notation of [20], \{ξ_5, ξ_6\}, \{ξ_7, ξ_8\}, \{ξ_{15}, ξ_{16}\}, \{ξ_{17}, ξ_{18}\} and \{ξ_{19}, ξ_{20}\}. By [20, Thm. 4.6 and Thm. 4.7] and Lemma 2.14, $σα_i^{-1}$ operates trivially on these 10 unipotent characters.

Looking at the other characters listed in CHEVIE, \{χ_{23}(k), χ_{24}(k)\} are $p$-exceptional for all parameters $k$. According to the Jordan decomposition, the characters $χ_{22}(k)$ through $χ_{25}(k)$ belong to the semisimple class type $t_1$. If $s$ is the representative of the $G$-conjugacy class of type $t_1$ corresponding to $k$, then $C_G(s) \cong ℤ_{q^2-1} \times 2B_2(q^2)$. Thus there are four possibilities for $λ$, namely the unipotent characters of $2B_2(q^2)$: 1, $μ$, $\bar{μ}$ and St. Therefore we have the correspondence $χ_{22}(k) \leftrightarrow χ_{s,1}$, $χ_{23}(k) \leftrightarrow χ_{s,μ}$, $χ_{24}(k) \leftrightarrow χ_{s,\bar{μ}}$ and $χ_{25}(k) \leftrightarrow χ_{s,St}$.

For a rational prime $l|(q^2 - \sqrt{2q} + 1)$, we have the following Brauer trees for such $s$ [19, Satz D.3.9]:

Thus, by Lemma 2.13, $σα_i^{-1}$ cannot induce the permutation $(χ_{s,μ}, χ_{s,\bar{μ}})$ which means that $σα_i^{-1}$ cannot permute $χ_{23}(k)$ and $χ_{24}(k)$.

There are similar Brauer trees for the semisimple class types $t_7$ and $t_9$. We have $C_G(t_7) \cong ℤ_{q^2 - \sqrt{2q} + 1} \times 2B_2(q^2)$, $C_G(t_9) \cong ℤ_{q^2 + \sqrt{2q} + 1} \times 2B_2(q^2)$. For each $G$-conjugacy class of $s$ of type $t_7$ or $t_9$, there are again 4 characters $χ_{s,1}$, $χ_{s,μ}$, $χ_{s,\bar{μ}}$ and $χ_{s,St}$. The Brauer trees are then as follows:

For $s$ of type $t_9$ and $l|(q^2 - \sqrt{2q} + 1)$, we get a tree analogous to the one for $s$ of type $t_1$ [19, Satz D.3.9].

For $s$ of type $t_7$ and $l|(q^2 + \sqrt{2q} + 1)$, the Brauer trees look like [19, Satz D.3.10]:

Thus, by Lemma 2.13, $σα_i^{-1}$ cannot induce the permutation $(χ_{s,μ}, χ_{s,\bar{μ}})$ which means that $σα_i^{-1}$ cannot permute $χ_{23}(k)$ and $χ_{24}(k)$.
Thus, by Lemma 2.13, $\sigma_\alpha^{-1}$ cannot permute $\chi_{s,\mu}$ and $\chi_{s,\bar{\mu}}$ for $s$ of type $t_7$ or $t_9$.

It remains to show that $\sigma_\alpha^{-1}$ operates trivially within each row of the ordinary generic character table.

This is clear for all the characters listed in CHEVIE except for those corresponding to the semisimple class type $t_1$. Thus we have to look at the characters corresponding to the semisimple types $t_1$, $t_7$, $t_9$ and $t_{12}$, $t_{13}$.

Since $t_{12}$ and $t_{13}$ are regular, i.e. the corresponding characters are of the form $\pm R_{T,\theta}$ with $T$ of type $T(6)$ for $t_{12}$ and of type $T(7)$ for $t_{13}$ and suitable $\theta$ in general position, the claim follows almost immediately.

For $s$ of type $t=t_1$, $t_7$ or $t_9$, respectively, we look at $\chi_{s,1} + \chi_{s,\text{St}}$.

If $s$ and $s'$ belong to different $G$-conjugacy classes of type $t$, it follows that $(\chi_{s,1} + \chi_{s,\text{St}})|_{G_{p'}} \neq (\chi_{s',1} + \chi_{s',\text{St}})|_{G_{p'}}$. Therefore $\sigma_\alpha^{-1}$ either does not permute $\chi_{s,1}$ and $\chi_{s',1}$ or it does not permute $\chi_{s,\text{St}}$ and $\chi_{s',\text{St}}$. Using the above Brauer trees, $\chi_{s,1}, \chi_{s,\mu}, \chi_{s,\bar{\mu}}$ and $\chi_{s,\text{St}}$ lie in the same Brauer tree of an $l$-block and are not $l$-exceptional. Thus $\sigma_\alpha^{-1}$ operates trivially on the rows of the generic ordinary character table corresponding to the semisimple class type $t$.

This proves Proposition 3.1(ii).

Remark 3.2. The conjecture (ZC) is also valid for the simple Tits group $2F_4(2)'$, which is the normal subgroup of $2F_4(2)$ of index 2. Since the proof uses different methods from those described here, we state this result without proof.

Proof of Theorem 1. The simple groups with abelian Sylow 2-subgroups have been classified by Walter [25, XI Thm. 13.7]. These are $\text{PSL}(2,2^f)$ for $f \geq 1$, $\text{PSL}(2,q)$ for $q = p^f$, $q \neq 3$, with $q \equiv 3 \pmod{8}$ or $q \equiv 5 \pmod{8}$, the Janko group $J_1$ and the finite Ree groups of type $G_2$.

The validity of (ZC) for $\text{PSL}(2,p^f)$ for arbitrary rational primes $p$ has been proved in [4, Prop. 3.2 and Prop. 4.1]. The theory of principal blocks with cyclic defect groups together with Lemma 2.14 shows that (ZC) holds for $J_1$. Thus, together with Proposition 3.1(i), Theorem 1 follows.

4. Finite groups of Lie type of rank 2.

In this section we want to deal with finite Chevalley groups and twisted groups of rank 2. This will especially prove Theorem 2.

4.1. The linear and unitary groups $\text{SL}(3,q)$, $\text{PSL}(3,q)$, $\text{SU}(3,q^2)$ and $\text{PSU}(3,q^2)$. The groups $\text{SL}(3,q)$ are simple for $q \neq 1(3)$, and $\text{SU}(3,q^2)$ are
simple for $q \neq -1\ (3)$. In these cases, the group automorphisms are generated by inner, graph and field automorphisms. In all other cases, there also exist diagonal automorphisms. The ordinary generic character tables have been determined in [41] and can also be found in CHEVIE. We first prove the following:

**Proposition 4.1.** The conjecture (ZC) is valid for $\text{SL}(3, q)$ and $\text{SU}(3, q^2)$ for all $q = p^m$, where $p$ is a rational prime and $m \geq 1$.

**Proof.** Let $G = \text{SL}(3, q)$ or $G = \text{SU}(3, q^2)$, respectively.

Part (A).

By [33, Thm. 1.1], all simple $kG$-modules with dimension smaller than or equal to 5 are of the form $M_{10}^{\alpha_j}$ or $M_{01}^{\alpha_j}$ for some $j$. Note that if $\gamma$ is the nontrivial graph automorphism for $G = \text{SL}(3, q)$, and if $\gamma$ is the field automorphism $\alpha_m$ for $G = \text{SU}(3, q^2)$, then $M_{10}^{\alpha_j} \cong M_{01}$. Furthermore, since in case $G = \text{SL}(3, q)$, $\gamma$ operates on the conjugacy classes as $(\text{Cl}(g))^\gamma = \text{Cl}(g^{-1})$ and since $\sigma$ preserves the power map, the action of $\sigma$ on the $p$-modular Brauer characters commutes with the action of $\gamma$. This implies hypothesis (i) of Proposition 2.11. The weak order $\prec$ on $X^+$ for hypothesis (ii) is defined as follows:

Let $\mu_1 = b_1b_2$ and $\mu_2 = c_1c_2$ be two dominant weights. Then $\mu_1 \prec \mu_2$ if and only if $b_1 + b_2 \leq c_1 + c_2$. To show that $\prec$ is allowable, we only have to show that $\mu_1 \leq \mu_2$, $\mu_1 \neq \mu_2$ implies $\mu_1 \prec \mu_2$ and $\mu_2 \prec \mu_1$. If $s_1$ and $s_2$ are the fundamental roots corresponding to 10 and 01, then we have

\[
\begin{align*}
10 &= \frac{1}{3}(2 \cdot s_1 + s_2), \\
01 &= \frac{1}{3}(s_1 + 2 \cdot s_2).
\end{align*}
\]

Thus it follows that $\mu_1 \leq \mu_2$ if and only if

\[
\begin{align*}
2b_1 + b_2 &\leq 2c_1 + c_2, \\
b_1 + 2b_2 &\leq c_1 + 2c_2,
\end{align*}
\]

and $2(c_1 - b_1) + (c_2 - b_2)$ and $(c_1 - b_1) + 2(c_2 - b_2)$ are multiples of 3. This implies that $b_1 + b_2 \leq c_1 + c_2$. If $b_1 + b_2 = c_1 + c_2$ and $\mu_1 \neq \mu_2$, then $\mu_1 \not\prec \mu_2$.

Part (B).

$G$ has only $p$-exceptional characters if $q \equiv 1\ (3)$ and $G = \text{SL}(3, q)$ or if $q \equiv -1\ (3)$ and $G = \text{SU}(3, q^2)$.

Let now $G = \text{SL}(3, q)$ and $q \equiv 1\ (3)$. In the notation of CHEVIE, there are 9 characters which are $p$-exceptional: \{\chi_6, \chi_7, \chi_8\}, \{\chi_{11}, \chi_{12}, \chi_{13}\}$ and \{\chi_{14}, \chi_{15}, \chi_{16}\}. The following actions on these characters are induced by group automorphisms: There exists a diagonal automorphism $\delta$ with operation $\tilde{\delta} = (\chi_6, \chi_7, \chi_8)(\chi_{11}, \chi_{12}, \chi_{13})(\chi_{14}, \chi_{15}, \chi_{16})$. Note that we use here cycle notation. For the graph automorphism $\gamma$ we get as operation
\(\tilde{\gamma} = (\chi_{11}, \chi_{14})(\chi_{12}, \chi_{15})(\chi_{13}, \chi_{16})\). If \(p \equiv 1 \pmod{3}\), we get for the field automorphism \(\alpha_1\) the same operation as for \(\gamma\). We want to show that the operations on the 9 \(p\)-exceptional characters generated by group automorphisms of \(G\) are the only operations that can be induced by \(\sigma\). Note that if \(\chi_{11}\) and \(\chi_{14}\) are permuted then this induces a nontrivial operation on the \(p\)-regular classes. On the other hand \(\delta\) operates trivially on the \(p\)-regular classes. Thus we only have to show that all operations on these 9 characters induced by \(\sigma\) are generated by \(\pi_1 = (\chi_6, \chi_7, \chi_8)(\chi_{11}, \chi_{12}, \chi_{13})(\chi_{14}, \chi_{15}, \chi_{16})\) and \(\pi_2 = (\chi_{11}, \chi_{14})(\chi_{12}, \chi_{15})(\chi_{13}, \chi_{16})\).

Note that \(|G| = q^3(q - 1)^2(q + 1)(q^2 + q + 1)\). By [24, II Satz 7.3], \(G\) has a cyclic subgroup of order \(q^2 + q + 1\). If \(l \neq 3\) is a rational prime which divides \(q^2 + q + 1\), then all Sylow \(l\)-subgroups are cyclic because \((q^2 + q + 1, q - 1) = 1\) and \((q^2 + q + 1, q + 1) = 1\). The characters \(\chi_{11}\) through \(\chi_{16}\) are not \(l\)-exceptional and lie in the following Brauer trees:

![Brauer tree](image)

By Lemma 2.13, it follows that all operations on the characters \(\chi_{11}, \chi_{12}, \chi_{13}, \chi_{14}, \chi_{15}, \chi_{16}\) induced by \(\sigma\) are generated by \(\pi_1\) and \(\pi_2\). Because of the corresponding operations of \(\pi_1\) and \(\pi_2\) on the conjugacy classes, this also follows for the characters \(\chi_6, \chi_7, \chi_8\).

For \(G = \text{SU}(3, q^3)\) and \(q \equiv -1 \pmod{3}\), the argumentation is similar. Here we use that \(\text{SU}(3, q^2)\) has a cyclic subgroup of order \(q^2 - q + 1\) [32, Satz 4.2].

This proves Proposition 4.1.

For the groups \(\text{PSL}(3, q)\) and \(\text{PSU}(3, q^2)\) we get a similar result. Note that \(\text{PSL}(3, q)\) and \(\text{PSU}(3, q^2)\) are simple except for \(\text{PSU}(3, 2^2) \cong (\mathbb{Z}_3 \times \mathbb{Z}_3) : Q_8\).

**Proposition 4.2.** The conjecture (ZC) is valid for \(\text{PSL}(3, q)\) and \(\text{PSU}(3, q^2)\) for all \(q = p^m\), where \(p\) is a rational prime and \(m \geq 1\).

**Proof.** Let \(\tilde{G} = \text{PSL}(3, q)\) or \(\tilde{G} = \text{PSU}(3, q^2)\), respectively. Let \(\tilde{\gamma}\) be the nontrivial graph automorphism if \(\tilde{G} = \text{PSL}(3, q)\), and \(\tilde{\gamma} = \tilde{\alpha}_m\) if \(\tilde{G} = \text{PSU}(3, q^2)\).

Since we can use the same argumentation as in Proposition 4.1 to deal with the characters which are \(p\)-exceptional, we only have to look at:

Part (A).

The modules \(M_{\lambda}, \lambda = a_1a_2 \in X_q\), are simple \(k\tilde{G}\)-modules if and only if \(a_1 + 2a_2 \equiv 0 \pmod{3}\). This follows by looking at the center of \(\text{SL}(3, q)\) or \(\text{SU}(3, q^2)\) as described in [7, §12.1].

Let now \(\lambda = a_1a_2\) with \(a_1 + 2a_2 \equiv 0 \pmod{3}\). Then \(\lambda = a_1 \cdot 11 + \frac{1}{3}(a_2 - a_1) \cdot 03\) if \(a_1 \leq a_2\) and \(\lambda = a_2 \cdot 11 + \frac{1}{3}(a_1 - a_2) \cdot 30\) if \(a_2 \leq a_1\). Note that \(|a_2 - a_1| = \frac{1}{3}a_1 + 2a_2 - 3a_1\) is divisible by 3. We have to show that there exists \(\alpha \in \text{Aut}(\tilde{G})\) with \(M_{\lambda}^0 \cong M_{\lambda}^0\) and \(M_{\lambda}^0 \cong M_{\lambda}^0\). This then implies hypothesis
isomorphic to the symmetric group on 6 elements. For even all other cases, there exist also diagonal automorphisms, but no nontrivial automorphisms are generated by inner, graph and field automorphisms. In 

4.2. The symplectic groups \( \text{Sp}(4,q) \) and \( \text{PSp}(4,q) \). After the linear and unitary groups of type \( A_2 \), we consider now the symplectic groups \( \text{Sp}(4,q) \) and \( \text{PSp}(4,q) \). Note that \( \text{Sp}(4,2^m) \) is simple for \( m \geq 2 \), and \( \text{Sp}(4,2) \) is isomorphic to the symmetric group on 6 elements. For even \( q \), the group automorphisms are generated by inner, graph and field automorphisms. In all other cases, there exist also diagonal automorphisms, but no nontrivial
Thus we get with Equation (4.1) $\beta_{01} \otimes \beta_{01} = 4 \cdot 1 + 2 \cdot \beta_{10} + \beta_{01}^{\alpha_1}$.

Because of dimensions, $\beta_{01}^\nu = \beta_{01}^{\nu \alpha_i}$ for some $\nu \in \{0, 1\}$, $0 \leq i \leq m - 1$. Thus we get with Equation (4.1) $4 \cdot 1 + 2 \cdot \beta_{10}^{\nu \alpha_i} + \beta_{01}^{\nu \alpha_{i+1}} = 4 \cdot 1 + 2 \cdot \beta_{10}^\nu + \beta_{01}^{\alpha_1}$ and thus $\beta_{10}^\nu = \beta_{10}^{\nu \alpha_i}$. So hypothesis (i) of Proposition 2.11 follows for the case $p = 2$.

Now let $p \neq 2$. Then $\dim(M_{01}) = 4$ and $\dim(M_{10}) = 5$. If $M$ is a simple $kG$-module with $1 < \dim(M) < 8$ then there exists an automorphism $\alpha$ of $G$, such that either $M \cong M_{01}^\alpha$ or $M \cong M_{10}^\alpha$ [33, Thm. 1.1]. This implies hypothesis (i) in case $p \neq 2$.

The weak order $\prec$ for hypothesis (ii) is defined as follows:

Let $\mu_1 = b_1b_2$ and $\mu_2 = c_1c_2$ be two different dominant weights. Then $\mu_1 \prec \mu_2$ if and only if $b_1 + b_2 < c_1 + c_2$, or $b_1 + b_2 = c_1 + c_2$ and $b_1 < c_1$. Similarly to $\text{SL}(3, q)$, it follows that $\mu_1 \leq \mu_2$, $\mu_1 \neq \mu_2$ implies $\mu_1 \prec \mu_2$ and $\mu_2 \not\prec \mu_1$.

Part (B).

When $p = 2$, $G$ has no $p$-exceptional characters. Therefore let $p \neq 2$.

We use here the notation given in [42] and [34]. The characters which are $p$-exceptional are exactly the following pairs: \{\xi_{21}(k), \xi_{22}(k)\} for each $k$, \{\xi'_{21}(k), \xi'_{22}(k)\} for each $k$, \{\xi_{41}(k), \xi_{42}(k)\} for each $k$, \{\xi'_{41}(k), \xi'_{42}(k)\} for each $k$, \{\Phi_1, \Phi_2\}, \{\Phi_3, \Phi_4\}, \{\Phi_5, \Phi_6\}, \{\Phi_7, \Phi_8\}, \{\theta_1, \theta_2\}, \{\theta_3, \theta_4\}, \{\theta_5, \theta_6\}$ and $\{\theta_7, \theta_8\}$. The nontrivial diagonal automorphism $\delta$ of $G$ permutes simultaneously the two characters of each such pair. Looking at the corresponding
conjugacy classes it follows that the only possible nontrivial operation induced by $\sigma$ on these pairs is the simultaneous permutation of each pair.

This proves Proposition 4.3.

For the groups $\text{PSp}(4,q)$ we get a similar result:

**Proposition 4.4.** The conjecture (ZC) is valid for $\text{PSp}(4,q)$ for all $q = p^m$ with $p$ a rational prime and $m \geq 1$.

**Proof.** Let $\tilde{G} = \text{PSp}(4,q)$. Because of Proposition 4.3, we only have to show Proposition 4.4 for $p \geq 3$. So let $p \geq 3$. Since we can use the same argumentation as in Proposition 4.3 to deal with the characters which are exceptional for $p$, we only have to look at:

**Part (A).**

The modules $M_\lambda$, $\lambda = a_1 a_2 \in X_q$, are simple $k\tilde{G}$-modules if and only if $a_2 \equiv 0 \pmod{2}$. This follows by looking at the center of $\text{Sp}(4,q)$ as described in [7, §12.1].

Let now $\lambda = a_1 \lambda_1 + a_2 \lambda_2 = a_1 a_2$ with $a_2 \equiv 0 \pmod{2}$. We have to show that there exists $\alpha \in \text{Aut}(\tilde{G})$ with $M_{10}^\alpha \cong M_{10}$ and $M_{20}^\alpha \cong M_{20}$. This then implies hypothesis (i) of Proposition 2.11. For hypothesis (ii) we can use the same weak order $\prec$ as for $\text{Sp}(4,q)$.

By [2, Thm. 1], all modules of dimension 5 are of the form $M_{10}^{\alpha_i}$ for some $i$, and all modules of dimension 10 are of the form $M_{02}^{\alpha_j}$ for some $j$. Thus we get the desired operation of $\tilde{\sigma}$ on these modules.

This proves Proposition 4.4. □

### 4.3. The finite Chevalley groups $G_2(q)$

After the classical groups of type $A_2$ and $B_2$, we want now to examine the exceptional groups of type $G_2$. The ordinary generic character tables of these groups can be found in [15] and in [9, 16] and have also been determined in CHEVIE.

$G_2(q)$ is simple except for $G_2(2) \cong \text{SU}(3,3^2).2 = \text{Aut}(\text{SU}(3,3^2))$. For $3 \nmid q$, the group automorphisms of $G_2(q)$ are generated by inner and field automorphisms. For $q = 3^n$ there exists also a nontrivial graph automorphism of order 2.

We want to prove the following proposition:

**Proposition 4.5.** The conjecture (ZC) is valid for $G_2(q)$ for all $q = p^m$ with $p$ a rational prime and $m \geq 1$.

**Proof.** Let $G = G_2(q)$.

**Part (A).**

If $p = 3$, then $X_3 = \{00, 01, 10, 02, 20, 11, 12, 21, 22\}$ with $\dim(M_{01}) = 7 = \dim(M_{10}), \dim(M_{02}) = 27 = \dim(M_{20}), M_{11} \cong M_{01} \otimes M_{10}, M_{22} \cong M_{02} \oplus M_{20}, M_{12} \cong M_{02} \otimes M_{10}$ and $M_{21} \cong M_{01} \otimes M_{20}$. If $\gamma$ denotes the nontrivial graph automorphism of $G$, then $M_{10} \cong M_{01}^\gamma$, $M_{20} \cong M_{02}^\gamma$ and
\[ M_{12} \cong M_{21}^2. \] Using the 3-modular character table of \( G_2(3) \) listed in GAP [39], it follows that for \( q = 3 \) the tensor product \( \beta_{01} \otimes \beta_{01} \) decomposes as

\[
\beta_{01} \otimes \beta_{01} = 1 \cdot 1 + 2 \cdot \beta_{01} + \beta_{10} + \beta_{02}.
\]

This is valid for arbitrary \( q = 3^m \), since the weights \( \lambda \in X^+ \) with \( \lambda \leq 2 \alpha \) are exactly \( \lambda \in \{00, 01, 10, 02\} \).

Because of dimensions, \( \beta_{01}^\sigma = \beta_{01}^{\gamma^\alpha i} \) for some \( \nu \in \{0, 1\} \), \( 0 \leq i \leq m - 1 \). Thus we get with Equation (4.2)

\[
1 \cdot 1 + 2 \cdot \beta_{01}^{\gamma^\alpha i} + \beta_{10}^{\gamma^\alpha i} + \beta_{02}^{\gamma^\alpha i} = 1 \cdot 1 + 2 \cdot \beta_{01}^\sigma + \beta_{10}^\sigma + \beta_{02}^\sigma
\]

and therefore

\[
\beta_{10}^{\gamma^\alpha i} + \beta_{02}^{\gamma^\alpha i} = \beta_{10}^\sigma + \beta_{02}^\sigma.
\]

Since the Brauer characters are linearly independent, it follows because of different dimensions that \( \beta_{10}^\sigma = \beta_{10}^{\gamma^\alpha i} \) and \( \beta_{02}^\sigma = \beta_{02}^{\gamma^\alpha i} \). Thus hypothesis (i) of Proposition 2.11 follows for the case \( p = 3 \).

Now let \( p \neq 3 \). Then \( \dim(M_{01}) = 7 - \delta_{p,2} \) and \( \dim(M_{10}) = 14 \). If \( M \) is a simple \( kG \)-module with \( 1 < \dim(M) < 18 \) then there exists an automorphism \( \alpha \) of \( G \), such that either \( M \cong M_{01}^\alpha \) or \( M \cong M_{10}^\alpha \). This follows from [31, Prop. 5.4.12], since the center of \( G_2(q) \) is trivial and thus every simple \( kG \)-module is also a simple projective \( kG \)-module. Therefore, hypothesis (i) follows for \( p \neq 3 \).

For the weak order \( \prec \) in hypothesis (ii) we take the natural order relation \( \leq \).

Part (B).

We have to show that if \( \alpha \) is the group automorphism from Part (A), then \( \sigma \alpha^{-1} \) operates trivially on the characters which are \( p \)-exceptional.

In CHEVIE, the ordinary generic character tables of \( G_2(q) \) are divided into five cases. In all five cases, only the characters \( \chi_8 \) and \( \chi_9 \) are \( p \)-exceptional.

Note that \( |G| = q^6(q - 1)^2(q^2 - 1)(q^2 + q + 1) \), and that \( G \) has maximal tori \( H_3 \cong \mathbb{Z}_{q^2+q+1} \) and \( H_6 \cong \mathbb{Z}_{q^2+q+1} \).

\( q \equiv 1(3) \):

Let \( l \) be a rational prime dividing \( q^2 - q + 1 \). Because of \( (q^2 - q + 1, q - 1) = 1 \), \( (q^2 - q + 1, q + 1) = 1 \) and \( (q^2 - q + 1, q^2 + q + 1) = 1 \), a Sylow \( l \)-subgroup of \( H_6 \) is also a Sylow \( l \)-subgroup of \( G \), i.e. all Sylow \( l \)-subgroups of \( G \) are cyclic. Since \( \chi_8 \) and \( \chi_9 \) belong to the principal \( l \)-block and are not \( l \)-exceptional, \( \sigma \alpha^{-1} \) fixes \( \chi_8 \) and \( \chi_9 \) by Lemma 2.14.

\( q \equiv -1(3) \) or \( q \equiv 0(3) \):

Let \( l \) be a rational prime dividing \( q^2 + q + 1 \). Similarly to \( q \equiv 1(3) \) it follows that all Sylow \( l \)-subgroups of \( G \) are cyclic. \( \chi_8 \) and \( \chi_9 \) belong to the principal \( l \)-block and are not \( l \)-exceptional. Thus by Lemma 2.14, \( \sigma \alpha^{-1} \) fixes \( \chi_8 \) and \( \chi_9 \).
This proves Proposition 4.5.

4.4. The finite twisted groups $3D_4(q^3)$. The finite groups of type $3D_4$ are the only missing simple exceptional groups of rank 2. $3D_4(q^3)$ is simple for all $q$, and all group automorphisms are generated by inner and field automorphisms.

The generic character tables of $3D_4(q^3)$ have been determined in [13] and can also be found in CHEVIE.

**Proposition 4.6.** The conjecture (ZC) is valid for $3D_4(q^3)$ for all $q = p^m$, $p$ a rational prime and $m \geq 1$.

**Proof.** Let $G = 3D_4(q^3)$. Since there exist no ordinary irreducible characters of $G$ which are $p$-exceptional, it suffices to look at:

*Part (A).*

The simple modules $M_{\lambda}$ satisfy $\dim(M_{\lambda_1}) = 8$ and $\dim(M_{\lambda_2}) = 28 - 2\delta_{p,2}$, $M_{\lambda_3} \cong M_{\lambda_1}^{\alpha_m}$ and $M_{\lambda_4} \cong M_{\lambda_1}^{\alpha_m}$. By [33, Thm. 1.1], for every simple $kG$-module $M$ with $1 < \dim(M) < 32$ there exists a group automorphism $\alpha$ of $G$ such that either $M \cong M_{\lambda_1}$ or $M \cong M_{\lambda_2}$. Note that [33, Thm. 1.1] shows this for the finite Chevalley groups of type $D_4$, but by Theorem 2.4, this must also be valid for $G$. This implies hypothesis (i) of Proposition 2.11.

The weak order $\prec$ for hypothesis (ii) is defined as follows:

Let $\mu_1 = b_1b_2b_3b_4$ and $\mu_2 = c_1c_2c_3c_4$ be two different dominant weights. Then $\mu_1 < \mu_2$ if and only if $b_1 + \frac{3}{2}b_2 + b_3 + b_4 < c_1 + \frac{3}{2}c_2 + c_3 + c_4$, or $b_1 + \frac{3}{2}b_2 + b_3 + b_4 = c_1 + \frac{3}{2}c_2 + c_3 + c_4$ and $b_2 < c_2$. Similarly to the case $SL(3, q)$, it follows that $\mu_1 \leq \mu_2$, $\mu_1 \neq \mu_2$ implies $\mu_1 < \mu_2$ and $\mu_2 \neq \mu_1$.

This proves Proposition 4.6.

The Propositions 3.1, 4.2, 4.4, 4.5 and 4.6 together with the results of [4] prove Theorem 2.

Altogether the conjecture (ZC) is valid for the following finite groups of Lie type:

$\text{SL}(2, p^m)$, $\text{PSL}(2, p^m)$, $p$ a rational prime

$2B_2(2^{2m+1})$, $2G_2(3^{2m+1})$, $2F_4(2^{2m+1})$

$\text{SL}(3, p^m)$, $\text{PSL}(3, p^m)$, $p$ a rational prime

$\text{SU}(3, p^{2m})$, $\text{PSU}(3, p^{2m})$, $p$ a rational prime

$\text{Sp}(4, p^m)$, $\text{PSp}(4, p^m)$, $p$ a rational prime

$G_2(p^m)$, $p$ a rational prime

$3D_4(p^{3m})$, $p$ a rational prime.
Acknowledgments. The results of this paper except for the results concerning the series of type $A_2$, $^2A_2$ and $B_2$ originate in my Ph.D. thesis [3]. I would like to express my thanks to Prof. Dr. W. Kimmerle and Prof. Dr. K.W. Roggenkamp for their continuous support.

References


Received August 21, 1997 and revised February 9, 1998.

SOUTHERN ILLINOIS UNIVERSITY
CARBONDALE, IL 62901-4408
E-mail address: fbleher@math.siu.edu
HECKE ALGEBRAS AND SEMIGROUP CROSSED PRODUCT C*-ALGEBRAS

B. Brenken

For an almost normal subgroup $\Gamma_0$ of a discrete group $\Gamma$, conditions are given which allow one to define a universal $C^*$-norm on the Hecke algebra $H(\Gamma, \Gamma_0)$. If $\Gamma$ is a semidirect product of a normal subgroup $N$ containing $\Gamma_0$ by a group $G$ satisfying some order relations arising from a naturally defined subsemigroup $T$, and if the normalizer of $N$ is also normal in $\Gamma$, then a presentation of $H(\Gamma, \Gamma_0)$ is given. In this situation the $C^*$-completion of $H(\Gamma, \Gamma_0)$ is $*$-isomorphic with the semigroup crossed product $C^*$-algebra $C^*(N/\Gamma_0) \rtimes T$.

In their paper introducing a number theoretical model of a quantum statistical system exhibiting a phase transition with symmetry breaking, Bost and Connes introduce the notion of an almost normal subgroup $\Gamma_0$ of a discrete group $\Gamma$, along with the associated Hecke algebra $H(\Gamma, \Gamma_0)$ and its reduced $C^*$-algebra completion $C^r(\Gamma, \Gamma_0)$ ([BC]). They also provide a presentation of the Hecke algebra in the context of the specific almost normal subgroup they consider in their model. A connection between these relations and some relations occurring in a stable $C^*$-algebra associated with certain examples of dynamical systems described in [B] provided the motivation for considering the Hecke algebras further.

An overview of the structure of the paper follows. After some preliminaries on almost normal subgroup pairs $(\Gamma, \Gamma_0)$ we introduce a fundamental semigroup $T$ in the group $\Gamma$, which contains the normalizer $N_{\Gamma_0}$ of $\Gamma_0$. A basic representation of this semigroup as isometries in the convolution Hecke algebra $H(\Gamma, \Gamma_0)$ is described. In the presence of a normal subgroup $N$ of $\Gamma$ containing $\Gamma_0$ and contained in $N_{\Gamma_0}$, a natural semigroup $C^*$-dynamical system occurs which possesses a universal property with respect to $*$-representations of the Hecke algebra.

In the second section we discuss some properties of group partial pre-order relations arising from a subsemigroup of the group in much the same spirit as Nica in [N]. Applying this to our situation, with $T$ as the subsemigroup of $\Gamma$, and introducing a notion of solvable least upper bounds, we obtain some conditions allowing a definition of a universal $C^*$-norm on the Hecke algebra. Assuming some more structure for the pair $(\Gamma, \Gamma_0)$, namely that $\Gamma$ is an extension of a normal subgroup $N$ containing $\Gamma_0$, we obtain that
the $C^*$-completion of the Hecke algebra is a quotient of a semigroup crossed product $C^*$-algebra.

The main focus of Section 3 is to obtain an identification of the $C^*$-completion of the Hecke algebra and the semigroup crossed product $C^*$-algebra if the group $\Gamma$ is a semidirect product of a normal subgroup $N$ containing $\Gamma_0$ and a subgroup $G$ which is both upward and downward directed. We also assume that solvable least upper bounds exist in the fundamental subsemigroup. This identification is proved by patterning our arguments after those of Bost and Connes to obtain a presentation of the Hecke algebra. A crucial role is played by the covariance relation Nica isolated in [N]. Once the identification is established, we can conclude that some of the relations were superfluous, as they are unnecessary in a presentation of the semigroup crossed product $C^*$-algebra.

Section 4 lists some examples pertaining to various stages in the structure of assumptions needed in the course of the paper.

Once a semigroup crossed product structure is available for these Hecke algebras, simplifications in the dynamical structure of the Hecke algebras can occur. For example, it is hoped that the study of the KMS state simplex and phase transitions under a one parameter automorphism group of the algebra, as first explored by Bost and Connes, can be extended to the other examples of Section 4.

As this paper was being prepared for submission we heard that results pertaining to Examples 4.1 and 4.5 discussed here were also being obtained in joint work of Arledge, Laca and Raeburn. The methods and approach employed are however different.

**Notation.** If $X$ is a set, $|X|$ denotes the cardinality of $S$. For sets $X$ and $Y$, $X \cong Y$ means that $X$ and $Y$ are isomorphic as sets. If $A$ is a set of transformations of a set $X$, then $A x = \{a(x) \mid a \in A\}$ for $x \in X$. If $H$ is a subgroup of group $G$, write $H \leq G$. Let $N_H$ be the normalizer of $H$ in $G$, and if $H$ is normal in $G$ write $H \trianglelefteq G$. Also $[g]$ is the left coset $g H$ in $G/H$, $(g \in G)$, and the index of $H$ in $G$ is $(G : H) = |G/H|$. For $g \in G$, $\text{ad } (g)$ is the group automorphism of a normal subgroup $H$ defined by $h \rightarrow g h g^{-1}$, $(h \in H)$. The unit element of a group or semigroup is $e$. The natural numbers with zero, a semigroup under addition, are denoted by $\mathbb{N}$, while $\mathbb{N}^\times$ denotes the non-zero elements of $\mathbb{N}$, an abelian semigroup under multiplication. If $R$ is a ring, $R^\times$ denotes the non-zero elements of $R$. For $d \in \mathbb{N}^\times$, $\mathcal{M}_d(R)$ denotes the $d \times d$ matrices with entries in $R$. For $F \in \mathcal{M}_d(R)$, $F_t$ denotes the transpose matrix.

1. Basics.

If $\Gamma_0$ is a subgroup of a discrete group $\Gamma$, then $\Gamma_0$ acts on the left on the coset space $\Gamma/\Gamma_0$. We say that $\Gamma_0$ is almost normal in $\Gamma$, or that $(\Gamma, \Gamma_0)$
form an almost normal subgroup pair, if the \( \Gamma_0 \)-orbits, \( \Gamma_0[\gamma] \), in \( \Gamma/\Gamma_0 \) are finite for all \( \gamma \in \Gamma \) ([BC]).

**Proposition 1.1.** If \( \Gamma_0 \) is a subgroup of a discrete group \( \Gamma \) then \( \Gamma_0[\gamma] \cong \Gamma_0/\Gamma_0 \cap \gamma \Gamma_0 \gamma^{-1}, \ (\gamma \in \Gamma) \).

**Proof.** For each \( \gamma \in \Gamma \) consider the map of \( \Gamma_0 \) onto the \( \Gamma_0 \)-orbit of \([\gamma]\) defined by \( h \mapsto [h \gamma] \). Since \( \{ h \in \Gamma_0 \mid [h \gamma] = [\gamma]\} = \{ h \in \Gamma_0 \mid h \gamma \in \gamma \Gamma_0 \} = \Gamma_0 \cap \gamma \Gamma_0 \gamma^{-1} \), this map defines a bijection of the coset space \( \Gamma_0/\Gamma_0 \cap \gamma \Gamma_0 \gamma^{-1} \) with \( \Gamma_0[\gamma] \).

Another set bijection is useful to note. Left multiplication by an element \( \alpha \) of \( \Gamma \) yields a bijection of \( \Gamma/\Gamma_0 \) with itself, so a subset \( M \) of \( \Gamma/\Gamma_0 \) is bijective with \( \alpha M \). Setting \( M \) to be the orbit \( \Gamma_0[\gamma] \) and \( \alpha \) to be \( \gamma^{-1} \), we have that \( \Gamma_0 \Gamma_0 \gamma^{-1} \cong \Gamma_0 \Gamma_0 \gamma^{-1} \Gamma_0/\Gamma_0 \).

For a given almost normal subgroup pair \( (\Gamma, \Gamma_0) \), there are \( \Gamma_0 \)-bivariant maps \( L \) and \( R \) from \( \Gamma \) to \( N^\times \) defined by \( L(\gamma) = |\Gamma_0[\gamma]| \) and \( R(\gamma) = L(\gamma^{-1}) \) ([BC]). The last proposition shows that \( L(\gamma) = (\Gamma_0 : \Gamma_0 \cap \gamma \Gamma_0 \gamma^{-1}) \) and that \( [\gamma] \) is a fixed point under the left \( \Gamma_0 \) action, i.e., \( L(\gamma) = 1 \), if and only if \( \Gamma_0 \subseteq \gamma \Gamma_0 \gamma^{-1} \).

**Definition.** For an almost normal subgroup pair \( (\Gamma, \Gamma_0) \) let \( T = \{ \gamma \in \Gamma \mid L(\gamma) = 1\} = \{ \gamma \in \Gamma \mid \Gamma_0 \gamma \subseteq \gamma \Gamma_0 \} \).

Since \( \Gamma_0 \gamma \subseteq \gamma \Gamma_0 \) implies \( \gamma \Gamma_0 \subseteq \gamma \Gamma_0 \Gamma_0 \subseteq \gamma \Gamma_0 \Gamma_0 = \gamma \Gamma_0 \), and since \( \Gamma_0 \gamma \subseteq \Gamma_0 \Gamma_0 \Gamma_0 \) it follows that \( T = \{ \gamma \in \Gamma \mid \Gamma_0 \gamma \Gamma_0 = \gamma \Gamma_0 \} \).

**Proposition 1.2.** If \( (\Gamma, \Gamma_0) \) is an almost normal subgroup pair then \( T \) is a subsemigroup of \( \Gamma \) and \( T \cap T^{-1} = N_{\Gamma_0} \), the normalizer of \( \Gamma_0 \) in \( \gamma \). The map \( R : T \to N^\times \) is a semigroup homomorphism.

**Proof.** Clearly \( e \in T \). For \( \alpha, \beta \in T \) we have \( \Gamma_0 \subseteq \alpha \Gamma_0 \alpha^{-1} \) and \( \Gamma_0 \subseteq \beta \Gamma_0 \beta^{-1} \). Applying the automorphism \( \text{ad}(\alpha) \) to the second inclusion shows that \( \alpha \Gamma_0 \alpha^{-1} \subseteq \alpha \beta \Gamma_0 \beta^{-1} \alpha^{-1} \), so \( \Gamma_0 \subseteq (\alpha \beta) \Gamma_0 (\alpha \beta)^{-1} \) and \( \alpha \in T \). Since \( T^{-1} = \{ \gamma \mid \Gamma_0 \gamma \Gamma_0 \subseteq \gamma \Gamma_0 \} \), the second claim is clear.

The last assertion follows from the elementary fact that \( (G : K) = (G : H)(H : K) \) for subgroups \( K \leq H \leq G \) of a group \( G \). For \( \alpha, \beta \in T \), we have \( R(\alpha \beta) = (\Gamma_0 : \Gamma_0 \cap (\alpha \beta)^{-1} \Gamma_0 \alpha \beta) \) which is equal to \( (\Gamma_0 : (\alpha \beta)^{-1} \Gamma_0 \alpha \beta) \) since \( \alpha \beta \in T \). This equals \( (\Gamma_0 : \beta^{-1} \Gamma_0 \beta)(\beta^{-1} \Gamma_0 \beta : \beta^{-1} \alpha^{-1} \Gamma_0 \alpha \beta) = (\Gamma_0 : \beta^{-1} \Gamma_0 \beta)(\Gamma_0 : \alpha^{-1} \Gamma_0 \alpha) = R(\beta)R(\alpha) \).

**Proposition 1.3.** The semigroup homomorphism \( R : T \to N^\times \) defines a map \( r : T/\mathbb{N}_{\Gamma_0} \to N^\times \) satisfying \( r([\alpha \beta]) = r([\alpha])r([\beta]) \) for \( \alpha, \beta \in T \).

**Proof.** For \( \alpha \in T \) set \( r([\alpha]) = R(\alpha) \). For \( \alpha, \beta \in T \) with \([\alpha] = [\beta]\) we have \( \beta^{-1} \alpha \in \mathbb{N}_{\Gamma_0} = T \cap T^{-1} \). Thus \( R(\alpha) = R(\beta \beta^{-1} \alpha) = R(\beta)R(\beta^{-1} \alpha) = R(\beta) \), the later equality a consequence of \( \beta^{-1} \alpha \in T^{-1} \). Thus \( r \) is a well defined map of sets.
Notice that \( T/N_{\Gamma_0} \) is in general only a coset space, so multiplication of elements is not well defined.

We now recall from [BC] that the Hecke algebra \( H(\Gamma, \Gamma_0) \) associated to an almost normal subgroup pair \((\Gamma, \Gamma_0)\) is the convolution algebra of \((\mathbb{C}-\text{valued say})\) functions with finite support on \(\Gamma_0\setminus\Gamma/\Gamma_0\), the space of \(\Gamma_0\)-orbits in \(\Gamma/\Gamma_0\). For \(f, h \in H(\Gamma, \Gamma_0)\) define
\[
    f * h(\gamma) = \sum \{ f(\alpha)h(\alpha^{-1}\gamma) \mid \alpha \in \Gamma/\Gamma_0 \},
\]
\[
    f^*(\gamma) = \overline{f(\gamma^{-1})}, \quad (\gamma \in \Gamma).
\]
Here we view \(f\) and \(h\) as \(\Gamma_0\)-bivariant functions on \(\Gamma\). We now proceed to define some elements of \(H(\Gamma, \Gamma_0)\) that will play a basic role in the rest of the paper. For a finite subset \(A\) of \(\Gamma_0\setminus\Gamma/\Gamma_0\) let \(\chi_A\) denote the characteristic function of the set \(A\). Also, for \(\gamma \in \Gamma\), let \(O_\gamma\) be the point \(\Gamma_0[\gamma] = \Gamma_0\gamma\Gamma_0\) in \(\Gamma_0\setminus\Gamma/\Gamma_0\).

**Definition.** If \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair and \(\gamma \in T\) let \(W_\gamma\) be the element of \(H(\Gamma, \Gamma_0)\) defined by
\[
    W_\gamma = R(\gamma)^{-1/2}\chi_{O_\gamma}.
\]

It will be useful to distinguish those elements \(W_\gamma\) with \(\gamma \in N_{\Gamma_0}\) from the others. Write \(U_\gamma = W_\gamma\) if \(\gamma \in N_{\Gamma_0}\), so \(U_\gamma = \chi_{O_\gamma}\). Note that \(O_\gamma = \gamma\Gamma_0\) for \(\gamma \in T\), so if \(\gamma, \gamma' \in T\) then \(O_\gamma = O_{\gamma'}\) if and only if \([\gamma] = [\gamma']\) in \(T/\Gamma_0\). Since \(R(\gamma) = R(\gamma')\) if \([\gamma] = [\gamma']\) in \(T/\Gamma_0\), we have that \(W_\gamma = W_{\gamma'}\) if and only if \([\gamma] = [\gamma']\) in \(T/\Gamma_0\).

**Theorem 1.4.** For \((\Gamma, \Gamma_0)\) an almost normal subgroup pair, \(W : T \to H(\Gamma, \Gamma_0)\) is a representation of the semigroup \(T\) by isometries.

**Proof.** We show first that \(W_\beta^* W_\gamma = W_e = I\). For \(\beta \in \Gamma\), \(W_\beta^* W_\gamma(\beta) = R(\gamma)^{-1}\sum \{ \chi_{O_\alpha}(\alpha^{-1})\chi_{O_\beta}(\alpha^{-1}\beta) \mid \alpha \in \Gamma/\Gamma_0 \}\). Since \(O_\alpha = \gamma\Gamma_0\) for \(\gamma \in T\), \(\alpha^{-1} \in O_\gamma\) if and only if \(\alpha \in \Gamma_0^{-1} = \Gamma_0\Gamma_0^{-1}\), a set with \(L(\gamma^{-1}) = R(\gamma)\) points in \(\Gamma/\Gamma_0\). Thus \(W_\beta^* W_\gamma(\beta) = R(\gamma)^{-1}\sum \{ \chi_{O_\alpha}(\beta) \mid \alpha \in \Gamma_0\Gamma_0^{-1} \} = \chi_{O_\beta}(\beta) = W_e(\beta)\). It is clear that \(W_e\) is the identity of \(H(\Gamma, \Gamma_0)\).

For \(\alpha, \beta \in T\) and \(\gamma \in \Gamma\) we have
\[
    W_\alpha W_\beta(\gamma) = (R(\alpha)R(\beta))^{-1/2} \sum \{ \chi_{O_\rho}(\rho)\chi_{O_\beta}(\rho^{-1}\gamma) \mid \rho \in \Gamma/\Gamma_0 \}
\]
\[
    = R(\alpha\beta)^{-1/2}\chi_{O_{\alpha\beta}}(\alpha^{-1}\gamma),
\]
since \(O_\alpha = \alpha\Gamma_0\) is a single point of \(\Gamma/\Gamma_0\). This expression equals \(R(\alpha\beta)^{-1/2}\) if and only if \(\gamma \in \alpha\Gamma_0\beta\Gamma_0 = \alpha\beta\Gamma_0 = O_{\alpha\beta}\), and is zero otherwise. Thus \(W_\alpha W_\beta = W_{\alpha\beta}\).

If \(\gamma \in N_{\Gamma_0}\), we have that \(\gamma^{-1} \in N_{\Gamma_0}\) also, so \(W_\gamma W_{\gamma^{-1}} = W_e = I\). Thus \(W_\gamma^* W_\gamma = W_\gamma W_\gamma^* (W_\gamma W_{\gamma^{-1}}) = W_\gamma W_{\gamma^{-1}} = I\) and \(U = W|_{N_{\Gamma_0}}\) is a unitary.

\[\text{BC}\]
representation of the group $N_{\Gamma_0}$ in $H(\Gamma, \Gamma_0)$. Since $W|_{\Gamma_0} = I$, $U$ should actually be viewed as a unitary representation of the group $N_{\Gamma_0}/\Gamma_0$. It follows from $W_{\gamma}W_{\gamma}^* = I$ (where $\gamma \in N_{\Gamma_0}$) that for $\alpha \in \mathcal{T}$ the selfadjoint idempotent $W_{\alpha}W_{\alpha}^*$ depends only on the equivalence class $[\alpha]$ of $\alpha$ in $\mathcal{T}/N_{\Gamma_0}$. Where convenient we denote $W_{\alpha}W_{\alpha}^*$ by $P[\alpha]$.

**Proposition 1.5.** Let $(\Gamma, \Gamma_0)$ be an almost normal subgroup pair, and choose $\alpha \in \mathcal{T}$, $g \in N_{\Gamma_0}$. Then

$$W_{\alpha}U_{g}W_{\alpha}^* = R(\alpha)^{-1}\sum\{\chi_{O}\beta \mid \text{ad} (\alpha^{-1})(\beta) \in g\Gamma_0\}$$

where this sum is over a set of $R(\alpha)$ points in $\Gamma_0 \setminus \Gamma/\Gamma_0$.

**Proof.** First note that $W_{\alpha}U_{g}W_{\alpha}^* = W_{\alpha}W_{g}W_{\alpha}^* = W_{\alpha g}W_{\alpha}^*$. Since $\alpha g \in \mathcal{T}$, the orbit $O_{\alpha g} = \Gamma_0 \alpha g \Gamma_0$ is the single point $\alpha g \Gamma_0$ in $\Gamma/\Gamma_0$, so the sum defining the product of $W_{\alpha g}$ with $W_{\alpha}^*$ consists of one non-zero term. We have $W_{\alpha g}W_{\alpha}^*(\gamma) = R(\alpha)^{-1}\chi_{O_{\alpha}}(\rho^{-1}g)^{-1}$ where $\rho = \alpha g \Gamma_0$. Since $\rho^{-1}g \in \Gamma_0\alpha^{-1}$ if and only if $\gamma \in \alpha g \Gamma_0\alpha^{-1}$, this expression is $R(\alpha)^{-1}$ if $\gamma \in \alpha g \Gamma_0\alpha^{-1}$ and zero otherwise. Since the set $\Gamma_0\alpha^{-1} = \Gamma_0\alpha^{-1}\Gamma_0$ consists of $L(\alpha^{-1}) = R(\alpha)$ points in $\Gamma/\Gamma_0$, so does the set $\alpha g \Gamma_0\alpha^{-1}\Gamma_0$. Each one of these points in $\Gamma/\Gamma_0$ is however also a point in $\Gamma_0 \setminus \Gamma/\Gamma_0$, since, as already noted, $\Gamma_0\alpha g \Gamma_0 = \alpha g \Gamma_0$. Thus $W_{\alpha}U_{g}W_{\alpha}^* = R(\alpha)^{-1}\sum\{\chi_{O}\beta \mid \beta \in \text{ad} (\alpha)(g\Gamma_0)\}$, where the sum has $R(\alpha)$ terms.

**Remark 1.6.** If we further stipulate that $N_{\Gamma_0}$ be normal in $\Gamma$, then the solutions $\beta \in \text{ad} (\alpha)(g\Gamma_0)$ occurring in the sum all occur in $N_{\Gamma_0}/\Gamma_0$, in fact, in $N_{\Gamma_0}/\Gamma_0$, and $\chi_{O}\beta = U_{\beta}$. Thus $W_{\alpha}U_{g}W_{\alpha}^* = R(\alpha)^{-1}\sum\{U_{\beta} \mid \text{ad} (\alpha^{-1})[\beta] = [g]\}$ in $N_{\Gamma_0}/\Gamma_0$.

This remark suggests that if $N_{\Gamma_0}$ is normal in $\Gamma$, a certain semigroup $C^*$-dynamical system associated to an almost normal subgroup pair $(\Gamma, \Gamma_0)$ should be considered. More generally, if $N$ is a normal subgroup of $\Gamma$ with $\Gamma_0 < N < N_{\Gamma_0}$, then $\text{ad} (\alpha^{-1}) N = N$ for $\alpha \in \Gamma$ and since $\text{ad} (\alpha^{-1}) \Gamma_0 \subseteq \Gamma_0$ for any $\alpha \in \mathcal{T}$, the map $\text{ad} (\alpha^{-1})$ defines a group homomorphism of $N/\Gamma_0$ to itself, for any $\alpha \in \mathcal{T}$. Thus, if $g \in N$, we may replace $N_{\Gamma_0}$ by $N$ in the above remark. Define an action $\Theta$ of the semigroup $\mathcal{T}$ on $l^1(N/\Gamma_0)$ by $\Theta_{\alpha}(f) = R(\alpha)^{-1} f \circ \text{ad} (\alpha^{-1})$, for $\alpha \in \mathcal{T}$, $f \in l^1(N/\Gamma_0)$. For $g \in N/\Gamma_0$ let $\delta_g$ be the element of $l^1(N/\Gamma_0)$ which is one at $g$ and zero elsewhere. Compute that

$$\Theta_{\alpha}(\delta_g) = R(\alpha)^{-1}\sum\{\delta_{\beta} \mid \text{ad} (\alpha^{-1})(\beta) = g\}$$

so $\|\Theta_{\alpha}(\delta_g)\|_1 \leq \|\delta_g\|_1 = 1$. Thus $\Theta$ is a continuous action of the semigroup $\mathcal{T}$ on the Banach $*$-algebra $l^1(N/\Gamma_0)$, so defines an action, again denoted by $\Theta$, of $\mathcal{T}$ on the $C^*$-completion, $C^*(N/\Gamma_0)$. Thus, to any almost normal subgroup pair $(\Gamma, \Gamma_0)$ and $N$ a normal subgroup of $\Gamma$ containing $\Gamma_0$, and
contained in the normalizer of \( \Gamma_0 \), there is a semigroup \( C^* \)-dynamical system 
\((C^*(N/\Gamma_0), \Theta, \mathcal{T})\).

For the reader’s convenience we recall some facts concerning semigroup dynamical systems. For further details see the results in [LR] and the references therein. A covariant representation of a semigroup \( C^* \)-dynamical system \((\mathcal{A}, \Theta, \mathcal{S})\) where \( \Theta \) is a representation of the semigroup \( \mathcal{S} \) as (possibly non-unital) endomorphisms of a unital \( C^* \)-algebra \( \mathcal{A} \), is a pair \((\pi, V)\) with \( \pi \) a unital representation of \( \mathcal{A} \) on a Hilbert space \( \mathcal{H} \) and \( V \) a representation of \( \mathcal{S} \) on \( \mathcal{H} \) as isometries with \( V_\alpha := I, \) such that \( \pi(\Theta_\alpha(a)) = V_\alpha \pi(a) V_\alpha^* \), \( a \in \mathcal{A} \) and \( \alpha \in \mathcal{S} \). If a semigroup dynamical system \((\mathcal{A}, \Theta, \mathcal{S})\) possesses a covariant representation \((\pi, V)\) then there is a unique unital \( C^* \)-algebra \( \mathcal{A} \times \mathcal{S} \) equipped with a unital homomorphism \( i : \mathcal{A} \to \mathcal{A} \times \mathcal{S} \) and a representation \( \nu : \mathcal{S} \to \mathcal{A} \times \mathcal{S} \) by isometries so that \((i, \nu)\) satisfies the covariance relation, their images generate \( \mathcal{A} \times \mathcal{S} \) as a \( C^* \)-algebra, and such that every covariant representation \((\rho, W)\) of \((\mathcal{A}, \Theta, \mathcal{S})\) yields a representation \( \rho \times W \) of \( \mathcal{A} \times \mathcal{S} \) with \((\rho \times W) \circ i = \rho \) and \((\rho \times W) \circ \nu = W \).

We return now to our context, namely an almost normal subgroup pair \((\Gamma, \Gamma_0)\) and a normal subgroup \( N \) of \( \Gamma \) with \( \Gamma_0 < N < N_{\Gamma_0} \). If \( \pi : H(\Gamma, \Gamma_0) \to \mathcal{B}(\mathcal{H}) \) is a \( \ast \)-representation of the Hecke algebra \( H(\Gamma, \Gamma_0) \) as bounded operators on a Hilbert space \( \mathcal{H} \), denote by \( \pi_W \) the representation of \( \mathcal{T} \) as partial isometries on \( \mathcal{H} \) given by \( \pi_W = \pi \circ W \), and denote by \( \pi_U \) the unitary representation of the group \( N/\Gamma_0 \) on \( \mathcal{H} \) given by \( \pi_U = \pi \circ U \). Combining Proposition 1.5 and Remark 1.6 with the above expression for \( \Theta_\alpha(\delta_\beta) \), \( \alpha \in \mathcal{T} \) and \( g \in N/\Gamma_0 \), gives the following result.

**Theorem 1.7.** If \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair and \( N \) a normal subgroup of \( \Gamma \) with \( \Gamma_0 < N < N_{\Gamma_0} \) then to every \( \ast \)-representation \( \pi : H(\Gamma, \Gamma_0) \to \mathcal{B}(\mathcal{H}) \) there corresponds a covariant representation \((\pi_U, \pi_W)\) of the semigroup \( C^* \)-dynamical system \((C^*(N/\Gamma_0), \Theta, \mathcal{T})\) on \( \mathcal{H} \).

**Corollary 1.8.** The \( C^* \)-algebra \( C^*(N/\Gamma_0) \times \mathcal{T} \) exists and for each \( \ast \)-representation \( \pi \) of \( H(\Gamma, \Gamma_0) \) there is a \( \ast \)-representation \( \rho \) of \( C^*(N/\Gamma_0) \times \mathcal{T} \) with image contained in the \( C^* \)-algebra generated by \( \text{Im}(\pi) \) and with \( \rho \circ i = \pi_U \) and \( \rho \circ \nu = \pi_W \).

**Proof.** To conclude that the \( C^* \)-semicrossed product algebra exists it is enough to show that there is at least one covariant representation of the dynamical system \((C^*(N/\Gamma_0), \Theta, \mathcal{T})\). This follows from Theorem 1.7 after noting that there is always a regular representation of \( H(\Gamma, \Gamma_0) \) on the Hilbert space \( l^2(\Gamma/\Gamma_0) \) ([BC]). \( \square \)

Under certain conditions, we may consider a slightly less cumbersome semigroup \( C^* \)-dynamical system. Consider the situation of a normal subgroup \( N \) of a discrete group \( \Gamma \). There is a commuting diagram of groups
with exact rows:

\[
\begin{array}{cccccc}
e & \rightarrow & N & \rightarrow & \Gamma & \xrightarrow{\rho} & G & \rightarrow & e \\
\downarrow & \downarrow & \downarrow & \text{ad} & \downarrow & \psi & \\
e & \rightarrow & C & \rightarrow & N & \xrightarrow{\text{ad}} & \text{Aut}(N) & \rightarrow & \text{Out}(N) & \rightarrow & e.
\end{array}
\]

Here \( C \) is the center of \( N, G = \Gamma/N, \) ad : \( \Gamma \rightarrow \text{Aut}(N) \) is defined by \( \gamma \rightarrow \text{ad}(\gamma) \mid_N \), and \( \psi \) is defined by the diagram. If \( \Gamma_0 \) is any normal subgroup of \( N \), then \( \text{ad}(\gamma)(\Gamma_0) = \Gamma_0 \) for any \( \gamma \in N \), so for \( g \in G \) we may denote by \( \psi_g(\Gamma_0) \) the well defined subgroup \( \text{ad}(\gamma)(\Gamma_0) \) where \( \gamma \in \Gamma \) is any element of \( \rho^{-1}(g) \).

**Lemma 1.9.** Given the above diagram and \( \Gamma_0 \) a normal subgroup of \( N \), the subgroup pair \((\Gamma, \Gamma_0)\) is almost normal if and only if the subgroup \( \psi_g(\Gamma_0)\Gamma_0/\Gamma_0 \) of \( N/\Gamma_0 \) is finite for each \( g \in G \).

**Proof.** The subgroup \( \gamma \Gamma_0\gamma^{-1}\Gamma_0/\Gamma_0 = \text{ad}(\gamma)(\Gamma_0)\Gamma_0/\Gamma_0 \) is finite for each \( \gamma \in \Gamma \) if and only if \( \psi_g(\Gamma_0)\Gamma_0/\Gamma_0 \) is finite for each \( g \in G \). The comment after Proposition 1.1 finishes the claim. \( \square \)

Assume now that \( (\Gamma, \Gamma_0) \) is an almost normal subgroup pair. Then \( \rho(T) = T/N = \{ g \in G \mid \Gamma_0 \subseteq \psi_g(\Gamma_0) \} \) is a semigroup of \( G \), denote it by \( T \). The normalizer of \( \Gamma_0 \) is equal to \( N \) if and only if \( \{ g \in G \mid \psi_g(\Gamma_0) = \Gamma_0 \} \), which is \( T \cap T^{-1} \), is equal to \( e \). Otherwise \( N_{\Gamma_0} \) is an extension of \( N \) by \( T \cap T^{-1} \). Proposition 1.3 shows that the map \( R : T \rightarrow \mathbb{N}^\times \) defines a semigroup homomorphism of \( T/N \) to \( \mathbb{N}^\times \). Also, for \( \alpha \in N \), the group endomorphism \( \text{ad}(\alpha^{-1}) \) of \( N/\Gamma_0 \) is actually an automorphism of \( N/\Gamma_0 \). If there is a splitting homomorphism \( \nu : G \rightarrow \Gamma \), in other words if \( \Gamma \) is the semidirect product of \( N \) by \( G \), then \( \nu(g) \in T \) for \( g \in T \), so we may define a semigroup \( C^*-\text{dynamical system} \) \( (C^*(N/\Gamma_0), \Theta, T) \) by setting \( \Theta = \Theta \circ \nu \). If \( \pi \) is a \( * \)-representation of \( H(\Gamma, \Gamma_0) \) in \( \mathcal{B}(\mathcal{H}) \) then set \( \pi_T = \pi \circ W \circ \nu \) and, as before, set \( \pi_U = \pi \circ U \). The pair \( (\pi_U, \pi_T) \) is a covariant representation of this dynamical system. Summarizing these observations in connection with the previous results gives the following result.

**Theorem 1.10.** If \( (\Gamma, \Gamma_0) \) is an almost normal subgroup pair and \( e \rightarrow N \rightarrow \Gamma \xrightarrow{\rho} G \rightarrow e \) is a split exact sequence with \( \Gamma_0 \trianglelefteq N \) then to every \( * \)-representation \( \pi : H(\Gamma, \Gamma_0) \rightarrow \mathcal{B}(\mathcal{H}) \) there corresponds a covariant representation \( (\pi_T, \pi_U) \) of the semigroup \( C^*\)-dynamical system \( (C^*(N/\Gamma_0), \Theta, T) \) on \( \mathcal{H} \). The \( C^* \)-algebra \( C^*(N/\Gamma_0) \rtimes T \) exists, and given \( \pi \) there is a \( * \)-representation \( \pi' \) of \( C^*(N/\Gamma_0) \rtimes T \) with image contained in the \( C^* \)-algebra generated by \( \text{Im}(\pi) \), and with \( \pi' \circ i = \pi_U, \pi' \circ \nu = \pi_T \).

For example, consider the case when \( \Gamma_0 \) is a normal subgroup of \( \Gamma \). It is clear that the Hecke algebra \( H(\Gamma, \Gamma_0) \) is the group algebra \( \mathbb{C}[\Gamma/\Gamma_0] \). Setting \( N = N_{\Gamma_0} \), so \( N = \Gamma \), we have \( G = \{ e \} \) and Theorem 1.10 gives a
representation \( \pi' \) of \( C^*(\Gamma/\Gamma_0) \) in the \( C^* \)-algebra generated by \( \text{Im}(\pi) \) for any representation \( \pi \) of \( \Gamma/\Gamma_0 \). Of course, in this context, Theorem 1.10 is far from the best possible.

2. Order Structure.

The semigroup \( T \) of \( \Gamma \) defines a pre-order type structure on \( \Gamma \). Some of the extra structure on this ordering that is of consequence for our context has been developed before in [N]. We develop some of the slightly more general results that are needed in our situation though.

Let \( S \) be a sub-semigroup of a discrete group \( G \) and define a relation \( \preceq \) on \( G \) by \( a \preceq b \) if and only if \( a^{-1}b \in S \). This relation is: Reflexive since \( e \in S \); transitive since \( SS \subseteq S \); and left invariant, i.e., \( a \preceq b \) implies \( ga \preceq gb \) for \( g \in G \). We do not specify that the subgroup \( S \cap S^{-1} = \{e\} \), so the relation is not a partial order on \( G \). The elements \( a, b \) satisfy both \( a \preceq b \) and \( b \preceq a \) if and only if \( a^{-1}b \in S \cap S^{-1} \). As noted in [N], the set \( SS^{-1} = \{g \in G \mid g \) has an upper bound in \( S \} \). If any two arbitrarily chosen elements of a subset \( A \) in \( G \) have a common upper bound (c.u.b) in \( G \) call \( A \) upward directed.

**Lemma 2.1.** A subset \( A \) of \( G \) is upward directed if and only if \( A^{-1}A \subseteq SS^{-1} \).

**Proof.** First suppose that \( A \) is upward directed. If \( l \) is a c.u.b. for a pair \( a, b \) in \( A \), then \( a^{-1}b = a^{-1}l(b^{-1}l)^{-1} \in SS^{-1} \). For the reverse implication first notice that if \( h = st^{-1} \) with \( s, t \in S \) then \( h \) and \( e \) have a c.u.b., namely \( s \). Given \( a, b \in A \) arbitrary, \( b^{-1}a \in SS^{-1} \) by hypothesis, so \( b^{-1}a \) and \( e \) have a c.u.b., say \( l \). By left invariance, \( a \preceq bl \) and \( b \preceq bl \), so \( bl \) is a c.u.b. for the pair \( a, b \) of \( A \).

Thus \( S \) is upward directed if and only if \( S^{-1}S \subseteq SS^{-1} \) (cf [N, 2.2.2]). Also \( G \) is upward directed if and only if \( G = SS^{-1} \).

For \( a, b \in G \), denote by \( a \vee b \) a least upper bound for \( a \) and \( b \), if it exists. The set of least upper bounds for \( a \) and \( b \) is \( a \vee b \) (\( S \cap S^{-1} \)). The following fact is a straightforward consequence of left invariance for the relation \( \preceq \).

**Lemma 2.2.** Given a pair \( a, b \) in \( G \) so that \( a \vee b \) exists, and \( g \in G \), then \( g(a \vee b) \) is a l.u.b. for the pair \( ga, gb \).

With an analogous definition for a subset \( A \) of \( G \) to be lower directed, it follows that \( A \) is lower directed if and only if \( A^{-1}A \subseteq S^{-1}S \). For example, if \( l \) is a common lower bound (c.l.b) for \( a, b \) in \( A \) then \( a^{-1}b = (l^{-1}a)^{-1}(l^{-1}b) \in S^{-1}S \). In the other direction, if \( h = s^{-1}l \in S^{-1}S \) then \( s^{-1} \) is a c.l.b. for \( h, e \). Thus, for \( a, b \in A \), \( a^{-1}b \in S^{-1}S \) so there is a c.l.b., say \( l \), for \( a^{-1}b, e \). Then \( al \) is a c.l.b. for \( a, b \).

Thus \( S^{-1} \) is lower directed if and only if \( SS^{-1} \subseteq S^{-1}S \) and \( G \) is lower directed if and only if \( G = S^{-1}S \). Note also that if \( A \subseteq G \) is lower directed
and if \( a \vee b \) exists for \( a, b \in S \) then \( g \vee h \) exists for all \( g, h \in A \). To see this, let \( c \) be a c.l.u.b. for \( g \) and \( h \). Then \( c^{-1}g, c^{-1}h \in S \), so \( c^{-1}g \vee c^{-1}h \) exists. Then \( c(c^{-1}g \vee c^{-1}h) \) is a l.u.b. for \( g \) and \( h \).

**Theorem 2.3.** Assume any pair of elements in \( S \) have a l.u.b. If \( SS^{-1} \subseteq S^{-1}S \) then \( e \vee g \) exists for any \( g \in SS^{-1} \). In particular if \( S \cap S^{-1} = \{e\} \), then \( (G, S) \) is a quasi-lattice ordered group.

**Proof.** Since \( SS^{-1} \subseteq S^{-1}S \), \( S^{-1} \) is lower directed. By the remarks preceding the theorem, \( c^{-1} \vee d^{-1} \) exists for any \( c, d \in S \). Thus \( c^{-1} \vee d^{-1} \) is a l.u.b. for \( e \) and \( cd^{-1} \). Set \( g = cd^{-1} \).

We may, following [N], define a partially pre-ordered group \((G, S)\) to be quasi-lattice pre-ordered if

1) \( e \vee g \) exists for \( g \in SS^{-1} \).
2) For \( a, b \in S \) with a c.u.b. \( a \vee b \) exists.

As an example, consider the following: \( G = \{g \in GL(d, \mathbb{Q}) \mid \det g > 0\} \) and \( S = \{g \in \mathcal{M}_d(\mathbb{Z}) \mid \det g \geq 1\} \). Then \( S \cap S^{-1} = SL(d, \mathbb{Z}) \) and \( G = SS^{-1} = S^{-1}S \). In fact \( G = N^{-1}S \) where \( n \in N \) is identified with the diagonal matrix \( n \otimes I_d \) and \( S = \{g \in G \mid g(\mathbb{Z}^d) \subseteq \mathbb{Z}^d\} \). Thus, for \( a, b \in G \), we have \( a \preceq b \) if and only if \( b(\mathbb{Z}^d) \subseteq a(\mathbb{Z}^d) \). It is known that \( a \vee b \) exists for any \( a, b \in S \), usually known as the least common multiple of \( a \) and \( b \).

**Theorem 2.3** shows that \((G, S)\) is a quasi-lattice pre-ordered group.

**Remark 2.4.** One last observation before we return to our own context. If \((G, S)\) is a pre-ordered group such that 1) above is satisfied, then any \( x \in SS^{-1} \) may be written as \( st^{-1} \) for some \( s, t \in S \) with \( s^{-1} \vee t^{-1} = e \). For example, set \( s \) to be a l.u.b. for \( e \) and \( x \) and set \( t = x^{-1}s \). Then \( s, t \in S \) and \( x = st^{-1} \). Since \( e \vee x = s \), it follows that \( s^{-1} \vee t^{-1} = s^{-1} \vee s^{-1}x = s^{-1}(e \vee x) = e \).

Returning to the situation of an almost normal subgroup pair \((\Gamma, \Gamma_0)\) with \( T \) the semigroup of \( \Gamma \) defined by \( T = \{\alpha \in \Gamma \mid \Gamma_0 \subseteq \text{ad}(\alpha)\Gamma_0\} \) it follows that \( \alpha \preceq \beta \) in \( \Gamma \) with the pre-order defined by \( T \) if and only if \( \alpha\Gamma_0\alpha^{-1} \subseteq \beta\Gamma_0\beta^{-1} \).

**Lemma 2.5.** Let \( \alpha, \beta \in \Gamma \) where \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair. Assume there is a \( \gamma \in \Gamma \) with \( \text{ad}(\gamma)\Gamma_0 = \text{ad}(\alpha)\Gamma_0 \text{ad}(\beta)\Gamma_0 \). Then \( \gamma \) is a l.u.b. for \( \alpha \) and \( \beta \). If \( \eta \) is any l.u.b. for \( \alpha \) and \( \beta \) then \( \text{ad}(\eta)\Gamma_0 = \text{ad}(\gamma)\Gamma_0 \). We also have \( \text{ad}(\alpha)\Gamma_0 \text{ad}(\beta)\Gamma_0 = \text{ad}(\beta)\Gamma_0 \text{ad}(\alpha)\Gamma_0 \).

**Proof.** Any common upper bound \( \delta \) for \( \alpha \) and \( \beta \) satisfies \( \text{ad}(\alpha)\Gamma_0 \subseteq \text{ad}(\delta)\Gamma_0 \) and \( \text{ad}(\beta)\Gamma_0 \subseteq \text{ad}(\delta)\Gamma_0 \), so \( \text{ad}(\alpha)\Gamma_0 \text{ad}(\beta)\Gamma_0 \subseteq \text{ad}(\delta)\Gamma_0 \). Thus \( \gamma \) is clearly a l.u.b. for \( \alpha \) and \( \beta \). If \( \eta \) is another l.u.b. for \( \alpha \) and \( \beta \), then \( \eta^{-1}\gamma \in T \cap T^{-1} = N_{\Gamma_0} \), so \( \text{ad}(\gamma)\Gamma_0 = \text{ad}(\eta)\Gamma_0 \). Since \( \text{ad}(\alpha)\Gamma_0 \text{ad}(\beta)\Gamma_0 \) is a subgroup \( H = \text{ad}(\gamma)\Gamma_0 \) of \( \Gamma \), \( H = H^{-1} = \text{ad}(\beta)\Gamma_0 \text{ad}(\alpha)\Gamma_0 \). □
A l.u.b. $\gamma \in \Gamma$ for $\alpha, \beta \in \Gamma$ that satisfies $\text{ad} \, (\alpha)\Gamma_0 \text{ad} \, (\beta)\Gamma_0 = \text{ad} \, (\gamma)\Gamma_0$ will be referred to as a solvable l.u.b. We write $\gamma = \alpha \vee_s \beta$. The previous results on least upper bounds hold in $(\Gamma, T)$ with l.u.b. replaced by solvable l.u.b. The example described after Theorem 2.3 actually has solvable least upper bounds.

**Lemma 2.6.** Let $(\Gamma, \Gamma_0)$ be an almost normal subgroup pair. If $\alpha, \beta \in T$ with $\alpha^{-1} \vee_s \beta^{-1} = e$, then $W_\alpha W_\beta^* = (R(\alpha\beta))^{-1/2} \chi_{O_{\alpha\beta^{-1}}}$.  

**Proof.** We have $W_\alpha W_\beta^*(\gamma) = (R(\alpha\beta))^{-1/2} \chi_{O_{\alpha\beta}}((\rho^{-1}\gamma)^{-1})$ where $\rho = \alpha\Gamma_0$. This is zero, except when $\gamma \in \alpha\Gamma_0\beta^{-1} = \text{ad} \, (\alpha)\Gamma_0\beta^{-1}$. Now $O_{\alpha\beta^{-1}} = \Gamma_0\alpha\beta^{-1}\Gamma_0 = \Gamma_0\text{ad} \, (\alpha\beta^{-1})\Gamma_0\alpha\beta^{-1}$. Thus, the equality holds if and only if $\text{ad} \, (\alpha)\Gamma_0 = \Gamma_0\text{ad} \, (\alpha\beta^{-1})\Gamma_0$, which is equivalent to $\alpha^{-1} \vee_s \beta^{-1} = e$.  

We now consider the problem of finding norms on $H(\Gamma, \Gamma_0)$. As in [BC], there is an $L^1$ norm on $H(\Gamma, \Gamma_0)$. For $f$ in $H(\Gamma, \Gamma_0)$ let  

$$\|f\|_1 = \sum_{\gamma \in \Gamma \setminus \Gamma_0} (R(\gamma)L(\gamma))^{1/2} |f(\gamma)|$$

$$= \sum_{\gamma \in \Gamma / \Gamma_0} \delta(\gamma)^{-1/2} |f(\gamma)|$$

where $\delta(\gamma) = L(\gamma)/R(\gamma)$. In order to define a universal $C^*$-norm on $H(\Gamma, \Gamma_0)$ the next lemma will be useful.

**Lemma 2.7.** If $\alpha, \beta \in T$ and $\alpha^{-1} \vee_s \beta^{-1} = e$ then $L(\alpha\beta^{-1}) = R(\beta)$ and $R(\alpha\beta^{-1}) = R(\alpha)$.  

**Proof.** Since $\alpha^{-1} \vee_s \beta^{-1} = e$, we have $\alpha\Gamma_0\beta^{-1} = \Gamma_0\alpha\beta^{-1}\Gamma_0$, so $L(\alpha\beta^{-1})$ is the number of left cosets in $\Gamma / \Gamma_0$ of $\alpha\Gamma_0\beta^{-1} = \alpha\Gamma_0\beta^{-1}\Gamma_0$, which in turn is the number of left cosets in $\Gamma / \Gamma_0$ of $\Gamma_0\beta^{-1}\Gamma_0$. This is of course $L(\beta^{-1}) = R(\beta)$. Since $\beta^{-1} \vee_s \alpha^{-1} = e$ also, it follows that $L(\beta\alpha^{-1}) = R(\alpha)$. So $R(\alpha\beta^{-1}) = R(\alpha)$.  

If we assume that $\Gamma$ is upward directed, in other words that $\Gamma = TT^{-1}$, then any $f \in H(\Gamma, \Gamma_0)$ may be written as a finite sum $\sum \{ a_i \chi_{O_{\alpha_i\beta_i^{-1}}} | \alpha_i, \beta_i \in T, a_i \in \mathbb{C} \}$. If we further assume that $T^{-1}T = \Gamma$ and that any pair of elements in $T$ have a solvable l.u.b., then by Theorem 2.3 and Remark 2.4, we can ensure that any $f \in H(\Gamma, \Gamma_0)$ may be written as a finite sum $\sum \{ a_i \chi_{O_{\alpha_i\beta_i^{-1}}} | \alpha_i, \beta_i \in T, \alpha_i^{-1} \vee_s \beta_i^{-1} = e, a_i \in \mathbb{C} \}$. If $\pi$ is a $*$-representation of $H(\Gamma, \Gamma_0)$ as bounded operators on a Hilbert space, then, for $f$ of this form, $\pi(f) = \sum a_i R(\alpha_i \beta_i)_{1/2} \pi [W_{\alpha_i} W_{\beta_i}^*]$ by Lemma 2.6. Thus $\|\pi(f)\| \leq \sum |a_i| R(\alpha_i)_{1/2} R(\beta_i)_{1/2} = \sum |a_i| R(\alpha_i \beta_i^{-1})_{1/2} L(\alpha_i \beta_i^{-1})_{1/2} = \|f\|_1$, etc.
the first equality following from Lemma 2.7. Summarizing this gives the next proposition.

**Proposition 2.8.** Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair satisfying \(\Gamma = T^{-1}T = TT^{-1}\) and so that any pair of elements of \(T\) have a solvable l.u.b. Then \(H(\Gamma, \Gamma_0) = \text{span}_C\{W_\alpha W_\beta^* | \alpha, \beta \in T, \alpha^{-1} \vee_s \beta^{-1} = e\}\). Also \(\|f\| = \sup\{\|\pi(f)\| : \pi : H(\Gamma, \Gamma_0) \to B(H)\text{ a }\ast\text{-representation}\} \) defines a \(C^*\)-norm on \(H(\Gamma, \Gamma_0)\). Denote the \(C^*\)-completion of \(H(\Gamma, \Gamma_0)\) in this norm by \(C^*(\Gamma, \Gamma_0)\).

**Remark 2.9.** One may define the almost normal subgroup pair \((\Gamma, \Gamma_0)\) to be amenable, under the hypothesis of the Proposition, if \(\|\| \leq \|\|_r\) on \(H(\Gamma, \Gamma_0)\), where \(\|\|_r\) is the norm on \(H(\Gamma, \Gamma_0)\) arising from the regular representation of \(H(\Gamma, \Gamma_0)\) on \(l^2(\Gamma/\Gamma_0)\) ([BC]).

The situation discussed previously in Section 1, with \(N\) a normal subgroup of \(\Gamma\) and \(\Gamma_0 \leq N\) leads to further conclusions for \((\Gamma, \Gamma_0)\) an almost normal subgroup pair satisfying the hypothesis of Proposition 2.8. We immediately have the following for example.

**Proposition 2.10.** If \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair with \(\Gamma = T^{-1}T = TT^{-1}\) and such that \(\alpha \vee_s \beta\) exists for \(\alpha, \beta \in T\), and if \(N\) is a normal subgroup of \(\Gamma\) with \(\Gamma_0 \leq N \leq N\Gamma_0\), then there is a natural surjective \(\ast\)-homomorphism of \(C^*(N/\Gamma_0) \rtimes T\) onto \(C^*(\Gamma, \Gamma_0)\).

**Proof.** Using the definition of \(C^*(\Gamma, \Gamma_0)\), Corollary 1.8 gives a \(\ast\)-representation \(\pi\) of \(C^*(N/\Gamma_0) \rtimes T\) into \(C^*(\Gamma, \Gamma_0)\). Proposition 2.8 shows that the image of \(H(\Gamma, \Gamma_0)\) in \(C^*(\Gamma, \Gamma_0)\) is contained in the image of \(\pi\), so \(\pi\) is a surjection. \(\square\)

Of course we can say more with the hypothesis of this proposition. With \(G = \Gamma/N = \rho(\Gamma)\) and \(T = T/N\) as before, we can carry the order structure of \((\Gamma, T)\) to \((G, T)\). Thus \(g \leq h\) in \(G\) if and only if \(\psi_g(\Gamma_0) \subseteq \psi_h(\Gamma_0)\), where \(\psi\) is defined after Corollary 1.8. In particular, the definition of solvable least upper bound remains compatible, so \(\alpha \vee_s \beta = \gamma\) in \(G\) if and only if \(\rho(\alpha) \vee_s \rho(\beta) = \rho(\gamma)\). In the particular case that \(N = N\Gamma_0\), then \(T \cap T^{-1} = \{e\}\), so \((G, T)\) is a partially ordered group (the order relation is antisymmetric) and least upper bounds, if they exist, are unique. Also note that \(G = T^{-1}T\) if and only if \(\Gamma = T^{-1}T\). To see this, it is enough to show that \(G = T^{-1}T\) implies \(\Gamma = T^{-1}T\). Choosing \(\gamma \in \Gamma\), we have \(\rho(\gamma) = \rho(\alpha^{-1}\beta)\) some \(\alpha, \beta \in T\). Thus \(\gamma \in \alpha^{-1}\beta N\) so \(\gamma \in T^{-1}T\), since \(N \leq N\Gamma_0 \leq T\). Similarly, \(G = TT^{-1}\) if and only if \(\Gamma = TT^{-1}\).

**Theorem 2.11.** If \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair and \(e \to N \to \Gamma \to G \to e\) is a split exact sequence with \(\Gamma_0 \leq N\), and if \(G = TT^{-1} = T^{-1}T\) so that the solvable least upper bound \(g \vee_s h\) exists for every pair \(g, h \in T\),
then there is a natural surjective $*$-homomorphism of $C^*(N/\Gamma_0) \rtimes T$ onto $C^*(\Gamma, \Gamma_0)$.

**Proof.** Since $G = TT^{-1} = T^{-1}T$, we have $\Gamma = T^{-1}T = TT^{-1}$. Also, the solvable l.u.b. exists for any pair $\alpha$, $\beta$ in $T$. Proposition 2.8 shows that $C^*(\Gamma, \Gamma_0)$ exists, so Theorem 1.10 yields a $*$-homomorphism $\pi : C^*(N/\Gamma_0) \rtimes T$ to $C^*(\Gamma, \Gamma_0)$. Proposition 2.8 allows us to see that the image of $H(\Gamma, \Gamma_0)$ in $C^*(\Gamma, \Gamma_0)$ is contained in the image of $\pi$, so $\pi$ is a surjection. $\square$

**3. Universal Properties.**

In this section our main goal is to provide certain conditions under which the Hecke $C^*$-algebra is isomorphic to a semigroup crossed product $C^*$-algebra. A crucial role is played here by the covariance condition of Nica, [N].

**Lemma 3.1.** If $\alpha, \beta \in T$ with $\alpha \lor_s \beta = \gamma$ then $R(\gamma) | \text{ad}(\alpha)\Gamma_0 \cap \text{ad}(\beta)\Gamma_0| = R(\alpha)R(\beta)$, where the cardinality is computed in $\Gamma/\Gamma_0$.

**Proof.** For $\alpha \in T$, the cardinality of $\text{ad}(\alpha)\Gamma_0 = \alpha\Gamma_0\alpha^{-1}\Gamma_0$ in $\Gamma/\Gamma_0$ is the same as the cardinality of $\Gamma_0\alpha\alpha^{-1}\Gamma_0$ in $\Gamma/\Gamma_0$, which is $R(\alpha)$. Since $\text{ad}(\gamma)\Gamma_0 = \text{ad}(\gamma)\Gamma_0 = \text{ad}(\gamma)\Gamma_0 = \bigcup \{ \eta \text{ad}(\beta)\Gamma_0 | \eta \in \text{ad}(\alpha)\Gamma_0 \}$. The left cosets $\eta \text{ad}(\beta)\Gamma_0$ are either disjoint or coincide as $\eta$ varies, and as sets, each is isomorphic to $\text{ad}(\beta)\Gamma_0$, which has $R(\beta)$ elements in $\Gamma/\Gamma_0$. Since two such cosets given by $\eta$ and $\eta'$ coincide if and only if $\eta^{-1}\eta' \in \text{ad}(\beta)\Gamma_0$, it follows that

\[
\text{ad}(\gamma)\Gamma_0 = \bigcup \{ \eta \text{ad}(\beta)\Gamma_0 | \eta \in \text{ad}(\alpha)\Gamma_0 / \text{ad}(\alpha)\Gamma_0 \cap \text{ad}(\beta)\Gamma_0 \}.
\]

Thus, in $\Gamma/\Gamma_0$, $R(\gamma) = | \text{ad}(\alpha)\Gamma_0 / \text{ad}(\alpha)\Gamma_0 \cap \text{ad}(\beta)\Gamma_0 | R(\beta)$, so $R(\gamma) | \text{ad}(\alpha)\Gamma_0 \cap \text{ad}(\beta)\Gamma_0 | = R(\alpha)R(\beta)$. $\square$

**Proposition 3.2.** Let $(\Gamma, \Gamma_0)$ be an almost normal subgroup pair. Suppose that $\alpha, \beta \in T$, with $\alpha \lor_s \beta = \gamma$. Then $P_{[\alpha]}P_{[\beta]} = P_{[\gamma]}$, where $P_{[\alpha]} = W_\alpha W_\alpha^*$ and $[ \ ]$ denotes the equivalence class in $T/N_{\Gamma_0}$.

**Proof.** Recall that $P_{[\alpha]} = W_\alpha W_\alpha^* = R(\alpha)^{-1} \sum \{ \chi_{0\delta} | \eta \in \text{ad}(\alpha)\Gamma_0 \}$ by Proposition 1.5, for $\alpha \in T$. Thus $P_{[\alpha]}P_{[\beta]} = R(\alpha)^{-1}R(\beta)^{-1} \sum \{ \chi_{0\delta} \chi_{0\delta} | \eta \in \text{ad}(\alpha)\Gamma_0, \delta \in \text{ad}(\beta)\Gamma_0 \} = R(\alpha)^{-1}R(\beta)^{-1} \sum \{ \chi_{0\delta} \chi_{0\delta} | \eta \in \text{ad}(\alpha)\Gamma_0, \delta \in \text{ad}(\beta)\Gamma_0 \}$ the convolution product of functions in $H(\Gamma, \Gamma_0)$. The later
expression
\[ R(\alpha)^{-1} R(\beta)^{-1} \sum \left\{ \sum \left\{ \chi_{O_{\gamma} O_{\delta}} | \delta \in \text{ad} (\beta) \Gamma_0 \right\} \right\} \eta \in \text{ad} (\alpha) \Gamma_0 / (\text{ad} (\beta) \Gamma_0 \cap \text{ad} (\alpha) \Gamma_0) \right\} \eta \in \text{ad} (\beta) \Gamma_0 \cap \text{ad} (\alpha) \Gamma_0 \right\} \right\} \]
\[ = R(\alpha)^{-1} R(\beta)^{-1} \sum \{ R(\gamma) P_{[\gamma]} | \eta \in \text{ad} (\beta) \Gamma_0 \cap \text{ad} (\alpha) \Gamma_0 \} \]
\[ = R(\alpha)^{-1} R(\beta)^{-1} | \text{ad} (\beta) \Gamma_0 \cap \text{ad} (\alpha) \Gamma_0 | R(\gamma) P_{[\gamma]} = P_{[\gamma]} \] by Lemma 3.1.

\[ \square \]

Remark 3.3. If \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair so that the solvable l.u.b. exists for any pair of elements in \(T\), and if \(\pi : H(\Gamma, \Gamma_0) \to B(\mathcal{H})\) is a \(\ast\)-representation of \(H(\Gamma, \Gamma_0)\), then Proposition 3.2 states that the map \(\pi_W = \pi \circ W\) is a representation of \(T\) satisfying the covariance condition
\[ \pi_W(\alpha) \pi_W(\alpha) \pi_W(\beta) \pi_W(\beta)^* = \pi_W(\alpha \vee_s \beta) \pi_W(\alpha \vee_s \beta)^*, \quad \alpha, \beta \in T \]
of Nica in [N].

Remark 3.4. For \(\alpha, \beta \in T\) we have that \(\alpha^{-1} \vee_s \beta^{-1} = e\) if and only if either one of the two equivalent conditions \(\alpha \beta \alpha^{-1} \vee_s \alpha = \alpha \beta, \beta \vee_s \beta \alpha \beta^{-1} = \beta \alpha\) holds. In particular, if \(\alpha \beta = \beta \alpha\) then \(\alpha^{-1} \vee_s \beta^{-1} = e\) if and only if \(\alpha \vee_s \beta = \alpha \beta\). Since \(W_\alpha W_\alpha^* W_\beta W_\beta^* = W_{\alpha \vee_s \beta} W_{\alpha \vee_s \beta}^*\) by Proposition 3.2, we have \(W_\alpha^* W_\beta = W_\alpha^* W_\alpha W_\alpha^{-1}(\alpha \vee_s \beta)(W_\beta W_\beta^{-1}(\alpha \vee_s \beta))^* W_\beta\), and so \(W_\alpha^* W_\beta = W_{\alpha^{-1}(\alpha \vee_s \beta)} W_{\beta^{-1}(\alpha \vee_s \beta)}\) which is equal to \(W_\beta W_\alpha^*\) if \(\alpha \beta = \beta \alpha\). Thus, for \(\alpha, \beta \in T\) with \(\alpha^{-1} \vee_x \beta^{-1} = e\) and \(\alpha \beta = \beta \alpha\) we have \(W_\alpha^* W_\beta = W_\beta W_\alpha^*\).

Definition. For \((\Gamma, \Gamma_0)\) an almost normal subgroup pair, let \(L\) be the linear span over \(\mathbb{C}\) of the set \(\{W_\alpha W_\beta^* | \alpha, \beta \in T\}\) in \(H(\Gamma, \Gamma_0)\).

Proposition 3.5. Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair and assume that the solvable l.u.b. exists for any pair of elements in \(T\). Then \(L\) is a \(\ast\)-subalgebra of \(H(\Gamma, \Gamma_0)\).

Proof. Let \(\alpha, \beta, \gamma, \delta \in T\). Proposition 3.2 shows that \(W_\beta W_\alpha^* W_\delta W_\delta^* = W_{\beta \vee_s \delta}^* W_{\beta \vee_s \delta}\). Thus
\[ W_\beta^* W_\delta = W_\beta^* W_\beta \beta \vee_s \delta W_\beta^* W_\delta \]
\[ = W_\beta^* W_\beta W_{\beta^{-1}(\beta \vee_s \delta)}(W_\delta W_{\delta^{-1}(\beta \vee_s \delta)})^* W_\delta = W_{\beta^{-1}(\beta \vee_s \delta)} W_{\delta^{-1}(\beta \vee_s \delta)}^*. \]

It follows that \(W_\alpha W_\beta^* W_\delta W_\gamma^* = W_{\alpha \beta^{-1}(\beta \vee_s \delta)} W_{\gamma \delta^{-1}(\beta \vee_s \delta)}^*\), which is in \(L\). Clearly \(L\) is closed under adjoints. \(\square\)
Lemma 3.6. Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair so that the solvable l.u.b. exists for any pair of elements in \(T\) and that \(\Gamma = TT^{-1} = T^{-1}T\). Suppose that \(\alpha, \beta, \gamma \in T\) satisfy \(\alpha^{-1} \vee_s \beta^{-1} = \eta^{-1} \vee_s \delta^{-1} = e\).

(a) If \(\alpha \beta^{-1} = \eta \delta^{-1}\) then there is an \(n \in \Gamma_{\Gamma_0}\) with \(\eta = \alpha n\) and \(\delta = \beta n\).

(b) If \(O_{\alpha \beta^{-1}} \subseteq O_{\eta \delta^{-1}}\) then \([\alpha] = [\eta]\) and \([\beta] = [\delta]\) in \(T/\Gamma_{\Gamma_0}\).

Proof. (a) Set \(x = \alpha \beta^{-1}\). Then \(e \vee_s x = \alpha (\alpha^{-1} \vee_s \beta^{-1}) = \eta (\eta^{-1} \vee_s \delta^{-1})\) by Lemma 2.2 and the comments preceding Theorem 2.3. Thus \(\alpha \eta = \alpha \delta\) and \(\eta = \delta\).

(b) Since \(\Gamma_0 \alpha \beta^{-1} \Gamma_0 \subseteq \Gamma_0 \eta \delta^{-1} \Gamma_0 \subseteq \eta \Gamma_0 \delta^{-1}\), there is an \(m \in \Gamma_0\) with \(\alpha \beta^{-1} = \eta \gamma \delta^{-1}\). By part (a) there is an \(n \in \Gamma_{\Gamma_0}\) with \(\alpha = \eta m \delta\) and \(\beta = \delta n\). Thus \([\alpha] = [\eta]\) and \([\beta] = [\delta]\) in \(T/\Gamma_{\Gamma_0}\). \(\square\)

Proposition 3.7. Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair with \(\Gamma = TT^{-1} = T^{-1}T\) and such that the solvable l.u.b. exists for any pair of elements of \(T\). Then \(L = H(\Gamma, \Gamma_0)\). If \(N\) is a normal subgroup of \(\Gamma\) with \(\Gamma_0 < N < \Gamma_{\Gamma_0}\) then \(L = \text{span}_{C}(W_sW_nW_t^* \mid s^{-1} \vee_s t^{-1} = e, \ n \in N/\Gamma_0\) and \(s,t \in F)\) where \(F\) is an arbitrarily chosen set in \(T\) of distinct coset representatives of \(T/N\).

Proof. That \(L = H(\Gamma, \Gamma_0)\) follows from Proposition 2.8. Let \(\alpha, \beta \in T\) be arbitrary. Then by Theorem 2.3 there are elements \(s, t \in T\) with \(e \vee_s \alpha \beta^{-1} = \alpha (\alpha^{-1} \vee_s \beta^{-1})\) and \(t = \beta (\alpha^{-1} \vee_s \beta^{-1})\). Then \(s^{-1} \vee_s t^{-1} = (\alpha^{-1} \vee_s \beta^{-1})^{-1}(\alpha^{-1} \vee_s \beta^{-1}) = e\). Since \(\alpha = s(\alpha^{-1} \vee_s \beta^{-1})^{-1}\) and \(\beta = t(\alpha^{-1} \vee_s \beta^{-1})^{-1}\), it follows that \(W_sW_nW_t^* W_q = W_sW_nW_qW_t^* W_q\) where \(q = (\alpha^{-1} \vee_s \beta^{-1})^{-1} \in T\). Since \(N\) satisfies \(\Gamma_0 \leq N \leq \Gamma\), Proposition 1.5 and Theorem 1.7 show that \(W_qW^* = R(q)^{-1}\sum \{U_n \mid n \in (\text{ad}(q))\Gamma_0/\Gamma_0\}\), a sum of \(R(q)\) terms over \(N/\Gamma_0\), and that \(W_sW_nW_t^* W_q\) is in the linear span over \(Q\) of \(W_sW_nW_t^*\) where \(n \in (\text{ad}(q))\Gamma_0 \subseteq N\) and \(s^{-1} \vee_s t^{-1} = e\). Note also that \((sn)^{-1} \vee_s t^{-1} = e\). \(\square\)

Proposition 3.8. Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair with \(\Gamma = TT^{-1} = T^{-1}T\) and such that the solvable l.u.b. exists for any pair of elements of \(T\). If \(\Gamma_{\Gamma_0}\) is normal in \(\Gamma\) then \(B = \{W_sW_nW_t^* \mid s^{-1} \vee_s t^{-1} = e, \ n \in \Gamma_{\Gamma_0}/\Gamma_0, \ s,t \in F_0\}\) is a basis for \(L\). Here \(F_0\) is a set in \(T\) of coset representatives of \(T/\Gamma_{\Gamma_0}\).

Proof. The preceding proposition shows that this set is a spanning set for \(L\). By Lemma 2.6 it is enough to show that the elements \(\{\chi_{O_{\alpha \beta^{-1}}} \mid s, t \in F_0, s^{-1} \vee_s t^{-1} = e, \ n \in \Gamma_{\Gamma_0}/\Gamma_0\}\) are linearly independent in \(H(\Gamma, \Gamma_0)\). If \(O_{\alpha \beta^{-1}} \subseteq O_{\eta \delta^{-1}}\) where \(s, t, p, q \in F, s^{-1} \vee_s t^{-1} = p^{-1} \vee_s q^{-1} = e\) and \(n,m \in \Gamma_{\Gamma_0}\), then Lemma 3.6 (b) implies that \([s] = [sn] = [pm] = [p]\) in \(T/\Gamma_{\Gamma_0}\) and similarly \([t] = [q]\). Thus \(s = p\) and \(t = q\). Again, using Lemma 2.6 and the fact that \(W_s\) and \(W_t\) are isometries, we have that \(W_n = W_m\) (up to a
scalar), so \( n = m \) in \( N_{\Gamma_0}/\Gamma_0 \). Since \( \{ O_{\gamma} \mid \gamma \in \Gamma \} \) are points in \( \Gamma_0 \backslash \Gamma / \Gamma_0 \), this shows that the elements \( \{ \chi_{O_{sn^{-1}}} \mid s, t \in \mathcal{F}_0, s^{-1} \vee_s t^{-1} = e, n \in N_{\Gamma_0}/\Gamma_0 \} \) are linearly independent.

In the presence of some more structure for the pair \((\Gamma, \Gamma_0)\) there is a slight strengthening of Proposition 3.8.

**Proposition 3.9.** Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair with \( \Gamma = T T^{-1} = T^{-1} T \) and such that the solvable l.u.b. exists for any pair of elements of \( T \). If \( \Gamma_0 \leq N \leq N_{\Gamma_0} \) with both \( N \) and \( N_{\Gamma_0} \) normal in \( \Gamma \), then \( B_N = \{ W_n W_n W_t^* \mid s^{-1} \vee_s t^{-1} = e, n \in N/\Gamma_0, s \in \mathcal{F}, t \in \mathcal{F}_0 \} \) is a basis for \( \mathcal{L} \). Here \( \mathcal{F}_0 \) is an arbitrary set in \( \mathcal{T} \) of coset representatives of \( \mathcal{T}/N_{\Gamma_0} \) and \( \mathcal{F} \) is an arbitrary set in \( \mathcal{T} \) of coset representatives of \( \mathcal{T}/N \).

**Proof.** We show that there is a bijective correspondence between the set \( \mathcal{B}_N \) and the basis \( \mathcal{B} = \mathcal{B}_{N_{\Gamma_0}} \) of Proposition 3.8. First notice that given \( s \in \mathcal{F}_0 \) and \( n \in N_{\Gamma_0} \), there is a unique \( u \in \mathcal{F} \) and a unique \( m \in N \) with \( sn = um \). To see this, observe \( snN = uN \) for a unique \( u \in \mathcal{F} \). Then \( sn \in uN \), so there is a unique \( m \in N \) with \( sn = um \). We then have that the element \( W_n W_n W_t^* \) of \( \mathcal{B} \) is equal to the element \( W_n W_m W_t^* \) of \( B_N \). Conversely, for \( u \in \mathcal{F} \) and \( m \in N \) given, there is a unique \( s \in \mathcal{F}_0 \) and \( n \in N_{\Gamma_0} \) with \( um = sn \). This follows as before, by first noting that \( umN_{\Gamma_0} = snN_{\Gamma_0} \) for a unique \( s \in \mathcal{F}_0 \).

**Theorem 3.10.** Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair with \( \Gamma = T T^{-1} = T^{-1} T \) and such that the solvable l.u.b. exists for any pair of elements in \( \mathcal{T} \). If \( \Gamma_0 \leq N \leq N_{\Gamma_0} \) with both \( N \) and \( N_{\Gamma_0} \) normal in \( \Gamma \), then \( H(\Gamma, \Gamma_0) \), respectively \( C^*(\Gamma, \Gamma_0) \), is the universal \(*\)-algebra, respectively \( C^* \)-algebra, generated by \( \{ V_{\alpha} \mid \alpha \in \mathcal{T} \} \) such that

1. \( V_{\alpha}^* V_{\alpha} = I = V_n \) \( (\alpha \in \mathcal{T}, n \in \Gamma_0) \)
2. \( V_{\alpha} V_{\beta} = V_{\alpha, \beta} \) \( (\alpha, \beta \in \mathcal{T}) \)
3. \( V_{\alpha} V_{\alpha}^* V_{\beta} = V_{\alpha \vee_s \beta} V_{\alpha, \beta}^* \) \( (\alpha, \beta \in \mathcal{T}) \)
4. \( V_{\alpha} V_{\alpha}^* = R(\alpha)^{-1} \sum \{ V_b \mid b \in N/\Gamma_0 \text{ such that } [\text{ad}(\alpha^{-1})b] = [n] \text{ in } N/\Gamma_0 \} \), \( (\alpha \in \mathcal{T}, n \in N) \).

**Proof.** If \( A \) is the universal \(*\)-algebra generated by \( \{ V_{\alpha} \mid \alpha \in \mathcal{T} \} \) with these relations, there is a natural \(*\)-homomorphism of \( A \) to \( H(\Gamma, \Gamma_0) \) mapping \( V_{\alpha} \) to \( W_{\alpha} \). This map is surjective since \( \mathcal{L} = H(\Gamma, \Gamma_0) \) by Proposition 3.7. Define \( \mathcal{L}' \) to be the linear subspace of \( A \) generated by \( \{ V_{\alpha} V_{\beta}^* \mid \alpha, \beta \in \mathcal{T} \} \). Using 1), 2) and 3), we see, as in Proposition 3.5, that \( \mathcal{L}' \) is a \(*\)-subalgebra of \( A \). Since \( \mathcal{L}' \) contains the generators of \( A \), \( \mathcal{L}' = A \). By 4), it follows as in Proposition 3.7, that \( \{ V_{\alpha} V_{\alpha}^* \mid s^{-1} \vee_s t^{-1} = e, n \in N/\Gamma_0, s, t \in \mathcal{F} \} \) spans \( \mathcal{L}' \). Now notice that condition 4) with \( n = e \) implies that \( V_{\alpha} \) is a unitary element of \( A \) for \( \alpha \in N_{\Gamma_0} \). If \( \alpha \in N_{\Gamma_0} \), then \( \text{ad}(\alpha) \Gamma_0 = \Gamma_0 \) and \( \text{ad}(\alpha) \) is an automorphism of \( N_{\Gamma_0}/\Gamma_0 \). The sum then has only \( R(\alpha) = 1 \) terms, and \( V_b \) for \( b = e \) is the
only term appearing. Thus \( V_\alpha V_\alpha^* = I \) for \( \alpha \in N_{\Gamma_0} \). Using this, we can show that the set \( \{ V_\alpha V_\beta V_\gamma^* \mid u^{-1} \vee_s w^{-1} = e, m \in N/\Gamma_0, u \in F, w \in F_0 \} \) spans \( \mathcal{L}' \). To see this note that \( tN_{\Gamma_0} = wN_{\Gamma_0} \) for some \( w \in F_0 \), so \( t = wp \) for some \( p \in N_{\Gamma_0} \). Then \( V_\alpha V_\beta V_\gamma^* = V_\alpha V_\beta_0 (V_\gamma V_\beta_0)^* = V_\alpha V_\beta V_\beta_0^{-1} V_\gamma^* = V_\beta_0^{-1} V_\gamma^* \). Now \( s np^{-1} N = uN \) for some \( u \in F \), so \( snp^{-1} = um \) for some \( m \in N \). Thus \( V_\alpha V_\beta V_\gamma^* = V_\beta V_\gamma V_\beta^* \) with \( u \in F, m \in N \) and \( w \in F_0 \). Note also that \( u^{-1} \vee_s w^{-1} = e \). Under the natural *-homomorphism above, the image of this spanning set in \( H(\Gamma, \Gamma_0) \) is, by Proposition 3.9, linearly independent, so it must also be linearly independent in \( A \), and so a basis for \( A \). Thus \( A \cong H(\Gamma, \Gamma_0) \).

We conclude this section by considering the case where \( \Gamma \) is a semidirect product of the normal subgroup \( N \) by \( G, \Gamma_0 \leq N \).

**Theorem 3.11.** Let \((\Gamma, \Gamma_0)\) be an almost normal subgroup pair with \( e \to N \to \Gamma \overset{\nu}{\to} G \to e \) a split exact sequence and \( \Gamma_0 \leq N \leq N_{\Gamma_0} \leq \Gamma \). Also assume that \( G = T^{-1}T = TT^{-1} \) and that the solvable l.u.b. exists for any pair of elements of \( T = \{ g \in G \mid \Gamma_0 \subseteq \text{ad} (g) \Gamma_0 \} \). Then \( H(\Gamma, \Gamma_0) \), respectively \( \mathcal{C}^*(\Gamma, \Gamma_0) \) is the universal *-algebra, respectively \( \mathcal{C}^* \)-algebra, generated by \( \{ V_g, U_n \mid g \in T, n \in N/\Gamma_0 \} \) such that for \( g, h \in T, n, m \in N/\Gamma_0 \)

1) \( V_g^* V_g = I \)
2) \( V_g V_h = V_{gh} \)
3) \( V_g V_g^* V_h V_h^* = V_{g \Delta h} V_{g \Delta h}^* \)
4) \( U_{nm} = U_n U_m \) and \( U_n^* = U_{n^{-1}}, U_e = V_e \)
5) \( U_n V_g = V_g U_{\text{ad}(\nu(g^{-1}))n} \)
6) \( V_g U_n V_g^* = R(g)^{-1} \sum \{ U_b \mid \text{ad}(\nu(g^{-1}))b = n \} \).

**Proof.** First note that \( V_e^2 = V_e \) by 2), so \( I = V_e^* V_e = V_e^* V_e V_e = V_e \). Note that the first three conditions state that \( V \) is a covariant representation by “isometries” of the semigroup \( T \), condition 4) states that \( U \) is a “unitary” representation of the group \( N/\Gamma_0 \) while condition 6) says that \( (U, V) \) is a “covariant pair”. We prove this result directly, rather than using Theorem 3.10. Let \( A \) be the universal *-algebra generated by \( V_g \) and \( U_n \) subject to the six conditions. Define \( \mathcal{L}' = \text{span}_C \{ V_s U_n V_s^* \mid s, t \in T, n \in N/\Gamma_0 \} \). We claim that \( \mathcal{L}' \) is a *-subalgebra of \( A \), thus, since it contains the generators of \( A, A = \mathcal{L}' \). To see this it is enough to show that \( V_s U_n V_t^* V_u U_m V_w^* \) is of the form \( V_a U_p V_b^* \). Condition 3) states \( V_t V_i^* V_u V_t^* = V_{t \vee s} V_{i \vee s} V_{t \vee s}^* \), so by 1), \( V_t^* V_u = V_t^* V_{t \vee s} V_{t \vee s}^* V_u \). Substituting this into \( V_s U_n V_t^* V_u U_m V_w^* \) and using condition 5) and its adjoint, finishes the claim.

One can define a *-homomorphism of \( A \) to \( H(\Gamma, \Gamma_0) \) by mapping \( V_g \) to \( W_{\nu(g)} \) and \( U_n \) to \( W_n \), since the six conditions are straightforward to verify for \( W_{\nu(g)} \) and \( W_n \).
The argument of Proposition 3.7 using condition 6) shows that \( L' = \text{span}_C \{V_s U_n V_t^* \mid s^{-1} \vee s t^{-1} = e, s, t \in T, n \in N/\Gamma_0 \} \). The argument in Theorem 3.10 using condition 6) with \( n = e \) shows that \( L' = \text{span}_C \{V_s U_n V_t^* \mid s^{-1} \vee s t^{-1} = e, s \in T, n \in N/\Gamma_0, t \in \Gamma_0 \} \) where \( \Gamma_0 \) is a set in \( T \) of coset representatives of \( T/\rho(\Gamma_0) \). The image of this spanning set in \( H(\Gamma, \Gamma_0) \) is a basis in \( H(\Gamma, \Gamma_0) \) by Proposition 3.9, so \( A \cong H(\Gamma, \Gamma_0) \).

Within the context of the preceding theorem, we saw earlier in Theorem 2.11 that there is a natural surjective \(*\)-homomorphism of \( C^*(N/\Gamma_0) \times T \) onto \( C^*(\Gamma, \Gamma_0) \). In fact this map is an isomorphism.

**Theorem 3.12.** Let \( (\Gamma, \Gamma_0) \) be an almost normal subgroup pair with \( e \rightarrow N \rightarrow \Gamma \xrightarrow{\nu} \rho \rightarrow e \) a split exact sequence. Assume also that \( \Gamma_0 \leq N \leq N_{\Gamma_0} \leq \Gamma \), \( G = T^{-1}T = TT^{-1} \) and that the solvable l.u.b. exists for any pair of elements of \( T = \{g \in G \mid \Gamma_0 \subseteq \text{ad}(\nu(g))\Gamma_0 \} \). Then the natural surjective \(*\)-homomorphism of \( C^*(N/\Gamma_0) \times T \) onto \( C^*(\Gamma, \Gamma_0) \) is an isomorphism.

**Proof.** As in Section 1, for \( n \in N/\Gamma_0 \) let \( \delta_n \) be the element of \( l^1(N/\Gamma_0) \) which is one at \( n \) and zero elsewhere. Let \( \eta : l^1(N/\Gamma_0) \rightarrow C^*(N/\Gamma_0) \) be the natural norm decreasing injective \(*\)-homomorphism and \( u_n = \eta(\delta_n) \) the natural unitaries generating \( C^*(N/\Gamma_0) \). Also denote by \( \{Y_g \mid g \in T\} \) the semigroup of isometries in \( C^*(N/\Gamma_0) \times T \) implementing the action of the semigroup \( T \) in the dynamical system \((C^*(N/\Gamma_0), \Theta, T)\) described before Theorem 1.10.

Define \( V_g = Y_g \) and \( U_n = u_n \) for \( g \in T \) and \( n \in N/\Gamma_0 \). We first show that the conditions of Theorem 3.11 are fulfilled. It is clear that conditions 1), 2), 4), and 6) hold for this family of elements in \( C^*(N/\Gamma_0) \times T \). Indeed, these are the defining relations for \( C^*(N/\Gamma_0) \times T \). We only need to show that conditions 3) and 5) also hold. By condition 6), \( V_g V_g^* = Y_g Y_g^* = R(g)^{-1} \sum \{U_b \mid \text{ad}(\nu(g^{-1}b = e)\} = \eta(R(g)^{-1} \sum \{\delta_n \mid \text{ad}(\nu(g))\Gamma_0, \text{ distinct in } N/\Gamma_0\} \). The argument of Proposition 3.2 along with the fact that \( \eta \) is a \(*\)-homomorphism shows that condition 3) holds.

We now check condition 5). First note that \( V_g V_g^* = \eta(\tilde{\Theta}_g(\delta_e)) \) by condition 6). Thus \( U_n V_g = \eta(\delta_n) V_g = \eta(\delta_n) Y_g Y_g^* Y_g = \eta(\tilde{\Theta}_g(\delta_e)) Y_g \). Also \( V_g U_{ad^{-1}(n)} = Y_g \eta(\delta_{ad^{-1}(n)}) = Y_g \eta(\delta_{ad^{-1}(n)}) Y_g^* Y_g = \eta(\tilde{\Theta}_g(\delta_{ad^{-1}(n)})) Y_g \) by condition 6). It is a straightforward calculation to check that the convolution product \( \delta_n \ast \tilde{\Theta}_g(\delta_e) = \tilde{\Theta}_g(\delta_{ad^{-1}(n)}) \), so condition 5) is verified.

Theorem 3.11 yields a \(*\)-homomorphism of \( C^*(\Gamma, \Gamma_0) \) to \( C^*(N/\Gamma_0) \times T \) which is easily seen to be surjective and the inverse of the surjective \(*\)-homomorphism of Theorem 2.11. \( \square \)
4. Examples.

In this section we illustrate and apply some of the above results to various examples. As we will see, many of the examples given are special cases of other examples.

4.1. As a first example, it is illustrative to see the Bost-Connes context. Their work provides the framework for much of this work. Here $\Gamma$ is the semidirect product of the abelian group $N= (\mathbb{Q}, +)$ and the abelian group $G = (\mathbb{Q}_+^\times, \cdot)$, the multiplicative group of nonzero positive rational numbers. The action $\psi : G \to \text{Aut} (N)$ is given by the ring structure of $\mathbb{Q}$, namely $\psi_g (r) = gr$ for $g \in G$, $r \in N$. For a subgroup $\Gamma_0$ of $N$ to be an almost normal subgroup of $\Gamma$, it is only necessary by Lemma 1.9 that the subgroup $\psi_g (\Gamma_0) \Gamma_0 / \Gamma_0$ of $N/\Gamma_0$ is finite for each $g \in G$. If $\Gamma_0 = \mathbb{Z}$, for $g = ab^{-1}$ with $a, b \in \mathbb{N}$, $b \neq 0$, we have $(\psi_g \mathbb{Z} + \mathbb{Z})/\mathbb{Z} \cong (\psi_a \mathbb{Z} + \psi_b \mathbb{Z})/\psi_b \mathbb{Z} = (a, b)\mathbb{Z}/b\mathbb{Z}$, a finite group. Since $\psi_g (\Gamma_0) \neq \Gamma_0$ for all $g \neq e$ in $G$, $N$ is the normalizer $N_{\Gamma_0}$ of $\Gamma_0$ in $\Gamma$. We compute that $T^{-1} = \{ g \in G \mid \psi_g (\mathbb{Z}) \subseteq \mathbb{Z} \} = \mathbb{N}$ and so $G = T^{-1} T = TT^{-1}$. For $a = n^{-1}$, $b = m^{-1}$ in $T = N^{-1}$, we have $a \leq b$ if and only if $n \mid m$, $a \vee b$ exists and is the least common multiple of $n$ and $m$. Thus $a^{-1} \vee b^{-1} = e$ if and only if $n$, $m$ are relatively prime. The $C^*$-Hecke algebra $C^*(\Gamma, \Gamma_0)$ is thus isomorphic to the semigroup crossed product $C^*$-algebra $C^* (\mathbb{Q}/\mathbb{Z}) \rtimes T \cong C^* (\mathbb{Q}/\mathbb{Z}) \times \mathbb{N}$, where we use the isomorphism of $G$ given by $n \to n^{-1}$, mapping the semigroup $T$ to $\mathbb{N}$.

4.2. With $N, G$ as in Example 4.1, there are different possible actions $\psi : G \to \text{Aut} (N)$, each giving rise to a split extension $\Gamma_{\psi}$ of $N$ by $G$. The group $\text{Aut} (N) = \text{GL}(1, \mathbb{Q}) = (\mathbb{Q}_+^\times, \cdot)$, so $\psi$ is determined by its effect on $P$, the prime members of $\mathbb{N}$, since $(\mathbb{Q}_+^\times, \cdot) \cong \bigoplus_P \mathbb{Z}$. If both $\psi^{-1} (1)$ and $\psi^{-1} (-1)$ are contained in $\{1\}$, the normalizer $N_{\Gamma_0}$ of $\Gamma_0 = \mathbb{Z}$ is $\mathbb{N}$ and a similar analysis to that of 4.1 may be carried through.

4.3. For $d \in \mathbb{N}$, let $G = \{ g \in \text{GL}(d, \mathbb{Q}) \mid \det g > 0 \} = \text{GL}(d, \mathbb{Q})_+$ and $N = (\mathbb{Q}^d, +)$, with $\psi : G \to \text{Aut} (N)$ the inclusion map. Setting $\Gamma$ to be an extension of $N$ by $G$ and $\Gamma_0 = \mathbb{Z}^d$ we check that $(\Gamma, \Gamma_0)$ is an almost normal subgroup pair. Again we need only check that $\psi_g (\mathbb{Z}^d) + \mathbb{Z}^d / \mathbb{Z}^d$ is finite for $g \in G$. Choosing $g \in G$, there is an $m \in \mathbb{N}$ with $mg \in \mathcal{W}_d (\mathbb{Z})$. For example, there is an $m \in \mathbb{N}$ so that the ideal $\{ r \in \mathbb{Z} \mid rg \in \mathcal{W}_d (\mathbb{Z}) \}$ of $\mathbb{Z}$ is $m\mathbb{Z}$. Then $\psi_g (\mathbb{Z}^d) + \mathbb{Z}^d / \mathbb{Z}^d = \psi_{m^{-1}} \psi_{mg} (\mathbb{Z}^d) + \mathbb{Z}^d / \mathbb{Z}^d \cong \psi_{mg} \mathbb{Z}^d + \psi_m \mathbb{Z}^d / \psi_m \mathbb{Z}^d \subseteq \mathbb{Z}^d / \psi_m (\mathbb{Z}^d)$, which has $m^d$ elements. The semigroup $T^{-1} = \{ g \in G \mid \psi_g (\mathbb{Z}^d) \subseteq \mathbb{Z}^d \} = \mathcal{W}_d (\mathbb{Z}) \cap G$, and $T \cap T^{-1} = \text{SL}(d, \mathbb{Z})$, which is not normal in $G$. Also $TT^{-1} = T^{-1} T = G$ and $a \vee b$ exists for each $a, b \in T$. Applying Proposition 2.8 gives us the universal $C^*$-algebra $C^* (\Gamma, \Gamma_0)$. Theorem 2.11 also applies.
4.4.

This example is a special case of Example 4.3, but designed to circumvent the problem of \( N_{\Gamma_0} \) not being normal in \( \Gamma \). It is also the example which motivated my work in this paper, cf. [B].

Choose \( M, F \in \mathfrak{M}_d(\mathbb{Z}) \) with \( MF = FM \), det \( F \) and det \( M \) both nonzero and relatively prime. Define an action \( \psi : \mathbb{Z} \oplus \mathbb{Z} \to \text{Aut} (N) \) by \( \psi(n, m) = F_t^{-n} M_t^{-m} \) where \( N \) is the subgroup of \( \mathbb{Q}^d \), endowed with the discrete topology, generated by \( \{ F_t^n M_t^m(\mathbb{Z}^d) \mid n, m \in \mathbb{Z} \} \). With \( \Gamma \) chosen so that \( e \to N \to \Gamma \to \mathbb{Z} \oplus \mathbb{Z} \to e \) is an extension inducing the given \( \psi \), and with \( \Gamma_0 = \mathbb{Z}^d \), the argument of Example 4.3 shows that \( (\Gamma, \Gamma_0) \) is an almost normal subgroup pair. The semigroup \( T = \{(n, m) \in \mathbb{Z} \oplus \mathbb{Z} \mid F_t^n M_t^m(\mathbb{Z}^d) \subseteq \mathbb{Z}^d\} \) contains \( \mathbb{N} \oplus \mathbb{N} \) and \( T \cap T^{-1} = \{(n, m) \in \mathbb{Z} \oplus \mathbb{Z} \mid F_t^n M_t^m \in GL(d, \mathbb{Z})\} \). Recall that \( (\text{det} F, \text{det} M) = 1 \), so if we stipulate that both \( |\text{det} F| \) and \( |\text{det} M| \) are not 1, then \( (n, m) \in T \cap T^{-1} \) if and only if \( n = m = 0 \). It also follows in this situation that \( \mathbb{N} \oplus \mathbb{N} = T \). Thus \( TT^{-1} = T^{-1} T = \mathbb{Z} \oplus \mathbb{Z} \).

We now show that any pair of elements of the additive group \( G = \mathbb{Z} \oplus \mathbb{Z} \) has a solvable l.u.b. The argument of Proposition 3.11 of [B] shows that \( F^aZ^d + M^bZ^d = Z^d \) for \( a, b \in \mathbb{N} \). If we denote the minimum of two integers \( a \) and \( b \) by \( a \wedge b \), it follows, since \( M \) and \( F \) commute, that \( F^aZ^d + M^bZ^d = F^{a \wedge 0} M^{b \wedge 0}(\mathbb{Z}^d) \). Thus, for \( (n, m), (p, q) \in G \), \( F^n M^m(\mathbb{Z}^d) + F^p M^q(\mathbb{Z}^d) = F^{n \wedge p} M^{m \wedge q}[F^{n - n \wedge p} M^{m - m \wedge q}(\mathbb{Z}^d) + F^{p - n \wedge p} M^{q - m \wedge q}(\mathbb{Z}^d)] = F^{n \wedge p} M^{m \wedge q}(\mathbb{Z}^d) \), so \( (n, m) \vee_s (p, q) = (n \wedge p, m \wedge q) \). Applying Theorem 3.10 we have, for \( \Gamma \) the semidirect product of \( N \) by \( \mathbb{Z} \) with respect to \( \psi \), that the Hecke \( C^* \)-algebra \( C^*(\Gamma, \Gamma_0) \) is isomorphic to the semigroup crossed product \( C^* \)-algebra \( C^*(\mathbb{N}/\Gamma_0) \rtimes T = C^*(\mathbb{N}/\Gamma_0) \rtimes (\mathbb{N} \oplus \mathbb{N}) \). There are other similar examples along these lines.

4.5.

Let \( K \) denote an algebraic number field; so a subfield of \( \mathbb{C} \) which is a finite dimensional extension of \( \mathbb{Q} \). Thus \( K = \mathbb{Q}[\alpha] \) for some algebraic number \( \alpha \), say of degree \( d \). Letting \( \Gamma_0 \) denote the ring of algebraic integers in \( K \), choose an integral basis \( B = \{ \beta_1, \ldots, \beta_d \} \) of \( \Gamma_0 \); so \( \Gamma_0 \) is the \( \mathbb{Z} \)-module generated by \( B \). Let \( \eta \) denote the norm for \( K \) over \( \mathbb{Q} \). The group of units of the ring \( \Gamma_0 \), denoted by \( \mathcal{U} \), is \( \{ g \in \Gamma_0 \mid \eta(g) = \pm 1 \} \) and \( K^\times \) denotes the multiplicative group of units of \( K \). There is an action \( \psi : K^\times \to \text{Aut} (K) \), where \( K \) is viewed as an additive abelian group, given by \( \psi_g(k) = gk, k \in K \) and \( g \in K^\times \). Also set \( T^{-1} = \{ g \in K^\times \mid \psi_g(\Gamma_0) \subseteq \Gamma_0 \} \). Since \( 1 \in \Gamma_0 \), the condition \( \psi_g \Gamma_0 \subseteq \Gamma_0 \) holds if and only if \( g \in \Gamma_0 \), so \( T^{-1} = \Gamma_0 \bigcap K^\times = \Gamma_0^\times \).

Lemma 4.5.1. \( K^\times = T^{-1} T = TT^{-1} \).

Proof. For \( g \in K^\times \) there is an \( m \in \mathbb{N} / \{0\} \) with \( mg \in \Gamma_0 \), so \( \psi_{mg}(\Gamma_0) \subseteq \Gamma_0 \) and \( mg \in T^{-1} \). Since \( m \in T^{-1} \) it follows that \( g = m^{-1}(mg) = (mg)m^{-1} \in TT^{-1} \bigcap T^{-1} T \).
The argument of Example 4.3 shows that \((\psi_g(\Gamma_0) + \Gamma_0)/\Gamma_0\) is finite for \(g \in K^\times\), so \((\Gamma, \Gamma_0)\) is an almost normal subgroup pair whenever \(e \to K \to \Gamma \to K^\times \to e\) is an extension inducing the action \(\psi\).

**Lemma 4.5.2.** \(T \cap T^{-1} = \mathcal{U}\).

**Proof.** If \(g \in T \cap T^{-1}\), then \(\psi_g \Gamma_0 = \Gamma_0\), so \(\psi_g\) as a matrix with respect to the basis \(B\) of \(\Gamma_0\), is in \(GL(d, \mathbb{Z})\). Thus \(\det(\psi_g) = \pm 1\). Since \(\eta(g) = \det(\psi_g)\), we have \(g \in \mathcal{U}\). Conversely, if \(g\) is a unit of \(\Gamma_0\), then \(\psi_g(\Gamma_0) = \Gamma_0\). \(\square\)

To be able to continue with this example, we need to show that solvable least upper bounds exist for pairs of elements from \(T\). Since \(K^\times\) is an abelian group, this is equivalent to any pair of elements from \(T^{-1}\) possessing a solvable l.u.b. For \(g, h \in T^{-1}\), \(g, h\) also belong to \(\Gamma_0\), so the subgroup \(\psi_g \Gamma_0 + \psi_h \Gamma_0\) is the sum of two principal ideals in the ring \(\Gamma_0\), so also an ideal. The question of whether the solvable l.u.b. of \(g\) and \(h\) exists in \(T^{-1}\) is then equivalent to whether every ideal of \(\Gamma_0\) is principle. For if \(c \in \Gamma_0\) with \(\psi_g \Gamma_0 + \psi_h \Gamma_0 = \psi_c \Gamma_0\) then \(\text{rank}(\psi_c \Gamma_0) \geq \text{rank}(\psi_g \Gamma_0) \geq d\). Thus \(\eta(c) \neq 0\) and \(c \in \Gamma_0 \cap K^\times = T^{-1}\). Since every ideal of \(\Gamma_0\) can be written as the sum of two principal ideals, the equivalence is established. Assuming then that every ideal of \(\Gamma_0\) is equivalent to a principle ideal, namely that the class group of the field \(K\) consists of the unit element only, i.e., that the class number \(h_K\) of \(K\) is 1, we have that solvable least upper bounds exist in \(T^{-1}\).

**Remark 4.5.3.** The subgroup \(K_+^\times = \{g \in K \mid \eta(g) > 0\}\) of \(K^\times\) is just \(\mathbb{Q}_+\) when \(K = \mathbb{Q}\), which is the group considered in [BC]. One could try the same approach as above in this situation, obtaining for example \(T^{-1} = \Gamma_0 \cap K_+^\times\) and \(K_+^\times = T T^{-1} = T^{-1} T\); however, there are examples where \(\psi_g \Gamma_0 + \psi_h \Gamma_0 = \psi_c \Gamma_0\) with \(g, h \in T^{-1}\), so \(\eta(g)\) and \(\eta(h)\) are both positive, but \(\eta(c) < 0\) for any such \(c\), so \(c \notin T^{-1}\). As an example, consider \(K = \mathbb{Q}[\sqrt{6}]\). Then \(\Gamma_0 = \mathbb{Z}[\sqrt{6}]\) and \(\mathcal{U}\), the units of \(\Gamma_0\), are \(\{\pm (5 + 2\sqrt{6})^n \mid n \in \mathbb{Z}\}\). These units all have norm 1. Setting \(g = 2\) and \(h = 4 + \sqrt{6}\) we have that the norms of \(g\) and \(h\) are both positive, equal to 4 and 10 respectively. Now write \(g = (2 + \sqrt{6})^2(5 - 2\sqrt{6})\) and \(h = (2 + \sqrt{6})(-1 + \sqrt{6})\). Since \(\eta(2 + \sqrt{6}) = -2\), the element \(2 + \sqrt{6}\) of \(\Gamma_0\) is indecomposable. Also \(\eta(-1 + \sqrt{6}) = -5\), so \(-1 + \sqrt{6}\) is indecomposable. Since \(2 + \sqrt{6} = -g + h\), it follows that \(\psi_g \Gamma_0 + \psi_h \Gamma_0 = \psi_c \Gamma_0\) where \(c = 2 + \sqrt{6}\) is an element of negative norm. If \(c \Gamma_0 = d \Gamma_0\) then \(d\) must be \(c\) up to multiplication by a unit of \(\Gamma_0\), so \(\eta(c) < 0\) for any \(c \in \Gamma_0\) with \(\psi_g \Gamma_0 + \psi_h \Gamma_0 = \psi_c \Gamma_0\). We also mention that \(\Gamma_0\) is Euclidean and so a principle ideal domain. Thus \(h_K = 1\) for this example.

It is a straightforward computation using that \(K^\times\) is abelian to show that \(N_{\Gamma_0}\) is a normal subgroup of \(\Gamma\), so Theorem 3.12 applies.
Proposition 4.5.4. Let $K$ be a number field with class number 1. Denoting the ring of algebraic integers by $\mathcal{O}_K$, $(\Gamma, \mathcal{O}_K)$ is an almost normal subgroup pair where $\Gamma$ is the semidirect product $K \times_{\psi} K^\times$ with $\psi : K^\times \to \text{Aut}(K)$ given by multiplication. If $T = \{g \in K^\times | \Gamma_0 \leq \psi_g(\Gamma_0)\}$ then $T^{-1} = \Gamma_0 \cap K^\times$, the Hecke algebra $H(\Gamma, \mathcal{O}_K)$ has a universal $C^*$-norm, and there is a natural $*$-isomorphism of the $C^*$-semigroup crossed product algebra $C^*(K/\mathcal{O}_K) \rtimes T$ with $C^*(\Gamma, \mathcal{O}_K)$, the $C^*$-completion of $H(\Gamma, \mathcal{O}_K)$. If $\Gamma_0$ has a unit of norm $-1$, the statement remains true if $K^\times$ is replaced with $K^\times_+ = \{g \in K^\times | \eta(g) > 0\}$ and $T = \{g \in K^\times_+ | \Gamma_0 \leq \psi_g(\Gamma_0)\}$.

Consider the multiplicative group $\mathcal{J}$ of fractional ideals of the Dedekind domain $\Gamma_0$. This is a free abelian group generated by the prime ideals of $\Gamma_0$ with unit element the ideal $\Gamma_0$. A fractional ideal is of the form $d^{-1}J$ for some integral ideal $J$ of $\Gamma_0$ and some $d \neq 0$ in $\Gamma_0$. The fractional principle ideals $k\Gamma_0$ with $k \in K^\times$ form a subgroup $\mathcal{K}$ of $\mathcal{J}$, namely the image of the group homomorphism $\varphi : K^\times \to \mathcal{J}$ mapping $k$ to $k\Gamma_0$. The kernel of this homomorphism is $\mathcal{U}$, the group of units of $\Gamma_0$, so $K^\times/\mathcal{U}$ is isomorphic to the subgroup $\mathcal{K}$ of fractional principle ideals of $\mathcal{J}$. Note that if the class number of $K$ is 1, all ideals of $\Gamma_0$ are principle, so $\varphi$ is surjective. Since $\mathcal{J}$ is a free abelian group, so is the subgroup of fractional principal ideals $\mathcal{K}$ and therefore the exact sequence $e \to \mathcal{U} \to K^\times \to \mathcal{K} \to e$ splits, yielding a subgroup $\mathcal{H}$ of $K^\times$, isomorphic with $\mathcal{K}$ and with $\mathcal{H} \cap \mathcal{U} = \{e\}$, $\mathcal{H} \mathcal{U} \cong K^\times$.

Theorem 4.5.5. Let $K$ be a number field with class number 1, $\Gamma_0$ the ring of algebraic integers in $K$. Then $(\Gamma, \mathcal{O}_K)$ is an almost normal subgroup pair where $\Gamma$ is the semidirect product $K \times_{\psi} \mathcal{H}$ with $\mathcal{H}$ a subgroup of $K^\times$ complementing $\mathcal{U}$, the group of units of $\Gamma_0$ and $\psi : \mathcal{H} \to \text{Aut}(K)$ defined by multiplication: $\psi_g(h) = gh$, $g \in \mathcal{H}$, $k \in K$. If $P = \{g \in \mathcal{H} | \Gamma_0 \leq \psi_g(\Gamma_0)\}$ then the Hecke algebra $H(\Gamma, \mathcal{O}_K)$ has a universal $C^*$-norm, and there is a natural $*$-isomorphism of the $C^*$-semigroup crossed product algebra $C^*(K/\mathcal{O}_K) \rtimes P$ with $C^*(\Gamma, \mathcal{O}_K)$, the $C^*$-completion of $H(\Gamma, \mathcal{O}_K)$.

4.6.

It seems worthwhile to include another example, as it encompasses all of the examples mentioned above and uses standard constructions in ring theory ([R]). Let $R$ be a unital ring and $\Gamma_0$ an $R$-module. For example, if $\Gamma_0$ is an abelian group, it can be viewed as an $R$-module where $R$ is any unital subring of the ring $\mathcal{R} = \text{Hom}(\mathcal{O}_K, \Gamma_0)$ by setting $f \cdot m = f(m)$ for $f \in \mathcal{R}$, $m \in \Gamma_0$. In general, the left regular representation of $R$ is a ring homomorphism $\rho : R \to \mathcal{R}$.

Now choose a unital multiplicatively closed subset $S$ of the center $Z(R)$ of $R$ and form $N$, the localization of the module $\Gamma_0$ at $S$, $N = S^{-1}\Gamma_0$. One construction of $N$ involves considering $S$ as a preordered directed set under $s \leq t$ if and only if $s$ divides $t$. Define $\varphi_s^t : \Gamma_s \to \Gamma_t$ for $s \leq t$ by restriction,
where $\Gamma_s$ is the abelian group $\text{Hom}_R(Rs, \Gamma_0)$ and set $S^{-1}\Gamma_0 = \lim\left(\Gamma_s, \varphi_s^t\right)$, a limit of $\mathbb{Z}$-modules. If $S^{-1}R$ denotes the ring obtained by localizing the ring $R$ at $S$, then $N$ becomes an $S^{-1}R$ module. Letting $G$ denote the group of units of the ring $S^{-1}R$ we have $S \subseteq G$ and we obtain an action of the group $G$ which extends the original action of $S$ on $\Gamma_0$. This construction is basically the one Cuntz used in forming the crossed product of a $C^*$-algebra by an endomorphism [C].

**Lemma 4.6.1.** If $\Gamma_0/s\Gamma_0$ is finite for each $s \in S$ then $(\Gamma, \Gamma_0)$ is an almost normal subgroup pair for any extension $e \to N \to \Gamma \to G \to e$.

*Proof.* Using Lemma 1.9 it suffices to show that $g\Gamma_0 + \Gamma_0/\Gamma_0$ is finite for each $g \in G$. Writing $g = s^{-1}r$ for some $s \in S$, $r \in R$ we have $g\Gamma_0 + \Gamma_0/\Gamma_0 \approx (r\Gamma_0 + s\Gamma_0)/s\Gamma_0 \subseteq \Gamma_0/s\Gamma_0$, which is finite. □

If we consider the smaller abelian subgroup $G_0 = S^{-1}S$ of $G$ and let $\Gamma$ be the split extension of $N = S^{-1}\Gamma_0$ by $G_0$, the setting of Theorem 2.11 begins to appear, with $T = S^{-1}$ in this case.

**References**


Received August 5, 1996. The author acknowledges support, in connection with this research, from the Natural Sciences and Engineering Research Council of Canada and the Fields Institute for Research in the Mathematical Sciences.

**University of Calgary**

Calgary, AB T2N 1N4

Canada

E-mail address: bbrenken@ucalgary.ca
QUANTITATIVE DEFORMATION THEOREMS AND CRITICAL POINT THEORY

JEAN-NOËL CORVELLEC

In the framework of critical point theory for continuous functionals defined on metric spaces, we show how quantitative deformation properties can be used to obtain saddle-point type results, even in the case when the usual geometric assumptions are not satisfied. We thus unify and extend to a nonsmooth setting some recent results of Schechter.

1. Introduction.

It is well known that deformation theorems are the basic tools in critical point theory. They can be derived under a condition of Palais-Smale type \((PS)\), for short. In the classical setting of a \(C^1\) functional \(f\) defined on a Banach space \(X\) (or a \(C^2\) Finsler manifold), we refer to [15]; for a continuous functional \(f\) defined on a complete metric space \(X\), we refer to [8], the results of which include the case of a \(C^1\) functional defined on a \(C^1\) Finsler manifold. On the other hand, some authors gave, in the smooth case, so-called quantitative deformation theorems which are based on a weaker condition than \((PS)\), see, e.g., [2, 21], and also [16] for a detailed account of the theory.

Now, it is also well known how the min-max principle yields the existence of a critical value of the functional \(f\), by combining a deformation theorem under the \((PS)\) condition, and some geometrical assumptions. Generally speaking, this is the case when a subset \(B\) of \(X\) links another subset \(A\) of \(X\), and the condition

\[-\infty < b_0 := \sup_B f \leq \inf_A f =: a < +\infty\]  

is fulfilled, providing a critical level \(b \geq a\) for \(f\). On the other hand, in this geometrical situation, quantitative deformation theorems yield the existence of a Palais-Smale sequence for \(f\) at the level \(b\) — so that both approaches essentially provide the same information.

It turns out that the quantitative results allow to draw some conclusion, in the presence of linking, even in the case when (1.1) is not satisfied. Roughly speaking, and using the terminology introduced in [9], one can estimate the weak slope of \(f\) at some point \(u\) with \(f(u) \in [a, b]\) in terms of the difference \(b_0 - a\) and the distance from \(A\) and \(B\); under appropriate assumptions, it
is then possible to find a critical level for $f$ in the interval $[a, b]$. This idea is developed in the recent papers of Schechter [17, 18], dealing with $C^1$ functionals on Banach spaces (see also [20]).

In this paper, we extend this approach to saddle-point type results, in the context of the critical point theory of [9, 8]. For this we use quantitative versions of the deformation theorems of [8], the former being straightforward consequences of the constructions performed in the latter. As for the notion of linking, we consider a variant of the one in [19], which proves convenient for our purposes, while allowing to recover all standard situations.

It is worth emphasizing that the deformations involved in our constructions are not homeomorphisms at each fixed time; indeed, when working in general metric spaces, we cannot expect this property to hold. Even in a Banach space setting, we do not know how to recover this property when dealing with functions which are only continuous. But, as it turns out, this is not needed to obtain essentially the desired generalizations of the results of [17, 18]. On the other hand, working in metric spaces offers considerable freedom, as, for example, variants or extensions of the basic results are readily obtained by a simple change of metric. This is the case for the mentioned extensions of the results of [18]. Moreover, avoiding *ad hoc* constructions of deformations, all our results are deduced from the basic deformation theorems.

The results of this paper are developed in view of applications to nonsmooth variational problems, such as quasilinear elliptic problems of the type studied in [3, 4] (see also the references in [4]).

We mention that other deformation results for continuous functionals on metric spaces can be found in [6] (with an application in [7]), and in [12, 13] where a similar approach to nonsmooth critical point theory has been developed independently.

In Section 2, we recall the notion of weak slope and state our quantitative deformation theorems. The notion of linking is introduced in Section 3, and the main results on estimating the weak slope in the presence of linking are established. Section 4 deals with a general change-of-metric procedure, yielding extensions of the results of Section 3. Finally, some corollaries in Banach spaces are given in Section 5.

2. Quantitative deformation theorems.

In this section, $X$ is a metric space endowed with the metric $d$, and $f : X \to \mathbb{R}$ is a continuous function. In the sequel, we denote respectively by $B(u; \delta)$, $\bar{B}(u; \delta)$, and $\partial B(u; \delta)$ the open, closed ball and sphere of radius $\delta > 0$ centered at $u \in X$.

We start with recalling the notion of weak slope from [9], as well as the associated notions of Palais-Smale sequence and Palais-Smale condition.
Definition 2.1 (See [9, Definition (2.1)]). For \( u \in X \) we denote by \(|df|(u)\) the supremum of the \( \sigma \)'s in \([0, +\infty[\) such that there exist \( \delta > 0 \) and \( \mathcal{H} : B(u; \delta) \times [0, \delta] \to X \) continuous with
\[
\begin{align*}
&d(\mathcal{H}(v, t), v) \leq t, \\
&f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.
\end{align*}
\]

This notion was also introduced independently in [14], after a similar notion was defined in [12].

We say that \( u \in X \) is a critical point of \( f \) if \(|df|(u) = 0\). For \( c \in \mathbb{R} \), we let
\[
K_c = \{u \in X : |df|(u) = 0, \ f(u) = c\}
\]
denote the set of critical points of \( f \) at level \( c \).

We recall that if \( X \) is a \( C^1 \) Finsler manifold and \( f \) is of class \( C^1 \), then \(|df|(u) = \|f'(u)\|\) for every \( u \in X \). See [4] and the references therein for further comparisons between the weak slope and various analytical notions involving functionals which are not, in general, (Gâteaux) differentiable.

Definition 2.2. Let \( f : X \to \mathbb{R} \) be a continuous function and let \( c \in \mathbb{R} \). We say that a sequence \((u_h) \subset X\) is a Palais-Smale sequence for \( f \) at level \( c \) if \(|df|(u_h) \to 0\) and \( f(u_h) \to c \). We say that \( f \) satisfies the Palais-Smale condition at level \( c \) ((PS)\(_c\) for short), if every Palais-Smale sequence \((u_h)\) for \( f \) at level \( c \) contains a subsequence \((u_{h_k})\) converging in \( X \).

Since, clearly, \( u \mapsto |df|(u) \) is lower semicontinuous, a limit point of a Palais-Smale sequence for \( f \) at level \( c \) is a critical point of \( f \) at level \( c \). Condition \((PS)_c\) is used in critical point theory to obtain deformations properties of the sublevel sets of the function \( f \), see [8, Theorems (2.14), (2.15)]. In fact, condition \((PS)_c\) readily implies the following, which is a condition of quantitative type used in practice to establish these deformation properties:

Given \( \gamma > 0 \) there exist \( \delta > 0 \) and \( \sigma > 0 \) such that
\[
c - \delta \leq f(u) \leq c + \delta, \quad d(u, K_c) \geq \gamma \implies |df|(u) > \sigma.
\]

Indeed, a close inspection of the proof of [8, Theorems (2.14), (2.15)] shows that the following two results hold.

Theorem 2.3 (Quantitative Deformation Theorem). Let \( X \) be a complete metric space, \( f : X \to \mathbb{R} \) a continuous function, \( c \in \mathbb{R} \), \( A \) a closed subset of \( X \), and \( \delta, \sigma > 0 \) such that
\[
c - 2\delta \leq f(u) \leq c + 2\delta, \quad d(u, A) \leq \delta/\sigma \implies |df|(u) > 2\sigma.
\]

Then, there exists \( \eta : X \times [0, 1] \to X \) continuous with:

(a) \( d(\eta(u, t), u) \leq (\delta/\sigma)t \);

(b) \( \eta(u, t) \neq u \implies f(\eta(u, t)) < f(u) \);

(c) \( u \in A, \ c - \delta \leq f(u) \leq c + \delta \implies f(\eta(u, t)) \leq f(u) - (f(u) - c + \delta)t \).
Theorem 2.4 (Quantitative Noncritical Interval Theorem). Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ be a continuous function, $a, b \in \mathbb{R}$ with $a < b$, $\delta > 0$, and $\sigma > 0$ such that

$$a - \delta \leq f(u) \leq b + \delta \implies |df|(u) > \sigma.$$ 

Then, there exists $\eta : X \times [0, 1] \to X$ continuous with:

(a) $d(\eta(u, t), u) \leq \frac{b-a}{\sigma} t$;
(b) $\eta(u, t) \neq u \implies f(\eta(u, t)) < f(u)$;
(c) $a \leq f(u) \leq b \implies f(\eta(u, t)) \leq f(u) - (f(u) - a)t$.

3. Estimating the weak slope in the presence of linking.

In this section, $X$ is a metric space endowed with the metric $d$ and $f : X \to \mathbb{R}$ is a continuous function. If $A, B$ are subsets of $X$, we denote by

$$d(A, B) := \inf\{d(u, v) : u \in A, v \in B\}$$

the distance between $A$ and $B$, with the convention that $d(A, \emptyset) = +\infty$.

For $c \in \mathbb{R}$, we use the notations

$$f^c := \{u \in X : f(u) < c\}, \quad f_c := \{u \in X : f(u) > c\},$$

to denote the open sublevel and upperlevel sets of $f$, respectively.

We say that a (nonempty) subset $B$ of $X$ is contractible in $X$ if there exists a continuous $\psi : B \times [0, 1] \to X$ such that $\psi(u, 0) = u$ for every $u \in B$, and $\psi(u, 1) = u_0$ for some $u_0 \in X$ and every $u \in B$. Such a continuous map $\psi$ will be called a contraction of $B$ in $X$.

We introduce the notion of linking of two subsets of $X$, which is a simplified version of the notion given in [19].

Definition 3.1. Let $X$ be a metric space, and $A, B$ two subsets of $X$. We say that $B$ links $A$ if $B$ is contractible in $X$, $B \cap A = \emptyset$, and $\psi(B \times [0, 1]) \cap A \neq \emptyset$ for every contraction $\psi$ of $B$ in $X$.

In the remainder of this section, we shall consider two subsets $A$ and $B$ of $X$ such that $B$ links $A$. We set:

$$a := \inf_A f, \quad b_0 := \sup_B f,$$

and, denoting by $\Phi_B$ the set of contractions of $B$ in $X$:

$$b := \inf_{\psi \in \Phi_B} \sup_{B \times [0, 1]} (f \circ \psi).$$

Of course, $b_0 \leq b$, and $a \leq b$ since $B$ links $A$.

Theorem 3.2. Assume that $b \in \mathbb{R}$, $b_0 \leq a$, and $b - a \leq \delta/2$ for some $\delta > 0$. Then, for any $\sigma > 0$ there exists $u \in X$ with

$$b - 2\delta \leq f(u) \leq b + 2\delta, \quad d(u, A) \leq 2\delta/\sigma, \quad \text{and} \quad |df|(u) \leq 2\sigma.$$
Proof. We argue by contradiction. Assume that for some \( \sigma > 0 \) we have
\[
\forall u \in X, \quad b - 2\delta \leq f(u) \leq b + 2\delta, \quad d(u, A) \leq 2\delta/\sigma \implies |df|(u) > 2\sigma.
\]
Let \( \eta : X \times [0, 1] \to X \) continuous be given by Theorem 2.3 (with \( A \) replaced by \{ \( u \in X : d(u, A) \leq \delta/\sigma \} \}) such that:
\[
d(\eta(u, t), u) \leq (\delta/\sigma)t, \quad \eta(u, t) \neq u \implies f(\eta(u, t)) < f(u),
\]
\[
[d(u, A) \leq \delta/\sigma, \quad b - \delta \leq f(u) \leq b + \delta] \implies f(\eta(u, t)) \leq f(u) - (f(u) - b + \delta)t.
\]
Let \( \psi_0 \in \Phi_B \) with \( \sup_{B \times [0, 1]} (f \circ \psi_0) \leq b + \delta \) and define \( \psi \in \Phi_B \) by
\[
\psi(u, t) = \begin{cases} u & \text{if } 0 \leq t < 3/4 \\ \psi(u, 4t - 3) & \text{if } 3/4 \leq t \leq 1,
\end{cases}
\]
so that, of course, \( \sup_{B \times [0, 1]} (f \circ \psi) \leq b + \delta \). Finally, define \( \tilde{\psi} \in \Phi_B \) by
\[
\tilde{\psi}(u, t) = \eta(\psi(u, t), t).
\]
Let \( (u, t) \in B \times [0, 1] \) such that \( \tilde{\psi}(u, t) \in A \). Then, \( t > 3/4 \) for, if \( t \in [0, 3/4) \) we have \( \tilde{\psi}(u, t) = \eta(u, t) \), so that if \( \tilde{\psi}(u, t) \neq u \) then \( f(\tilde{\psi}(u, t)) < f(u) \leq b_0 \leq a \) and \( \tilde{\psi}(u, t) \notin A \). On the other hand, we have \( d(\psi(u, t), A) \leq \delta/\sigma \), hence
\[
f(\tilde{\psi}(u, t)) = f(\eta(\psi(u, t), t)) \leq f(\psi(u, t)) - (f(\psi(u, t)) - b + \delta/2)t < b - \delta/2 \leq a,
\]
contradicting the fact that \( \tilde{\psi}(u, t) \in A \).

\[\square\]

**Theorem 3.3.** Let \( (X, d) \) be a complete metric space, \( f : X \to \mathbb{R} \) a continuous function, and \( A, B \) two subsets of \( X \) such that \( B \) links \( A \). Set:
\[
a := \inf_A f, \quad b_0 := \sup f, \quad b := \inf_{\psi \in \Phi_B} \sup_{B \times [0, 1]} (f \circ \psi),
\]
and assume that \( a, b \in \mathbb{R} \).

(a) Assume that \( b_0 \leq a \). Then, there exists a sequence \((u_h) \subset X\) with
\[(3.4) \quad f(u_h) \to b \quad \text{and} \quad |df|(u_h) \to 0;\]
if, moreover, \( b = a \) we can require \((u_h)\) to satisfy also \( d(u_h, A) \to 0 \).
Consequently, if \( f \) satisfies condition (PS)_b, there exists \( u \in X \) such that \( f(u) = b \), \( |df|(u) = 0 \) (and \( u \in \bar{A} \) if \( b = a \)).

(b) Assume that \( b_0 > a \). If either: (i) \( 0 < \alpha \leq d(B \cap f_a, A) \) or: (ii) \( 0 < \alpha \leq d(B, f^{b_0} \cap A) \), and if \( \sigma > (b_0 - a)/\alpha \), then, for any \( \delta > 0 \) there exists \( u \in X \) with
\[
a - \delta \leq f(u) \leq b + \delta \quad \text{and} \quad |df|(u) \leq \sigma.
\]
Proof. (a) Apply Theorem 3.2 with \( \delta = \delta_h := 1/h^2 \), \( \sigma = \sigma_h := 1/h \), and either the given \( A \) if \( b = a \), or \( A \) replaced by \( \hat{A} := f_{b-1/2h^2} \) if \( b > a \). In the latter case, it is indeed clear that \( B \cap \hat{A} = \emptyset \) for large \( h \) because \( b_0 \leq a < b \), while \( \psi(B \times [0,1]) \cap \hat{A} \neq \emptyset \) for every \( \psi \in \Phi_B \), by the definition of \( b \). (This shows that (3.4) holds whenever \( b \in \mathbb{R}, b > b_0 \), without any explicit reference to a set \( A \) such that \( B \) links \( A \).)

(b) If \( b > b_0 \), the conclusion is given by (a). Hence, we now assume that \( b = b_0 \). Observe that \( b_0 > a \) means that \( B \cap f_a \) and \( A \cap f^{b_0} \) are nonempty, so that \( \alpha < +\infty \) — while \( \sigma > 0 \).

Arguing by contradiction, we assume that for some \( \delta > 0 \)
\[
  u \in X, \quad a - 2\delta \leq f(u) \leq b + 2\delta \implies |df|(u) > \sigma,
\]
and that \( \delta \) is so small that \( b - a + 2\delta \leq \sigma \alpha \). Let \( \eta : X \times [0,1] \to X \) be given by Theorem 2.4 (with \( a \) replaced by \( a - \delta \) and \( b \) by \( b + \delta \)) such that:
\[
  d(\eta(u,t),u) \leq \frac{b - a + 2\delta}{\sigma} t, \quad \eta(u,t) \neq u \implies f(\eta(u,t)) < f(u), \text{ and}
\]
\[
  a - \delta \leq f(u) \leq b + \delta \implies f(\eta(u,t)) \leq f(u) - (f(u) - a + \delta)t.
\]
Let \( (b-a+\delta)/(b-a+2\delta) < \varepsilon < 1 \) be fixed; then,
\[
  (3.5) \quad f(u) \leq b + \delta, \quad \varepsilon \leq t \leq 1 \implies f(\eta(u,t)) < a.
\]
Indeed, if \( a \leq f(u) \leq b + \delta \) and \( t \in [\varepsilon,1] \), we have:
\[
  f(\eta(u,t)) \leq (1-t)f(u) + (a-\delta)t \leq b + \delta - (b-a+2\delta)\varepsilon < a,
\]
while (3.5) clearly holds whenever \( f(u) < a \).

Now, let \( \psi_0 \in \Phi_B \) with \( \sup_{B \times [0,1]}(f \circ \psi_0) \leq b + \delta \), and define \( \psi \in \Phi_B \) by
\[
  \psi(u,t) = \begin{cases} 
    u & \text{if } 0 \leq t \leq \varepsilon \\
    \psi_0 \left( u, \frac{t - \varepsilon}{1 - \varepsilon} \right) & \text{if } \varepsilon \leq t \leq 1;
  \end{cases}
\]
finally, define \( \tilde{\psi} \in \Phi_B \) by
\[
  \tilde{\psi}(u,t) = \eta(\psi(u,t),t).
\]
We show that \( \tilde{\psi}(B \times [0,1]) \cap A = \emptyset \), contradicting the fact that \( B \) links \( A \).

Let \( u \in B \). For \( t \in [0,\varepsilon] \) we have \( \tilde{\psi}(u,t) = \eta(u,t) \), so that
\[
  (3.6) \quad d(\tilde{\psi}(u,t),u) = d(\eta(u,t),u) \leq \frac{b - a + 2\delta}{\sigma} \varepsilon < \alpha,
\]
while
\[
  (3.7) \quad \tilde{\psi}(u,t) = \eta(u,t) \neq u \implies f(\tilde{\psi}(u,t)) < f(u).
\]
In case (i), (3.6) shows that \( \tilde{\psi}(u,t) \notin A \) whenever \( f(u) > a \), while (3.7) shows that \( \tilde{\psi}(u,t) \notin A \) whenever \( f(u) \leq a \) (recall that \( B \cap A = \emptyset \)). In case
(ii), (3.6) and (3.7) show that \( \tilde{\psi}(u,t) \notin A \) since \( f(u) \leq b_0 \). For \( t \in [\varepsilon, 1] \) we have:
\[
f(\tilde{\psi}(u,t)) = f \left( \eta \left( \psi_0 \left( u, \frac{t - \varepsilon}{1 - \varepsilon} \right), t \right) \right) < a
\]
according to (3.5), so that again, \( \tilde{\psi}(u,t) \notin A \).

Theorem 3.3 (a) is similar to the “usual” min-max principle in the presence of linking, compare with [8, Theorem 3.7]. We now show how Theorem 3.3 can be used to obtain the existence of critical points of \( f \), even in the case when (1.1) is not satisfied.

**Corollary 3.8.** Let \( X \) be a complete metric space, \( f : X \to \mathbb{R} \) continuous, and \((A_h),(B_h)\) two sequences of subsets of \( X \) such that for each \( h \):

\[
d(B_h, A_h) > 0 \quad \text{and} \quad B_h \text{ links } A_h.
\]

Set:
\[
a_h := \inf_{A_h} f, \quad b_{h0} := \sup_{B_h} f, \quad b_h := \inf_{\psi \in \Phi_h} \sup_{B_h A \times [0,1]} (f \circ \psi),
\]
where \( \Phi_h \) denotes the set of contractions of \( B_h \) in \( X \),
\[
d_h := d(B_h \cap f_{a_h}, A_h), \quad d_{h0} := d(B_h, f^{b_{h0}} \cap A_h).
\]
Assume that \( b_h, a_h \in \mathbb{R} \), eventually, and that either:
\[
\limsup_{h \to \infty} \frac{b_{h0} - a_h}{d_h} \leq 0, \quad \text{or} \quad \limsup_{h \to \infty} \frac{b_{h0} - a_h}{d_{h0}} \leq 0
\]
(with the convention that \( \frac{1}{+\infty} = 0 \)). Finally, let
\[
b := \liminf_{h \to \infty} b_h, \quad a := \limsup_{h \to \infty} a_h,
\]
and assume that \( a, b \in \mathbb{R} \) and that \( f \) satisfies condition \((PS)_c\) for all \( c \in [a,b] \).

Then, there exists \( u \in X \) such that \( |df|(u) = 0 \) and \( f(u) \in [a,b] \).

**Proof.** Let \( h \) be so large that \( b_h, a_h \in \mathbb{R} \). Of course, \( d_h = +\infty \) (resp., \( d_{h0} = +\infty \)) if and only if \( b_{h0} \leq a_h \). In this case, apply Theorem 3.3 (a) to obtain \( u_h \in X \) with \( |f(u_h) - b_h| \leq 1/h \) and \( |df|(u_h) \leq 1/h \).

Whenever \( d_h < +\infty \) (resp., \( d_{h0} < +\infty \)), observe first that \( d_h > 0 \) (resp., \( d_{h0} > 0 \)) since \( d(B_h, A_h) > 0 \). Let \( \varepsilon_h > 0 \) with \( \varepsilon_h \to 0 \) be such that
\[
\frac{b_{h0} - a_h}{d_h} \leq \varepsilon_h, \quad \text{resp.,} \quad \frac{b_{h0} - a_h}{d_{h0}} \leq \varepsilon_h,
\]
and apply Theorem 3.3 (b) with \( \alpha = \alpha_h := d_h \) (resp., \( \alpha = \alpha_h := d_{h0} \)) and \( \sigma = \sigma_h := \varepsilon_h + 1/h \) to obtain \( u_h \in X \) with
\[
a_h - 1/h \leq f(u_h) \leq b_h + 1/h \quad \text{and} \quad |df|(u_h) \leq \sigma_h.
\]
The conclusion follows, using the Palais-Smale condition (see Definition 2.2).

\[\square\]

**Remark 3.9.** The results of this section are nonsmooth variants of results of [17].

### 4. Changing the metric.

In this section, \(X\) is a metric space endowed with the metric \(d\), and \(f : X \to \mathbb{R}\) is a continuous function. We describe a general procedure which allows to deduce new results from those of the preceding section, by a change of the metric of \(X\).

**Theorem 4.1.** Let \((X, d)\) be a metric space, \(\tilde{A}\) be a nonempty subset of \(X\), and \(\beta : [0, +\infty[ \to [0, +\infty[\) be continuous. Then, there exists a metric \(\tilde{d}\) on \(X\) which is topologically equivalent to \(d\) and such that:

(a) For any subset \(B\) of \(X\), it holds

\[
\tilde{d}(B, \tilde{A}) \geq \int_0^{d(B, \tilde{A})} \beta(t) \, dt.
\]

Consequently, if \(\int_0^{+\infty} \beta(t) \, dt = +\infty\), then \((X, \tilde{d})\) is complete if (and only if) \((X, d)\) is complete;

(b) if \(f : X \to \mathbb{R}\) is a continuous function, and we denote by \(|\tilde{df}|\) the weak slope of \(f\) with respect to the metric \(\tilde{d}\), it holds

\[
|\tilde{df}|(u) = \frac{|df|(u)}{\beta(d(u, \tilde{A}))} \quad \text{for every } u \in X.
\]

**Proof.** Let \(Y\) be a Banach space such that \(X\) is isometrically embedded in \(Y\), according to the Arens-Eells Theorem (see, e.g., [11, in 2.5.16, p. 110]). For \(u, v \in Y\), we denote by \(\Gamma_{u,v}\) the set of \(C^1\)-paths \(\gamma : [0,1] \to Y\) with \(\gamma(0) = u, \gamma(1) = v\), and we set

\[
\tilde{d}(u,v) := \inf_{\gamma \in \Gamma_{u,v}} \int_0^1 \beta(d(\gamma(t), \tilde{A})) \|\gamma'(t)\| \, dt,
\]

where \(d\) also stands for the distance associated to the norm. It is easy to verify that \(\tilde{d}\) is a metric on \(Y\). Considering line segments, we see that if \(B \subset Y\) is norm-bounded there exists \(\delta > 0\) such that

\[
\tilde{d}(u,v) \leq \delta \|u - v\| \quad \text{for every } u, v \in B.
\]

Also, if \(u \in Y\) and \(r > 0\) are fixed there exists \(\tilde{\delta} > 0\) such that

\[
\|u - v\| \geq r \quad \implies \quad \tilde{d}(u,v) \geq \tilde{\delta}.
\]

It follows that the restriction of \(\tilde{d}\) to \(X\) is topologically equivalent to \(d\), and that \((X, \tilde{d})\) is complete whenever \((X, d)\) is.
To prove (a), we may assume that \( B \neq \emptyset \) (otherwise, (4.2) is trivially satisfied). Let \( u \in \tilde{A}, v \in B, \) and \( \gamma \in \Gamma_{u,v}. \) For \( t \in [0,1] \) set \( \rho(t) := d(\gamma(t), \tilde{A}), \) so that \( \rho(0) = 0, \rho(1) = d(v, A). \) Being locally Lipschitz continuous on \([0,1], \rho\) is almost everywhere differentiable, and \( |\rho'(t)| \leq \|\gamma'(t)\| \) for almost every \( t \in ]0,1[. \) We thus have, through a change of variable (see, e.g., [11, Theorem 3.2.5]):
\[
\int_0^1 \beta(d(\gamma(t), \tilde{A})))\|\gamma'(t)\| \, dt \geq \int_0^1 \beta(\rho(t))|\rho'(t)| \, dt \geq \int_0^{d(v, \tilde{A})} \beta(t) \, dt.
\]
From the definition of \( \tilde{d} \) and since \( \gamma \) is arbitrary in \( \Gamma_{u,v} \), we obtain that
\[
\tilde{d}(u, v) \geq \int_0^{d(v, \tilde{A})} \beta(t) \, dt \geq \int_0^{d(B, \tilde{A})} \beta(t) \, dt.
\]
Since \( u \) and \( v \) are arbitrary in \( \tilde{A} \) and \( B \), respectively, (4.2) follows.

We now observe that for every \( R > 0 \) there exists \( \tilde{\delta} > 0 \) such that
\[
(4.3) \quad d(u_i, \tilde{A}) \leq R, \ i = 1, 2 \implies \tilde{d}(u_1, u_2) \geq \tilde{\delta} \min\{1, \|u_1 - u_2\| \}.
\]
Indeed, for \( \gamma \in \Gamma_{u_1,u_2} \) let \( t_\gamma := \sup\{t \in [0,1] : d(\gamma(s), \tilde{A}) \leq R + 1 \text{ for all } s \in [0,t] \} \in [0,1] \), and let \( \tilde{\delta} := \min_{[0,R+1]} \beta. \) Then,
\[
\int_0^{t_\gamma} \beta(d(\gamma(t), \tilde{A})))\|\gamma'(t)\| \, dt \geq \tilde{\delta}\|u_1 - \gamma(t_\gamma)\| \geq \tilde{\delta} \min\{1, \|u_1 - u_2\| \},
\]
considering the cases \( t_\gamma < 1, t_\gamma = 1, \) and (4.3) follows since \( \gamma \) is arbitrary in \( \Gamma_{u_1,u_2}. \)

Thus, if \( \int_0^{+\infty} \beta(t) \, dt = +\infty \) and \((u_h)\) is a Cauchy sequence in \((X, \tilde{d}), (4.2)\) shows that \( (d(u_h, \tilde{A})) \) is bounded, and (4.3) shows that \((u_h)\) is a Cauchy sequence in \((X, d)\). Hence, \((X, \tilde{d})\) is complete if \((X, d)\) is complete.

Assertion (b) is derived from the definitions of \( \tilde{d} \) and \( |\tilde{d} f| \) (see Definition 2.1); we give the proof for the reader’s convenience. In a standard way, we need to establish both inequalities. We still consider \( X \) as embedded in the Banach space \( Y. \)

Fix \( u \in X, \varepsilon > 0, \) and assume that \( |df|(u) > \sigma > 0. \) Let \( \delta > 0 \) and \( \mathcal{H} : B(u; \delta) \times [0, \delta] \to X \) such that
\[
\|\mathcal{H}(v, t) - v\| \leq t, \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t;
\]
we may assume that \( \delta \) is so small that
\[
\beta(d(v, \tilde{A})) \leq \beta(d(u, \tilde{A})) + \varepsilon \quad \text{for all } v \in B(u; 2\delta).
\]
Let \( 0 < \tilde{\delta} \leq \delta(\beta(d(u, \tilde{A}))) + \varepsilon \) such that \( \tilde{B}(u; \tilde{\delta}) \subset B(u; \delta) \) (where \( \tilde{B} \) stands for the ball with respect to the metric \( \tilde{d} \)), and define \( \overline{\mathcal{H}} : \tilde{B}(u; \tilde{\delta}) \times [0, \delta] \to X \)
by
\[ \tilde{\mathcal{H}}(v, t) = \mathcal{H}(v, \frac{t}{\beta(d(u, A)) + \varepsilon}). \]
For \( (v, t) \in \hat{B}(u; \delta) \times [0, \delta] \) and \( s \in [0, 1] \), let \( \gamma(s) := v + s(\tilde{\mathcal{H}}(v, t) - v) \in B(u; 2\delta) \). Then,
\[ \tilde{d}(\tilde{\mathcal{H}}(v, t), v) \leq \int_0^1 \beta(d(\gamma(s), \tilde{\gamma}))\|\gamma'(s)\| \, ds \leq t \]
and
\[ f(\tilde{\mathcal{H}}(v, t)) \leq f(v) - \frac{\sigma}{\beta(d(u, A)) + \varepsilon} t. \]
It follows that \( |\tilde{d}f|(u) \geq \frac{\sigma}{\beta(d(u, A)) + \varepsilon} \), hence \( |\tilde{d}f|(u) \geq \frac{\sigma}{\beta(d(u, A))} \) since \( \varepsilon > 0 \) is arbitrary. Consequently, \( |\tilde{d}f|(u) \geq \frac{|df|(u)}{\beta(d(u, A))} \), the inequality being obvious whenever \( |df|(u) = 0 \).
Conversely, let \( u \in X \) and \( 0 < \varepsilon < \beta(d(u, \tilde{A})) \) be fixed, and let \( \delta_0 > 0 \) such that
\[ \beta(d(v, \tilde{A})) \geq \beta(d(u, \tilde{A})) - \varepsilon \]
for all \( v \in B(u; 3\delta_0) \).
Assume that \( |\tilde{d}f|(u) > \sigma > 0 \). Let \( \tilde{\delta} > 0 \) and \( \tilde{\mathcal{H}} : \hat{B}(u; \tilde{\delta}) \times [0, \tilde{\delta}] \to X \) such that
\[ \tilde{d}(\tilde{\mathcal{H}}(v, t), v) \leq t, \quad f(\tilde{\mathcal{H}}(v, t)) \leq f(v) - \sigma t; \]
we may assume that \( \tilde{\delta} \) is so small that \( \hat{B}(u; 2\tilde{\delta}) \subset B(u; \delta_0) \). For \( (v, t) \in \hat{B}(u; \tilde{\delta}) \times [0, \tilde{\delta}] \) and \( \gamma \in \Gamma_{v, \mathcal{H}(v, t)} \), let \( \tilde{s} := \sup\{s \in [0, 1] : \gamma(r) \in B(u; 3\delta_0) \} \) for all \( 0 \leq r \leq s \} \in [0, 1] \). We have:
\[ \int_0^{\tilde{s}} \beta(d(\gamma(s), \tilde{A}))\|\gamma'(s)\| \, ds \geq (\beta(d(u, \tilde{A})) - \varepsilon)\|\gamma(\tilde{s}) - v\| \]
\[ \geq (\beta(d(u, \tilde{A})) - \varepsilon)\|\tilde{\mathcal{H}}(v, t) - v\|. \]
We deduce that
\[ (\beta(d(u, \tilde{A})) - \varepsilon)\|\tilde{\mathcal{H}}(v, t) - v\| \leq \tilde{d}(\tilde{\mathcal{H}}(v, t), v) \leq t. \]
Let now \( \delta > 0 \) such that \( (\beta(d(u, \tilde{A})) - \varepsilon)\delta \leq \tilde{\delta} \) and \( B(u; \delta) \subset \hat{B}(u; \tilde{\delta}) \), and define \( \mathcal{H} : B(u; \delta) \times [0, \delta] \to X \) by
\[ \mathcal{H}(v, t) = \tilde{\mathcal{H}}(v, (\beta(d(u, \tilde{A})) - \varepsilon)t). \]
Then,
\[ \|\mathcal{H}(v, t) - v\| \leq t \quad \text{and} \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma(\beta(d(u, \tilde{A})) - \varepsilon)t. \]
It follows that $|df|(u) \geq \sigma(\beta(d(u, \tilde{A})) - \varepsilon)$, hence $|df|(u) \geq \sigma \beta(d(u, \tilde{A}))$ since $\varepsilon > 0$ is arbitrary. Consequently, $|df|(u) \geq \beta(d(u, \tilde{A}))[df](u)$, the inequality being obvious whenever $|df|(u) = 0$. \hfill \Box

**Remark 4.4.** In the case when $X$ is a Banach space, $\tilde{A}$ is the origin, and $\beta(t) = 1/(1 + t)$, the metric $\tilde{d}$ defined in the above proof is the Cerami metric, see [5], [10, p. 138], where it is used in the context of critical point theory for “smooth” functions. This metric is used in [1] in the context of the critical point theory for continuous functions defined on complete metric spaces developed in [9, 8]. Since $(X, \tilde{d})$ is complete (see Theorem 4.1 (a)), and the Palais-Smale condition “with weight” $\beta(\|u\|)^{-1} = 1 + \|u\|$ (see, e.g., [16] for the terminology) is just the condition of Definition 2.2 in $(X, \tilde{d})$, existence results of critical points of a continuous function $f : X \to \mathbb{R}$ readily follow from the general theory (see Theorem 3.3 (a)).

Here, we show how the main abstract results of [18] can be obtained from the results of Section 3, just changing metric according to the previous theorem.

**Theorem 4.5.** Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ continuous, and $A, B$ two subsets of $X$ such that $B$ links $A$. Set:

$$a := \inf_A f, \quad b_0 := \sup_B f, \quad b := \inf_{\psi \in \Phi_B} \sup_{B \times [0,1]} (f \circ \psi),$$

where $\Phi_B$ is the set of contractions of $B$ in $X$, and assume that $a, b \in \mathbb{R}$ and $b_0 > a$. If $0 < \rho < d(B \cap f_a, A)$ (resp., $0 < \rho < d(B, A \cap f_{b_0})$) and $\beta : [0, +\infty[ \to [0, +\infty[ \to \mathbb{R}$ is a continuous function such that

$$\int_0^{+\infty} \beta(t) \, dt = +\infty \quad \text{and} \quad \int_0^\rho \beta(t) \, dt \geq b_0 - a,$$

then, for any $\delta > 0$ there exists $u \in X$ with

$$a - \delta \leq f(u) \leq b + \delta \quad \text{and} \quad |df|(u) \leq \beta(d(u, A)) \quad \text{(resp., } |df|(u) \leq \beta(d(u, B))\text{)}).$$

**Proof.** We consider the case $0 < \rho < d(B \cap f_a, A)$. Let $\tilde{d}$ be the metric given by Theorem 4.1, with $\tilde{A} := A$. Then, $(X, \tilde{d})$ is complete and

$$\tilde{d}(B \cap f_a, A) \geq \int_0^{d(B \cap f_a, A)} \beta(t) \, dt > \int_0^\rho \beta(t) \, dt \geq b_0 - a > 0,$$

while $|\tilde{d}f|(u) = \frac{|df|(u)}{\beta(d(u, A))}$ for every $u \in X$. Applying Theorem 3.3 (b) in $(X, \tilde{d})$ with $\alpha := \tilde{d}(B \cap f_a, A)$ and $\sigma := 1$ yields the conclusion.

In the case $0 < \rho < d(B, A \cap f_{b_0})$, consider the metric $\tilde{d}$ associated with $\beta$ and $\tilde{A} := B$. \hfill \Box

**Remark 4.6.** Letting $\beta(t) \equiv \sigma$, a positive constant, we recover Theorem 3.3 (b).
Corollary 4.7. Let $X$ be a complete metric space, $f : X \to \mathbb{R}$ continuous, and $(A_h), (B_h)$ two sequences of subsets of $X$ such that for each $h$:

$$d(B_h, A_h) > 0 \quad \text{and} \quad B_h \text{ links } A_h.$$ 

Set:

$$a_h := \inf_{A_h} f, \quad b_{h0} := \sup_{A_h} f, \quad b_h := \inf_{B_h} \sup_{\psi \in \Phi_h} (f \circ \psi),$$

where $\Phi_h$ denotes the set of contractions of $B_h$ in $X$,

$$d_h := d(B_h \cap f_{a_h}, A_h), \quad d_{h0} := d(B_h, f_{h0} \cap A_h).$$

Assume that $b_h, a_h \in \mathbb{R}$, eventually, and that for some $\mu \geq 0$:

$$\limsup_{h \to \infty} \frac{b_{h0} - a_h}{h^{\mu+1}} \leq 0, \quad \left( \text{resp., } \limsup_{h \to \infty} \frac{b_{h0} - a_h}{h^{\mu+1}} \leq 0 \right)$$

(with the convention that $\frac{1}{+\infty} = 0$).

Then, there is a sequence $(u_h)$ in $X$ such that

$$a_h - 1/h \leq f(u_h) \leq b_h + 1/h$$

and

$$\frac{|df|(u_h)}{(d(u_h, A_h) + 1)^\mu} \to 0, \quad \left( \text{resp., } \frac{|df|(u_h)}{(d(u_h, B_h) + 1)^\mu} \to 0 \right).$$

Proof. As in the proof of Corollary 3.8, we let $h$ be so large that $b_h, a_h \in \mathbb{R}$ and we observe that $d_h = +\infty$ (resp., $d_{h0} = +\infty$) if and only if $b_{h0} \leq a_h$. In this case, apply Theorem 3.3 (a) to obtain $u_h \in X$ with $|f(u_h) - b_h| \leq 1/h$ and

$$|df|(u_h) \leq 1/h \leq (d(u_h, A_h)+1)^\mu/h \quad \text{(resp., } |df|(u_h) \leq (d(u_h, B_h)+1)^\mu/h).$$

If $d_h < +\infty$ (resp., $d_{h0} < +\infty$), then $d_h > 0$ (resp., $d_{h0} > 0$) because $d(B_h, A_h) > 0$. Define (similarly as in the proof of [18, Corollary 2.3]) a continuous nondecreasing function $\beta_h : [0, +\infty[ \to [0, +\infty[$ by

$$\beta_h(t) = (\mu + 1)(b_{h0} - a_h)(t + 1)^\mu/D_h^{\mu+1},$$

where $D_h := d_h/2$ (resp., $D_h := d_{h0}/2$). We have:

$$\int_0^{D_h} \beta_h(t) dt = (b_{h0} - a_h)(D_h + 1)^{\mu+1} - 1/D_h^{\mu+1} \geq b_{h0} - a_h > 0.$$

Applying Theorem 4.5, we obtain $u_h \in X$ with $a_h - 1/h \leq f(u_h) \leq b_h + 1/h$ and

$$|df|(u_h) \leq \beta_h(d(u_h, A_h)) \quad \text{(resp., } |df|(u_h) \leq \beta_h(d(u_h, B_h))).$$

Dividing by $(d(u_h, A_h) + 1)^\mu$ (resp., $(d(u_h, B_h) + 1)^\mu$), and taking the lim sup as $h \to \infty$ yields the conclusion. \qed
Remark 4.8. Theorem 4.5 extends Schechter’s [18, Theorems 2.1, 2.2], where $X$ is a Banach space, $f$ is of class $C^1$, and $\beta$ is a positive nondecreasing function (observe that if $\inf \beta > 0$, Theorem 4.5 holds without assuming $b_0 > a$, thanks to Theorem 3.3 (a)). Corollary 4.7 extends [18, Corollary 2.3].

5. Some corollaries in Banach spaces.

In this section, $X$ is a Banach space and $f : X \to \mathbb{R}$ is continuous. We give some corollaries of the results of Sections 3 and 4, which are useful in applications. First, we give standard examples of linking subsets, corresponding to the geometrical situations encountered in Rabinowitz’ Saddle Point and Generalized Mountain Pass Theorems, see [15].

Example 5.1. Let $X$ be a Banach space which splits into the direct sum $X = Y \oplus Z$, with $Y$ finite dimensional. Set

$$A := Z; \quad B_R := \partial B(0; R) \cap Y, \quad R > 0.$$ 

Then, $B_R$ links $A$. Moreover, if $f : X \to \mathbb{R}$ and we set:

$$b := \inf_{\psi \in \Phi_{B_R}} \sup_{B_R \times [0,1]} (f \circ \psi), \quad \tilde{b} := \inf_{\varphi \in \Gamma_{B(0;R) \cap Y}} \sup_{\varphi} (f \circ \varphi),$$

where $\Phi_{B_R}$ is the set of contractions of $B_R$ in $X$ and

$$\Gamma := \{ \varphi : \bar{B}(0; R) \cap Y \to X : \varphi \text{ is continuous and } \varphi(u) = u \text{ for } u \in B_R \},$$

then $b = \tilde{b}$.

Proof. The fact that $B_R$ links $A$ follows from a standard application of Brouwer’s topological degree, similar to the proof that $\varphi(B(0; R) \cap Y) \cap A \neq \emptyset$ for any $\varphi \in \Gamma$, see, e.g., [15, Chapter 5]. Now, if $\psi \in \Phi_{B_R}$ with $\psi(u, 1) = u_0$, we can define $\varphi \in \Gamma$ by

$$\varphi(u) = \begin{cases} \psi(Ru/\|u\|, 1 - \|u\|/R) & \text{if } u \neq 0 \\ u_0 & \text{if } u = 0, \end{cases}$$

so that $\varphi(\bar{B}(0; R) \cap Y) = \psi(B_R \times [0, 1])$ and $\tilde{b} \leq b$. Conversely, if $\varphi \in \Gamma$ define $\psi \in \Phi_{B_R}$ by $\psi(u, t) = \varphi((1 - t)u)$, so that $\psi(B_R \times [0, 1]) = \varphi(B(0; R) \cap Y)$, and $b \leq \tilde{b}$. \hfill $\square$

In a similar way, we have:

Example 5.2. Let $X$ be a Banach space, $X = Y \oplus Z$ with $Y$ finite dimensional, and $v_0 \in Z$, $v_0 \neq 0$. For $\rho > 0$, $R > 0$, set:

$$A_\rho := \partial B(0; \rho) \cap Z,$$

$$B'_R := \{ sv_0 + u : s \geq 0, u \in Y, \|sv_0 + u\| = R \},$$

$$B_R := B'_R \cup (B(0; R) \cap Y) \subset Y \oplus \mathbb{R}v_0.$$
Then, $B_R$ links $A_R$ if $R > \rho$.

**Theorem 5.3.** Let $X$ be a Banach space which splits into the direct sum $X = Y \oplus Z$, with $Y$ finite dimensional, and let $f : X \to \mathbb{R}$ be a continuous function. Set:

$$m_Y := \sup_{u \in Y} \inf_{v \in Z} f(u + v),$$

$$m_Z := \inf_{v \in Z} \sup_{u \in Y} f(u + v).$$

Assume that $m_Y, m_Z \in \mathbb{R}$, and that $f$ satisfies condition $(PS)_c$ for every $c \in [m_Y, m_Z]$.

Then, there exists $u \in X$ with $|df|(u) = 0$ and $f(u) \in [m_Y, m_Z]$.

**Proof.** Based on Corollary 3.8, the proof is formally the same as in [17, Theorem 2.11]. For $h \in \mathbb{N}$, let $u_h \in Y$ and $v_h \in Z$ such that

$$\inf_{v \in Z} f(u_h + v) \geq m_Y - 1/h, \quad \sup_{u \in Y} f(u + v_h) \leq m_Z + 1/h,$$

and set:

$$A_h := u_h \oplus Z, \quad B_h := \{u + v_h : u \in Y, \|u - u_h\| = h\} \subset Y \oplus v_h.$$

Then, $B_h$ links $A_h$ for every $h$: See Example 5.1.

Now, set:

$$a_h := \inf_{A_h} f, \quad b_h := \inf_{\psi \in \Phi_B, B_h \times [0,1]} \sup_{u \in Y} (f \circ \psi),$$

$$a := \lim_{h \to \infty} \sup_{A_h} a_h, \quad b := \lim_{h \to \infty} \inf_{B_h} b_h.$$

It is immediately seen that $m_Y \leq a \leq b \leq m_Z$. Since $d(B_h, A_h) = h$, the conclusion follows from Corollary 3.8, and the Palais-Smale condition. \( \Box \)

**Theorem 5.4.** Let $X$ be a Banach space which splits into the direct sum $X = Y \oplus Z$, with $Y$ finite dimensional, and let $f : X \to \mathbb{R}$ be a continuous function. Assume that there exist $\rho > 0$ and $m_0 \in \mathbb{R}$ such that

$$f(u) \leq m_0 \leq f(v) \quad \text{for all } u \in Y \text{ and all } v \in \partial B(0; \rho) \cap Z,$$

and that there is some $v_0 \in Z$, $v_0 \neq 0$, such that for $R > 0$ there exists $m_R \in \mathbb{R}$ with

$$f(sv_0 + u) \leq m_R \quad \text{for all } s \geq 0, u \in Y, \|sv_0 + u\| = R.$$

(a) Assume that $m := \sup_{R > 0} m_R \in \mathbb{R}$ and that $f$ satisfies condition $(PS)_c$ for every $c \in [m_0, m]$. Then, there exists $u \in X$ with $|df|(u) = 0$ and $f(u) \in [m_0, m]$.

(b) Assume that for some $\mu \geq 0$, $m_R/R^{\mu+1} \to 0$ as $R \to \infty$. Then, there is a sequence $(u_h) \subset X$ such that

$$\lim_{h} \inf_{h} f(u_h) \geq m_0 \quad \text{and} \quad \frac{|df(u_h)|}{\|u_h\| + 1} \to 0.$$
Proof. Let $A_\rho$, $B'_R$, and $B_R$, $R > 0$, be defined as in Example 5.2, so that $B_R$ links $A_\rho$ as soon as $R > \rho$. Let

$$a_\rho := \inf_{A_\rho} f \geq m_0, \quad b_{R0} := \sup_{B_R} f \leq \max\{m_0, m_R\},$$

$$b_R := \inf_{\psi \in \Phi_B} \sup_{BR \times [0,1]} (f \circ \psi).$$

We have $B_R \cap f a_\rho \subset B'_R$ and $d(B'_R, A_\rho) = R - \rho$.

(a) Observe that $m \geq m_0$ since $B'_R \cap A_\rho \neq \emptyset$, and that $b_R \leq m$ for all $R > 0$ (consider $\psi \in \Phi_B$ with $\psi(B_R \times [0,1]) \subset \bigcup_{0 < r \leq R} B'_r$). Since

$$\frac{b_{R0} - a_\rho}{d(B_R \cap f a_\rho, A_\rho)} \leq \frac{m - m_0}{R - \rho} \to 0 \quad \text{as } R \to \infty,$$

the conclusion follows from Corollary 3.8 and the Palais-Smale condition.

(b) We have:

$$\frac{b_{R0} - a_\rho}{d(B_R \cap f a_\rho, A_\rho)} \leq \frac{m_R - m_0}{(R - \rho)^{\mu+1}} \to 0 \quad \text{as } R \to \infty,$$

and, for every $u \in X$,

$$\frac{|df|(u)}{\|u\| + 1} \leq (\rho + 1)^\mu \frac{|df|(u)}{(d(u, A_\rho) + 1)^\mu};$$

so, the conclusion follows from Corollary 4.7.

\[\square\]

Remark 5.5. Theorems 5.3 and 5.4 are variants of results of Silva [20] and Schechter [17, 18] (see also the references in [18]), dealing with a $C^1$ functional $f$. We observe that in the corresponding results of [17, 18], it is assumed that either $Y$ or $Z$ is finite dimensional. For the $C^1$ version of Theorem 5.3 this is possible, of course, because the statement is “symmetric” with respect to the splitting of the space, and $f$ and $-f$ have the same critical points — which is no more the case, in general, if $f$ is only continuous. As for the $C^1$ version of Theorem 5.4, the notion of linking introduced in [19] ensures that $B_R$ links $A_\rho$ ($R > \rho$) also when $Z$, rather than $Y$, is finite dimensional; basically, this is due to the fact that the notion of [19] involves contractions $\psi \in \Phi_X$ such that $\psi(\cdot, t)$ is a homeomorphism onto $X$ for each $t \in [0,1]$. As mentioned in the introduction, we cannot, in our setting, restrict the class of admissible deformations in this direction.

Remark 5.6. Theorem 5.4 (b) does not assert existence of a critical point of the function $f$. In applications, one has to show that the kind of “Palais-Smale sequence” obtained has a subsequence converging to such a critical point.

Acknowledgment. I thank Marco Degiovanni for a helpful discussion about Section 4.
References


Received September 12, 1997 and revised March 5, 1998.

Université de Perpignan
66860 PERPIGNAN Cedex
FRANCE
E-mail address: corvellec@univ-perp.fr
GENERALIZED NUCLEI OF COMPLEX SURFACES

Terry Fuller

For each \( g \geq 1 \), we study a family \( Y_g(n) \) of complex surfaces which admit a singular fibration over \( \mathbb{C}P^1 \) by complex curves of genus \( g \). By examining a handlebody description for \( Y_g(n) \), we show that these complex surfaces can be smoothly decomposed as the Milnor fiber of a Brieskorn homology 3-sphere union a small submanifold, termed a “nucleus”. This description generalizes known decompositions for elliptic surfaces.

0. Introduction.

Elliptic surfaces have long been an important source of examples in the study of smooth 4-manifolds. One important step in their study was the discovery by Gompf of “nuclei” of elliptic surfaces, that is, of small submanifolds which carry much of the differential topological information of the larger elliptic surface \([G]\). Furthermore, Gompf identified the complements of nuclei as familiar examples (a certain family of Milnor fibers of Brieskorn homology 3-spheres), yielding a decomposition for elliptic surfaces that has proven useful in a variety of contexts. (See \([FS]\), \([G]\), \([GM]\), \([LM]\) for applications.)

The situation for higher genus fibrations, however, is less settled. In this paper, for each \( g \geq 1 \) we study a family \( Y_g(n) \) of complex surfaces admitting a singular fibration over \( \mathbb{C}P^1 \) by genus \( g \) Riemann surfaces, describing their topology by means of Kirby calculus. These complex surfaces are shown to have a decomposition analogous to Gompf’s decomposition for elliptic surfaces: each \( Y_g(n) \) contains a small (\( b_2 = 2 \)) submanifold, also referred to as a nucleus, whose complement is diffeomorphic to the Milnor fiber of the Brieskorn homology 3-sphere \( \Sigma(2, 2g + 1, 2(2g + 1)n - 1) \). Indeed, for \( g = 1 \), we recover precisely the decomposition of \([G]\) for elliptic surfaces.

1. A Family of Branched Covers.

For all \( k \geq 0 \), let \( F_k \) denote the \( k \)th Hirzebruch surface, the holomorphic \( \mathbb{C}P^1 \)-bundle over \( \mathbb{C}P^1 \) with a holomorphic section of self-intersection number \(-k\). As a smooth 4-manifold, we will find it convenient to think of \( F_k \) as the double of a \( D^2 \)-bundle over \( S^2 \) with Euler number \( k \), and hence \( F_k \) can be described by the handlebody in Figure 1. \( F_k \) admits two disjoint holomorphic sections \( \Delta_k \) and \( \Delta_{-k} \), with \( \Delta_{\pm k}^2 = \pm k \).
For all $g \geq 1$ and $n \geq 1$, we define $Y_g(n)$ to be the 2-fold cover of $\mathbb{P}_{2n}$, branched over the disjoint union of a smooth curve in $|(2g+1)\Delta_{2n}|$ and $\Delta_{-2n}$. $Y_g(n)$ admits a singular fibration $Y_g(n) \to \mathbb{C}P^1$ obtained by composing the branched cover map with the bundle map $\mathbb{P}_{2n} \to \mathbb{C}P^1$. Since a typical nonsingular fiber appears as the double cover of a sphere branched over $2g + 2$ points, this is a singular fibration by genus $g$ curves. This fibration admits a holomorphic section $S$ with $S^2 = -n$, obtained as the lift of $\Delta_{-2n}$ to $Y_g(n)$.

2. A handlebody description of $Y_g(n)$.

Our results follow by obtaining a handlebody description for $Y_g(n)$, which is used to understand the structure of the singular fibration $Y_g(n) \to \mathbb{C}P^1$. Because of the large Euler characteristics involved, we discuss the development of handlebody pictures for the family $Y_2(n)$ of genus 2 fibrations in detail, and comment later on the obvious modifications for $Y_g(n)$. This handlebody description is also discussed in [Fu] for the special case $Y_2(1)$.

Special Case: $Y_2(n)$. Setting $g = 2$, we begin by constructing a smooth complex curve representing $[5\Delta_{2n}] \in H^2(\mathbb{P}_{2n})$. To do this, we construct the $D^2$-bundle over $S^2$ with Euler number $2n$ by gluing two copies of $\mathbb{C}^2$ with coordinates $(w, \eta)$ and $(z, \xi)$ according to the identifications

$$w = z^{-1},$$

$$\eta = z^{-2n}\xi.$$ 

We can describe five copies of $\Delta_{2n}$ in the $(w, \eta)$ chart as the set of all points $(w, \eta_1), \ldots, (w, \eta_5)$, where $\eta_1, \ldots, \eta_5$ are the five roots of unity, and extend these over the $(z, \xi)$ chart by setting $\xi_i = \eta_i z^{2n}$ for $1 \leq i \leq 5$. These five copies intersect at $z = 0$. These five copies are thus given locally by the complex curve

$$(\xi - \eta_1 z^{2n}) \cdots (\xi - \eta_5 z^{2n}) = \xi^5 - z^{10n} = 0,$$

and we can smooth out the intersection point by deforming this curve into

$$\xi^5 = z^{10n} + \epsilon.$$

Figure 2 shows $\mathbb{P}_{2n}$ along with a piece of the smoothed branch surface, which is known to be given by the fibered Seifert surface for the $(5, 10n)$ torus link ([Mi]). We label the 2-handles of $\mathbb{P}_{2n}$ as $h$ and $h'$, for later reference. The boundary of this surface is five circles; the rest of the surface is five disjoint smoothly embedded disks $D_1, \ldots, D_5$ in $\partial(4$-handle), obtained from parallel copies of the core of $h'$. The other component of the branch set is $\Delta_{-2n}$, which is seen in Figure 2 as the cocore of $h$ union an unseen disk $D_6$ in the 4-handle. Sliding $h' 2n$ times over $h$ produces Figure 3. The images of the disks $D_1, \ldots, D_5$ (which we continue to call $D_1, \ldots, D_5$) under this
diffeomorphism are parallel copies of the $-2n$-framed 2-handle (which we continue to call $h'$). Moreover, since the handle slides give a diffeomorphism from $F_{2n}$ to its upside down handlebody description, the disk $D_6$ is also seen in Figure 3 as a parallel copy of the core of $h'$. Figure 4 shows the result of an isotopy, where the five flat disks in Figure 3 have been placed horizontally across the page. (We leave it to the reader to check that this isotopy untwists the half-twisted bands of Figure 3.)

Figure 5 shows the 2-fold branched cover $Y_2(n)$ of $F_{2n}$, drawn from Figure 4 using the methods of [AK]. To understand Figure 5, it is best to picture it being built in stages. We begin by picturing the branch surface as the six 0-handles $\cup 40n$ 1-handles in $\partial(0$-handle) of $F_{2n}$ visible in Figure 4, union the 2-handles $D_1, \ldots, D_6$ in the 4-handle of $F_{2n}$. To form the branched cover $Y_2(n)$, we lift the handles of $F_{2n}$ one at a time. To lift the 0-handle of $F_{2n}$, we isotop the interior of the six 0-handles $\cup 40n$ 1-handles into the interior of the 0-handle of $F_{2n}$. We then cut each of two copies of the 0-handle of $F_{2n}$ along the track of this isotopy, and glue the resulting manifolds together by attaching handles. Gluing two copies of the 0-handle of $F_{2n}$ along the six 0-handles of the branch set is accomplished by attaching six 1-handles with each component of the attaching region in a separate copy, or equivalently by attaching five 1-handles to a single 4-ball. These five 1-handles are the dotted circles in Figure 5. Additional gluings along the $40n$ 1-handles of the branch set are obtained by attaching the 40n 0-framed 2-handles in Figure 5.

To lift the 2-handles $h$ and $h'$, we use the method of [AK], Section 3. The attaching circle of $h'$ intersects the branch set geometrically in $2n$ points. We lift $h'$ to two 2-handles $h'_1$ and $h'_2$ that are attached to circles obtained by cutting the attaching circle of $h'$ along the branch set, and gluing the endpoints together as we perform the gluing of the 0-handles of $F_{2n}$. We compute, as in [AK], that the framings of $h'_1$ and $h'_2$ are $-n$. Similarly, $h$ lifts to 2-handles $h_1$ and $h_2$, each with framing $-3$. Lastly, we must lift the 4-handle of $F_{2n}$. However, the lift is obtained from two copies of the 4-handle glued along the (pushed in) disks $D_1, \ldots, D_6$, hence by turning the initial part of our argument upside down it appears as five 3-handles $\cup 4$-handle.

We next modify this picture of $Y_g(n)$. We begin by isotoping the wheels of 0-framed 2-handles so that the two half twists per wheel are turned into one full twist, resulting in Figure 6. The 1-handles are then isotoped by sliding each dotted circle over the other dotted circles to its right; this rearranges the dotted circles so that each wheel runs through a single dotted circle. Cancelling four of the 1-handles with the outermost 2-handle from each wheel, and cancelling the fifth 1-handle with one of the $-3$-framed 2-handles produces Figure 7. Figure 8 is obtained by sliding one of the $-n$-framed 2-handles over the other, splitting off an unknotted 0-framed 2-handle from
the rest of the picture, which is used to cancel one of the 3-handles. The handlebody in Figure 9 is diffeomorphic to that in Figure 8, as can be seen by cancelling each 1-handle with the extra 2-handle in each wheel (and isotoping the 0-framed (2,5) torus knot).

From Figure 9, we can extract a description of the singular fibers of $Y_2(n) \to \mathbb{C}P^1$. Viewing Figure 9 in stages, attaching first the 1-handles and the 0-framed 2-handle (see Figure 10) describes $\Sigma_2 \times D^2$, a trivial fibration over $D^2$ by genus 2 surfaces. The $40n$ $-1$-framed 2-handles are attached next with attaching circles lying in fibers $\Sigma_2 \times \{pt.\} \subset \Sigma_2 \times S^1 = \partial (\Sigma_2 \times D^2)$. This is known (see [HKK], [K]) to be a handlebody description of a singular fibration over $D^2$ with $40n$ singular fibers, each of which is an immersed genus 2 surface with one transverse self-intersection, obtained from nearby fibers by identifying one of the circles $c_1, c_2, c_3, c_4$ (see Figure 11) to a point. The monodromy around each is given by $D(c_i)$, a right-handed Dehn twist around $c_i$. The boundary is an $\Sigma_2$-bundle over $S^1$ whose global monodromy is the product of those Dehn twists. Since a meridional circle to the 0-framed 2-handle in Figure 9 intersects each fiber in the boundary once, traversing this circle allows us to read off this monodromy as $(D(c_1)D(c_2)D(c_3)D(c_4))^{10n}$.

Finally, the submanifold given by the $-n$-framed 2-handle $\cup$ four 3-handles $\cup$ 4-handle in Figure 9 is diffeomorphic (by tracing it back to Figure 5) to the 2-fold cover of the submanifold $h' \cup$ 4-handle branched over the pushed in disks $D_1, \ldots, D_6$. Since these disks are all parallel copies of $h'$, this is diffeomorphic to the 2-fold cover of a trivial $D^2$-bundle over $S^2$ branched over six fibers, namely $\Sigma_2 \times D^2$. Hence adding the $-n$-framed 2-handle $\cup$ four 3-handles $\cup$ 4-handle to complete Figure 9 describes a $\Sigma_2 \times D^2$ attached to the boundary, preserving the $\Sigma_2$-bundle structure. In particular, the global monodromy above must be trivial. (The word $(D(c_1)D(c_2)D(c_3)D(c_4))^{10n}$ is known to be trivial in the mapping class group of a genus 2 surface [B].)

Figure 12(a) shows a regular neighborhood $C$ of four consecutive singular fibers, whose singularities are defined by the circles $c_1, c_2, c_3, c_4$, respectively. Cancelling the 1-handles produces a 4-manifold obtained by attaching a 2-handle to $B^4$ along a 0-framed (2,5) torus knot $K$, as in Figure 12(b). With this description, since $K$ is a fibered knot with genus 2 fiber, we have a fibration on $C$ with only one singular fiber, called a cusp fiber. (We have described the genus 2 analog of the familiar statement from elliptic surfaces that two fishtail fibers can be deformed into one cusp fiber.) Examining the global monodromy $(D(c_1)D(c_2)D(c_3)D(c_4))^{10n}$, we see that the original fibration with $40n$ singular fibers is diffeomorphic to a fibration with $10n$ cusp fibers.

Following the terminology coined in [G], we make the following definition.
Definition 1. The nucleus $N_2(n) \subset Y_2(n)$ is a regular neighborhood of a single cusp fiber union the section $S$.

Since $S$ intersects each fiber in $C$ once, Figure 13 gives a Kirby calculus picture of $N_2(n)$. We next show that the decompositions of elliptic surfaces in [G] apply here as well.

Theorem 2. The complement of $N_2(n)$ in $Y_2(n)$ is diffeomorphic to $B(2, 5, 10n-1)$, the Milnor fiber of the Brieskorn homology 3-sphere $\Sigma(2, 5, 10n-1)$.

Proof. The argument in [G] generalizes intact to our pictures. It follows from the above discussion that drawing Figure 9 without the $-n$-framed 2-handle $\cup$ four 3-handles $\cup$ 4-handle describes $Y_g(n) - \overset{\circ}{C}$, we further leave out one $-1$-framed 2-handle per wheel. To get the complement of $N_g(n)$, we must additionally delete a neighborhood of $S$ restricted to $Y_g(n) - \overset{\circ}{C}$. However, the cocore of the 0-framed 2-handle is a properly embedded disk in $Y_g(n) - \overset{\circ}{C}$ which extends to $S$ in $Y_g(n)$. Hence deleting this disk, or equivalently omitting the 0-framed 2-handle, describes the complement of $N_g(n)$ in $Y_g(n)$. Figure 14 shows what has survived from Figure 9. Cancelling the 1-handles produces Figure 15.

The Milnor fiber $B(2, 5, 10n-1)$ is by definition the solution set of $x^2 + y^5 + z^{10n-1} = \varepsilon$ in the unit ball of $\mathbb{C}^3$. This is easily seen (see the discussion in [AK]) to be diffeomorphic to the $(10n-1)$-fold cover of $B^4$, branched over the fibered Seifert surface for a $(2, 5)$ torus knot whose interior has been pushed into $B^4$. Applying the methods of [AK] to this branched cover also yields Figure 15.

The General Case: $Y_g(n)$. We can easily extend this discussion to describe the analogous pictures for arbitrary $g$. To depict a smooth complex curve in $|(2g+1)\Delta_{2n}|$, we replace the Seifert surface of a $(5, 10)$ torus link in Figure 2 with the Seifert surface for a $(2g+1, 2(2g+1)n)$ torus link, drawn as $2g+1$ disks connected by $4gn(2g+1)$ twisted bands. Carrying through the same analysis as before produces the picture of $Y_g(n)$ in Figure 16. In this case, we consider a regular neighborhood $C$ of $2g$ consecutive singular fibers whose monodromies are given by Dehn twists around the circles $c_1, \ldots, c_{2g}$ (see Figure 17). $C$ can be obtained by attaching a 2-handle to $B^4$ along a 0-framed $(2, 2g+1)$ torus knot, which is taken to be the genus $g$ analog of the neighborhood of a cusp fiber. As before, if we delete the $-n$-framed 2-handle $\cup$ $2g$ 3-handles $\cup$ 4-handle from Figure 16, then the boundary is a $\Sigma_g$-bundle over $S^1$ with global monodromy

$$(D(c_1)D(c_2) \cdots D(c_{2g}))^{2(2g+1)n},$$
and we see that $Y_g(n)$ is diffeomorphic to the total space of a fibration with $2(2g + 1)n$ cusp fibers. Once again, we define the *nucleus* $N_g(n) \subset Y_g(n)$ to be a regular neighborhood of a cusp fiber union the section $S$, so that $N_g(n)$ appears as in Figure 18. With these definitions, the following theorem follows as before.

**Theorem 3.** The complement of $N_g(n)$ in $Y_g(n)$ is diffeomorphic to $B(2, 2g + 1, 2(2g + 1)n − 1)$, the Milnor fiber of the Brieskorn homology 3-sphere $\Sigma(2, 2g + 1, 2(2g + 1)n − 1)$.

Since the Milnor fibers $B(2, 2g + 1, 2(2g + 1)n − 1)$ have an even intersection form, we have also shown the following corollary.

**Corollary 4.** $Y_g(n)$ is spin if and only if $N_g(n)$ is spin.

Indeed, this fact is apparent from comparing Figures 16 and 18. In fact, from these Figures we see that $Y_g(n)$ and $N_g(n)$ are spin precisely when $n$ is even.

In addition, examining Figure 16 more carefully, we note as before that the $-n$-framed 2-handle $\cup 2g$ 3-handles $\cup 4$-handle represent a $\Sigma_g \times D^2$ attached to the boundary of its complement, preserving the $\Sigma_g$-bundle structure. Thus the global monodromy recorded above must be trivial. In particular, setting $n = 1$, this gives a geometric explanation for the following well-known fact [B].

**Corollary 5.** The product of Dehn twists $(D(c_1)D(c_2)\cdots D(c_{2g}))^{2(2g + 1)}$ represents the trivial element in the mapping class group of $\Sigma_g$. 
Figure 1.

Figure 2.
Figure 3.

Figure 4.
Figure 5.

Figure 6.
Figure 7.

10n - 1 -2-framed 2-handles per wheel

∪ five 3-h, 4-h

Figure 8.

10n - 1 -2-framed 2-handles per wheel

∪ four 3-h, 4-h
Figure 9.

Figure 10.

Figure 11.
Figure 12.

Figure 13.
Figure 14.

10n - 1 -1-framed 2-handles per wheel

Figure 15.

10n - 2 -2-framed 2-handles per wheel
Figure 16.

Figure 17.

Figure 18.
References


Received June 10, 1997 and revised November 9, 1997.

UNIVERSITY OF TEXAS
AUSTIN, TX 78712

INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540
SINGULAR LIMIT OF SOLUTIONS OF
THE EQUATION \( u_t = \Delta \left( \frac{u^m}{m} \right) \) AS \( m \to 0 \)

Kin Ming Hui

We will show that for \( n = 1, 2 \), as \( m \to 0 \) the solution \( u^{(m)} \) of the fast diffusion equation \( \partial u / \partial t = \Delta (u^m / m) \), \( u > 0 \), in \( R^n \times (0, \infty) \), \( u(x, 0) = u_0(x) \geq 0 \) in \( R^n \), where \( u_0 \in L^1(R^n) \cap L^\infty(R^n) \) will converge uniformly on every compact subset of \( R^n \times (0, T) \) to the maximal solution of the equation

\[
\begin{align*}
vt &= \Delta \log v, \\
v(x, 0) &= u_0(x),
\end{align*}
\]

where \( T = \infty \) for \( n = 1 \) and \( T = \int_{R^2} u_0 dx / 4\pi \) for \( n = 2 \).

The degenerate parabolic equation

\[
\begin{cases}
u_t = \Delta \left( \frac{u^m}{m} \right), & u > 0, \quad \text{in } R^n \times (0, \infty) \\
u(x, 0) = u_0(x) \geq 0 & \forall x \in R^n
\end{cases}
\]

where \( m > 0 \) arises in the modelling of many physical phenomenon such as the flow of gases through a porous medium or the flow of viscous fluid on a surface. When \( m > 1 \), the above equation is called the porous medium equation. When \( 0 < m < 1 \), it is called the fast diffusion equation and when \( m = 1 \), it becomes the famous heat equation. We refer the reader to the papers by Aronson \([A]\) and Peletier \([P]\) for extensive reference on the above equation.

In this paper we will investigate the convergence of solution \( u^{(m)} \) of the fast diffusion equation \((0.1)\) as \( m \to 0 \). We will show that for \( n = 1, 2 \), if \( u_0 \in L^1(R^n) \cap L^\infty(R^n) \), then as \( m \to 0 \) the solution \( u^{(m)} \) of the above fast diffusion equation will converge uniformly on every compact subset of \( R^n \times (0, T) \) to the maximal solution of

\[
\begin{cases}
v_t &= \Delta \log v, v > 0, \quad \text{in } R^n \times (0, T) \\
v(x, 0) &= u_0(x) & \forall x \in R^n
\end{cases}
\]

where

\[
T = \begin{cases}
\infty & \text{if } n = 1 \\
\frac{1}{4\pi} \int_{R^2} u_0 dx & \text{if } n = 2.
\end{cases}
\]

As proved in \([ERV]\) for \( u_0 \in L^1(R) \) when \( n = 1 \) and in \([DP]\), \([H]\) for \( u_0 \in L^1(R^2) \cap L^p(R^2) \) for some \( p > 1 \) when \( n = 2 \), there exists infinitely
many solutions of (0.2). However by the results of [ERV] for \( n = 1 \) and [DD] for \( n = 2 \) there exists only a single maximal solution of (0.2) for \( n = 1, 2 \). So the limit \( \lim_{m \to 0} u^{(m)} \) is unique.

When \( n = 2 \) L.F.Wu [W1], [W2] showed that (0.2) is related to the study of Ricci flow and when \( n = 1 \) K.G. Kurtz [K] showed that (0.2) is related to the limiting density of two type of particles moving against each other obeying the Boltzmann equation. Similar type of singular limits for solutions of (0.1) as \( m \to \infty \) and \( m \to 1 \) are obtained by L.A. Caffarelli and A. Friedman [CF2], P.L. Lions, P.E. Souganidis and J.L. Vazquez [LSV] and for the p-laplacian equation which are related to the sandpile model by L.C. Evans, M. Feldman, etc. [AEW], [EFG], [EG].

For any domain \( \Omega \subset R^n \), \( T > 0 \), \( \phi(u) = u^m / m \) for some \( 0 < m < 1 \) or \( \phi(u) = \log u \), we say that \( u \) is a solution (respectively subsolution, supersolution) of

\[
\frac{\partial u}{\partial t} = \Delta \phi(u)
\]

in \( \Omega \times (0, T) \) if \( u \in C(\overline{\Omega} \times (0, T)) \cap C^\infty(\Omega \times (0, T)) \), \( u > 0 \) on \( \overline{\Omega} \times (0, T) \), and satisfies (0.4) (respectively \( \leq \), \( \geq \)) in \( \Omega \times (0, T) \) in the classical sense and we say that \( u \) has initial value \( u_0 \) if the following holds

\[
\lim_{t \to 0} \int_{\Omega} u(x, t) \eta(x) dx = \int_{\Omega} u_0(x) \eta(x) dx \quad \forall \eta \in C_0^\infty(\Omega).
\]

We say that \( u \) is a solution (respectively subsolution, supersolution) of

\[
\begin{cases}
  u_t = \Delta \phi(u), & u > 0, \quad \text{in} \quad R^n \times (0, \infty) \\
  u(x, 0) = u_0(x) \geq 0, & \forall x \in R^n
\end{cases}
\]

if \( u \) is a solution (respectively subsolution, supersolution) of (0.4) in \( R^n \times (0, T) \) with initial value \( u_0 \). We say that \( \tilde{v} \) is a maximal solution of (0.2) if \( \tilde{v} \geq v \) for any solution \( v \) of (0.2). For any \( x_0 \in R^n \), \( R > 0 \), we let \( B_R(x_0) = \{ x \in R^n : |x - x_0| < R \} \) and \( B_R = B_R(0) \).

The plan of the paper is as follows. In Section 1 we will recall some existence results and construct some subsolutions of (0.1). In Section 2 we will use the subsolutions obtained in Section 1 to obtain lower bound estimates for the solution of (0.1). We will also prove a Harnack type inequality for solution of (0.1) and a convergence result for the case \( n = 1, 2 \), under the assumption that \( u_0 \in L^1(R^n) \cap L^\infty(R^n) \) and \( \varepsilon(1 + |x|^2)^{-\alpha} \leq u_0(x) \leq M \) for some constants \( \varepsilon > 0 \), \( M \geq 1 \), and \( 1 < \alpha < 2 \) for all \( x \in R^n \). The general convergence result for \( u_0 \in L^1(R^n) \cap L^\infty(R^n) \) then follows by an approximation argument. Since we are concerned with the limit \( m \to 0 \), we will assume \( 0 < m < 1 \) and let \( u^{(m)} \) be the solution of (0.1) for the rest of the paper.
Section 1.

In this section we will recall some existence results and comparison principles for solutions of (0.1) in [AB], [BBC], [DK]. We will also construct some subsolutions of (0.1). We first start with an existence result of [AB], [DK]:

**Theorem 1.1** ([AB], [DK]). If \( u_0 \in L^1(\mathbb{R}^n) \) and \((n - 2)/n < m < 1\), then (0.1) will have a unique solution \( u^{(m)}(t) \in L^\infty((0, \infty); L^1(\mathbb{R}^n)) \cap C^\infty(\mathbb{R}^n \times (0, \infty)) \), \( u^{(m)} > 0 \) on \( \mathbb{R}^n \times (0, \infty) \) and satisfying

\[
  u_t^{(m)} \leq \frac{u^{(m)}}{(1 - m)t} \quad \text{in} \quad \mathbb{R}^n \times (0, \infty).
\]

**Proof.** Existence of solution to (0.1) is proved in [DK], [AB]. Let \( p = u_t^{(m)} / u^{(m)} \). Then as in [AB], [CF1], [ERV] \( p \) will satisfy the equation

\[
p_t = u^{(m)m-1}p + 2mu^{(m)m-2}\nabla u^{(m)} \cdot \nabla p - (1 - m)p^2.
\]

Since \( 1/(1 - m)t \) also satisfies the above equation but with initial value \( \infty \). By the maximum principle

\[
  u_t^{(m)} \leq \frac{u^{(m)}}{(1 - m)t}
\]

and the theorem follows. \( \square \)

**Theorem 1.2** ([BBC], [E]). If \((n - 2)/n < m < 1\) and \( u_1^{(m)} \), \( u_2^{(m)} \), are subsolution and supersolution of (0.1) with initial values \( u_{0,1}, u_{0,2} \in L^1(\mathbb{R}^n) \) respectively, then

\[
  \int_{\mathbb{R}^n} (u_1^{(m)}(x,t) - u_2^{(m)}(x,t)) + dx \leq \int_{\mathbb{R}^n} (u_{0,1} - u_{0,2}) + dx \quad \forall t > 0
\]

and if \( u_1^{(m)} \) and \( u_2^{(m)} \) are two solutions of (0.1) with initial values \( u_{0,1}, u_{0,2} \in L^1(\mathbb{R}^n) \) respectively, then

\[
  \int_{\mathbb{R}^n} |u_1^{(m)}(x,t) - u_2^{(m)}(x,t)| dx \leq \int_{\mathbb{R}^n} |u_{0,1} - u_{0,2}| dx \quad \forall t > 0.
\]

In particular if \( u_{0,1} \leq u_{0,2} \), then \( u_2 \leq u_1 \).

**Lemma 1.3.** If \( T_1 > 0 \), \( 0 < m < 1 \), \( \alpha > 1 \), \( 0 < k < 1 \) are constants satisfying the condition \((1 - m)^{-1} < \alpha < 2(1 - m)^{-1}\) and \( 0 < k < (4\alpha)^{-1/(\alpha(1 - m) - 1)}\), then the function

\[
  w(x, t) = \frac{[(1 - m)(T_1 - t) + k]}{(k + |x|^2)^\alpha}
\]

is a subsolution of

\[
  u_t = \Delta(u^m/m)
\]
in \(R^n \times (0,T_1), n = 1, 2\), with initial value \([(1-m)T_1]^{1/1-m}(k+|x|^2)^{-\alpha}\).

**Proof.** Write \(w(x,t) = f(t)g(x)\) where \(f(t) = [(1-m)(T_1-t)_+]^{1/1-m}\) and \(g(x) = g(r) = (k+|x|^2)^{-\alpha}\) where \(r = |x|\). For \(n = 1\) by direct computation,

\[
\begin{align*}
(g^m)'(r) &= -2\alpha mr(k+r^2)^{-\alpha m-1} \\
(g^m)'(r) &= 4\alpha m(\alpha m + 1)r^2(k+r^2)^{-\alpha m-2} - 2\alpha m(k+r^2)^{-\alpha m-1}.
\end{align*}
\]

Hence

\[
\Delta(g^m) + mg
= m(k+|x|^2)^{-\alpha}(1 + 4\alpha(\alpha m + 1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2
- 2\alpha(k+|x|^2)^{\alpha(1-m)-1})
= m(k+|x|^2)^{-\alpha}(1 + (2\alpha + 4\alpha^2 m)(k+|x|^2)^{\alpha(1-m)-1}
- 4\alpha(\alpha m + 1)(k+|x|^2)^{\alpha(1-m)-2}k)
geq m(k+|x|^2)^{-\alpha}(1 + (2\alpha + 4\alpha^2 m)k^{\alpha(1-m)-1} - 4\alpha(\alpha m + 1)k^{\alpha(1-m)-1})
geq m(k+|x|^2)^{-\alpha}(1 - 2\alpha k^{\alpha(1-m)-1}) \geq 0.
\]

Similarly for \(n = 2\), we have

\[
\Delta(g^m) + mg
= m(k+|x|^2)^{-\alpha}(1 + 4\alpha(\alpha m + 1)(k+|x|^2)^{\alpha(1-m)-2}|x|^2
- 4\alpha(k+|x|^2)^{\alpha(1-m)-1})
geq m(k+|x|^2)^{-\alpha}(1 - 4\alpha k^{\alpha(1-m)-1}) \geq 0.
\]

Hence

\[
\Delta\left(\frac{w^m}{m}\right) - w_t = f^m\Delta\left(\frac{g^m}{m}\right) - f_t g = \frac{f^m}{m} \cdot (\Delta(g^m) + mg) \geq 0
\]

Thus \(w\) is a subsolution of (0.1) with initial value \([0,1-m)(T_1]^{1/1-m}(k+|x|^2)^{-\alpha}\) and the lemma follows. \[\square\]

As a consequence of Theorem 1.2 and Lemma 1.3 we have the following corollary.

**Corollary 1.4.** If \(u_0 \in L^1(R^n)\) and \(u_0(x) \geq \varepsilon(k+|x|^2)^{-\alpha}\) for some constants \(\varepsilon > 0\), \(0 < k < 1\), \(0 < m < 1\), satisfying \((1-m)^{-1} < \alpha < 2(1-m)^{-1}\) and \(0 < k < (4\alpha)^{-1/(\alpha(1-m)-1)}\), then the solution \(u^{(m)}\) of (0.1) is bounded below by \((\varepsilon^{1-m} - (1-m)t)^{1/1-m}(k+|x|^2)^{-\alpha}\).

**Theorem 1.5.** If \(u_0 \in L^1(R^n)\) for \(n = 1\) and \(u_0 \in L^1(R^2) \cap L^p(R^2)\) for some \(p > 1\) for \(n = 2\), then there exists a unique maximal solution \(v\) of (0.2) in \(R^n \times (0,T)\) where \(T\) is given by (0.3).
Proof. For $n = 1$ the theorem is proved in [ERV]. For $n = 2$ by Theorem 4.3 of [H] (cf. [DP] Theorem 1.2) there exists a solution $v$ of (0.2) in $R^2 \times (0, T)$ satisfying the condition

\[(1.3) \quad \int_{R^2} v(x,t)dx = \int_{R^2} u_0dx - 4\pi t \quad \forall 0 < t \leq T
\]

where $T$ is given by (0.3). By Prop 3.1 and Prop 4.1 of [DD], there exists a maximal solution $v_1$ of (0.2) in $R^2 \times (0, T)$. By Theorem 1.3 of [DP],

\[
\int_{R^2} v_1(x,t)dx \leq \int_{R^2} u_0dx - 4\pi t \quad \forall 0 < t \leq T \\
\Rightarrow \quad \int_{R^2} v_1(x,t)dx \leq \int_{R^2} v(x,t)dx \quad \forall 0 < t \leq T \text{ by (1.3)} \\
\Rightarrow \quad v(x,t) = v_1(x,t) \quad \forall x \in R^2, 0 < t \leq T \text{ since } v \leq v_1.
\]

Hence $v$ is the maximal solution solution of (0.2). \qed

**Lemma 1.6.** There exists a constant $0 < m_1 \leq 1/2$ such that the function

\[u_1(x,t) = \frac{A m_1^{2m^4/1-m}}{|x|^{2/(1-m)}(|x|^m - 1)^2}\]

is a subsolution of (1.2) in $R^2 \setminus B_2 \times (0, \infty)$ for all $0 < m \leq m_1$ where $0 < A \leq (2/3)^4$ is a constant.

Proof. Let $b = 2/(1-m)$ and $g(x) = g(r) = r^{-mb}(r^m - 1)^{-2m}$ where $r = |x|$. Then

\[(1.4) \quad \Delta g = g'' + \frac{1}{r}g' = \frac{(mb)^2r^{-mb-2}(r^m - 1)^{-2m} + 4m^3br^{-mb+m-2}(r^m - 1)^{-2m-1}}{r^{-mb+m-2}(r^m - 1)^{-2m-1}} + 2m^3(2m + 1)r^{-mb+m-2}(r^m - 1)^{-2m-2} \geq 2m^3r^{-mb+m-2}(r^m - 1)^{-2m-2} \quad \forall r \geq 2
\]

\[
\Rightarrow \quad \Delta \left( \frac{u_1}{m} \right) - u_{1,t} = Am^m m^{-2m-1}t^{-m/(1-m)}\Delta g - Am^2(1-m)^{-1}t^{m/1-m}r^{-b}(r^m - 1)^{-2} \\
\geq 2Am^m m^{-2m-1}t^{-m/(1-m)}r^{-mb+m-2}(r^m - 1)^{-2m-2} - Am^2(1-m)^{-1}t^{m/1-m}r^{-b}(r^m - 1)^{-2} \\
= m^2Am^m m^{-1-m}r^{-b}(r^m - 1)^{-2m-2} - (1-m)^{-1}A^{1-m}(r^m - 1)^{2m}.
\]
Since \( \lim_{m \to 0} m^m = 1 \), there exists \( 0 < m_1 \leq 1/2 \) such that \( m^m > 2/3 \) for all \( 0 < m \leq m_1 \). Thus

\[
2m^{2m}r^m - (1-m)^{-1}A^{1-m}(r^m - 1)^{2m} \\
\geq 2(2/3)^{2r^m} - 2A^{1/2}r^{2m^2} \\
\geq 2(2/3)^{2r^{2m^2}}(r^{m(1-2m)} - 1) \geq 0 \quad \forall 0 < A \leq (2/3)^{4}, 0 < m \leq m_1, r \geq 2.
\]

Hence the right hand side of (1.4) is positive for all \( 0 < m \leq m_1, r \geq 2 \), and the lemma follows. \( \Box \)

By direct computation we also have the follow result:

**Lemma 1.7.** For any \( 0 < A < 1 \), \( 0 < m < 1 \), the function

\[
u_2(x,t) = A \left( \frac{t}{|x|^2} \right)^{1/(1-m)}
\]

is a subsolution of (1.2) in \( \mathbb{R} \setminus B_2 \times (0, \infty) \).

**Theorem 1.8.** If \( v \) is a solution of (0.2) in \( \mathbb{R}^2 \times (0, T') \) and satisfies the condition

\[
v(x,t) \geq \frac{Ct}{(|x| \log |x|)^2} \quad \forall |x| \geq R, 0 < t \leq T'
\]

for some constants \( C > 0 \) and \( R > 0 \), then \( v \) is the unique maximal solution of (0.2) in \( \mathbb{R}^2 \times (0, T') \).

**Proof.** For any \( 0 < t_1 < T' \), since

\[
v(x,t) \geq \frac{Ct_1}{(|x| \log |x|)^2} \quad \forall |x| \geq R, t_1 < t \leq T'
\]

by Prop 2.1 of [DD] \( v \) is the unique maximal solution of

\[
(1.5) \begin{cases} \rho_t = \Delta \log \rho, \rho > 0, & \text{in } \mathbb{R}^n \times (t_1, T') \\ \rho(x,t_1) = v(x,t_1) & \forall x \in \mathbb{R}^n \end{cases}
\]

with \( n = 2 \). By Theorem 4.3 of [H] or Theorem 1.2 of [DP] there exist a solution \( \overline{v} \) of (1.5) satisfying

\[
\int_{\mathbb{R}^2} \overline{v}(x,t)dx = \int_{\mathbb{R}^2} v(x,t_1)dx - 4\pi(t - t_1) \quad \forall 0 < t_1 \leq t \leq T'.
\]

Since \( v \) is the unique maximal solution of (1.5), \( v \geq \overline{v} \). Thus

\[
\int_{\mathbb{R}^2} v(x,t)dx \geq \int_{\mathbb{R}^2} \overline{v}(x,t)dx = \int_{\mathbb{R}^2} v(x,t_1)dx - 4\pi(t - t_1) \quad \forall 0 < t_1 \leq t \leq T'.
\]
We next let $R > 0$ and let $\eta_R \in C_0^\infty(R^2)$ be such that $0 \leq \eta_R \leq 1$, $\eta_R(x) \equiv 1$ for $|x| \leq R$, $\eta_R(x) \equiv 0$ for $|x| \geq 2R$. Then

$$\int_{R^2} v(x,t_1)dx \geq \int_{R^2} v(x,t_1)\eta_R(x)dx$$

$$\Rightarrow \liminf_{t_1 \to 0} \int_{R^2} v(x,t_1)dx \geq \liminf_{t_1 \to 0} \int_{R^2} v(x,t_1)\eta_R(x)dx$$

$$= \int_{R^2} u_0(x)\eta_R(x)dx$$

$$\Rightarrow \liminf_{t_1 \to 0} \int_{R^2} v(x,t_1)dx \geq \int_{R^2} u_0(x)dx \quad \text{as} \quad R \to \infty.$$

Hence letting $t_1 \to 0$ in (1.6) we have

$$\int_{R^2} v(x,t)dx \geq \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0 < t \leq T'.$$

By Theorem 4.3 of [H] and the proof of Theorem 1.5 there exists a unique maximal solution $\tilde{v}$ of (0.2) which satisfies

$$\int_{R^2} \tilde{v}(x,t)dx = \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0 < t \leq T'.$$

Since $\tilde{v} \geq v$, by (1.7) and (1.8) we get

$$\int_{R^2} v(x,t)dx = \int_{R^2} \tilde{v}(x,t)dx = \int_{R^2} u_0(x)dx - 4\pi t \quad \forall 0 < t \leq T'$$

$$\Rightarrow v \equiv \tilde{v} \quad \text{on} \quad R^2 \times (0,T').$$

Hence $v$ is the unique maximal solution of (0.2). □

**Theorem 1.9.** If $v$ is a solution of (0.2) in $R \times (0,T')$ and satisfies the condition

$$v(x,t) \geq \frac{Ct}{|x|^2} \quad \forall |x| \geq R, 0 < t \leq T'$$

for some constants $C > 0$ and $R > 0$, then $v$ is the unique maximal solution of (0.2) in $R \times (0,T')$.

**Proof.** For any $0 < t_1 < T'$, since $v$ satisfies (1.9)

$$\log (1/v) \leq -\log Ct + 2\log |x| \leq o(|x|) \quad \forall |x| \geq R, t_1 < t \leq T'.$$

By the result in Section 3 of [ERV] $v$ is the unique maximal solution of (1.5) with $n = 1$. By an argument similar to the proof of Theorem 1.8 we get that $v$ is the unique maximal solution of (0.2) and the theorem follows. □
Section 2.

In this section we will first prove the convergence of solution \( u^{(m)} \) of (0.1) as \( m \to 0 \) for \( n = 1, 2 \), under the assumption that \( u_0 \in L^1(R^n) \cap L^\infty(R^n) \) and 
\[
\varepsilon(1 + |x|^2)^{-\alpha} \leq u_0(x) \leq M \quad \text{for some constants } \varepsilon > 0, \quad M \geq 1, \quad \text{and } \quad 1 < \alpha < 2
\]
for all \( x \in R^n \). The general convergence theorem then follows by an approximation argument. We will first start with a Harnack type inequality.

**Lemma 2.1.** If \( u^{(m)} \in L^\infty((0, \infty); L^1(R^n)) \cap L^\infty(R^n \times (0, \infty)) \) and \( 0 \leq u^{(m)} \leq M \) for some constant \( M \), then for any \( (n - 2)_+/n < m < 1, \quad 0 < t_1 < t_2, \quad t > 0, \quad R > 0, \) and \( x_0 \in R^n \), the following inequalities hold,

\[
\begin{align*}
(i) \quad & \frac{1}{|B_R|} \int_{B_R(x_0)} \frac{u^{(m)}(x,t)}{m} dx \leq \frac{u^{(m)}(x_0,t)}{m} + CMR^2(1 - m)t \\
(ii) \quad & \frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)}m}{m} dx dt \leq I_2 + CMR^2 \\
(iii) \quad & I_1 \leq \frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)}m}{m} dx dt + CMR^2
\end{align*}
\]

where \( C > 0 \) is a constant independent of \( m \) and \( n \).

\[
(2.1)
\]

\[
I_i = \frac{(t_2 - t_1)}{m} \left( 1 - (1 - m)u^{(m)}m(x_0, t_i)t_i^{-m/1-m} \left( \frac{t_2^{1/1-m} - t_1^{1/1-m}}{t_2 - t_1} \right) \right), \quad i = 1, 2.
\]

**Proof.** We will use a modification of the argument of Lemma 6 of [V] and Lemma 4 of [GH] to prove the lemma. Let (cf. [V] p. 509, [DD] p. 653)

\[
G_R(x) = \begin{cases} 
|x - x_0|^{2-n} - R^{2-n} + \frac{n-2}{2}R^{-n}(|x - x_0|^2 - R^2) & \text{if } n \geq 3 \\
\log \frac{R}{|x - x_0|} + \frac{1}{2}R^{-2}(|x - x_0|^2 - R^2) & \text{if } n = 2 \\
R - |x - x_0| + R^{-1}(|x - x_0|^2 - R^2) & \text{if } n = 1
\end{cases}
\]

be the Green’s function for \( B_R(x_0) \). Then \( G_R \geq 0, \quad G_R(R) = G'_R(R) = 0, \) and

\[
\Delta G_R = \begin{cases} 
(n(n-2)R^{-n} - (n-2)|\partial B_1|\delta_{x_0} & \text{if } n \geq 3 \\
2R^{-2} - 2\pi\delta_{x_0} & \text{if } n = 2 \\
R^{-1} - 2\delta_{x_0} & \text{if } n = 1
\end{cases}
\]
where $\delta_{x_0}$ is the delta mass at $x_0$. Then by Theorem 1.1 we have
\[
\begin{align*}
a_n & \left( \frac{1}{|B_R|} \int_{B_R(x_0)} u^{(m)m}(x,t) \frac{1}{m} dx - \frac{u^{(m)m}(x_0,t)}{m} \right) \\
& = \int_{B_R(x_0)} G_R(x) \Delta \left( \frac{u^{(m)m}}{m} \right)(x,t) dx \\
& \leq \frac{1}{(1-m)} \int_{B_R(x_0)} G_R(x) u(x,t) dx \\
& = \int_{B_R(x_0)} G_R(x) u_t(x,t) dx \\
\end{align*}
\]
(2.2)

where $a_1 = 2$, $a_2 = 2\pi$, and $a_n = (n-2)|\partial B_1|$ for $n \geq 3$. Thus (i) follows. Integrating (2.2) over $(t_1, t_2)$ we get
\[
\frac{1}{|B_R|} \int_{t_1}^{t_2} \int_{B_R(x_0)} \frac{1 - u^{(m)m}}{m} dx dt = \int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt - \frac{1}{a_n} \int_{B_R(x_0)} G_R(x) u(x,t) dx \bigg|_{t_1}^{t_2}.
\]

By Theorem 1.1,
\[
\frac{u(x,t_2)}{t_2^{1/1-m}} \leq \frac{u(x,t_1)}{t_1^{1/1-m}} \leq \frac{u(x,t)}{t^{1/1-m}} \quad \forall t_1 \leq t \leq t_2.
\]
Hence
\[
\int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt \leq \int_{t_1}^{t_2} \frac{1 - u^{(m)m}(x_0,t_2)(t/t_2)^{m/1-m}}{m} dt \leq I_2
\]
where $I_2$ is given by (2.1). Since $G_R \geq 0$ and
\[
\int_{B_R(x_0)} G_R(x) u(x,t) dx \leq CMR^2 \quad \forall t > 0
\]
(ii) follows. Similarly
\[
\int_{t_1}^{t_2} \frac{1 - u^{(m)m}}{m} dt \geq I_1
\]
where $I_1$ is given by (2.1) and (iii) follows. \qed

Lemma 2.2. \ Let $u^{(m)} \in L^\infty((0,\infty);L^1(R^n)) \cap L^\infty(R^n \times (0,\infty))$ be the solution of (0.1). If there exists a constant $M \geq 1$ such that $0 \leq u^{(m)} \leq M$ for all $0 < m < 1$ and
\[
\liminf_{m \to 0} u^{(m)}(x_0,t_0) > 0
\]
(2.3)
for some \( x_0 \in \mathbb{R}^n, n = 1, 2, 0 < t_0 < T, T > 0. \) Then for any \( R > 0, \)
\( 0 < t_1 < t_2 < t_0, \) there exists \( C > 0 \) and \( 0 < m_0 < 1 \) depending on \( t_1, t_2, \)
\( t_0 - t_2, \) and \( R > 0 \) such that
\[
(2.4) \quad u^{(m)}(x,t) \geq C \quad \forall 0 < m \leq m_0, x \in B_R, t_1 \leq t \leq t_2.
\]

Proof. Without loss of generality we may assume that \( R > |x_0|. \) By (2.3) there exists \( 0 < m_1 < 1 \) and \( \delta > 0 \) such that
\[
u^{(m)}(x_0,t_0) \geq \delta \quad \forall 0 < m \leq m_1.
\]
Since \( B_1(y) \subset B_{2R+1}(x_0) \) for any \( y \in B_R, \) by Lemma 2.1 we have for any \( y \in B_R, 0 < t_1 \leq t \leq t_2 < t_0, \)
\[
(2.5) \quad \frac{t_0 - t}{m} \left\{ 1 - (1 - m)u^{(m)}(y,t) t_{1/m} - t_{1/m} \left( \frac{t_{1/m} - t_{1/m}}{t_0 - t} \right) \right\}
\]
\[
\leq \frac{1}{|B_1|} \int_t^{t_0} \int_{B_1(y)} \frac{1 - u^{(m)}(y,t)}{m} dxd s + CMR^2
\]
\[
\leq \frac{1}{|B_1|} \int_t^{t_0} \int_{B_1(y) \cap \{u^{(m)} \leq 1\}} \frac{1 - u^{(m)}(y,t)}{m} dxd s + CMR^2
\]
\[
\leq \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} \leq 1\}} \frac{1 - u^{(m)}(x,t)}{m} dxd s + CMR^2
\]
\[
\leq \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} > 1\}} \frac{u^{(m)}(x,t) - 1}{m} dxd s + CMR^2
\]
\[
\leq \frac{(2R + 1)^n}{|B_{2R+1}|} \left\{ 1 - (1 - m)u^{(m)}(x_0,t_0) t_{1/m} - t_{1/m} \left( \frac{t_{1/m} - t_{1/m}}{t_0 - t} \right) \right\}
\]
\[
\leq \frac{(2R + 1)^n}{|B_{2R+1}|} \int_t^{t_0} \int_{B_{2R+1}(x_0) \cap \{u^{(m)} > 1\}} \frac{u^{(m)}(x,t) - 1}{m} dxd s + CMR^2
\]
where \( C > 0 \) is a constant. Since
\[
z^{m-1} \leq z \log z \leq M \log M \quad \forall 1 \leq z \leq M.
\]
The second term on the right hand side above is bounded above by
\[
(2.6) \quad (t_0 - t)(2R + 1)^n M \log M.
\]
By the mean value theorem there exists $t_3 \in (t, t_0)$ and $0 < \theta_1 < 1$ such that

$$
(2.7) \quad \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)}(x_0, t_0)t_0^{m/1 - m} \left( t_0^{1/1-m} - t^{1/1-m} \right) \right\}
$$

$$
= \frac{(t_0 - t)}{m} \left\{ 1 - u^{(m)}(x_0, t_0)(t_3/t_0)^{m/1 - m} \right\}
$$

$$
= (t_0 - t)(u^{(m)}(x_0, t_0)(t_3/t_0)^{1/1-m}) \theta_1 m \log \frac{(t_0/t_3)^{1/1-m}}{u^{(m)}(x_0, t_0)}
$$

$$
\leq TM \log \frac{(T/t_1)^{1/1-m}}{\delta} \quad \forall 0 < m \leq m_1.
$$

Similarly there exists $t_4 \in (t, t_0)$ such that

$$
(2.8) \quad \frac{(t_0 - t)}{m} \left\{ 1 - (1 - m)u^{(m)}(y, t)t_0^{m/1 - m} \left( t_0^{1/1-m} - t^{1/1-m} \right) \right\}
$$

$$
= \frac{(t_0 - t)}{m} \left\{ 1 - u^{(m)}(y, t)(t_4/t)^{m/1 - m} \right\}
$$

$$
\geq \frac{(t_0 - t)}{m} \left\{ 1 - u^{(m)}(y, t)(T/t_1)^{m/1 - m} \right\}.
$$

By (2.5), (2.6), (2.7), (2.8), there exists a constant $c_1 > 0$ such that for all $0 < m \leq m_1$ we have

$$
(2.9) \quad \frac{(t_0 - t_2)}{m} \left\{ 1 - u^{(m)}(y, t)(T/t_1)^{1/1-m} \right\} \leq c_1
$$

$$
\Rightarrow \quad u^{(m)}(y, t) \geq \left( 1 - \frac{mc_1}{t_0 - t_2} \right)^{1/m} (t_1/T)^{1/1-m} \quad \forall 0 < m \leq m_1.
$$

Since the right hand side of (2.9) tends to some positive constant independent of $y \in B_R, t_1 \leq t \leq t_2$, there exists constants $0 < m_0 \leq m_1$ and $C > 0$ such that

$$
u^{(m)}(y, t) \geq C > 0 \quad \forall y \in B_R, t_1 \leq t \leq t_2, 0 < m \leq m_0
$$

and the lemma follows. \qed

**Theorem 2.3.** For $n = 1, 2$, if $u_0 \in L^1(R^n) \cap L^\infty(R^n)$ and $\varepsilon(k + |x|^2)^{-\alpha} \leq u_0(x) \leq M$ for some constants $\varepsilon > 0, 0 < k < 1, 1 < \alpha < 2, M \geq 1$, satisfying $\alpha > (1-m_2)^{-1}$ and $0 < k < (4\alpha)^{-1/(\alpha(1-m_2)-1)}$ for some constant $0 < m_2 < 1/2$, then as $m \to 0$ the solution $u^{(m)}$ of (0.1) will converge uniformly on every compact subset of $R^n \times (0, T)$ to the unique maximal solution $v$ of (0.2) where $T$ is given by (0.3).
Proof. We first consider the case \( n = 2 \). For any \( 0 < m \leq m_2 \), we have \((1 - m)^{-1} < \alpha < 2(1 - m)^{-1}\) and \(\alpha(1 - m) - 1 \geq \alpha(1 - m_2) - 1\). Thus \(0 < k < (4\alpha)^{-1/(\alpha(1-m_2) - 1)} \leq (4\alpha)^{-1/(\alpha(1-m) - 1)}\) for all \(0 < m \leq m_2\). By Theorem 1.2 and Corollary 1.4,

\[
(\varepsilon^{1-m} - (1 - m)t)^{1/1-m}(k + |x|^2)^{-}\alpha \\
\leq u^{(m)}(x,t) \leq M \quad \forall x \in \mathbb{R}^2, t > 0, 0 < m \leq m_2.
\]

Since \((\varepsilon^{1-m} - (1 - m)t)^{1/1-m} \geq \varepsilon/2\) for all \(0 < t \leq \varepsilon/2\) and \(0 < m \leq 1/2\),

\[
M \geq u^{(m)}(x,t) \geq C_K > 0 \quad (x,t) \in K
\]

for any compact subset \(K\) of \(\mathbb{R}^2 \times (0, \varepsilon/2)\). Let

\[
T_0 = \max \left\{ s > 0 : \liminf_{m \to 0} u^{(m)}(x_0,s) > 0 \text{ for some } x_0 \in \mathbb{R}^n \right\}.
\]

Then \(T_0 \geq \varepsilon/2 > 0\). For any \(0 < t_1 < t_2 < T_0\), let \(t_0 = (t_2 + T_0)/2\). Then there exists \(x_0 \in \mathbb{R}^n\) such that

\[
\liminf_{m \to 0} u^{(m)}(x_0, t_0) > 0.
\]

By Lemma 2.2 for any \(R > 0\) there exists \(0 < m_0 < m_2\) such that (2.4) holds. Then there exists constants \(C_2 > 0, C_3 > 0\) independent of \(0 < m < m_0\) such that

\[
C_2 \leq u^{(m)m^{-1}}(x,t) \leq C_3 \quad \forall x \in B_R, t_1 \leq t \leq t_2.
\]

Hence (0.1) is uniformly parabolic for \(\{u^{(m)}\}_{0 < m < m_0}\) on any compact subset of \(\mathbb{R}^2 \times (0, T_0)\). By standard parabolic theory [LSU], \(\{u^{(m)}\}_{0 < m \leq m_0}\) is uniformly equi-Holder continuous on any compact subset of \(\mathbb{R}^2 \times (0, T_0)\). Let \(\{u^{(m_i)}\}, m_i \to 0\) as \(i \to 0\), be a sequence of \(u^{(m)}\). By the Ascoli Theorem and a diagonalization argument \(u^{(m_i)}\) will have a subsequence \(u^{(m_i')}\) converging uniformly on every compact subset \(K\) of \(\mathbb{R}^2 \times (0, T_0)\) to a continuous function \(v\) satisfying \(M \geq v \geq C_K > 0\) on \(K\) for some constant \(C_K > 0\). Without loss of generality we may assume \(u^{(m_i)}\) converges uniformly on every compact subset \(K\) of \(\mathbb{R}^2 \times (0, T_0)\) to \(v\) as \(i \to \infty\). Since \(u^{(m_i)}\) satisfies (0.1), for any \(\eta \in C_0^\infty (\mathbb{R}^2 \times (0, T_0))\) we have

\[
\int_{t_1}^{t_2} \int_{\mathbb{R}^2} \left( u^{(m_i)} \eta_t + u^{(m_i)m_i} - 1 \right) dx ds
\]

\[
= \int_{\mathbb{R}^2} u^{(m_i)} \eta dx \bigg|_{t_1}^{t_2} \quad \forall 0 < t'_1 < t'_2 < T_0.
\]
Now by the mean value theorem, for each \((x, t) \in K\), there exists \(\theta = \theta(x, t)\) such that

\[
\frac{u^{(m_i)}(x, t) - 1}{m_i} - \log v
\leq \left| \frac{u^{(m_i)}(x, t) - 1}{m_i} - \log u^{(m_i)} \right| + |\log u^{(m_i)} - \log v|
\leq \left| \frac{e^{m_i \log u^{(m_i)}} - 1}{m_i} - \log u^{(m_i)} \right| + |\log u^{(m_i)} - \log v|
\leq u^{(m_i)} \theta \log u^{(m_i)} - \log u^{(m_i)} + |\log u^{(m_i)} - \log v|
\leq \max(M^{m_i} - 1, [C^{m_i}_K - 1]) \cdot \max(|\log M|, |\log C_K|) + |\log u^{(m_i)} - \log v|
\to 0 \quad \text{as} \quad i \to \infty
\]

uniformly on \(K\). Hence letting \(i \to \infty\) in (2.11), for any \(\eta \in C^\infty_0(R^2 \times (0, T_0))\) we have

\[
\int_{t_1}^{t_2} \int_{R^2} \left( v_\eta + \log v \Delta \eta \right) dxds = \int_{R^2} v_\eta dx \bigg|_{t_1}^{t_2} \quad \forall 0 < t_1 < t_2 < T_0.
\]

By standard parabolic theory \([\text{LSU}]\) \(v \in C^\infty(R^2 \times (0, T_0))\). Thus \(v\) is a solution of (0.3) in \(R^2 \times (0, T_0)\) with \(\phi(v) = \log v\). We next claim that \(v\) has initial value \(u_0\). To prove the claim we let \(\eta \in C^\infty_0(R^2)\) be such that \(\text{supp} \ \eta \subset B_R\) for some constant \(R > 2\) and fix a \(0 < t_2 < T_0\). Then by the definition of \(T_0\), if \(t_0 = (t_2 + T_0)/2\), then there exists \(x_0 \in R^2\) such that (2.10) holds. By Lemma 2.2 there exists \(\delta > 0\) and \(0 < M_0 < \min(m_1, m_2)\) such that

\[
u^{(m)}(x, t_2) \geq \delta \quad \forall |x| \leq R, 0 < m < M_0
\]

where \(m_1\) is as in Lemma 1.6. By Theorem 1.1, we have for any \(0 < m < M_0\),

\[
u^{(m)}(x, t) \geq \frac{u^{(m)}(x, t_2)}{t_2^{1/1-m}} t^{1/1-m}
\geq \frac{\delta}{t_2^{1/1-m}} t^{1/1-m} \geq c_1 t^{1/1-m} \quad \forall |x| \leq R, 0 < t \leq t_2
\]
where \( c_1 = \delta/(1 + T_0)^2 \). Hence

\[
\left| \int_{R^2} u(m)(x,t) \eta(x) dx - \int_{R^2} u_0 \eta(x) dx \right|
= \left| \int_0^t \int_{R^2} u_t^{(m)} \eta(x) dx ds \right|
= \left| \int_0^t \int_{R^2} \Delta \left( \frac{u^{(m)} - 1}{m} \right) \eta(x) dx ds \right|
= \left| \int_0^t \int_{BR} \left( \frac{u^{(m)} - 1}{m} \right) \Delta \eta(x) dx ds \right|
\leq \| \Delta \eta \|_{L^\infty} \left( \int_0^t \int_{BR \cap \{u^{(m)} \geq 1\}} \frac{u^{(m)} - 1}{m} dx ds \right)
+ \int_0^t \int_{BR \cap \{u^{(m)} < 1\}} \frac{1 - u^{(m)} m}{m} dx ds
\leq \| \Delta \eta \|_{L^\infty} \left( \int_0^t \int_{BR \cap \{u^{(m)} \geq 1\}} \frac{M - 1}{m} dx ds \right)
+ \int_0^t \int_{BR \cap \{u^{(m)} < 1\}} \frac{1 - (c_1 s^{1/m})^m}{m} dx ds
\leq \| \Delta \eta \|_{L^\infty} |B_R| \left( t M \log M - \int_0^t (c_1 s^{1/m}) \theta m \log (c_1 s^{1/m}) ds \right)
\leq C(t + t \log t) \quad \forall 0 < t < t_2
\]

for some constants \( C > 0 \) and \( 0 < \theta < 1 \) by the mean value theorem. Letting \( m = m_i \to 0 \), we get

\[
\left| \int_{R^2} v(x,t) \eta(x) dx - \int_{R^2} u_0 \eta(x) dx \right| \leq C(t + t \log t) \to 0 \quad \text{as} \quad t \to 0.
\]

Hence \( v \) have initial value \( u_0 \) and is thus a solution of (0.2) in \( R^2 \times (0, T_0) \).

We next claim that \( v \) is the unique maximal solution of (0.2). To prove the claim we observe that by \( (2.12) \)

\[
\frac{2^{1-m} \delta}{(1 + T_0)^2} \left( \frac{2m - 1}{m} \right)^2 \frac{t^{2(1-m)}}{|x|^{2(1-m)}(|x|^m - 1)^2}
\geq \frac{A m^{2(1-m)}}{|x|^{2(1-m)}(|x|^m - 1)^2} \quad \forall |x| = 2, 0 < t \leq t_2, 0 < m \leq M_0
\]

where

\[
A = \min \left( \frac{(2/3)^4}{16^8(1 + T_0)^2}, \frac{(\log 2)^2 \delta}{16^8(1 + T_0)^2} \right)
\]
then by Theorem \( (2.14) \) Lemma since \( (2m_1 < t < m_2) \) any 0
and choose
By the same argument as the proof of \((2.14)\) we get
\[
\frac{A_m}{2} \leq \frac{A_1}{2} \leq \frac{A_1}{2}\]
and choose \( t_3 > 0 \) such that \( T_0/(1 + A_1) < t_3 < T_0 \). Since \( u_1 \to v \) as \( m \to 0 \) uniformly on \( B_2 \times \{ t_3 \} \). There exists 0
\[
\alpha_m(1 - m) - 1 = (1 + m)(1 - m) = 1 - m^2 > \frac{1}{2} \quad \forall 0 < m < 1/2
\]
\[
\Rightarrow \frac{1}{\alpha_m(1 - m) - 1} < 2 < \frac{8}{\alpha_m}
\]
\[
\Rightarrow 0 < k_1 = (2A_1(1 + T_0)^2/\delta_1) \leq 16^{-8/\alpha_m} \leq (4\alpha_m)^{-1/\alpha_m(1-m)-1}.
\]
Since by the mean value theorem there exists 0 < \( \theta < 1 \) such that
\[
\frac{x^m - 1}{m} = |x|^m \log |x| \leq |x|^m \log |x| \leq |x|^{1 + m} \quad \forall |x| \geq 2
\]
and

\[(2.16)\]
\[T_1 = (1 + A_1) t_3 \Rightarrow ((1 - m)(T_1 - t_3)_+)^{1/1 - m} \leq (A_1 t_3)^{1/1 - m} \leq A_1 t_3^{1/1 - m}\]

hence for all \(|x| \geq 2, 0 < m < M_1\), we have

\[
\frac{A_1 m^2 t_3^{1/1 - m}}{|x|^{2/(1 - m)}(|x|^m - 1)^2} \geq \frac{A_1 t_3^{1/1 - m}}{|x|^{2 + 2m + 2/(1 - m)}} \geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1 - m}}{(k_1 + |x|^2)^{\alpha_m}}
\]

\[
\Rightarrow u^{(m)}(x, t_3) \geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1 - m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3)
\]

\[\forall |x| \geq 2, 0 < m \leq M_1.\]

Since \(k_1^{\alpha_m} = 2A_1(1 + T_0)^2/\delta_1\), by (2.16) for any \(|x| \leq 2, 0 < m < M_1\), we have

\[
u^{(m)}(x, t_3) \geq \frac{\delta_1}{2} = \frac{A_1 (1 + T_0)^2}{k_1^{\alpha_m}} \geq \frac{A_1 t_3^{1/1 - m}}{(k_1 + |x|^2)^{\alpha_m}} \geq \frac{((1 - m)(T_1 - t_3)_+)^{1/1 - m}}{(k_1 + |x|^2)^{\alpha_m}} = w(x, t_3).
\]

Hence

\[u^{(m)}(x, t_3) \geq w(x, t_3) \forall x \in R^2.\]

By the maximum principle,

\[u^{(m)}(x, t) \geq w(x, t) \forall x \in R^2, t_3 \leq t < T_1, 0 < m \leq M_1.\]

Since \(T_1 = (1 + A_1) t_3 > T_0\), we have \((T_0 + T_1)/2 > T_0\) and

\[u^{(m)}(x, (T_0 + T_1)/2) \geq w(x, (T_0 + T_1)/2) > 0 \forall x \in R^2.\]

This contradicts the maximality of \(T_0\). Hence \(T_0 = T\) where \(T\) is given by (0.3) and the theorem for the case \(n = 2\) follows. For \(n = 1\) by an argument similar to the case \(n = 2\) but with the subsolution \(u_2\) of Lemma 1.7 in place of \(u_1\) of Lemma 1.6 in the argument we get that for \(n = 1\) as \(m \to 0\) the solution \(u_t^{(m)}\) also converges uniformly on every compact subset of \(R^1 \times (0, \infty)\) to the maximal solution \(v\) of (0.2) in \(R^1 \times (0, \infty)\) and the theorem follows. □

**Theorem 2.4.** If \(n = 1, 2,\) and \(u_0 \in L^1(R^n) \cap L^\infty(R^n),\) then as \(m \to 0,\)
the solution \(u_t^{(m)}\) of (0.1) will converge uniformly on every compact subset of \(R^n \times (0, T)\) to the unique maximal solution \(v\) of (0.2) where \(T\) is given by (0.3).

**Proof.** Choose \(1 < \alpha < 2, 0 < m_2 < 1/2, 0 < k < 1,\) satisfying the conditions \(\alpha > (1 - m_2)^{-1}\) and \(0 < k < (4\alpha)^{-1/(\alpha(1 - m_2) - 1)}\) and let \(u_t^{(m)}\) be
the solution of (0.1) with initial value
\[ u_{0j}(x) = u_0(x) + \frac{1}{j} (k + |x|^2)^{-\alpha} \quad j = 1, 2, \ldots \]

Then by Theorem 2.3 for each \( j = 1, 2, \ldots \), \( u_{j}^{(m)} \) will converge uniformly on every compact subset of \( R^n \times (0, T_j) \) to the unique maximal solution \( v_j \) of (0.2) with initial value \( u_{0j} \) as \( m \to 0 \) where
\[
T_j = \begin{cases} 
\infty & \text{if } n = 1 \\
\frac{1}{4\pi} \int_{R^2} u_{0,j}(x) \, dx & \text{if } n = 2.
\end{cases}
\]

Suppose the theorem is false. Then there exists \( \varepsilon > 0 \), \( R > 0 \), \( 0 < t_1 < t_2 < T \) and a sequence \( u^{(m)}, m_i \to 0 \) as \( i \to \infty \), of \( u^{(m)} \), such that
\[
(2.17) \quad \|u^{(m)}_i - v\|_{L^\infty(B_R \times (t_1, t_2))} \geq \varepsilon \quad \forall i = 1, 2, \ldots
\]

where \( v \) is the unique maximal solution of (0.2) with initial value \( u_0 \). Since \( \|u^{(m)}\|_{L^\infty} \leq \|u_0\|_{L^\infty}, u^{(m)} \) will have a subsequence \( u^{(m_i)} \) converging weakly in \( L^\infty(R^n \times (0, T)) \) and a.e. \( (x, t) \in R^n \times (0, T) \) to some function \( \bar{v} \) as \( i \to \infty \). By the uniqueness of maximal solution and comparison principle for maximal solution of (0.2) (Lemma 4.2 of [H] or Lemma 4.1 of [DP] for \( n = 2 \) and [ERV] for \( n = 1 \)) we have \( v_1 \geq v_2 \geq \ldots \geq v > 0 \) in \( R^n \times (0, T) \)

Hence \( v_j \) are uniformly bounded below by some positive constant on any compact subset of \( R^n \times (0, T) \). Thus (0.4) with \( \phi(u) = \log u \) is uniformly parabolic for \( v_j \) on any compact subset of \( R^n \times (0, T) \). Hence \( v_j \) are equi-

By Theorem 1.2 and Fatou’s lemma, for any \( 0 < t < T \) we have
\[
\int_{R^n} |u_{j}^{(m)}(x,t) - u^{(m)}(x,t)| \, dx \leq \int_{R^n} |u_{0j} - u_0| \, dx \\
\leq \frac{1}{j} \int_{R^n} (k + |x|^2)^{-\alpha} \, dx \leq \frac{C}{j} \\
\Rightarrow \int_0^T \int_{R^n} |u_{j}^{(m)}(x,t) - u^{(m)}(x,t)| \, dxdt \leq \frac{CT}{j} \\
\Rightarrow \int_0^T \int_{R^n} |v_j(x,t) - \bar{v}(x,t)| \, dxdt \leq \frac{CT}{j} \quad \text{as } m \to m_i' \\
\Rightarrow \int_0^T \int_{R^n} |\bar{v}(x,t) - \bar{v}(x,t)| \, dxdt = 0 \quad \text{as } j \to \infty \\
\Rightarrow \bar{v}(x,t) = \bar{v}(x,t) \quad \text{a.e. } (x, t) \in R^n \times (0, T).
Since $\tau \geq v > 0$ in $R^n \times (0, T)$, $\tilde{v}(x, t) > 0$ for a.e. $(x, t) \in R^n \times (0, T)$. Thus there exists a set $E \subset R^n \times (0, T)$ of measure zero such that
\[
\lim_{i \to \infty} u^{(m_i)}(x, t) = \tilde{v}(x, t) = \overline{v}(x, t) \geq v(x, t) > 0 \quad \forall (x, t) \in (R^n \times (0, T)) \setminus E.
\]
Hence for any $R' > 0$, $0 < t'_1 < t'_2 < T$, there exist $x_0 \in R^n$, $t'_2 < t_0 < T$, such that
\[
\lim_{i \to \infty} u^{(m_i)}(x_0, t_0) > 0.
\]
By Lemma 2.2 there exists a constant $C > 0$ such that
\[
\|u_0\|_{L^\infty} \geq u^{(m_i)}(x, t) \geq C > 0 \quad \forall x \in B_{R'}, t'_1 \leq t \leq t'_2, i = 1, 2, \ldots.
\]
Then there exists constants $C_1 > 0$, $C_2 > 0$, independent of $m'_i$ such that
\[
C_1 \leq u^{(m'_i)}(x, t) \leq C_2 \quad \forall x \in B_{R'}, t'_1 \leq t \leq t'_2, i = 1, 2, \ldots.
\]
Hence (1.2) is uniformly parabolic for $u^{(m'_i)}$. By standard parabolic theory \[\text{LSU}, \]\ \{
$u^{(m'_i)}$\} is uniformly equi-Hölder continuous on any compact subset of $R^2 \times (0, T)$. By the Ascoli Theorem and a diagonalization argument similar to the proof of Theorem 2.3 $u^{(m'_i)}$ will have a subsequence converging uniformly on every compact subset of $R^2 \times (0, T)$ to the maximal solution $v$ of (0.2). This contradicts (2.17). Hence the theorem must be true and $u^{(m)}$ converges uniformly on every compact subset of $R^2 \times (0, T)$ to the maximal solution $v$ of (0.2) as $m \to 0$ and we are done. $\square$

By the proof of Theorems 2.3 and 2.4 we have the following corollary:

**Corollary 2.5.** If $v$ is the maximal solution of (0.2) in $R^n \times (0, T)$, $n = 1, 2$, where $T$ is given by (0.3), then for any $0 < T_0 < T$ there exists a constant $C > 0$ such that

\[
v(x, t) \geq \begin{cases} 
\frac{C_1}{|x|^2} & \forall |x| \geq 2, 0 < t \leq T_0 \quad \text{if } n = 1 \\
\frac{C_1}{(|x| \log |x|)^2} & \forall |x| \geq 2, 0 < t \leq T_0 \quad \text{if } n = 2.
\end{cases}
\]

**References**


[H] Kin Ming Hui, Existence of solutions of the equation $u_t = \Delta \log u$, Nonlinear analysis TMA, to appear.


Received September 4, 1997.

Academia Sinica
Nankang, Taipei, 11529
Taiwan, R. O. C.

E-mail address: makmhui@ccvax.sinica.edu.tw
The objective of this paper is to develop criteria that guarantee that a finite group $G$ which acts faithfully on a vector space $V$ possesses “many” orbits of different sizes. This has consequences on (classical and modular) character degrees of finite solvable groups.

Introduction.

The starting point of this investigation was the well-known problem of bounding the derived length $dl(G)$ of a finite solvable group $G$ by the number $|cd(G)|$ of its irreducible complex character degrees. As well-known, in 1985 D. Gluck established the bound

$$dl(G) \leq 2 |cd(G)|$$

for arbitrary solvable groups (cf. [2]).

I.M. Isaacs was the first to doubt that this bound asymptotically is the right bound, and there are indications that a much better bound should exist. For example, for the $p$-Sylow subgroups of some classical groups logarithmic bounds have been established (cf. [6], [12]), and on the other hand for A-groups, i.e. solvable groups all Sylow subgroups of which are abelian, we found a bound that is even stronger than logarithmic (cf. [9]).

Unfortunately, despite the results on some well-known $p$-groups mentioned above, the abundance of $p$-groups seems to make it impossible to attack the problem for $p$-groups with today’s methods in general; for instance, no reasonable induction argument is available (cf. [11]). So in order to avoid the problems occuring with $p$-groups, B. Huppert and I.M. Isaacs suggested to consider the Fitting height instead of the derived length. For instance, a bound that is even stronger than logarithmic for A-groups, the result in [9] is already a result on the Fitting height. So we aimed at getting a similar bound for more general groups.

Now by standard character theory, in minimal counterexamples one often has a minimal normal subgroup $N$ of $G$ that has a complement $H$ in $G$, and then character degrees that are induced from the extension of a linear character of $N$ to its inertia group correspond to orbit sizes in the action of $H$ on $Irr(N)$. This is where the question of the number of different orbit
sizes comes into play, a question, that surely is of independent interest and that we are going to study in this paper as generally as possible.

In the first section we prove that any finite group $G$ that possesses a normal subgroup $N$ of a certain type (almost extra-special) possesses (almost) as many distinct orbit sizes in its action on a vector space $V$ as $N$ does, provided that the center of $N$ acts fixed point freely on $V$ (which is a natural and necessary hypothesis); and for $N$ this number is well–known. This result is not far from being best possible and one key step for the proof of our results in Section 2; namely it enables us to find many orbit sizes in quasiprimitive group actions of solvable groups. To handle the imprimitive case, we have to restrict ourselves to groups of odd order (which then are solvable by Feit-Thompson), and we make use of a recent result of A. Seress (cf. [13]) stating that the minimal base size of a solvable primitive permutation group of odd order is at most 3. This is why our inductive argument only works for primes greater than 3 whence we have an additional hypothesis in Theorem 2.1, where we examine the relationship between the number of orbit sizes of $G$ on a vector space over a finite field of odd characteristic and the number of distinct orbit sizes of a suitable normal subgroup of $G$. But still this is strong enough to deal with our original subject, the dependence of the Fitting height $f(G)$ on $|cd(G)|$, at least for groups of odd order, and we find absolute constants $C_1, C_2$ such that

$$f(G) \leq C_1 \log |cd(G)| + C_2$$

for any group of odd order (see Corollary 2.4). This improves the former linear bound by Broline-Garrison (cf. [7, Corollary (12.21)]) and in fact is the asymptotically best possible bound. We conjecture that such a bound is true for arbitrary solvable groups.

Finally in the third section we discuss a result of I. M. Isaacs on bounding the derived length of a finite solvable group by the number of its irreducible Brauer characters in the light of our results obtained so far.

**Results and Proofs.**

**Notation.** All groups considered in this paper are finite. By $f(G)$ we denote the Fitting height (or nilpotent length) of $G$, and $dl(G)$ is the derived length of $G$. $\text{Irr}(G)$ is the set of irreducible complex characters of $G$, and $cd(G) := \{\chi(1) | \chi \in \text{Irr}(G)\}$.

For any group $G$ we denote by $d(G)$ the minimal number of generators of $G$. If $p$ is a prime and $n \in \mathbb{N}$, then $n_p$ is the $p$-part of $n$. For $x \in \mathbb{R}$ we let $\lceil x \rceil = \max\{n \in \mathbb{N} | n - 1 < x\}$. All other notation is standard.
1. Orbit sizes of quasiprimitive groups.

Definition 1.1. Let $G$ be a group acting on a vector space $V$. If $0 \neq v \in V$ such that $C_G(v) = 1$, then we call $v^G$ a regular orbit of $G$ on $V$.

If no non-identity element of $G$ fixes any non-zero vector, then we say that $G$ acts fixed point freely on $V$.

For a prime $p$ and any integer $m \geq 0$ we say that a $p$-group $E$ is of type $E(p,m)$ if

either $p > 2$ and $E$ is extra-special of exponent $p$ and order $p^{2m+1}$
or $p = 2$ and $E = E_1Z(E)$ (central product) with an extra-special group $E_1$ of order $2^{2m+1}$ and $Z(E)$ cyclic of order 2 or 4.

This notion differs from [1] only in that we also allow $m = 0$ (in which case $E = Z(E)$ is cyclic). Note that the Lemmas 2, 3, 4, 5 in [1] remain true for $m = 0$, so that we may apply them also to our groups of type $E(p,m)$.

The following lemma and its proof were brought to my attention by Thomas R. Wolf.

Lemma 1.2. Let $G$ be a group acting on a vector space $V$ over $K := GF(q)$ for a prime $q$. Suppose that $E \trianglelefteq G$ is of type $E(p,m)$ for a prime $p$ and an integer $m \geq 0$ such that $Z(E)$ acts fixed point freely on $V$. If $g \in G$ with $g \notin C_G(E)$, then

$$\dim_K C_V(g) \leq \frac{p+1}{2p} \dim_K V.$$  

Proof. As $Z(E)$ acts fixed point freely on $V$, $E$ acts faithfully on $V$ and $p \neq q$. So $V_E$ is completely reducible, i.e. there is an $n \in \mathbb{N}$ such that

$$V_E = V_1 \oplus \cdots \oplus V_n$$

with irreducible and obviously also faithful $E$–modules $V_i$. Now if $1 \neq x \in E$, then by [1, Lemma 3] we have $\dim_K C_{V_i}(x) \leq (1/p) \dim_K V_i$ for $i = 1, \ldots, n$, and thus we easily obtain

$$\dim_K C_V(x) \leq \frac{1}{p} \dim_K V.$$  

To prove the lemma, let $g \in G$ with $g \notin C_G(E)$. Then there exists a $1 \neq x \in E$ such that $1 \neq [g, x] \in E$ whence we have $\dim_K C_V([g, x]) \leq (1/p) \dim_K V$. So as clearly $C_V(g^x) \cap C_V(g) \leq C_V([g, x])$, we find that

$$\dim_K V \geq \dim_K (C_V(g^x) + C_V(g))$$
$$= \dim_K C_V(g^x) + \dim_K C_V(g) - \dim_K (C_V(g^x) \cap C_V(g))$$
$$\geq 2 \dim_K C_V(g) - \dim_K C_V([g, x])$$
$$\geq 2 \dim_K C_V(g) - \frac{1}{p} \dim_K V.$$
Hence the assertion follows. □

Next we prove a lemma on the existence of $p$–regular orbits. We do not try here to figure out the minimal hypothesis we need to establish our claim, but simply give a straightforward argument that suffices to achieve our asymptotic results later on.

**Lemma 1.3.** Let $G$ be a group acting on a vector space $V$ over $K := GF(q)$ for a prime $q$. Suppose that $E \trianglelefteq G$ is of type $E(p,m)$ for a prime $p$ and an integer $m \geq 0$ such that $C_G(E)$ acts fixed point freely on $V$.

If $m \geq 10$, then $G$ has a regular orbit on $V$. If $p$ is odd, then $G$ has a $p$–regular orbit on $V$ even for $m \geq 5$.

**Proof.** Clearly $E$ acts faithfully on $V$ and so $(|E|, |V|) = 1$. Consequently $V/E$ is completely reducible so that for an $n \in \mathbb{N}$ we have

$$V/E = V_1 \oplus \cdots \oplus V_n$$

with irreducible and obviously also faithful $E$-modules $V_i$.

Now put $\delta = 0$ if $p$ is odd, and $\delta = 1$ in case of $p = 2$. Then it is well-known that there is an abelian $A \trianglelefteq E$ with $A \cap Z(E) = 1$ of order $p^{m-\delta}$ (cf. [5, III, §13]). Now with [1, Lemma 3] we see that $|C_{V_i}(A)| = |V_i|^{(1/p^{m-\delta})}$ for all $i = 1, \ldots, n$ and thus $|C_V(A)| = |V|^{(1/p^{m-\delta})}$. Moreover $C_G(E)$ normalizes (and so acts fixed point freely on) $C_V(A)$ which implies $|C_G(E)| | |C_V(A)| - 1$, in particular $|C_G(E)| \leq |C_V(A)|$.

Furthermore, if we put $B := C_G(Z(E))$, by [10, Corollary 1.6] $B/E C_B(E) = B/C_B(E/Z)$ is isomorphic to a subgroup of $Sp(2m, p)$, so that we find the elementary estimation $|B/E C_B(E)| \leq p^{2m^2 + m}$ (note that $C_B(E) = C_G(E)$). Finally observe that obviously $|G/B| \leq |Aut(Z(E))| \leq p$.

Now assume that $G$ has no $p$–regular orbit on $V$. Then obviously

$$V = \bigcup_{g \in G \setminus \{1\}} C_V(g),$$

and thus as $C_G(E)$ acts fixed point freely on $V$, we obtain with Lemma 1.2 that

$$|V| \leq \sum_{g \in G \text{ with } o(g) = p} |C_V(g)| \leq |G| \cdot |V|^{\frac{p+1}{p^2}}$$

and hence

$$|V|^{\frac{p+1}{p^2}} \leq |G| \cdot |G/B| \cdot |B/E C_B(E)| \cdot |E/Z(E)| \cdot |C_G(E)| \leq p \cdot p^{2m^2 + m} \cdot p^{2m} \cdot |V|^{(1/p^{m-\delta})}.$$ 

Now by [1, Lemma 2] for $i = 1, \ldots, n$ we have $\dim_K V_i = s p^m$, where $s \geq 1$ is the smallest integer such that $|Z(E)|$ divides $q^s - 1$. Hence $p \leq |Z(E)| \leq q^s$. 

and

$$|V| \geq |V_i| = q^{sp^m} \geq p^m.$$ 

So we conclude that

$$p^{m^2+3m+1} \geq |V| \left( \frac{p^{m-\delta - p^{m-1-\delta} - 2}}{2p^{m-4}} \right) \geq p \left( \frac{p^{m-\delta - p^{m-1-2p^\delta}}}{2} \right)$$

and further that

$$p^{m-1}(p-1) \leq 4m^2 + 6m + 2 + 2p^\delta.$$ 

If $p = 2$, then this is only true for $m \leq 9$ contradicting our hypothesis, and if $p \geq 3$, we have $3^{m-1} \leq 2m^2 + 3m + 2$ which only holds for $m \leq 4$, so that again we have a contradiction. Hence the lemma is shown. □

**Lemma 1.4.** Let $G$ be a group acting on a vector space $V$ over $K := GF(q)$ for a prime $q$. Suppose that $E \leq G$ is of type $E(p, m)$ for a prime $p$ and an integer $m \geq 0$ such that $Z(E)$ acts fixed point freely on $V$. Now define

$$m' = \begin{cases} 
m & \text{if } p \text{ is odd or } E \text{ is the central product of } m \\
m - 1 & \text{if } m \geq 1 \text{ and } E \text{ is the central product of } m - 1 \\
\text{dihedral groups of order } 8 \text{ and } Z(E) \\
\text{a quaternion group of order } 8 \text{ and } Z(E). \end{cases}$$

Fix $i \in \{0, \ldots, m'\}$. Moreover let $Z(E) \leq E_i \leq E$ be of type $E(p, i)$. In case of $m' = m-1$ and $i \geq 1$ we also assume that $E_i$ contains a quaternion subgroup. Let $A_i \leq E$ be elementary abelian of order $p^m - i$ such that $E_i \cap A_i = 1$. (Such $E_i$, $A_i$ exist, as one can see with the proof of [1, Lemma 5].) Hence $[E_i, A_i] = 1$, and (by [1, Lemma 3]) $C_V(A_i) > 0$.

Now let

$$C_i = C_G(E_i, A_i/A_i)$$

$$:= \{ g \in G \mid (xA_i)^g = x^g A_i = x A_i \text{ for all } x \in E_i \}$$

and let $A_i \leq N_i \leq C_i$ be maximal subject to $C_V(N_i) > 0$.

Put $V_i = C_V(N_i)$ and $H_i = N_G(N_i)$, and for any $U \leq H_i$ we write $\overline{U} = UN_i/N_i$.

Then the following holds:

a) $H_i$ acts on $V_i$, and $E_i \cong \overline{E_i} \leq \overline{H_i}$ and $E_i N_i \leq H_i$. Furthermore $E \cap H_i = E_i \times A_i \leq H_i$, $E \cap N_i = A_i \leq H_i$, and $Z(\overline{E_i}) = \overline{Z(E_i)} = \overline{Z(E)}$ acts fixed point freely on $V_i$; in particular, $\overline{E_i}$ acts faithfully on $V_i$.

b) $C_{\overline{H_i}}(\overline{E_i})$ acts fixed point freely on $V_i$.

c) Assume now that $0 \leq i \leq m'-1$ and that $A_i > A_{i+1}$ and $E_i < E_{i+1}$. If $N_{i+1}$ is given, then we can choose $N_i$, $V_i$ such that $N_i > N_{i+1}$ and $V_i < V_{i+1}$.
Proof. a) First observe that \( C_i \leq N_{\mathcal{G}}(A_i) \) and that \( E_i \cap N_i = 1 \).

Now of course \( \overline{H_i} \) normalizes \( V_i \). As \( N_i \leq C_i \), for any \( x \in N_i, y \in E_i \) there is an \( a \in A_i \) such that \( xyx^{-1} = y^r = ya \). Hence \( x^y = y^{-1}xy = ax \in N_i \). This shows that \( E_i \leq H_i \). So \( Z(E) \leq H_i \) acts fixed point freely on \( V_i \), and so does \( Z(E_i) = Z(E_i) \cong Z(E) \). Hence clearly \( E_i \cong E_i \).

Now we have \( A_i \times E_i \leq H_i \). If \( y \in E \backslash (A_i \times E_i) \), then there is an \( x \in A_i \) with \( 1 \neq [x, y] \in Z(E) \); hence the assumption \( y \in H_i \) yields \([x, y] \in N_i \) which contradicts \( C_V(N_i) > 0 \). This shows \( A_i \times E_i = E \cap H_i \leq H_i \) and then \( \overline{E_i} = \overline{E_i} \times A_i \leq \overline{H_i} \) and \( E_i N_i \leq H_i \). Clearly \( \overline{E_i} \) acts faithfully on \( V_i \).

Now finally we have \( A_i \leq E \cap N_i \leq E \cap H_i = A_i \times E_i \), so we have \( E \cap N_i = A_i \), because \( E_i \cap N_i = 1 \). So a) is shown.

b) It suffices to show that all elements of prime order in \( C_{\mathcal{H}_i}(E_i) \) act fixed point freely on \( V_i \). Let \( 1 \neq g N_i \in C_{\mathcal{H}_i}(E_i) \) with \( g \in H_i \). Then we have \( 1 = [\langle g N_i \rangle, \overline{E_i}] = [\langle g \rangle, E_i] N_i / N_i \), and so a) yields \( [\langle g \rangle, E_i] \leq N_i \cap E = A_i \).

As furthermore \( g \) normalizes \( A_i \), it follows that \( g \in C_i \).

Let \( N_i^* = \langle N_i, g \rangle = N_i(g) > N_i \). Then we have \( A_i \leq N_i^* \leq C_i \), so the maximal choice of \( N_i \) forces \( C_V(N_i^*) = 0 \). Consequently also \( C_{V_i}(N_i^*) = 0 \), and as \( N_i \) acts trivially on \( V_i \), we find \( C_{V_i}(g) = 0 \). Hence \( g N_i \) acts fixed point freely on \( V_i \), as desired.

c) Let \( A_i \cap E_{i+1} = \langle a \rangle \). By a) we have \( A_{i+1} = A_i \times \langle a \rangle \), and thus for \( h \in N_{i+1} \) we have \( a^h = ab \) with a suitable \( b \in A_{i+1} \), as \( N_{i+1} \leq C_{i+1} \). So it follows that \( N_{i+1} \leq N_{\mathcal{G}}(A_i) \). As \( N_{i+1} \leq C_{i+1} \), moreover we have that \( N_{i+1} \leq C_G(E_i \times \langle a \rangle A_{i+1} / A_{i+1} = C_G(E_i A_i / A_{i+1}) \). Hence it follows that \( N_{i+1} \leq C_i \).

Now we build the semidirect product \( K_{i+1} := N_{i+1} A_i = N_{i+1} \langle a \rangle \times N_{i+1} \). Then we have \( A_i \leq K_{i+1} \leq C_i \). Moreover by a) we know that \( E_{i+1} \leq H_{i+1} \) acts faithfully on \( V_{i+1} \). Hence again with [1] we see that \( V_{i+1} \) is completely reducible as \( E_{i+1} \)–module and then that \( C_{V_{i+1}}(a) > 0 \). Hence \( 0 < C_{V_{i+1}}(N_{i+1}(a)) = C_{V_{i+1}}(K_{i+1}) \) and so \( C_V(K_{i+1}) > 0 \). Thus \( K_{i+1} \) has all the properties that we demand of \( N_i \) (except maximality), i.e. we can choose \( N_i \) such that \( N_i \geq K_{i+1} > N_{i+1} \), as wanted. Then clearly \( V_i \leq V_{i+1} \), and as \( E_{i+1} \) acts faithfully on \( V_{i+1} \) whereas \( a \) centralizes \( V_i \), it follows that \( V_i \leq C_{V_{i+1}}(a) < V_{i+1} \), and finally c) is proven.

\[ \square \]

**Theorem 1.5.** Let \( G \) be a group acting on a vector space \( V \) over \( K := GF(q) \) for a prime \( q \). Suppose that \( E \leq G \) is of type \( E(p, m) \) for a prime \( p \) and an integer \( m \geq 0 \) such that \( Z(E) \) acts fixed point freely on \( V \). Then the following holds:

a) Suppose that \( p \) is odd and that \( m \geq 5 \). For \( i = 5, \ldots, m \) take subgroups \( N_i \) of \( G \) as in Lemma 1.4 such that \( N_5 > \cdots > N_m \) (which is possible by Lemma 1.4c)). Then there exist \( v_i \in V \) (\( i = 5, \ldots, m \)) such that \( C_G(v_i) = \)
$N_i > 1$. Hence by Lemma 1.4 the $|v_i^G|_p$ $(i = 5, \ldots, m)$ are strictly increasing; in particular, $G$ has at least $m - 4$ different nontrivial orbit sizes on $V$.

b) Suppose that $p = 2$, let $m'$ be defined as in Lemma 1.4, and suppose that $m' \geq 10$. For $i = 10, \ldots, m'$ take subgroups $N_i$ of $G$ as in Lemma 1.4 such that $N_{i10} > \cdots > N_{m'}$. Then there exist $v_i \in V$ ($i = 10, \ldots, m'$) such that $C_G(v_i) = N_i > 1$, whence the 2-parts of the corresponding orbit sizes are strictly increasing; in particular, $G$ has at least $m' - 9$ different nontrivial orbit sizes on $V$.

**Proof.** a) Fix $i \in \{5, \ldots, m\}$, and let the $A_i$, $E_i$, $H_i$ and $V_i$ be defined as in Lemma 1.4. First note that $V_i \leq C_V(A_i)$. By Lemma 1.4a) $E_i$ acts faithfully (and clearly completely reducibly) on $V_i$, so by [1, Lemma 4] it has a regular orbit on $V_i$. Now for every $v \in V_i$ that lies in a regular orbit of $E_i$ on $V_i$ obviously we have $C_E(v) = A_i$. Consequently $A_i = E \cap C_G(v) \leq C_G(v)$, so with $M_i := N_G(A_i)$ we have $C_G(v) \leq M_i$. Hence we have shown that for every $v \in V_i$ that lies in a regular orbit of $E_i$ we have $C_G(v) = C_M(v)$ ($\ast$).

Now clearly $E_iA_i = E_i \times A_i = E \cap M_i \leq M_i$ and $C_i \leq M_i$ and thus $C_i = C_M(E_iA_i/A_i) \leq M_i$, and so we have $N_i \leq C_i \leq M_i$.

Next let $v \in V_i$ be an element out of a regular orbit of $E_i$ on $V_i$ and put $S_i = C_M(v) \cap C_i \leq C_M(v)$. We claim that $S_i = N_i$. Obviously $N_i \leq S_i$. On the other hand we have $A_i \leq S_i \leq C_i$ and $v \in C_V(S_i)$, in particular $C_V(S_i) > 0$. Hence our maximal choice of $N_i$ forces $S_i = N_i$, establishing our claim.

So now by ($\ast$) we have $N_i = S_i \leq C_M(v) = C_G(v)$, whence $C_G(v) \leq N_G(N_i) = H_i$, and as $v$ was chosen arbitrarily, now we know that for every $v \in V_i$ that lies in a regular orbit of $E_i$ on $V_i$ we have $C_G(v) = C_{H_i}(v)$ ($\ast\ast$).

Now by Lemma 1.4a), b) the action of $H_i := H_i/N_i$ on $V_i$ fulfills the hypotheses of Lemma 1.3, so that Lemma 1.3 yields a $v_i \in V_i$ that lies in a regular orbit of $H_i$ on $V_i$, i.e. $|C_{H_i}(v_i)| = 1$. In particular, as $E_i \cong \overline{E_i}$, $v_i$ lies in a regular orbit of $E_i$ on $V_i$, so that with ($\ast\ast$) we obtain $C_G(v_i) = C_{H_i}(v_i) = C_{N_i}(v_i) = N_i$, as desired.

Finally as $N_i \geq N_{i+1}$ and as by Lemma 1.4a) we have $A_i \leq N_i$ and $A_i \not\leq N_{i+1}$, it readily follows that $|N_i|_p > |N_{i+1}|_p$, and the proof of a) is complete.

b) is proved analogously with the help of the results in the Lemmata 1.3 and 1.4 for the case $p = 2$. □

Remember that if $G$ is a group and $V$ a $G$–module such that $V_N$ is homogeneous for every $N \leq G$, then $V$ is called a quasiprimitive $G$–module. In this case the structure of $G$ is well–known (cf. [10, §1]).

As a special case of Theorem 1.5 we get a result on quasiprimitive actions.

**Corollary 1.6.** Let $G$ be a solvable group acting faithfully and quasiprimi-

tively on a vector space $V$ over a finite field. Then there is an $E \leq G$ of type

$E(p, m)$ for a prime $p$ and an integer $m \geq 0$, and the following hold:
a) If \( p \) is odd and \( m \geq 5 \), then \( G \) has at least \( m - 4 \) different nontrivial orbit sizes on \( V \).

b) Let \( p = 2 \) and \( m' \) be defined as in Lemma 1.4. If \( m' \geq 10 \), then \( G \) has at least \( m' - 9 \) different nontrivial orbit sizes on \( V \).

Proof. The existence of \( E \) is clear from [10, Corollary 1.10]. Moreover as \( Z(E) \trianglelefteq G \), \( V_{Z(E)} \) is homogeneous, and so \( Z(E) \) acts fixed point freely on \( V \). Thus by Theorem 1.5 the assertion follows.

2. The main results for groups of odd order.

Theorem 2.1. Let \( G \) be a (solvable) group of odd order that acts faithfully on a vector space \( V \) over a finite field \( K \) of odd characteristic \( q \). Suppose that \( N \trianglelefteq G \) is a \( p \)-group of class 1 or 2 and of exponent \( p \) for a prime \( p \geq 5 \) with \( p \neq q \). Let \( A \leq N \) be an abelian subgroup of \( N \), and put \( s = d(A) \). Then \( G \) has on \( V \) at least \( m := \left\lceil \frac{s}{6} \right\rceil \) orbits, whose sizes have mutually distinct nontrivial \( p \)-parts. In particular, \( G \) has at least \( m \) different nontrivial orbit sizes on \( V \).

Proof. We prove the result by induction on \( |G| + |V| \).

Note that \( A \) is elementary abelian. If \( s \leq 5 \), the assertion is trivial, so let \( s \geq 6 \).

First suppose that \( V_N \) is homogeneous. Then \( V \) is a multiple of a faithful and irreducible \( N \)-module. Hence (by [3, Theorem 3.2.2]) \( Z(N) \) is cyclic, and as \( N' \leq Z(N) \), we see that either \( N' = 1 \) and \( N \) is cyclic of order \( p \) or \( |N'| = |Z(N)| = p \). In the first case our assertion is trivially fulfilled, so we only have to consider the latter. As \( \exp(N) = p \), we also have \( N' = \phi(N) \) (which is the Frattini subgroup of \( N \)). Consequently \( N \) is extra–special and thus of type \( E(p, n) \) for an integer \( n \). Thus \( 6 \leq s \leq n + 1 \), and by Theorem 1.5 \( G \) has on \( V \) at least \( n - 4 \geq s - 5 \geq \left\lceil \frac{s}{6} \right\rceil = m \) orbits, whose sizes have mutually distinct nontrivial \( p \)-parts. So we are done in this case.

It remains to consider the case that \( V_N \) is inhomogeneous.

First assume that \( V \) is not irreducible, i.e. \( V = V_1 \oplus V_2 \) for nontrivial \( G \)-modules \( V_i \) \((i = 1, 2)\). Let \( C = C_G(V_i) \), \( s_1 = d(AC/C) \) and \( m_1 = \left\lceil \frac{s_1}{6} \right\rceil \). Applying induction to the action of \( G/C \) on \( V_1 \) yields \( v_1, \ldots, v_{m_1} \in V_1 \) such that \( |G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{m_1})|_p \). Now let \( H = C_G(v_{m_1}) \), \( D = C_H(V_2) \), \( s_2 = d((A \cap H)D/D) \) and \( m_2 = \left\lceil \frac{s_2}{6} \right\rceil \). This time induction, applied to the action of \( H/D \) on \( V_2 \), yields \( w_1, \ldots, w_{m_2} \in V_2 \) such that \( |H|_p > |C_H(w_1)|_p > \cdots > |C_H(w_{m_2})|_p \). Now put \( v_{m_1 + i} = v_{m_1} + w_i \) \( i = 1, \ldots, m_2 \). Then \( C_G(v_{m_1 + i}) = C_G(v_{m_1}) \cap C_G(w_i) = C_H(w_i) \) \( i = 1, \ldots, m_2 \). Thus we obtain \( |G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{m_1 + m_2})|_p \). Now as \( C_A(V_1) \leq H \) surely acts faithfully on \( V_2 \), we see that \( s_1 + s_2 \geq s \) and consequently \( m_1 + m_2 \geq \left\lceil \frac{s_1}{6} + \frac{s_2}{6} \right\rceil \geq \left\lceil \frac{s}{6} \right\rceil \) which shows that we have established our assertion.

So now we may assume that \( V \) is an irreducible imprimitive \( G \)-module. By [10, Proposition 0.2(ii)] there exists a \( N \leq C \leq G \) such that \( V_C = \)}
\( V_1 \oplus \cdots \oplus V_n \) (with \( n > 1 \)) with \( C \)-invariant \( V_i \) \( (i = 1, \ldots, n) \) that are faithfully and primitively permuted by \( G/C \).

If for any \( M \subseteq \{V_1, \ldots, V_n\} \) we put \( \text{stab}_{G/C}(M) = \{ hC \in G/C \mid M^{hC} = M \} \), then by [13, Theorem 1.3] we may assume that

\[
(*) \quad \bigcap_{j=1}^{3} \text{stab}_{G/C} \{\{V_j\}\} = 1, \text{ i.e. } \bigcap_{j=1}^{3} N_G(V_j) = C.
\]

Now \( H = N_G(V_1) \) is known to be a maximal subgroup of \( G \), and \( V \) is induced from the \( H \)-module \( V_1 \), i.e. if \( G = \bigcup_{j=1}^{[G:H]} Hg_j \) with suitable \( g_j \in G \) (and \( g_1 := 1 \)) is the decomposition of \( G \) into right cosets of \( H \), then \( V = V_1 \otimes_{KH} KG \), and thus (by identifying \( V_1 \) and \( V_1 \otimes 1 \)) we have \( V_i = V_1 \otimes g_i \) for \( i = 1, \ldots, n \).

Next observe that (by [10, Lemma 4.1]) \( v^H \neq (-v)^H \) for all \( v \in V_1 \). Hence there is an \( X \subseteq V_1 \) such that with \( Y := -X \) we have \( V_1 \setminus \{0\} = X \cup Y \) and \( X \cap Y = \emptyset \). Now we define \( B \subseteq V \) as follows: If \( v \in V \) and \( v = \sum_{i=1}^{n} x_i \otimes g_i \) with \( x_i \in V_1 \) for all \( i \), then \( v \in B \) if and only if

1. \( x_i \in X \) for \( i = 1, \ldots, \min\{3, n\} \) and \( x_i \in Y \) for \( i = 4, \ldots, n \), and
2. whenever \( x_i = 0 \) for an \( i < n \), then \( x_{i+1} = 0 \).

Now let \( v = \sum_{i=1}^{n} x_i \otimes g_i \in B \), and define \( S(v) := \bigcap_{i=1}^{n} C_G(x_i \otimes g_i) \). Then we claim that \( S(v) \leq C_G(v) \) such that

\[ (**) \quad |C_G(v)/S(v)| \in \{1, 3\}. \]

To see this, first let \( j \in \{1, \ldots, n\} \) be maximal subject to \( x_j \neq 0 \). If \( j = 1 \), clearly \( C_G(v) = S(v) \), and this is also true for \( j = 2 \), as \( |G| \) is odd. So let \( n \geq j \geq 3 \). Now note that for any \( g \in G \) we have \( v^g \in \bigoplus_{i=1}^{j} x_i^{gH} \otimes g_{\sigma(i)} \), where \( \sigma \in S_n \) is the permutation defined by the permutation action of \( g \) on the \( V_i \), i.e. \( V_i^{gH} = V_{\sigma(i)} \) for \( i = 1, \ldots, n \). So as \( X \) and \( Y \) are \( H \)-invariant, by (*) and our definition of \( B \) we find that if \( g \in C_G(v) \), then \( gC \in T/C := \text{stab}_{G/C}(\{V_1, V_2, V_3\}) \). Now by (*) we have that \( T/C \) is isomorphic to a subgroup of \( S_3 \), so as \( |G| \) is odd, either \( T = C \) or \( |T/C| = 3 \). Furthermore as \( j \geq 3 \) and as naturally \( C_G(x_i \otimes g_i) \leq H^{g_i} = N_G(V_i) \) for all \( i \), (*) also implies that \( S(v) \leq C \). So we conclude that \( S(v) = C \cap C_G(v) \leq C_G(v) \) and further that \( |C_G(v)/S(v)| = |C_G(v)/C/G| = |T/C| \in \{1, 3\} \) whence (***) is shown.

Now we can construct representatives of the orbits that we are looking for. We will do this by an inductive process, namely we put \( v_0 = 0 \in V \), \( t_0 = 0 \) and for all \( i = 1, \ldots, n \) proceed as follows:
(***)

Suppose that we have already \( t_{i-1} \in \mathbb{N} \) and \( v_0, \ldots, v_{t_{i-1}} \in B \) with
\[
|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_{i-1}})|_p,
\]
then let \( H_i = C_G(v_{t_{i-1}}) \) and \( C_i = C_{H_i}(V_i) \). Furthermore let \( s_i = d((A \cap H_i)C_i/C_i) \) and \( m_i = \left\lceil \frac{s_i}{6} \right\rceil \). First suppose that \( m_i \geq 1 \). Apply induction to the action of \( N_{H_i}(V_i)/C_i \) on \( V_i \). As obviously \( C_{H_i}(w) \leq N_{H_i}(V_i) \) for all \( w \in V_i \), this yields elements \( w_{i,1}, \ldots, w_{i,m_i} \in V_i \) such that
\[
|H_i|_p > |C_{H_i}(w_{i,1})|_p > \cdots > |C_{H_i}(w_{i,m_i})|_p.
\]
If \( i \leq 3 \), we choose the \( w_{i,j} (j = 1, \ldots, m_i) \) to be out of \( X \otimes g_i \), if \( i \geq 4 \), we take them out of \( Y \otimes g_i \) (which is obviously possible). Then put \( v_{t_{i-1}+j} = v_{t_{i-1}} + w_{i,j} \in B \) (\( j = 1, \ldots, m_i \)) and put \( t_i := t_{i-1} + m_i \). Hence as \( p \geq 5 \) and because of (**) we conclude
\[
|G|_p > |C_G(v_{t_{i-1}+j})|_p = |S(v_{t_{i-1}+j})|_p = |S(v_{t_{i-1}}) \cap C_G(w_{i,j})|_p = |C_G(v_{t_{i-1}}) \cap C_G(w_{i,j})|_p = |C_{H_i}(w_{i,j})|_p < |H_i|_p
\]
for \( j = 1, \ldots, m_i \). Thus we have
\[
|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_i})|_p.
\]
Now suppose that \( m_i = 0 \). If \( i \leq 3 \), take an arbitrary \( v \in X \otimes g_i \), if \( i \geq 4 \), then take an arbitrary \( v \in Y \otimes g_i \). Put \( t_i = t_{i-1} \) and simply replace \( v_{t_i} \) by \( v_{t_i} + v \). Then still \( v_{t_i} \in B \) and
\[
|G|_p > |C_G(v_1)|_p > \cdots > |C_G(v_{t_i})|_p.
\]
If \( i = n \), then we are done, else we repeat the procedure (***)

So this process yields \( t_n \) orbits on \( V \) the sizes of which have mutually distinct \( p \)-parts \( > 1 \). It remains to show that \( t_n \geq m \). If we show that
\[
t := \sum_{i=1}^{n} s_i \geq s,
\]
then we obtain
\[
t_n = \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} \left\lceil \frac{s_i}{6} \right\rceil \geq \left\lceil \sum_{i=1}^{n} \frac{s_i}{6} \right\rceil \geq \left\lceil \frac{s}{6} \right\rceil = m,
\]
as wanted.

So we have to prove \( t \geq s \). For this let \( 1 \neq a \in A \), and surely it suffices to show that there is an \( i \in \{1, \ldots, n\} \) such that \( a \notin H_i \) and \( a \notin C_i \), i.e. \( 1 \neq aC_i \in (A \cap H_i)C_i/C_i \). To see this note first that if \( a \notin C_1 \), then we are done with \( i = 1 \), as \( a \in P \leq N \leq C \leq H = H_1 \). So let \( a \in C_1 \), and let \( j \in \{1, \ldots, n\} \) be maximal such that \( a \in C_l \) for \( l = 1, \ldots, j \). Surely \( j \leq n - 1 \) because else we would get the contradiction \( 1 \neq a \in \bigcap_{k=1}^{n} C_G(V_k) = 1 \). Then clearly \( a \in H_{j+1} \) and \( a \notin C_{j+1} \). So we choose \( i := j + 1 \). This completes the proof of the theorem.

Notice that Theorem 2.1 is also true for \( p = 3 \) if \( K \) is not too small, more precisely, if besides \( -1 \) there is another \( a \in K^{\times} \) whose order does not divide \( |G| \). Namely then in an obvious way we can define \( B \) in the proof of Theorem 2.1 even such that \( |C_G(v)/S(v)| = 1 \) for all \( v \in B \). Details are left to the reader.

Notice furthermore that obviously with the proof of Theorem 2.1 one even can get elements \( w_1, \ldots, w_m \in V \) whose centralizers have orders with increasing \( 3^k \)-parts (but this is not relevant for our purposes).

In the following for any solvable group \( G \) we shall make use of subgroups
\[
K_i(G) (i \in \mathbb{N} \cup \{0\})
\]
which are defined as follows:
\( K_0(G) = G \), and for \( i \geq 1 \) and given \( K_{i-1}(G) \) let \( K_i(G) \) be the smallest normal subgroup of \( K_{i-1}(G) \) such that \( K_{i-1}(G)/K_i(G) \) is nilpotent.

Then by [5, III, Satz 4.6] we know that the \( K_i(G) \) are characteristic subgroups of \( G \) and that if \( n \) is the minimal number such that \( K_n(G) = 1 \), then \( n = f(G) \). Furthermore, by the definition of the \( K_i \) in [5] it is clear that for any \( N \leq G \) and for all \( i \in \mathbb{N} \) we have

\[
K_i(G/N) = K_i(G)N/N.
\]

Hence for any \( N \leq G \) with \( K_{n-1}(G)N/N > 1 \) we have \( f(G/N) = f(G) \).

**Lemma 2.2.** Let \( G \) be a solvable group, \( p \) a prime and \( H \) a Hall-\( p' \)-subgroup of \( G \). Put \( n = f(G) \) and \( K = K_{n-1}(G) \). Then

\[
dl(H) \geq \begin{cases} 
\frac{f(G)}{2} & \text{if } K \text{ is not a } p\text{-group} \\
\frac{f(G)-1}{2} & \text{if } K \text{ is a } p\text{-group}.
\end{cases}
\]

**Proof.** We prove the result by induction on \( |G| \). It is obviously true if \( G \) is nilpotent. So let \( f(G) \geq 2 \).

First suppose that \( K \) is a \( p\)-group. Then, as \( K_{n-2}(G) \) is not nilpotent, \( K_{n-2}(G/K) = K_{n-2}(G)/K \) is not a \( p\)-group, and thus induction yields \( dl(H) = dl(HK/K) \geq \frac{f(G/K)}{2} = \frac{f(G)-1}{2} \), as wanted.

Hence it remains that \( K \) is not a \( p\)-group. If \( p \mid |K| \), then let \( P \) be a Sylow-\( p\)-subgroup of \( K \). As \( K \) is nilpotent, we have \( P \leq G \), and as \( K/P > 1 \), we have \( f(G/P) = n = f(G) \) and \( K_{n-1}(G/P) = K/P \) is not a \( p\)-group, whence induction yields \( dl(H) = dl(HP/P) \geq \frac{f(G/P)}{2} = \frac{f(G)}{2} \). Hence we may assume that \( K \) is a \( p'\)-group.

Now if there is an \( N \leq G \) with \( 1 < N < K \), then induction yields \( dl(H) \geq \frac{f(G/N)}{2} = \frac{f(G)}{2} \). Thus we may assume that \( K \) is a minimal normal subgroup of \( G \), and as such it is an elementary abelian \( q\)-group for a prime \( q \neq p \).

Let \( k = dl(H) \) and \( T = H^{(k-1)} \) the \((k-1)\)-th commutator subgroup of \( H \), then \( T > 1 \) and \( T \) is abelian. Put \( C := C_G(K) \). Clearly \( K_{n-2}(G) \nleq C \) and thus \( K_{n-2}(G/C) = K_{n-2}(G)/C \geq 1 \) and so \( f(G/C) = n - 1 = f(G) - 1 \). Thus if \( T \leq C \), by induction we conclude \( dl(H) \geq dl(HC/C) + 1 \geq \frac{f(G/C)-1}{2} + 1 = \frac{f(G)-2}{2} + 1 = \frac{f(G)}{2} \), as desired.

Hence we may assume that \( T := TC/C > 1 \). We will lead this to a contradiction. Now as \( K \) may be regarded as a faithful and irreducible \( G/C \)-module over \( GF(q) \), we have that \( T \) is a \( q\)-group of automorphisms of \( K \) (cf. [5, V, Satz 5.17]). Hence by [3, Theorem 5.3.6] for \( 1 < R := [T, K] \leq K \) we have \([T, R] = R \). Going back to \( G \), this obviously means that \( R = [T, K] \) and \([T, R] = R \) (*).

Next we claim that \( R \leq H^{(i)} \) for \( i = 0, \ldots, k-1 \). For \( i = 0 \) this is trivial, and if we already have \( R \leq H^{(i)} \) for an \( i \in \{0, \ldots, k-2\} \), then, as also
Let $G$ be a (solvable) group of odd order that acts faithfully and irreducibly on a vector space $V$ over a finite field of odd characteristic $q$. Put $n = f(G)$ and let $m$ be the number of different orbit sizes of $G$ on $V$. Assume further that $K := K_{n-1}(G)$ is not a 3-group. Then

\[ f(G) \leq 8 \log_2 m + 30. \]

\begin{proof}
We may assume that $n \geq 2$. Let $p \geq 5$ be a prime dividing $|K|$. Then $p \neq q$ (by [5, V, Satz 5.17]). Let $P \in \text{Syl}_p(K)$ and put $C = C_G(P)$. Then $P \leq G$ and $K_{n-2}(G) \leq C$, because else $K_{n-2}(G)/O_{p'}(K)$ would be nilpotent contradicting the definition of $K$. Hence we have $f(G/C) \geq f(G) - 1$. Now let $H$ be a Hall-$p'$-subgroup of $G/C$. Then $H$ acts faithfully on $P$ with $(|H|, |P|) = 1$. Hence by [3, Theorem 5.3.13] there is a characteristic subgroup $N$ of $P$ of class at most 2 and of exponent $p$ such that $H$ acts faithfully on $N$. Note that $N \leq G$. By [3, Theorem 5.1.4] $H$ acts also faithfully on $N/\phi(N)$ (which is of order $p^{d(N)}$) whence by [10, Corollary 3.12(b)] we have $dl(H) \leq 2\log_2 d(N) + 2$. Now by Lemma 2.2 we know that $dl(H) \geq \frac{f(G/C)-1}{2} \geq \frac{f(G)}{2} - 1$, so that we obtain $f(G) \leq 2dl(H) + 2 \leq 4\log_2 d(N) + 6$. As $p$ is odd, by [5, III, Satz 12.3] there is an abelian $A \leq N$ with $d(N) \leq \frac{d(A)(d(A)+1)}{2} \leq d(A)^2$, so that $f(G) \leq 8\log_2 d(A) + 6$. By Theorem 2.1 we have $m \geq \frac{d(A)}{6}$, whence we finally conclude $f(G) \leq 8\log_2 (6m) + 6 \leq 8\log_2 m + 30$, as desired.
\end{proof}

Actually we have not tried to optimize the constants in Theorem 2.3.

**Corollary 2.4.** Let $G$ be a group of odd order. Then

\[ f(G) \leq 8 \log_2 |cd(G)| + 32. \]

\begin{proof}
Put $n = f(G)$, $K = K_{n-1}(G)$ and (in case of $n \geq 2$) $L = K_{n-2}(G)$ and $F = F(L)$, the Fitting subgroup of $L$. First we show:

(*) If $n \leq 3$ or if $n \geq 4$ and $L/F$ is not a 3-group, then $f(G) \leq 8\log_2 |cd(G)| + 31$.

To prove (*), we argue by induction on $|G|$. For $n \leq 3$ there is nothing to prove. Let $n \geq 4$ and let $p \geq 5$ be a prime dividing $|L/F|$, and let $P \in \text{Syl}_p(L)$. Surely $PF/F > 1$, and there is a prime $q \neq p$ such that for $Q_0 \in \text{Syl}_q(K)$ we have $[P, Q_0] > 1$ because else $P$ would centralize $O_{p'}(K)$ and thus $P \times O_{p'}(K) = PK \leq F$ against our choice of $P$. Clearly $Q_0 \leq G$, and with [3, Theorem 5.3.6] we conclude that for $Q := [P, Q_0] \leq Q_0$ we have $[P, Q] = Q > 1$.

\end{proof}
Now if there is a $1 < N \leq G$ such that $QN/N > 1$, then surely we have 
\[ f(G/N) = f(G) \text{ and } QN/N \leq QnN/N \leq F(LN/N) = F(K_{n-2}(G/N)), \]
and as $[PN/N, QN/N] = [P, Q]N/N = QN/N > 1$, it follows that $1 < PN/N \leq LN/N = K_{n-2}(G/N)$ and $PN/N \not\leq F(K_{n-2}(G/N))$. Consequently we see that $K_{n-2}(G/N)/F(K_{n-2}(G/N))$ is not a 3–group whence induction yields $f(G) = f(G/N) \leq 8 \log_2 |cd(G/N)| + 31 \leq 8 \log_2 |cd(G)| + 31$, as desired. Hence we may assume that no such $N$ exists ($+$); in particular, $O_q'(G) = 1$, thus $K = Q_0$ and $F(G)$ is a $q$-group. Now as $PK/K \leq F(G/K)$ and $F(G)/K \leq F(G/K)$, we see that $PF(G)/K$ is nilpotent and hence
\[ 1 = [PK/K, F(G)/K] = [PK, F(G)]/K/K \text{ and therefore } [P, F(G)] \leq K. \]
Thus with [3, Theorem 5.3.6] we see that $Q \leq [P, F(G)] = [P, [P, F(G)]] \leq [P, K] = Q$ and hence $1 < Q = [P, F(G)] \not\leq \phi(F(G))$ (by [3, Theorem 5.3.5]). Thus $Q\phi(F(G))/\phi(F(G)) > 1$, and by (+) we conclude that $\phi(F(G)) = 1$, i.e., $F(G)$ is elementary abelian, and now with (+) we easily see that $f(G) = K = Q$ is a minimal normal subgroup of $G$.

Hence by [5, III, Hilfssatz 4.4] let $H$ be a complement of $K$ in $G$; then $K$ is an irreducible and faithful $H$-module. Now let $V = \text{Irr}(K)$. By [10, Proposition 12.1] $V$ is a faithful and irreducible $H$-module. As by [7, Problem (6.18)] every $\lambda \in V$ can be extended to its inertia group in $G$, we conclude by standard character theory that every orbit size in the action of $H$ on $V$ is an irreducible complex character degree of $G = HK$. So if $m$ is the number of different orbit sizes of $H$ on $V$, then we have $m \leq |cd(G)|$. Now as $K_{n-2}(H) \cong K_{n-2}(G)/K = K_{n-2}(G)/F(K_{n-2}(G))$ is not a 3–group and $f(H) = n - 1$, by Theorem 2.3 we have $f(H) \leq 8 \log_2 m + 30 \leq 8 \log_2 |cd(G)| + 30$, and we get $f(G) \leq 8 \log_2 |cd(G)| + 31$. So (*) is shown.

To prove the corollary, note first that by (*) we may assume that $n \geq 4$ and that $L/F$ is a 3–group. Let $\overline{G} = G/F$. Then $f(\overline{G}) = n - 1$, and surely $K_{n-3}(\overline{G})/F(K_{n-3}(\overline{G})) \cong (K_{n-3}(G)/F)/F(K_{n-3}(G)/F)$ is not a 3–group, because else $K_{n-3}(\overline{G})/O_3'(K_{n-3}(\overline{G}))$ would be a 3–group, hence $K_{n-3}(\overline{G})$ would be isomorphic to a subgroup of $K_{n-3}(\overline{G})/O_3'(K_{n-3}(\overline{G})) \times K_{n-3}(\overline{G})/K_{n-2}(\overline{G})$ and as such be nilpotent against $f(\overline{G}) = n - 1$. Hence application of (*) to $G/F$ yields $f(G) = f(G/F) + 1 \leq 8 \log_2 |cd(G/F)| + 31 + 1 \leq 8 \log_2 |cd(G)| + 32$, and we are done. \[\square\]

The result in Corollary 2.4 is asymptotically best possible, as the following trivial example shows (which also can be used to see that Theorem 2.3 is asymptotically best possible, too).

**Example 2.5.** Let $A, B$ be groups of order 3 and 5, respectively. For $i \in \mathbb{N}$ we let $Z_i = A$ if $i$ is even, and $Z_i = B$ if $i$ is odd, and for $n \in \mathbb{N}$ we define $G_n = Z_1 \ast \cdots \ast Z_n$ to be iterated (regular) wreath products of the $Z_i$. Then obviously $f(G_n) = n$, and if $t$ is the number of divisors of $|G_n|$, then clearly
\[ |cd(G_n)| \leq t. \] Now if \( B_n = B \cdot \cdots \cdot B \) (with \( n \) factors), then it is well-known that \( |B_n| = 5^s \) with \( s = \sum_{i=0}^{n-1} 5^i = \frac{1}{4}(5^n - 1) \leq 5^n - 1. \)

Now \( |G_n| \leq |B_n| \leq 5^{5n-1} \) and thus obviously \( |G_n| \mid (3 \cdot 5)^{5n-1} \). This implies that \( t \leq 5^{2n} \). Thus we have
\[
f(G_n) = n \geq \frac{1}{2} \log_5 t \geq \frac{1}{2} \log_5 |cd(G_n)|,
\]
whence asymptotically Corollary 2.4 is the right bound.

3. An application to modular character theory.

In this section we want to discuss briefly a modification of results obtained by Isaacs in [8] on orbit sizes and modular character degrees. Because of the hypotheses in Theorem 2.1, we have to restrict ourselves to groups of odd order and to primes greater than 3. In this case, for groups whose \( p \)-Sylow subgroups have bounded exponent, we can asymptotically improve the bounds obtained in [8] considerably. The reason why we need that bounded exponent is that with Theorem 2.1 we only get many different orbit sizes of a \( p \)-group \( P \) acting on a vector space, if \( d(A) \) is large for some abelian subgroup of \( P \). But, as well-known, there are \( p \)-groups of arbitrary large derived length all of whose abelian normal subgroups are generated by only three elements (cf. [4]).

**Theorem 3.1.** Let \( G \) be a (solvable) group of odd order and \( p \geq 5 \) be a prime. Suppose that \( G \) acts on an abelian \( p' \)-group \( V \) of odd order, and let \( b \) be the number of different \( p \)-parts among the orbit sizes of \( G \) on \( V \). Let \( P = O_p(G) \) and \( \exp(P) = p^k \). Then
\[
dl(P/C_P(V)) \leq 2 \log_2 b + 2 \log_2 e + 8.
\]

**Proof.** As in the proof of [8, Theorem (2.1)] we see that we may assume that \( V \) is elementary abelian and a faithful and irreducible \( G \)-module. Now let \( k = dl(P) \). If \( P \) is abelian, there is nothing to prove, hence let \( k \geq 2 \).

Now let \( P^{(k-1)} \leq D \) be a maximal characteristic abelian subgroup of \( P^{(k-2)} \). Then we define subgroups \( C, A, M \) of \( G \) as follows:

If \( D \) is also a maximal normal abelian subgroup of \( P^{(k-2)} \), then let \( C = D = M \) and \( A = \Omega_1(M) \leq C \); in particular \( d(A) = d(M) \).

If not, then let \( D < M \leq P^{(k-2)} \) be a maximal normal abelian subgroup of \( P^{(k-2)} \). Thus \( M \leq C_{P^{(k-2)}}(D) =: H. \) By the choice of \( D \) clearly \( H/D > 1 \) is abelian, so let \( D \leq C \leq P^{(k-2)} \) such that \( C/D = H/D \cap \Omega_1(P^{(k-2)}/D) = \Omega_1(H/D) \). Then \( \Omega_1(M/D) \leq C/D \) and hence obviously \( A := \Omega_1(M) \leq C \) and \( d(A) = d(M) \).

This shows that in any case there is a subgroup \( C \) of \( P^{(k-2)} \) such that \( \Omega_1(C) \) contains an abelian subgroup \( A \) with \( d(A) = d(M) \) for some maximal abelian normal subgroup \( M \) of \( P^{(k-2)} \). Now observe that by our construction
of $C$ and the proof of [3, Theorem 5.3.12] we see that $C$ is a critical subgroup of $P^{(k-2)}$ in the sense of [3]; so by the proof of [3, Theorem 5.3.13] we know that $\Omega_1(C) \leq G$ is a $p$-group of class 1 or 2 and of exponent $p$ which contains the abelian subgroup $A$. Hence we may apply 2.1 to the action of $G$ on $V$ which yields that

$$b \geq \frac{1}{6} d(A).$$

Now by the proof of [5, III, Satz 7.11] we have $|P^{(k-2)}| = p^n$ with $n \geq 2^{k-2} + 2$, so that with [5, III, Satz 7.3b)] we have $|M| = p^a$ with $2a^2 \geq a(a + 1) \geq 2n \geq 2^{k-1} + 4$ whence we conclude that $a \geq 2^{(k-2)/2}$ and consequently $d(A) \geq a/e \geq e^{-1} 2^{(k-2)/2}$. Thus altogether we have

$$b \geq \frac{1}{6e} 2^{k-2}$$

from which we easily deduce the assertion of the theorem.

The following corollary is the counterpart of Theorem B in [8].

**Corollary 3.2.** Let $p \geq 5$ be a prime and $P$ be a $p$-group that acts faithfully on an abelian $p'$-group $V$ of odd order. Let $\exp(P) = p^e$ and $b$ be the number of different orbit sizes in the action of $P$ on $V$. Then

$$dl(P) \leq 2 \log_2 b + 2 \log_2 e + 8.$$

**Proof.** This follows from Theorem 3.1 with $G = P$. $\square$

Now we can also derive a counterpart of Theorem A of [8].

**Theorem 3.3.** Let $G$ be a (solvable) group of odd order and $p \geq 5$ be a prime. Let $cd_p(G)$ denote the set of the degrees of the irreducible Brauer characters at the prime $p$. Put $c = |cd_p(G)|$ and let $p^e$ be the exponent of a $p$-Sylow subgroup of $G$. Then

$$dl(G/O_p(G)) \leq 3 (c - 1) (2 \log_2 c + 2 \log_2 e + 8) + 1.$$

**Proof.** The proof runs completely the same as the proof of Theorem A in [8], i.e. first one proves a counterpart of [8, Theorem (3.2)] by using Theorem 3.1 instead of [8, Theorem (2.1)], and then one completes the proof with the help of [8, Theorem (3.3)]. $\square$

**Acknowledgments.** This paper was written while I had a visiting position and later was a Research Fellow, supported by a DFG-fellowship, at Ohio University. I would like to thank the Department of Mathematics for its hospitality and T.R. Wolf, G. Navarro and D.S. Passman for stimulating discussions on the subject. I am also grateful to the Deutsche Forschungsgemeinschaft for granting me the research fellowship.
References


Received August 27, 1997.

University of Leicester
Leicester, LE1 7RH
England
E-mail address: tmk99@mcs.le.ac.uk
MAXIMALITY OF THE MICROSTATES FREE ENTROPY FOR R-DIAGONAL ELEMENTS

ALEXANDRU NICA, DIMITRI SHLYAKHTENKO, AND ROLAND SPEICHER

A non-commutative non-self adjoint random variable $z$ is called $R$-diagonal, if its $*$-distribution is invariant under multiplication by free unitaries: if a unitary $w$ is $*$-free from $z$, then the $*$-distribution of $z$ is the same as that of $wz$. Using Voiculescu’s microstates definition of free entropy, we show that the $R$-diagonal elements are characterized as having the largest free entropy among all variables $y$ with a fixed distribution of $y^*y$. More generally, let $Z$ be a $d \times d$ matrix whose entries are non-commutative random variables $X_{ij}$, $1 \leq i, j \leq d$. Then the free entropy of the family $\{X_{ij}\}_{ij}$ of the entries of $Z$ is maximal among all $Z$ with a fixed distribution of $Z^*Z$, if and only if $Z$ is $R$-diagonal and is $*$-free from the algebra of scalar $d \times d$ matrices. The results of this paper are analogous to the results of our paper [3], where we considered the same problems in the framework of the non-microstates definition of entropy.

1. Introduction.

Let $(M, \tau)$ be a tracial non-commutative $W^*$-probability space. A (non-self-adjoint) element $z \in M$ is called $R$-diagonal if its $*$-distribution is invariant under multiplication by free unitaries; i.e., if $w$ is a unitary, $*$-free from $z$, the $*$-distributions of $wz$ and $z$ coincide. The concept of $R$-diagonality was introduced in [4], where it was shown to be equivalent to several conditions; we mention that if $z^*z$ has a (possibly unbounded) inverse (in particular, if the distribution of $z^*z$ is non-atomic), then $z$ is $R$-diagonal if and only if in its polar decomposition $z = u(z^*z)^{1/2}$, $u$ is $*$-free from $(z^*z)^{1/2}$ and satisfies $\tau(u^k) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$.

In our recent paper [3] $R$-diagonal elements appeared in connection with certain maximization problems in free entropy. Free entropy was introduced by Voiculescu in [8]; later, a different definition was given by him in [10]. The first definition involves approximating the given $n$-tuple of variables using finite-dimensional matrices (so-called microstates); the normalized limit of the logarithms of volumes of all such possible microstates is then the free entropy. On the other hand, Voiculescu’s definition in [10] does not involve
microstates, but uses free Fisher information measure and non-commutative Hilbert transform. At present it is not known whether the two definitions of free entropy always give the same quantity. Our approach in [3] used the second definition of Voiculescu.

In this paper we prove two theorems for the microstates free entropy, which are analogous to our results in [3] for the second (non-microstates) definition of entropy. One of our results can be interpreted as saying that $R$-diagonal elements $z$ are characterized by the statement that the free entropy $\chi(z)$ is maximal among all possible $\chi(y)$, so that the distributions of $y^*y$ and $z^*z$ are the same.

When this paper was almost finished we received a preprint of Hiai and Petz [1], where the same kind of problems were considered.

If $Y_1,\ldots,Y_n \in M$ (not necessarily self-adjoint), we denote by $\chi(Y_1,\ldots,Y_n)$ the free entropy of $Y_1,\ldots,Y_n$ as defined by Voiculescu in [11]. We denote by $\chi^{s\alpha}(X_1,\ldots,X_n)$ for $X_i \in M$ self-adjoint the free entropy of a self-adjoint $n$-tuple as defined in [8]; we give a brief review of these quantities below in §2.3. A unitary $u$ in a non-commutative probability space $(M,\tau)$ is called a Haar unitary if $\tau(u^k) = 0$ for all $k \in \mathbb{Z} \setminus \{0\}$.

**Theorem 1.** Let $y \in M$, and let $u \in M$ be a Haar unitary which is *-free from $b = (y^*y)^{1/2}$. Let $x$ be an element such that $\tau(x^{2k}) = \tau(b^{2k})$ and $\tau(x^{2k+1}) = 0$, for all $k \in \mathbb{N}$ (i.e., $x$ is symmetric). Then:

(a) $\chi(y) \leq \chi(ub)$.

(b) $\chi(ub) = \chi^{s\alpha}(b^2/2) + 3/4 + 1/2 \log 2\pi = 2\chi^{s\alpha}(2^{-1/2}x)$.

(c) If $\chi(y) = \chi(ub) > -\infty$, then $y$ is $R$-diagonal, i.e., in the polar decomposition $y = vb$ we have: $v$ is a Haar unitary and is *-free from $b$.

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter; i.e., a homomorphism from the algebra $C(\mathbb{N})$ of all bounded (continuous) functions on $\mathbb{N}$ to $\mathbb{C}$, which is not given by the evaluation at a point in $\mathbb{N}$. For $d \in \mathbb{N}$ we write $d\omega$ for the free ultrafilter corresponding to the functional $f \mapsto \lim_{n \to +\omega} f(dn)$. Given $\omega$, one can construct (see [11] and see also a brief review below) free entropy quantities $\chi^{s\alpha\omega}$ and $\chi^\omega$, which have properties similar to those of $\chi^{s\alpha}$ and $\chi$; it is in fact not known whether these quantities are different. It is known that in the one-variable case, $\chi^{s\alpha}(X) = \chi^{s\alpha\omega}(X)$.

**Theorem 2.** Let $X_{ij}, 1 \leq i,j \leq d$ be a family of non-commutative random variables in a tracial non-commutative probability space $(M,\hat{\tau})$. Let $Z \in M \otimes M_d$ be given by

$$Z = \sum_{i,j=1}^d X_{ij} \otimes e_{ij},$$

where $e_{ij}$ are matrix units in the algebra of $d \times d$ matrices. We denote by $\tau$ the normalized trace on $M \otimes M_d$. Let $\omega$ be a free ultrafilter. Let $X$
be a self-adjoint variable with \( \tau(X^{2n+1}) = 0 \) for all \( n \in \mathbb{N} \), and such that \( \tau(X^{2n}) = \tau((Z^*Z)^n) \), \( \forall n \in \mathbb{N} \). Then we have:

(a) \( \chi^\omega(\{X_{ij}\}_{1 \leq i,j \leq d}) \leq d^2 \chi^{sa}(Z) - d^2 \log d \leq 2d^2 \chi^{sa}(2^{-\frac{1}{n}}X) - d^2 \log d. \)

(b) If \( Z \) is \( R \)-diagonal and \( * \)-free from the algebra \( 1 \otimes M_d \), then \( \chi^{da}(\{X_{ij}\}_{1 \leq i,j \leq d}) = d^2 \chi^{sa}(Z) - d^2 \log d = 2d^2 \chi^{sa}(2^{-\frac{1}{n}}X) - d^2 \log d. \)

(c) If \( \chi^\omega(\{X_{ij}\}_{1 \leq i,j \leq d}) = 2d^2 \chi^{sa}(2^{-\frac{1}{n}}X) - d^2 \log d \) and \( \chi^{sa}(2^{-\frac{1}{n}}X) \neq -\infty \), then \( Z \) is \( R \)-diagonal and is \( * \)-free from the algebra \( 1 \otimes M_d \).

The proof of the first theorem is quite different in nature than our proof in [3] (the microstates-free proof relied on the notion of free entropy with respect to a completely-positive map introduced in [6]). On the other hand, the proof of the second theorem is analogous to the one we gave in [3], and relies on the microstates analog [5] of the relative entropy [10] that we used in the microstates-free approach.

### 2. Maximality of microstates free entropy for \( R \)-diagonal pairs.

Let \( (M, \tau) \) be a tracial \( W^* \)-probability space, and \( b \in M \) be a fixed positive element. Let \( u \in M \) be a Haar unitary which is \( * \)-free from \( b \). Lastly, let \( x \in M \) be such that for all \( k \in \mathbb{N} \), \( \tau(x^{2k+1}) = 0 \) and \( \tau(x^{2k}) = \tau(b^{2k}) \). The main result of the section is:

**Theorem 2.1.** Let \( u \), \( b \) and \( x \) be as above. Assume that \( y \in M \) satisfies \( (y^*y)^{1/2} = b \). Then:

(a) \( \chi(y) \leq \chi(ub) \).

(b) \( \chi(u\bar{b}) = \chi^{sa}(b^2/2) + 3/4 + 1/2 \log 2\pi = 2\chi^{sa}(2^{-\frac{1}{n}}x) \).

(c) If \( \chi(y) = \chi(ub) > -\infty \), then \( y \) is \( R \)-diagonal, i.e., in the polar decomposition \( y = vb \), we have: \( v \) is a Haar unitary and is \( * \)-free from \( b \).

The same conclusions hold for \( \chi^\omega \) in place of \( \chi \).

Before starting the proof of the theorem, we fix some notation and definitions.

**Notation 2.2.** We use the following notation:

- \( U(k) \) is the unitary group of \( k \times k \) unitary matrices.
- \( M_k \) is the set of all \( k \times k \) matrices; \( M_k^{sa} \) is the set of all self-adjoint matrices in \( M_k \).
- \( M_k^+ \subset M_k \) is the set of all positive \( k \times k \) matrices.
- \( \mu_k \) is the normalized Haar measure on \( U(k) \); thus \( \mu_k(U(k)) = 1 \).
- \( \lambda_k \) is the measure on \( M_k \), coming from its Euclidean structure \( \langle a, b \rangle = \text{Re } \text{Tr}(ab^*) \), where \( \text{Tr} \) is the usual matrix trace, \( \text{Tr}(I) = k \); \( \lambda_k^{sa} \) is the Lebesgue measure on \( M_k^{sa} \) coming from its Euclidean structure \( \langle a, b \rangle = \text{Re } \text{Tr}(ab^*) \).
• \( \lambda^+_k \) is the measure on \( M^+_k \) coming from its structure of a cone in the Euclidean space of \( k \times k \) matrices.

• \( P : U(k) \times M^+_k \to M_k \) is given by \( (v, p) \mapsto vp \).

• \( \Omega_k \) is the canonical volume form on \( M_k \) giving rise to Lebesgue measure.

• \( \Omega_k^\vee \wedge \Omega_k^+ \) is the canonical volume form on \( U(k) \times M^+_k \), giving rise to the product measure \( \mu_k \times \lambda^+_k \).

• \( u(k) \) is the Lie algebra of \( U(k) \).

• \( C_k \) is the volume of \( U(k) \) with respect to the bi-invariant volume form arising from the Euclidean structure on \( u(k) \) coming from the Killing form \( \langle a, b \rangle = \text{Re} \, \text{Tr}(ab) \).

2.3. Definitions of free entropy. Let \( X_1, \ldots, X_n \in M \) be self-adjoint, and \( Y_1, \ldots, Y_n \in M \) be not necessarily self-adjoint. Let \( \epsilon > 0, R > 0 \) be real numbers and \( k > 0, m > 0 \) be integers. Then define the sets (cf. [8, 11])

\[
\Gamma_R^{sa}(X_1, \ldots, X_n; m, k, \epsilon) = \left\{ (x_1, \ldots, x_n) \in (M^{sa}_k)^n : \left| \frac{1}{k} \text{Tr}(x_{i_1} \ldots x_{i_p}) - \tau(X_{i_1} \ldots X_{i_p}) \right| < \epsilon \right. \\
\text{for all } p \leq m, 1 \leq i_j \leq n, 1 \leq j \leq p \};
\]

\[
\Gamma_R(Y_1, \ldots, Y_n; m, k, \epsilon) = \left\{ (y_1, \ldots, y_n) \in (M_k)^n : \left| \frac{1}{k} \text{Tr}(y_{i_1}^{g_1} \ldots y_{i_p}^{g_p}) - \tau(Y_{i_1}^{g_1} \ldots Y_{i_p}^{g_p}) \right| < \epsilon \right. \\
\text{for all } p \leq m, 1 \leq i_j \leq n, g_j \in \{*, \cdot\}, 1 \leq j \leq p \}.
\]

Define next

\[
\chi^{sa}(X_1, \ldots, X_n; m, \epsilon) = \limsup_{k \to \infty} \left[ \frac{1}{k^2} \log \lambda_k \Gamma_R^{sa}(X_1, \ldots, X_n; m, k, \epsilon) + \frac{n}{2} \log k \right]
\]

and similarly

\[
\chi(Y_1, \ldots, Y_n; m, \epsilon) = \limsup_{k \to \infty} \left[ \frac{1}{k^2} \log \lambda_k \Gamma_R(Y_1, \ldots, Y_n; m, k, \epsilon) + n \log k \right].
\]

For \( \omega \) a free ultrafilter on \( \mathbb{N} \), the quantities \( \chi^\omega(Y_1, \ldots, Y_n; m, \epsilon) \) and \( \chi^{sa\omega}(X_1, \ldots, X_n; m, \epsilon) \) are defined in exactly the same way, except that
\[
\limsup_{k \to \infty} \text{ is replaced by } \lim_{k \to \omega}. \quad \text{Next, the free entropy is defined by}
\]
\[
\chi_{\text{sa}}(X_1, \ldots, X_n) = \sup_{R} \inf_{m, \epsilon} \chi_{\text{sa}}(X_1, \ldots, X_n; m, \epsilon);
\]
the quantities \(\chi_{\text{sa}}\), \(\chi\), \(\chi^\omega\) are defined in exactly the same way, using in the place of \(\chi_{\text{sa}}(\ldots ; m, \epsilon)\) the quantities \(\chi_{\text{sa}}(\ldots ; m, \epsilon)\), \(\chi(\ldots ; m, \epsilon)\), and \(\chi^\omega(\ldots ; m, \epsilon)\), respectively.

**Definition 2.4.** Let \((X_R(k, m, \epsilon), \mu_{X_R(k, m, \epsilon)})\) and \((Y_R(k, m, \epsilon), \mu_{Y_R(k, m, \epsilon)})\) be two sequences of measure spaces depending on \(k, m \in \mathbb{N}\) and \(R, \epsilon \in (0, +\infty)\). We shall say that \(X\) is asymptotically included in \(Y\), if for all \(m, \epsilon, R, \epsilon'\) there is \(k_0, m' \geq m, \epsilon' \leq \epsilon, R' > R\), such that for all \(k > k_0\), there is a map
\[
\phi = \phi_{R', k, m', \epsilon'} : X_{R'}(k, m', \epsilon') \to Y_R(k, m, \epsilon),
\]
which is measure preserving. We say that \(X\) and \(Y\) are asymptotically equal, if both \(X\) is asymptotically included in \(Y\) and \(Y\) is asymptotically included in \(X\).

**Remark 2.5.** Note that if \(X\) is asymptotically included into \(Y\), we obtain
\[
\sup_{R} \inf_{m, \epsilon} \limsup_{k} \alpha_k \log \mu_{X_R(k, m, \epsilon)}(X_R(k, m, \epsilon)) + a_k
\]
\[
\leq \sup_{R} \inf_{m, \epsilon} \limsup_{k} \alpha_k \log \mu_{Y_R(k, m, \epsilon)}(Y_R(k, m, \epsilon)) + a_k,
\]
for all sequences \(a_k, \alpha_k\).

It is not hard to see that the sets
\[
\Gamma_R(Y_1, \ldots, Y_n; k, m, \epsilon)
\]
and
\[
\Gamma_{R}(\text{Re}(Y_1), \text{Im}(Y_1), \ldots, \text{Re}(Y_n), \text{Im}(Y_n); k, m, \epsilon)
\]
are asymptotically equal; the relevant maps \(\phi\) send the \(n\)-tuple \((y_1, \ldots, y_n)\) of non-self-adjoint matrices to the \(2n\)-tuples of self-adjoint matrices \((\text{Re}(y_1), \text{Im}(y_1), \ldots, \text{Re}(y_n), \text{Im}(y_n))\). This implies (using the Remark 2.5) that
\[
\chi(Y_1, \ldots, Y_n) = \chi_{\text{sa}}(\text{Re}(Y_1), \text{Im}(Y_1), \ldots, \text{Re}(Y_n), \text{Im}(Y_n)).
\]

We proceed to prove several lemmas that will be used in the proof of the main theorem.

**Lemma 2.6.** Let \(\Gamma \subset M_k^+\) and \(U_k \subset U(k)\) be measurable sets. Let
\[
U_k \Gamma = \{vp : v \in U_k, p \in \Gamma\} \quad \text{and} \quad S(\Gamma) = \left\{ \frac{p^2}{2} : p \in \Gamma \right\}.
\]
Then
\[
\lambda_k(U_k \Gamma) = C_k \mu_k(U_k) \lambda_k^+(S(\Gamma)).
\]
In other words, the map $Q : (v,p) \mapsto v \sqrt{2p}$ from $U(k) \times M_k^+$, endowed with the measure $\mu_k \times C_k \lambda_k^+$, to $M_k$, endowed with the measure $\lambda_k$, is measure preserving.

**Proof.** Since invertible matrices are a set of comeasure zero in $M_k$, we see by existence of polar decomposition that $P : (v,p) \mapsto vp$ is invertible as a map of measure spaces. We start by computing the pull-back of Lebesgue measure on $M_k$ to $U(k) \times M_k^+$. Note that since $P$ is equivariant with respect to the actions of $U(k)$ by left multiplication, and Lebesgue measure is invariant under this action (since the Euclidean structure is), the resulting measure on $U(k) \times M_k^+$ is the product of Haar measure on $U(k)$ and some measure $\nu_k$ on $M_k^+$, hence $\lambda_k(U_k \Gamma) = \mu_k(U_k) \nu_k(\Gamma)$. It remains to identify $\nu_k$.

We have the equation

$$d\mu_k(v) d\nu_k(p) = (P^* (\Omega_k) : \Omega_k^u \wedge \Omega_k^e) d\mu_k(v) d\lambda_k^+(p),$$

where $P^* (\Omega_k) : \Omega_k^u \wedge \Omega_k^e$ is the ratio of the two volume forms. Furthermore, in view of the mentioned invariance under an action of $U(k)$, it is sufficient to compute $(P^* (\Omega_k) : \Omega_k^u \wedge \Omega_k^e)$ in (1) at the point $(1, p) \in U(k) \times M_k^+$.

Note that the tangent space $T_{1,p}(U(k) \times M_k^+)$ is isomorphic to the direct sum $u(k) \times M_k^{sa}$, where $u(k) = iM_k^{sa}$ is the Lie algebra of $U(k)$. Identify $T_{1,p}(U(k) \times M_k^+) = iM_k^{sa} \oplus M_k^{sa}$ with $M_k = T_p(M_k)$. Then the inner product given by the trace $\langle a, b \rangle = \text{Re} \text{Tr}(ab^*)$ defines on $T_{1,p}$ a Euclidean structure, for which the subspaces $M_k^{sa}$ and $iM_k^{sa}$ are perpendicular. Since the restriction of this inner product to $u(k)$ is the Killing form on this Lie algebra, and the restriction to $T_p M_k^+$ is the inner product we chose before on this space, $\Omega_k$ (which via the above identification is a volume form on $U(k) \times M_k^+$) has the form $C_k \Omega_k^u \wedge \Omega_k^e$. Further, $C_k$ is the ratio of the volume form on $U(k)$ arising from the Euclidean structure on $u(k)$ coming from the Killing form and the volume form corresponding to the normalized Haar measure. Hence $C_k$ is just the volume of $U(k)$ with respect to the volume form arising from the Euclidean structure on $u(k)$ coming from the Killing form.

Thus from (1) we get that

$$d\nu_k(p) d\mu_k(v) = C_k d\mu_k(v) \text{det}(DP)(p) d\lambda_k^+(p).$$

It remains to compute $DP$. We note that $P$ is the identity map restricted to $M_k^+$. Choose a basis in which $p$ is diagonal with eigenvalues $l_1, \ldots, l_k$, and let $e_{ij} \in M_k$ be the matrix all of whose entries are zero, except that the $i, j$-th entry is 1. Consider the orthonormal basis $\xi_{\alpha\beta}$ for $iM_k^{sa}$, given by:

$$\xi_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{2}}(e_{\alpha\beta} - e_{\beta\alpha}) & \text{if } \alpha < \beta \\ ie_{\alpha\alpha} & \text{if } \alpha = \beta \\ i\frac{1}{\sqrt{2}}(e_{\alpha\beta} + e_{\beta\alpha}) & \text{if } \alpha > \beta. \end{cases}$$
Then
\[ DP(\xi_{\alpha\beta})p = \xi_{\alpha\beta}p = \frac{1}{2}(l_\alpha + l_\beta)\xi_{\alpha\beta} + \frac{1}{2}(l_\alpha - l_\beta)\eta_{\alpha\beta}, \quad \eta_{\alpha\beta} \in M_k^{sa}. \]

It follows that
\[ \det(DP)(p) = \frac{1}{2^k} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta). \]

Hence we record the final answer:
\[ d\nu_k(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta) d\lambda_k^+(p) \]

where \( l_i \) are the eigenvalues of \( p \).

Consider the map \( S : p \mapsto \frac{p^2}{2} \) from \( M_k^+ \) to itself. This map is a.e. invertible; moreover, its Jacobian \( \det(DS) \) at \( p \) is given by \( \det(\frac{1}{2}(1 \otimes p + p \otimes 1)) \), where \( 1 \otimes p \) and \( p \otimes 1 \) are viewed as elements of \( M_k \otimes M_k \cong M_k^2 \) (see e.g. [8]). To compute this determinant, let \( \zeta_i, i = 1, \ldots, k \) be orthonormal eigenvectors of \( p \), such that \( p\zeta_i = l_i \zeta_i \). Then \( \zeta_i \otimes \zeta_j \) is an orthonormal basis for \( C_k^2 \), on which \( M_k^2 \) acts naturally. Moreover, \( \frac{1}{2}(1 \otimes p + p \otimes 1)(\zeta_i \otimes \zeta_j) = \frac{1}{2}(l_i + l_j)\zeta_i \otimes \zeta_j \). So the determinant is \( 2^{-k^2} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta) \). Hence the push-forward of \( \nu_k \) by \( S \) is given by
\[ d(S_*\nu_k)(p) = C_k 2^{-k^2} \prod_{\alpha,\beta=1}^{k} (l_\alpha + l_\beta) d\lambda_k^+(p) \cdot \det(DS)^{-1}(p) = C_k d\lambda_k^+(p). \]

Thus we have
\[ S_*\nu_k = C_k \lambda_k^+, \]

which is our assertion. \( \square \)

We have the following standard lemma (see [8]).

**Lemma 2.7.** Let \( p \) be a positive element in \( M \). Then the sequences of sets \( \Gamma_{R^+}(p, m, k, \epsilon) \) and \( \Gamma_{R^+}(p, m, k, \epsilon) \cap M_k^+ \), each taken with the measure \( \lambda_k \), are asymptotically equal.

**Lemma 2.8.** \( \lim_k \frac{1}{k^2} \log(C_k) + \frac{1}{2} \log k = \frac{3}{4} + \frac{1}{2} \log 2\pi. \)

In this exact form this lemma can be found, for example, in [2] (the reader is cautioned that the cited paper uses a slightly different normalization of the Killing form, different from ours by a factor).

**Lemma 2.9.** Let \( y \in (M, \tau) \) be a (not necessarily self-adjoint) random variable. Then
\[ \chi(y) \leq \chi^{sa} \left( \frac{y^*y}{2} \right) + \frac{3}{4} + \frac{1}{2} \log 2\pi. \]
Proof. Denote by $S : M_k \to M_k^+$ the map

$$y \mapsto \frac{y^* y}{2}.$$  

Note that

$$S(\Gamma_R(y; m, k, \epsilon)) \subset \Gamma_{R^2}^\times \left(\frac{y^* y}{2}; m/2, k, \epsilon\right),$$

hence the former is asymptotically included in the latter. Note that

$$\Gamma_R(y; m, k, \epsilon) \subset U(k)\, \Gamma_R(y; m, k, \epsilon).$$

We therefore get

$$\lambda_k(\Gamma_R(y; m, k, \epsilon)) \leq \lambda_k(U(k)\, \Gamma_R(y; m, k, \epsilon)) \leq \lambda_k(U(k)\, \{(a^* a)^{1/2} : a \in \Gamma_R(y; m, k, \epsilon)\}) \leq C_k \lambda_k(S(\Gamma_R(y; m, k, \epsilon))) \leq C_k \lambda_k \left(\Gamma_{R^2}^\times \left(\frac{y^* y}{2}; m/2, k, \epsilon\right)\right).$$

Taking the logarithm and passing to the limits gives the result. \hfill \Box

**Lemma 2.10.** Let $u, b \in (M, \tau)$ be such that $u$ is a Haar unitary $*$-free from the positive element $b$. Let $z = ub$. Given $\delta > 0$, there exists $k_0$, such that for all $k > k_0$, there is a subset $X_k \subset U(k) \times (\Gamma_{R^2}^\times (\frac{z^* z}{2}; m/2, k, \epsilon) \cap M_k^+)$,

$$\log \frac{\mu_k \times \lambda_k^+(X_k)}{\mu_k \times \lambda_k^+(U_k \times \Gamma_{R^2}^\times (\frac{z^* z}{2}; m/2, k, \epsilon))} \geq -\delta,$$

such the map

$$Q : (v, p) \mapsto v\sqrt{2p}$$

is an asymptotic inclusion of $X_k$, endowed with the measure $\mu_k \times C_k \lambda_k^+$, into $\Gamma_R(z; m, k, \epsilon)$, endowed with the measure $\lambda_k$.

Proof. Note that by Lemma 2.6, the map defined in Equation (2) is measure preserving.

Let $R > 0$, $\epsilon > 0$ and $\delta > 0$ be fixed. For $x \in M_k^+$, let $U_k(x, \epsilon) \subset U(k)$ be the maximal set of unitaries, for which $U_k(x, \epsilon) \cdot x \in \Gamma_R(wx; m, k, \epsilon)$, where $w$ is a Haar unitary $*$-free from $x$ (in other words, “elements of $U_k(x, \epsilon)$ and $x$ are $*$-free to order $m, \epsilon$”). Note that $U_k(x, \epsilon)$ is open. By Corollary 2.12 of [11], there exists $k_0$, such that for all $k > k_0$, and any $x \in M_k^+$, $\|x\| < R$, $\log \mu_k(U_k(x, \epsilon/2)) > -\delta$. Let

$$\hat{X}_k = \bigcup_{x \in \Gamma_R^\times (\frac{z^* z}{2}; m, k, \epsilon) \cap M_k^+} U_k(x, \epsilon) \times \{x\}.$$  

Since whenever $x \in \Gamma_R^\times (\frac{z^* z}{2}; m, k, \epsilon) \cap M_k^+$, $U_k(x) \cdot \sqrt{2x} \subset \Gamma_R(z; m, k, \epsilon)$, $Q(\hat{X}_k)$ lies in $\Gamma_R(z; m, k, \epsilon)$. 

\hfill \Box
We claim that there exists a measurable subset $X_k \subset \hat{X}_k$ of measure at least $\exp(-\delta)$ times that of $\Gamma_R^{sa}(z^2; m, k, \epsilon)$. First, it is sufficient to show (because of Lemma 2.7) that the measure of $X_k$ is at least $\exp(-\delta)$ times that of $\Gamma_R^{sa}(z^2; m, k, \epsilon) \cap M_k^+$. Next, let $x \in \Gamma_R^{sa}(z^2; m, k, \epsilon) \cap M_k^+$, and let $V(x)$ be an open neighborhood of $x$ for the norm topology. Then if $V(x)$ is sufficiently small, for all $x' \in V$, $U_k(x, \epsilon/2) \subset U_k(x', \epsilon)$. Hence

$$O(x) = U_k(x, \epsilon/2) \times V(x) \subset \hat{X}_k.$$  

Moreover, the volume of $O(x)$ is at least $\exp(-\delta)$ times the volume of $V(x)$. Let

$$X_k = \bigcup_x O(x).$$

Then $X_k$ is open, and its volume is at least $\exp(-\delta)$ times that of $\Gamma_R^{sa}(z^2; m, k, \epsilon) \cap M_k^+$. □

**Proof of 2.1(a) and 2.1(b) in Theorem 2.1.** Assume that $x$, $u$ and $b$ are as in the statement of Theorem 2.1(b) and let $z = ub$; note that $z$ is $R$-diagonal. By Lemma 2.10 and Lemma 2.8, we have that

$$\chi^{sa}(\frac{z^*z}{2}) + \frac{3}{4} + \frac{1}{2} \log 2\pi \leq \chi(z).$$

Since, by Lemma 2.9, we always have the other inequality, we obtain

(3) $$\chi(z) = \chi^{sa}(\frac{z^*z}{2}) + \frac{3}{4} + \frac{1}{2} \log 2\pi.$$ 

This can be expressed in terms of the free entropy of the symmetric variable $x$ as follows (by using the explicit formula for $\chi^{sa}$ of one variable given by Voiculescu in [8]):

$$\chi(z) = \chi^{sa}(\frac{z^*z}{2}) + \frac{3}{4} + \frac{1}{2} \log 2\pi = 2 \left(\frac{3}{4} + \frac{1}{2} \log 2\pi\right) + \int \int \log |s - t|d\mu_{2z}(s)d\mu_{2z}(t) = 2 \left(\frac{3}{4} + \frac{1}{2} \log 2\pi\right) + 2 \int \int \log |s - t|d\mu_{2z-1z}(s)d\mu_{2z-1z}(t) = 2 \chi^{sa}(2^{-1/2}x).$$

This proves 2.1(b).
Combining the above with Lemma 2.9 we get 2.1(a):
\[
\chi(y) \leq \chi^{sa}\left(\frac{y^*y}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi
\]
\[
= \chi^{sa}\left(\frac{y^*y}{2}\right) + \frac{3}{4} + \frac{1}{2} \log 2\pi
\]
\[
= \chi(z).
\]

\[\square\]

**Proposition 2.11** (A change of variables formula for polar decomposition).

Let \(y_1, \ldots, y_n\) be elements of a \(W^*\)-probability space \((M, \tau)\), and let \(y_i = v_i(y_i^*y_i)^{1/2}\) be their polar decompositions. Assume that \(f_i : [0, +\infty) \rightarrow [0, +\infty)\) are \(C^1\)-diffeomorphisms, and let \(z_i = v_i[2f_i(y_i^*y_i/2)]^{1/2}\). Then

\[
\chi(z_1, \ldots, z_n) = \chi(y_1, \ldots, y_n) + \sum_{j=1}^n \int\int \log \left| \frac{f(s) - f(t)}{s - t} \right| d\mu_i(s)d\mu_i(t),
\]

where \(\mu_i\) is the distribution of \(y_i^*y_i/2\) for \(i = 1, \ldots, n\). The same statement holds for \(\chi^w\) in the place of \(\chi\).

**Proof.** If for some \(i\) the distribution of \(y_i^*y_i\) contains atoms, then so does the distribution of \(z_i^*z_i\). In this case we have

\[
\chi(y_1, \ldots, y_n) \leq \sum_j \chi(y_j) = -\infty,
\]

since by Lemma 2.9, \(\chi(y_i) \leq \chi^{sa}(y_i^*y_i/2) + \text{const} = -\infty\). Similarly, \(\chi(z_1, \ldots, z_n) = -\infty\), and there is nothing to prove. Hence we may assume that the distributions of \(y_i^*y_i\), and thus the distributions of \(z_i^*z_i\), are non-atomic for all \(i\); in particular, that \(v_i\) are unitaries.

We may also assume that \(f_i\) for \(i \neq 1\) are the identity diffeomorphisms; moreover, by replacing \(f_i\) with \(f_i^{-1}\), we only need to prove that the left-hand side of the statement of Equation (4) is greater than or equal to the right hand side. We write \(f = f_1\).

Consider the mappings

\[
T : M_k \ni x \mapsto v[2f(x^*x/2)]^{1/2} \in M_k,
\]

where \(x = v(x^*x)^{1/2}\) is the polar decomposition of \(x\), and

\[
\hat{T} : M_k^n \ni (x_1, \ldots, x_n) \mapsto (T(x_1), x_2, \ldots, x_n) \in M_k^n.
\]

Note that the set \(\hat{T}(\Gamma_R(y_1, \ldots, y_n; m, k, \epsilon))\), taken with the measure \(\lambda_k \times \cdots \times \lambda_k\) is asymptotically included into the set \(\Gamma_R(z_1, \ldots, z_n; m, k, \epsilon)\), taken with the same measure. Moreover, the infimum of the Jacobian of \(\hat{T}\) on the set \(\Gamma_R(y_1, \ldots, y_n; m, k, \epsilon)\) is not less than the infimum of the Jacobian of \(T\) on the set \(\Gamma_R(y_1; m, k, \epsilon)\). View \(T\) as a map from \(U(k) \times M_k^+\) to itself, using
the identification of measure spaces $U(k) \times M^+_k \cong M_k$, $(v, p) \mapsto vp$. Then $T$ acts trivially on the unitary component. Recall that the measure on $M^+_k$, arising from the identification of $M_k$ with $U(k) \times M^+_k$, is the push-forward of Lebesgue measure on $M^+_k$ to $M^+_k$ by the map $p \mapsto p^2/2$. Hence the infimum of the Jacobian of $T$ is equal to the infimum of the Jacobian of the map $p \mapsto [2f(p^2/2)]^{1/2}$ viewed as a map from $M^+_k$ endowed with Lebesgue measure to itself, on the set $\Gamma_R(y_1^2/2; m, k, \epsilon)$. The rest of the computation is exactly as in the proof of Proposition 3.1 of [9].

\textbf{Remark 2.12.} Let $B \subset M$ be a subalgebra of $M$. The proof of the proposition above also works if we replace $\chi(\cdot)$ with the relative entropy $\chi(\cdot|B)$ introduced in [5]; we leave the details to the reader.

\textbf{Proof of 2.1(c) of Theorem 2.1.} Assume that $\chi(y) = \chi(ub) > -\infty$. Because of part 2.1(b), we conclude that $\chi(b) > -\infty$; in particular, the distribution of $b$ is non-atomic (see [8]). Since $(y^*y)^{1/2} = b$, this implies that in the polar decomposition of $y = v(y^*y)^{1/2}$, $v$ is a unitary.

Arguing as in Lemma 4.2 of [9], we may assume that there exists a family $f_i$ of $C^1$ diffeomorphisms on $[0, +\infty)$, and a continuous function $f : [0, +\infty) \to [0, +\infty)$, such that $f(y_i^2/2)$ is the square of a $(0, 1)$-semicircular random variable, $\|f_j(y^*y) - f(y^*y)\| \to 0$ as $j \to \infty$, $W^*(y^*y) = W^*(f(y^*y))$, and $\lim_j \chi^{sa}(f_j(y^*y)) = \lim_j \chi^{sa}(f(y^*y))$. Let $y = v(y^*y)^{1/2}$ be the polar decomposition of $y$; let $z = v[2f(y^*y/2)]^{1/2}$, and similarly $z_j = v[2f_j(y^*y/2)]^{1/2}$. Then by Proposition 2.11 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [8]), we get for all $j$,

$$
\chi(z_j) = \chi(y) + \chi^{sa}(f_j \left( \frac{y^*y}{2} \right)) - \chi^{sa} \left( \frac{y^*y}{2} \right).
$$

Applying Proposition 2.6 of [8], we get that

$$
\chi(z) \geq \limsup_j \chi(z_j)
$$

$$
= \limsup_j \left[ \chi(y) + \chi^{sa} \left( f_j \left( \frac{y^*y}{2} \right) \right) - \chi^{sa} \left( \frac{y^*y}{2} \right) \right]
$$

$$
= \chi(y) + \chi^{sa} \left( f \left( \frac{y^*y}{2} \right) \right) - \chi^{sa} \left( \frac{y^*y}{2} \right).
$$

Since $\chi(y) = \chi(ub)$ by assumption, and $\chi(ub) = \chi^{sa}(y^*y/2) + \frac{3}{4} + \frac{1}{2} \log 2\pi$ by Theorem 2.1(b) we get that

$$
\chi(z) \geq \chi^{sa} \left( f \left( \frac{y^*y}{2} \right) \right) + \frac{3}{4} + \frac{1}{2} \log 2\pi.
$$

By assumption, the distribution of $(z^*z)^{1/2}$ is quarter-circular (i.e., it is the absolute value of a $(0, 2)$-semicircular). Let $c$ be a circular variable (i.e., its
real and imaginary parts are free \((0,1)\)-semicircular variables. Then, since \(c\) is \(R\)-diagonal (see \([4]\)), we have by \(2.1(b)\), that
\[
\chi(c) = \chi^{sa}\left(\frac{c^*c}{2}\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi
\]
\[
= \chi^{sa}\left(f\left(\frac{y^*y}{2}\right)\right) + \frac{3}{4} + \frac{1}{2}\log 2\pi,
\]
since \(c^*c\) has the same distribution as \(z^*z = 2f(y^*/2)\). Hence \(\chi(z) \geq \chi(c)\).

On the other hand, \(c\) is \(R\)-diagonal, with the same distribution of the positive part as \(z\), so by \(2.1(a)\), we have \(\chi(z) \leq \chi(c)\). So \(\chi(z) = \chi(c)\).

We claim that \(z\) is circular. This will prove the proposition, since then the polar and positive parts of \(z\) are \(\ast\)-free (see \([7]\) or \([4]\)), and thus the polar and positive parts of \(y\) are \(\ast\)-free, since the polar part of \(y\) is the same as the polar part of \(z\), and the positive part of \(y\) is some function of the positive part of \(z\).

Now, for the claim that \(z\) is circular, let \(\gamma\) be a complex number of modulus one; then \(\chi(\gamma z) = \chi(z)\).

Let
\[
X_\gamma = \frac{1}{2}(\gamma z + \gamma^*z), \quad Y_\gamma = \frac{1}{2i}(\gamma z - \gamma^*z).
\]
Then
\[
\tau(X_\gamma^2) = \frac{1}{4}\left[2\tau(zz^*) + \gamma^2\tau(z^2) + \gamma^2 \cdot \tau(z^2)\right].
\]
Similarly,
\[
\tau(Y_\gamma^2) = \frac{1}{4}\left[2\tau(zz^*) - \gamma^2\tau(z^2) - \gamma^2 \cdot \tau(z^2)\right].
\]
We choose \(\gamma\) such that \(\gamma^2\tau(z^2)\) is purely imaginary. Since \(\tau(z^*z) = 2\), we have then \(\tau(X_\gamma^2) = \tau(Y_\gamma^2) = 1\). But \(\chi(z) = \chi(c) = \chi^{sa}(x_1, x_2)\), where \(x_i\) are free \((0,1)\) semicircular variables. Hence we have
\[
\chi(z) = \chi^{sa}(X_\gamma, Y_\gamma) = \chi(\gamma z) = \chi^{sa}(x_1, x_2),
\]
where \(X_\gamma\) and \(Y_\gamma\) are some self-adjoint random variables of covariance 1. But then by Voiculescu’s Proposition 2.4 of \([9]\), \(X_\gamma\) and \(Y_\gamma\) are both semicircular and free, so that \(\gamma z\) is circular, so \(z\) is circular.


**Theorem 3.1.** Let \(X_{ij}, 1 \leq i, j \leq d\) be non-commutative random variables in a tracial non-commutative probability space \((M, \tau)\). Let \(Z \in M \otimes M_d\) be given by
\[
Z = \sum_{i,j=1}^d X_{ij} \otimes e_{ij},
\]
where \(e_{ij}\) are matrix units in the algebra of \(d \times d\) matrices. We denote by \(\tau\) the normalized trace on \(M \otimes M_d\). Let \(\omega\) be a free ultrafilter. Let \(X\)
be a self-adjoint variable with \( \tau(X^{2n+1}) = 0 \) for all \( n \in \mathbb{N} \), and such that
\[
\tau(X^{2n}) = \tau((Z^*Z)^n), \quad \forall n \in \mathbb{N}.
\]
Then we have:

(a) \( \chi^{d\omega}(\{X_{ij}\}_{1 \leq i,j \leq d}) \leq d^2\chi^\omega(Z) - d^2\log d \leq 2d^2\chi^{sa}(2^{-\frac{1}{2}}X) - d^2\log d. \)

(b) If \( Z \) is \( R \)-diagonal and \( * \)-free from the algebra \( 1 \otimes M_d \), then
\[
\chi^{d\omega}(\{X_{ij}\}_{1 \leq i,j \leq d}) = d^2\chi^\omega(Z) - d^2\log d = 2d^2\chi^{sa}(2^{-\frac{1}{2}}X) - d^2\log d.
\]

(c) If \( \chi^{d\omega}(\{X_{ij}\}_{1 \leq i,j \leq d}) = 2d^2\chi^{sa}(2^{-\frac{1}{2}}X) - d^2\log d, \) and \( \chi^{sa}(2^{-\frac{1}{2}}X) \neq -\infty, \) then \( Z \) is \( R \)-diagonal and is \( * \)-free from the algebra \( 1 \otimes M_d \).

**Proof.** Let \( B = 1 \otimes M_d \). We have by [5] that
\[
\chi^{d\omega}(\{X_{ij}\}) = d^2\chi^\omega(Z|B) - d^2\log d \leq d^2\chi^\omega(Z) - d^2\log d.
\]
(We have the summand \(-d^2\log d \) rather than \(-\frac{d^2}{2} \log d \) appearing above because we are dealing with \( \chi \), not \( \chi^{sa} \).) Moreover, \( d^2\chi^\omega(Z) \leq 2d^2\chi^{sa}(2^{-\frac{1}{2}}X) \) by Theorem 2.1, hence 3.1(a).

If \( Z \) is \( * \)-free from \( B \), then, by [5], we have \( \chi^\omega(Z|B) = \chi^\omega(Z) \). Moreover, if \( Z \) is \( R \)-diagonal, we have, by Theorem 2.1, that \( \chi^\omega(Z) = 2\chi^{sa}(X/\sqrt{2}) \), which proves 3.1(b).

Assuming the conditions in 3.1(c) are satisfied, we get that \( \chi^\omega(Z) = 2\chi^{sa}(2^{-\frac{1}{2}}X) > -\infty \), so \( Z \) is \( R \)-diagonal by Theorem 2.1(c), i.e. \( Z \) has polar decomposition \( Z = v(Z^*Z)^{1/2} \), where \( v \) is a Haar unitary, which is \( * \)-free from \( Z^*Z \). Note also that we are given that \( \chi^\omega(Z|B) = \chi^\omega(Z) \).

We may assume, as in the proof of statement 2.1(c) of Theorem 2.1 that there exists a family \( f_i \) of \( C^1 \) diffeomorphisms on \([0, +\infty)\), and a continuous function \( f : [0, +\infty) \to [0, +\infty) \), such that \( f(\frac{Z^*Z}{2}) \) is the square of a \((0, 1)\)-semicircular random variable, \( \|f_j(Z^*Z) - f(Z^*Z)\| \to 0 \) as \( j \to \infty \), \( W^*(Z^*Z) = W^*(f(Z^*Z)) \), and \( \lim_j \chi^{sa}(f_j(Z^*Z)) = \chi^{sa}(f(Z^*Z)) \).

Given the polar decomposition \( Z = v(Z^*Z)^{1/2} \), let \( z = v[2f(Z^*Z/2)]^{1/2} \), and similarly \( z_j = v[2f_j(Z^*Z/2)]^{1/2} \). Notice that \( z \) is circular; moreover, since \( W^*(Z^*Z) = W^*(f(Z^*Z)) = W^*(z^*z) \), we have that \( Z \in W^*(z) \). Hence it will suffice to prove that \( z \) is \( * \)-free from \( B \), as then also \( Z \) is \( * \)-free from \( B \).

By Remark 2.12 and the explicit formula for the free entropy of one variable given by Voiculescu (Proposition 4.5 in [8]), we get for all \( j \),
\[
\chi^\omega(z_j|B) = \chi^\omega(Z|B) + \chi^{sa}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{sa}\left(\frac{Z^*Z}{2}\right).
\]
We get
\[
\chi^\omega(z|B) \geq \limsup_j \chi^\omega(z_j|B)
= \limsup_j \left[\chi^\omega(Z|B) + \chi^{sa}\left(f_j\left(\frac{Z^*Z}{2}\right)\right) - \chi^{sa}\left(\frac{Z^*Z}{2}\right)\right]
= \chi^\omega(Z|B) + \chi^{sa}\left(f\left(\frac{Z^*Z}{2}\right)\right) - \chi^{sa}\left(\frac{Z^*Z}{2}\right).
\]
By assumption, we have that \( \chi^\omega(Z|B) = \chi^\omega(Z) \); moreover, by \( R \)-diagonality of \( Z \) we get by Theorem 2.1(b) that
\[
\chi(Z) = \chi^{sa}(Z^*Z/2) + 3/4 + (1/2) \log 2\pi.
\]
Therefore, we get that
\[
\chi^\omega(z|B) \geq \chi^{sa}(Z^*Z/2) + 3/4 + (1/2) \log 2\pi + \chi^{sa}(f(Z^*Z/2)) - \chi^{sa}(Z^*Z/2) = 3/4 + (1/2) \log 2\pi + \chi^{sa}(f(Z^*Z/2)).
\]
But \( z \) is circular, in particular \( R \)-diagonal; moreover, \( z^*z/2 = f(Z^*Z/2) \). So from the formula in 2.1(b), we get that
\[
\chi^\omega(z) = \frac{3}{4} + \frac{1}{2} \log 2\pi + \chi^{sa}(f(Z^*Z/2)).
\]
Thus \( \chi^\omega(z|B) \geq \chi^\omega(z) \). Since \( \chi^\omega(z|B) \leq \chi^\omega(z) \) in general, we get that \( \chi^\omega(z|B) = \chi^\omega(z) \).

Now let \( S_1, S_2 \) be the real and imaginary parts of \( z \). Then we have that \( \chi^{sa}(S_1, S_2|B) = \chi^{sa}(S_1, S_2) \). Since \( S_1 \) and \( S_2 \) are two free semicircular variables, it follows by Theorem 4.5 from [5] that \( W^*(S_1, S_2) \) is free from \( B \). Hence \( z \) is *-free from \( B \); hence \( Z \) is *-free from \( B \).

\[\square\]

References


Received September 30, 1998. The first author was supported by a grant from the Natural Sciences and Engineering Research Council, Canada. The second author was supported in part by a National Science Foundation USA postdoctoral fellowship DMS-9804625. The third author was supported by a Heisenberg Fellowship of the DFG, Germany.

University of Waterloo
Ontario N2L 3G1
Canada
E-mail address: anica@math.uwaterloo.ca

University of California
Los Angeles, CA 90095
E-mail address: shlyakht@member.ams.org

Universität Heidelberg
D-69120 Heidelberg
Germany
E-mail address: roland.speicher@urz.uni-heidelberg.de
ON THE DUAL PAIRS \((O(p, q), SL(2, \mathbb{R})), (U(p, q), U(1, 1))\)
AND \((Sp(p, q), O^*(4))\)

Annegret Paul and Eng-Chye Tan

We describe the Howe quotient and theta lift for one-dimensional representations of the dual pairs \((O(p, q), SL(2, \mathbb{R})), (U(p, q), U(1, 1)), (Sp(p, q), O^*(4))\), by explicitly constructing the Howe quotients (of the representations in correspondence) using the Fock model.

1. Introduction.

Let \(Sp(2k, \mathbb{R})\) be the symplectic group on \(\mathbb{R}^{2k}\) and \(\widetilde{Sp}(2k, \mathbb{R})\) be the metaplectic group. If \(H\) is a subgroup of \(Sp(2k, \mathbb{R})\), we shall let \(\widetilde{H}\) be the pullback of \(H\) by the covering map from \(\widetilde{Sp}(2k, \mathbb{R})\) to \(Sp(2k, \mathbb{R})\). The oscillator representation \(\omega\) of \(\widetilde{Sp}(2k, \mathbb{R})\) may be realized on a space of holomorphic functions on \(C^k\), using the Fock model.

Let \((G, G')\) be a reductive dual pair in \(Sp(2k, \mathbb{R})\) (in the sense of \([Ho1]\)). The maximal compact subgroup of \(\widetilde{Sp}(2k, \mathbb{R})\) is \(\widetilde{U}(k)\), the half-determinant cover of \(U(k)\). In the Fock model, the space of \(\widetilde{U}(k)\)-finite vectors of the oscillator representation is \(\mathcal{P} = \mathcal{P}(C^k)\), the set of complex-valued polynomials on \(C^k\). We can also assume that \(K = U(k) \cap G\) and \(K' = U(k) \cap G'\) are maximal compact subgroups in \(G\) and \(G'\) respectively. We shall let lower gothic symbols denote Lie algebras of Lie groups, e.g., \(\mathfrak{g}\) and \(\mathfrak{g}'\) will be the Lie algebras of \(G\) and \(G'\) respectively.

For a reductive subgroup \(H\) (with maximal compact subgroup \(K_H = U(k) \cap H\)) of \(Sp(2k, \mathbb{R})\), we denote by \(\mathcal{R}(\mathfrak{h}, \widetilde{K}_H, \omega)\) the set of infinitesimal equivalence classes of irreducible \((\mathfrak{h}, \widetilde{K}_H)\) modules realizable as quotients of \(\mathcal{P}\). Consider the dual pair \((G, G')\). For \(\rho \in \mathcal{R}(\mathfrak{g}', \widetilde{K}', \omega)\), the Howe quotient corresponding to \(\rho\) is defined by (see \([Ho2]\))

\[\Omega(\rho) = \mathcal{P}/\mathcal{N}_\rho,\]

where \(\mathcal{N}_\rho\) is the intersection of all \((\mathfrak{g}', \widetilde{K}')\)-invariant subspaces \(\mathcal{N}\) of \(\mathcal{P}\) such that \(\mathcal{P}/\mathcal{N} \cong \rho\) as \((\mathfrak{g}', \widetilde{K}')\) modules. It is known (see \([Ho2]\)) that

\[\Omega(\rho) \simeq \rho' \otimes \rho,\]

349
where \( \rho' \) is a \( (\mathfrak{g}, \widetilde{K}) \) module of finite length, with a unique irreducible quotient \( \theta(\rho) \). The correspondence

\[
\rho \mapsto \theta(\rho)
\]

is commonly known as the (local) theta correspondence, and \( \theta(\rho) \) is often called the theta lift of \( \rho \).

The pullback \( \widetilde{H} \) of a Lie subgroup of \( Sp(2k, \mathbb{R}) \) is a split or non-split extension by \( \mathbb{Z}/2\mathbb{Z} \) depending on the dual pair under consideration. The representations which occur in the theta correspondence are genuine, i.e., they do not factor to \( H \). In the case where the cover of \( \widetilde{H} \) is split, this just means that they are of the form \( \pi \otimes sgn \), where \( \pi \) is a representation of \( H \), and \( sgn \) is the non-trivial character of \( \mathbb{Z}/2\mathbb{Z} \). If \( H = O(p, q) \), the non-split cover \( \widetilde{H} \) may be realized as \( H \times \mathbb{Z}/2\mathbb{Z} \) with group law \((g, \epsilon)(h, \delta) = (gh, \epsilon\delta(\det(g), \det(h))_{\mathbb{R}})\), where \((\cdot, \cdot)_{\mathbb{R}}\) is the Hilbert symbol of \( \mathbb{R} \). The character \( \chi \) of \( \widetilde{H} \) given by

\[
\chi(g, \epsilon) = \epsilon \cdot \begin{cases} 
\sqrt{-1} & \text{if } \det(g) = -1, \\
1 & \text{otherwise}
\end{cases}
\]

is genuine, and a genuine representation of \( \widetilde{H} \) is of the form \( \pi \otimes \chi \) for \( \pi \in H^\sim \). In either case we will only refer to \( \pi \).

It is a central problem in the theory of dual pairs to describe the theta correspondence. There are several techniques used to obtain explicit correspondences, however they are not elementary.

The theta correspondence for the pairs \((O(p, q), SL(2, \mathbb{R}))\), \((U(p, q), U(1, 1))\), \((Sp(p, q), O^*(4))\) is known mostly to experts in the field. Early results dealing with the theta correspondence of \((O(p, q), SL(2, \mathbb{R}))\) could be found in [Ho4] which built on results of Rallis and Schiffman [RS] and Strichartz [St]. Literature on the last two pairs is quite difficult to locate. The objective of this paper is to study the Howe quotient corresponding to a small representation by explicit construction using the Fock model. The representations dealt with here are the one-dimensional representations of \( SL(2, \mathbb{R}), U(1, 1) \), unitary finite-dimensional representations of \( O^*(4) \simeq (SU(2) \times SL(2, \mathbb{R}))/\{\pm I\} \) (see [Lm]) as well as the one-dimensional representations of \( O(p, q), U(p, q) \) and \( Sp(p, q) \). We believe that in the stable range (see [Ho3]), the Howe quotient (corresponding to a unitary representation or “small” representation) is irreducible. Evidences in support of this can be found in this paper as well as [LZ1], [LZ2], [ZH] and [TZ]. The Howe quotient also features prominently in many applications; see [KV2] and [Zh] (and the references therein) for applications to invariant distributions and [KR2] (and the references therein) for applications to the construction of automorphic forms. The setup used to study the duality correspondence enables one to have control on the Howe quotients, and it is our hope to try to extend it to other dual pairs.
The study of Howe quotients was initiated by Kudla and Rallis (see [KR1]; for the dual pairs \((O(p, q), Sp(2n, \mathbb{R}))\)). Their technique is to embed the Howe quotient in an appropriate degenerate principal series of \(Sp(2n, \mathbb{R})\) and using the work of Guillemonat (see [Gu]) to understand the structure of these representations, thereby extracting the theta lift in some cases. Recently, S.T. Lee and C.B. Zhu used similar ideas to describe the theta lift of a class of one-dimensional representations of \(O(p, q)\) (for the dual pairs \((O(p, q), Sp(2n, \mathbb{R}))\); see [LZ2]) and one-dimensional representations of \(U(p, q)\) (for the dual pairs \((U(p, q), U(n, n))\); see [LZ1]) using S.T. Lee’s work on the degenerate principal series of \(\tilde{SL}(2, \mathbb{R}), \tilde{U}(1, 1)\) and \(O^*(4)\) are easily understood. However, our approach is elementary - we simply work with concrete objects, i.e., polynomials.

It is easy to see that the theta lift of the trivial representation of \(Sp(2n, \mathbb{R})\) exists only if \(p + q\) is even, and in that case, it has a multiplicity-free \(\tilde{O}(p) \times \tilde{O}(q)\) spectrum (see [ZH]). The theta lifts of one-dimensional representations of \(U(n, n)\) also give irreducible \(U(p, q)\) representations with multiplicity-free \(U(p) \times U(q)\) spectrum (see [TZ]) if \(\min(p, q) \geq 2n\). These representations are “small” in the sense that they have small Gelfand-Kirillov dimensions and small rank (in the sense of [Ho3]). They “should” arise from appropriate quantization of nilpotent orbits (see [TZ] and [ZH]) and are generalizations of representations dealt with in [BZ], [Ko1] and [Ko2]. Another objective for the computations in this paper is to provide a basis of \(K\) highest weight vectors (where \(K\) is a maximal compact subgroup) for the representations treated in [TZ] and [ZH]. With such a basis, irreducibility and perhaps unitarity of these representations result from similar considerations as in [HT2]. Of course, irreducibility and unitarity would follow from [Li]’s results (see [ZH]). But our technique has invariant-theoretic flavour and has the advantage of providing a model of the representation space which might be useful to those who would like to make explicit calculations on these representations. Due to the length of the computations involved, we shall discuss these in a separate paper (see [Ta]).

2. The Dual Pairs \((O(p, q), SL(2, \mathbb{R}))\).

Let \(p \geq 2\) and consider the dual pair \((O(p, q), SL(2, \mathbb{R}))\) acting on the \(\tilde{U}(p)\)-finite vectors of the associated Fock space \(\mathbb{C}[x_1, \ldots, x_p]\) as follows:

(a) Action of \(o(p)\): 
\[ x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i < j \leq p. \]

(b) Action of \(sl(2) = \{H_1, r^2_1, \Delta_1\}\):
\[
H_1 = \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} + \frac{p}{2}, \quad r_1^2 = \sum_{i=1}^{p} x_i^2, \quad \Delta_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}.
\]

It is easy to see that the duality correspondence is as follows:

\[
\mathbb{C}[x_1, \ldots, x_p]|_{O(p) \times \widetilde{SL}(2, \mathbb{R})} = \sum_{m=0}^{\infty} \mathcal{H}_m^{(p)} \otimes V_{m+\frac{p}{2}},
\]

where

\[
\mathcal{H}_m^{(p)} = \{ f \in \mathbb{C}[x_1, \ldots, x_p] \mid \deg f = m, \Delta_1 f = 0 \}
\]

is the irreducible \(O(p)\) module spanned by spherical harmonics in variables \(x_1, \ldots, x_p\) of degree \(m\) and \(V_{m+\frac{p}{2}}\) is the \(\widetilde{SL}(2, \mathbb{R})\) lowest weight module of lowest weight \(m + \frac{p}{2}\) spanned by \(\{(r_1^2)^i(x_1 - \sqrt{-1}x_2)^m \mid i = 0, 1, \ldots \}\).

The duality correspondence enables us to write

\[
\mathbb{C}[x_1, \ldots, x_p]|_{O(p)} = \sum_{i,m=0}^{\infty} (r_1^2)^i \mathcal{H}_m^{(p)},
\]

where \((r_1^2)^i \mathcal{H}_m^{(p)}\) are \(O(p)\) modules isomorphic to \(\mathcal{H}_m^{(p)}\).

We note an interesting computation which makes the computability of this problem even easier:

**Lemma 2.1.** Let \(\phi \in \mathcal{H}_m^{(p)}\) where \(m \geq 1\). If

\[
(x_i \phi)^\sim = x_i \phi - \frac{1}{(2m + p - 2)} r_1^2 \frac{\partial \phi}{\partial x_i},
\]

then

\[
x_i \phi = (x_i \phi)^\sim + \frac{1}{(2m + p - 2)} r_1^2 \frac{\partial \phi}{\partial x_i}
\]

gives the projection of \(x_i \phi\) into the \(O(p)\) modules \(\mathcal{H}_{m+1}^{(p)}\) and \(r_1^2 \mathcal{H}_m^{(p)}\).

**Remark.** This is a special case of (2.1).

**Proof.** Easy. \(\Box\)

For convenience, we shall let

\[
c_{p,m} = \frac{1}{2m + p - 2}.
\]

We note that when \(m \geq 1\), \(c_{p,m} > 0\).

Likewise, for \(q \geq 2\), the dual pair \((O(q), SL(2, \mathbb{R}))\) acting on \(\mathbb{C}[y_1, \ldots, y_q]\) gives rise to the following decomposition of the Fock space as an \(O(q)\) module:

\[
\mathbb{C}[y_1, \ldots, y_q]|_{O(q)} = \sum_{j,n=0}^{\infty} (r_2^2)^j \mathcal{H}_n^{(q)},
\]
where \( r_2^2 = \sum_{j=1}^{q} y_j^2 \); \((r_2^2)^j \mathcal{H}_n^{(q)}\) is isomorphic (as \(O(q)\) modules) to \(\mathcal{H}_n^{(q)}\), the spherical harmonics (i.e., killed by \(\Delta_2 = \sum_{j=1}^{q} \frac{\partial^2}{\partial y_j^2}\)) of degree \(n\) in the variables \(y_1, \ldots, y_q\).

Consider \(\mathcal{P} = \mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q]\). This is the space of \(\mathcal{U}(p+q)\)-finite vectors of the associated Fock model for the dual pair \((O(p, q), SL(2, \mathbb{R}))\), and the actions of the complexified Lie algebras of \(O(p, q)\) and \(\widetilde{SL}(2, \mathbb{R})\) can be described as follows:

\[(2.3)\]

(a) Action of \(\mathfrak{o}(p, q)_{\mathbb{C}} = \mathfrak{o}(p)_{\mathbb{C}} \oplus \mathfrak{o}(q)_{\mathbb{C}} \oplus \mathfrak{p}\)

(i) Action of \(\mathfrak{o}(p)_{\mathbb{C}}\): 
\[x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad 1 \leq i < j \leq p,\]

(ii) Action of \(\mathfrak{o}(q)_{\mathbb{C}}\): 
\[y_i \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial y_i}, \quad 1 \leq i < j \leq q,\]

(iii) Action of \(\mathfrak{p}\): 
\[x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}, \quad 1 \leq i \leq p, \quad 1 \leq j \leq q.\]

(b) Action of \(\mathfrak{sl}(2)_{\mathbb{C}} = \text{Span } \{H, E, F\}\):

(i) \(H = \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} - \sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j} + \frac{p - q}{2}\),

(ii) \(E = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 = r_1^2 - \Delta_2\),

(iii) \(F = \sum_{j=1}^{q} y_j^2 - \sum_{i=1}^{p} x_i^2 = r_2^2 - \Delta_1\).

We note that \(\mathcal{O}(p, q)\) is a non-split extension by \(\mathbb{Z}_2\) while \(\widetilde{SL}(2, \mathbb{R})\) is split if and only if \(p + q\) is even.

Because of the decompositions \((2.1)\) and \((2.2)\), we have the following decomposition of \(\mathcal{P}\) as an \(O(p) \times O(q)\) module:

\[(2.4)\] 
\[\mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q]|_{O(p) \times O(q)} = \sum_{i,j,m,n=0}^\infty (r_1^2)^i(r_2^2)^j \mathcal{H}_m^{(p)} \mathcal{H}_n^{(q)}.\]

We will take as a “basis” for \(\mathcal{P}\) elements of the form

\[(2.5)\] 
\[[i, j, m, n] = (r_1^2)^i(r_2^2)^j \phi_1 \phi_2\]

where \(\phi_1 \in \mathcal{H}_m^{(p)}\) and \(\phi_2 \in \mathcal{H}_n^{(q)}\).

To be strictly correct, we should take \(\phi_1\) from a basis of \(\mathcal{H}_m^{(p)}\) and likewise for \(\phi_2\). But our computations basically disregard this. In other words, we can disregard the actions of \(\mathcal{O}(p) \times \mathcal{O}(q)\) when we study the \(\mathcal{O}(p, q)\) structure of modules in \(\mathcal{P}\). The reason is simple: The action of \(\mathcal{O}(p) \times \mathcal{O}(q)\) leaves
an element as in (2.5) in the same \( \tilde{O}(p) \times \tilde{O}(q) \) type, whilst an operator from \( p \) (the operators transverse to those coming from the Lie algebra of the maximal compact subgroup \( \tilde{O}(p) \times \tilde{O}(q) \)) moves elements from one \( \tilde{O}(p) \times \tilde{O}(q) \) type to another.

**Lemma 2.2.** Assume that \( p, q \geq 2 \). The actions of \( \mathfrak{sl}(2)_\mathbb{C} \) and \( p \subset \mathfrak{o}(p,q)_\mathbb{C} \) on the basis in (2.5) are as follows:

\[
H \cdot [i, j, m, n] = \left( 2i - 2j + m - n + \frac{p - q}{2} \right) [i, j, m, n];
\]
\[
E \cdot [i, j, m, n] = [i + 1, j, m, n] - 2j(q + 2n + 2j - 2)[i, j - 1, m, n];
\]
\[
F \cdot [i, j, m, n] = [i, j + 1, m, n] - 2i(p + 2m + 2i - 2)[i - 1, j, m, n];
\]
\[
\left( x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j} \right) \cdot [i, j, m, n] = [i, j, m + 1, n + 1]
\]
\[
+ c_{p,m}[i + 1, j, m - 1, n + 1] + c_{q,n}[i, j + 1, m + 1, n - 1] + c_{p,m}c_{q,n}[i + 1, j + 1, m - 1, n - 1]
\]
\[
- 4ij[i - 1, j - 1, m + 1, n + 1] - 2i(2j + 1)[i - 1, j, m + 1, n - 1]
\]
\[
- 2j(2ic_{p,m} + 1)[i - 1, j - 1, m - 1, n + 1]
\]
\[
- (2ic_{p,m} + 1)(2jc_{q,n} + 1)[i, j, m - 1, n - 1].
\]

**Proof.** Follows from Lemma 2.1 and the commutation relations

(a) \([\Delta_1, r_1^2] = 4 \left( \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i} + \frac{p}{2} \right),\)

(b) \([\Delta_2, r_2^2] = 4 \left( \sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j} + \frac{q}{2} \right),\)

(c) \([\Delta_1, (r_1^2)^i] = 2i(p + 2i - 2)(r_1^2)^{i-1} + 4i(r_1^2)^{i-1} \sum_{i=1}^{p} x_i \frac{\partial}{\partial x_i},\)

(d) \([\Delta_2, (r_2^2)^j] = 2j(q + 2j - 2)(r_2^2)^{j-1} + 4j(r_2^2)^{j-1} \sum_{j=1}^{q} y_j \frac{\partial}{\partial y_j}.\)

\(\square\)

**Proposition 2.3.** Let \((G, G') \subset Sp(2k, \mathbb{R})\) be a reductive dual pair and \( \mathcal{P} \) be the space of \( \tilde{U}(k) \)-finite vectors of the Fock model. Let \( \chi \) be a one-dimensional representation of \( G' \) with differential \( d\chi \), then

\[
\mathcal{N}_\chi = \text{Span} \{ Xf - d\chi(X)f, \omega(k)f - \chi(k)f \mid X \in g', k \in \tilde{K}', f \in \mathcal{P} \}.
\]
is \((\mathfrak{g} \times \mathfrak{g}', \widetilde{K} \times \widetilde{K}')\) invariant and \(\mathcal{P}/\mathcal{N}_\chi\) is the Howe quotient corresponding to the \((\mathfrak{g}', \widetilde{K}')\)-module of \(\chi\).

**Remark.** If \(G'\) is connected, one could omit the terms \(\omega(k)f - \chi(k)f\). If \(G'\) is disconnected, there may be more than one character with the same differential and the term \(\omega(k)f - \chi(k)f\) would capture the piece with the correct \(\widetilde{K}'\) action.

**Proof.** Take \(\sum (X_i - d\chi(X_i))f_i \in \mathcal{N}_\chi\) where \(X_i \in \mathfrak{g}'\) and \(f_i \in \mathcal{P}\). Let \((A, B) \in \mathfrak{g} \times \mathfrak{g}'\) and \((a, b) \in \widetilde{K} \times \widetilde{K}'\). Then

\[
(A, B) \sum (X_i - d\chi(X_i))f_i = \sum AB(X_i - d\chi(X_i))f_i = \sum (BX_i - d\chi(X_i)B)(Af_i) \quad (\text{since } A \text{ commutes with } B \text{ and } X_i)
\]

\[
= \sum (X_iB - [X_i, B] - d\chi(X_i)B)(Af_i)
\]

\[
= \sum (X_i - d\chi(X_i))\left(BAf_i - \sum [X_i, B](Af_i)\right).
\]

Now fix a choice of the Cartan subalgebra \(\mathfrak{h}\) of \(\mathfrak{g}'\). If \(B, X_i \in \mathfrak{h}, [X_i, B] = 0\), so we don’t have a problem here. If \(B \in \mathfrak{h}\) and \(X_i\) is a non-zero root vector (so that \(d\chi(X_i) = 0\)), then \([X_i, B]\) is a multiple of \(X_i\), and we still have \([X_i, B](Af_i) \in \mathcal{N}_\chi\). Likewise we have no problem if \(X_i\) and \(B\) are both non-zero root vectors. For the general case, extend by linearity to see that \(\mathcal{N}_\chi\) is \(\mathfrak{g} \times \mathfrak{g}'\) invariant.

For the action of \(\widetilde{K} \times \widetilde{K}'\),

\[
(a, b) \sum (X_i - d\chi(X_i))f_i = \sum ab(X_i - d\chi(X_i))f_i = \sum (bX_i - d\chi(X_i)b)(af_i) \quad (\text{since } a \text{ commutes with } b \text{ and } X_i)
\]

\[
= \sum (bX_ib^{-1} - d\chi(X_i))(baf_i)
\]

\[
= \sum (bX_ib^{-1} - d\chi(bX_ib^{-1}))(baf_i) \quad (\text{since } d\chi(bX_ib^{-1}) = d\chi(X_i))
\]

\[
\in \mathcal{N}_\chi.
\]

The argument for the term \(\omega(k)f - \chi(k)f\) is similar. So \(\mathcal{N}_\chi\) is \((\mathfrak{g} \times \mathfrak{g}', \widetilde{K} \times \widetilde{K}')\) invariant.

If \(\mathcal{N} \subset \mathcal{P}\) is such that

\[
\mathcal{P}/\mathcal{N} \simeq \chi \quad (\text{as a } (\mathfrak{g}', \widetilde{K}') \text{ module}),
\]

then for \(X \in \mathfrak{g}'\) and \(\bar{f} \in \mathcal{P}/\mathcal{N}\), we have

\[
X \bar{f} = d\chi(X)\bar{f} \Rightarrow (X - d\chi(X))\bar{f} = 0
\]
\[ (X - d\chi(X)) f = 0 \]
\[ (X - d\chi(X)) f \in \mathcal{N}. \]

Likewise for \( k \in \tilde{K}' \),
\[ \omega(k) \tilde{f} = \chi(k) \tilde{f} \Rightarrow (\omega(k) - \chi(k)) \tilde{f} = 0. \]

Thus, \( \mathcal{N}_\chi \subset \mathcal{N} \) and \( \mathcal{P}/\mathcal{N}_\chi \) is the Howe quotient corresponding to the representation \( \chi \) of \((\mathfrak{g}', \tilde{K}')\). \( \square \)

**Lemma 2.4.** Assume \( p, q \geq 2 \). Consider the basis of \( \mathcal{P} \) as in (2.5), and take \( \chi = 1 \), the trivial \((\mathfrak{sl}(2), \tilde{U}(1))\) module, then \( \mathcal{P}/\mathcal{N}_1 \) is
\[
\begin{cases} 
\text{Span of (images of)} & \{[0, 0, m, n] \mid m - n + \frac{p-q}{2} = 0, m \geq 0, n \geq 0\} \text{ if } \frac{p-q}{2} \in \mathbb{Z}; \\
\{0\} & \text{otherwise.}
\end{cases}
\]

**Proof.** We note that from Proposition 2.3, 
\[ \mathcal{N}_1 = \text{Span } \{Hf, Ef, Ff \mid f \in \mathcal{P}\}. \]

From Lemma 2.2, we infer that

(a) Action of \( H \Rightarrow [i, j, m, n] \in \mathcal{N}_1 \) if \( 2i - 2j + m - n + \frac{p-q}{2} \neq 0 \);

(b) Action of \( E \Rightarrow [i, 0, m, n] \in \mathcal{N}_1 \) if \( i > 0 \), and
\[ [i, j, m, n] \equiv 2j(q + 2n + 2j - 2)[i - 1, j - 1, m, n] \mod \mathcal{N}_1; \]

(c) Action of \( F \Rightarrow [0, j, m, n] \in \mathcal{N}_1 \) if \( j > 0 \), and
\[ [i, j, m, n] \equiv 2i(p + 2m + 2i - 2)[i - 1, j - 1, m, n] \mod \mathcal{N}_1. \]

Thus,
\[ [i, j, m, n] \equiv c_1[i - j, 0, m, n] \equiv 0 \mod \mathcal{N}_1 \text{ if } i > j; \]
\[ [i, j, m, n] \equiv c_2[0, j - i, m, n] \equiv 0 \mod \mathcal{N}_1 \text{ if } j > i; \]
\[ [i, j, m, n] \equiv c_3[0, 0, m, n] \mod \mathcal{N}_1 \text{ if } i = j > 0. \]

Here \( c_1, c_2 \) and \( c_3 \) are non-zero constants. The result follows. \( \square \)

**Theorem 2.5.** Assume \( p, q \geq 2 \). The trivial \((\mathfrak{sl}(2), \tilde{U}(1))\) module belongs to \( \mathcal{R}(\mathfrak{sl}(2), \tilde{U}(1), \omega) \) if and only if \( \frac{p-q}{2} \in \mathbb{Z} \), and if \( \frac{p-q}{2} \in \mathbb{Z} \), the theta lift of the trivial representation is the irreducible and unitary ladder representation \( L_{p,q} = \text{Span } \{[0, 0, m, n] \mid m - n + \frac{p-q}{2} = 0\} \).

**Remark.** These representations of \( \tilde{O}(p, q) \) are known as ladder representations in the Physics literature (see [AFR] and [BZ]). They have Gelfand-Kirillov dimension \( p + q - 3 \) (in the sense of [Vo]) and correspond to the quantization of certain minimal orbits (see [BZ], [Ko1] and [Ko2]).

**Proof.** The first part is immediate from Lemma 2.4.
Assume that $\frac{p-q}{2} \in \mathbb{Z}$. We note the action of $\mathfrak{p} \subset o(p,q)_{\mathbb{C}}$ using Lemma 2.2:
\[
\left(x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}\right) [0,0,m,n] = [0,0,m+1,n+1] - [0,0,m-1,n-1] \mod \mathcal{N}_{\mathbb{H}}.
\]
A simple computation on the $O(p)$ and $O(q)$ weights shows that $[0,0,m,n] \neq [0,0,m',n'] \mod \mathcal{N}_{\mathbb{H}}$ unless both are in $\mathcal{N}_{\mathbb{H}}$. Starting from $[0,0,2p^2,0]$ or $[0,0,0,\frac{p-q}{2}]$ which is clearly not in $\mathcal{N}_{\mathbb{H}}$, the transition formula above shows that $L_{p,q}$ is irreducible as an $(\mathfrak{o}(p,q), \tilde{O}(p) \times \tilde{O}(q))$ module. Unitarity follows from [Li]'s results.

Next we compute the Howe quotient corresponding to the trivial representation of $O(p,q)$ for the dual pair $(O(p,q), SL(2,\mathbb{R}))$. We remark that [LZ1] has treated these theta lifts in a different way (and for the dual pairs $(O(p,q), Sp(2n,\mathbb{R}))$ and have explicit information on the structure of the corresponding Howe quotients.

Assume $p, q \geq 2$. If $\mathcal{P} = \mathbb{C}[x_1, \ldots, x_p, y_1, \ldots, y_q]$ as before, let
\[
\mathcal{N}'_{\mathbb{H}} = \text{Span}\{Xf | f \in \mathcal{P}, X \in o(p,q)_{\mathbb{C}}\}.
\]
Then by the remark following proposition 2.3, the Howe quotient is contained in $\mathcal{P}/\mathcal{N}'_{\mathbb{H}}$. Recall that the action of $o(p,q)_{\mathbb{C}}$ and $\mathfrak{sl}(2)_{\mathbb{C}}$ is given in (2.3).

**Lemma 2.6.** Let $\tilde{1}, X$, and $Y$ be the elements of $\mathcal{P}/\mathcal{N}'_{\mathbb{H}}$ given by
\[
\begin{align*}
\tilde{1} &= 1 + \mathcal{N}'_{\mathbb{H}}, \\
X &= x_1^2 + \mathcal{N}'_{\mathbb{H}}, \\
Y &= y_1^2 + \mathcal{N}'_{\mathbb{H}}.
\end{align*}
\]
Then $B = \{\tilde{1}, X^k, Y^l | k, l \in \mathbb{Z}_{\geq 1}\}$ is a basis of $\mathcal{P}/\mathcal{N}'_{\mathbb{H}}$.

**Proof.** For $\lambda = (\lambda_1, \ldots, \lambda_p)$ and $\mu = (\mu_1, \ldots, \mu_q)$ with $\lambda_i, \mu_j \in \mathbb{Z}_{\geq 0}$, let $x^\lambda y^\mu$ be the monomial $\prod_{i=1}^p x_i^{\lambda_i} \prod_{i=1}^q y_i^{\mu_i}$. For $1 \leq i < j \leq p$ and arbitrary $x^\lambda y^\mu$, we have that
\[
\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right) x_i x_j x^\lambda y^\mu = (x_j + 1)x_i x_j x^\lambda y^\mu \in \mathcal{N}'_{\mathbb{H}},
\]
so that
\[
(2.6a) \quad x_i^2 x^\lambda y^\mu \equiv \frac{\lambda_i+1}{\lambda_j+1} x_j^2 x^\lambda y^\mu \mod \mathcal{N}'_{\mathbb{H}}.
\]
Similarly, we have for $1 \leq i < j \leq q$,
\[
(2.6b) \quad y_i^2 x^\lambda y^\mu \equiv \frac{\mu_i+1}{\mu_j+1} y_j^2 x^\lambda y^\mu \mod \mathcal{N}'_{\mathbb{H}}.
\]
If $\lambda_i = 0$, then
\[ (x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}) x_j x^\lambda y^\mu = (\lambda_j + 1)x_i x^\lambda y^\mu. \]

This implies that $x^\lambda y^\mu \in \mathcal{N}_1^\prime$ whenever $\lambda_i = 1$ for some $i$. Similarly, $x^\lambda y^\mu \in \mathcal{N}_1^\prime$ whenever $\mu_i = 1$ for some $i$. Applying (2.6) repeatedly if necessary yields
\begin{equation}
\lambda_i \text{ odd for some } i \text{ or } \mu_i \text{ odd for some } i \Rightarrow x^\lambda y^\mu \in \mathcal{N}_1^\prime.
\end{equation}

For $1 \leq i \leq p$ and $1 \leq j \leq q$, applying the operator $x_i y_j - \frac{\partial^2}{\partial x_i \partial y_j}$ to the monomial $x_i y_j x^\lambda y^\mu$ yields
\begin{equation}
x_i^2 y_i^2 x^\lambda y^\mu \equiv (\lambda_i + 1)(\mu_j + 1)x^\lambda y^\mu \mod \mathcal{N}_1^\prime.
\end{equation}

Using (2.6), (2.7), and (2.8), we see that every monomial in $\mathcal{P}$ is either in $\mathcal{N}_1^\prime$ or in one of the cosets listed in $B$. So we have that $B$ spans $\mathcal{P}/\mathcal{N}_1^\prime$. To see that the elements of $B$ are linearly independent, we notice that each is a weight vector for $\mathfrak{u}(1)_C \subset \mathfrak{sl}(2)_C$, of the following weights:
\begin{equation}
\begin{array}{ll}
\tilde{I} & \text{has weight } \frac{p-q}{2}, \\
X^k & \text{has weight } \frac{p-q}{2} + 2k, \\
Y^k & \text{has weight } \frac{p-q}{2} - 2k.
\end{array}
\end{equation}

Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. \hfill \Box

Remark. The proof of Lemma 2.6 also shows that all $\tilde{U}(1)$-types in $\mathcal{P}/\mathcal{N}_1^\prime$ have multiplicity one.

Theorem 2.7. Suppose $p, q \geq 2$. The trivial $(\mathfrak{a}(p, q), \tilde{O}(p) \times \tilde{O}(q))$-module $\mathbb{I}$ belongs to $\mathcal{R}(\mathfrak{a}(p, q), \tilde{O}(p) \times \tilde{O}(q), \omega)$ and no other one-dimensional module does, so that $\mathcal{P}/\mathcal{N}_1^\prime$ is the Howe quotient corresponding to the trivial module.

(i) If $p$ and $q$ are both odd then $\mathcal{P}/\mathcal{N}_1^\prime$ is irreducible and isomorphic to the principal series of the split double cover of $SL(2, \mathbb{R})$ with infinitesimal character $-\frac{p+q}{2} + 1$, and even $U(1)$-types if $\frac{p-q}{2}$ is even, odd $U(1)$-types otherwise.

(ii) If $p$ and $q$ are both even then $\mathcal{P}/\mathcal{N}_1^\prime$ has two irreducible $(\mathfrak{sl}(2), U(1))$-submodules $V_1$ and $V_2$ spanned by $\{X^k|k \geq \frac{q}{2}\}$ and $\{Y^k|k \geq \frac{p}{2}\}$ respectively, which are discrete series representations with minimal $U(1)$-types $\frac{p+q}{2}$ and $-\frac{p+q}{2}$ respectively. The theta-lift of $\mathbb{I}$ is the irreducible quotient of $\mathcal{P}/\mathcal{N}_1^\prime$ by $V_1 \oplus V_2$, which is isomorphic to the unique $(\mathfrak{sl}(2), U(1))$-module of dimension $\frac{p+q}{2} - 1$.

(iii) If $p$ is even and $q$ is odd then $\mathcal{P}/\mathcal{N}_1^\prime$ has an irreducible $(\mathfrak{sl}(2), \tilde{U}(1))$-submodule $V$ spanned by $\{Y^k|k \geq \frac{p}{2}\}$, which is the Harish-Chandra module of the discrete series representation of $SL(2, \mathbb{R})$ with minimal $\tilde{U}(1)$-type $-\frac{p+q}{2}$. 
The theta-lift of $\mathfrak{h}$ is the irreducible quotient of $P/N'_1$ by $V$ which is a lowest weight module with lowest weight $-\frac{p+q}{2} + 2$.

(iv) If $p$ is odd and $q$ is even then $P/N'_1$ has an irreducible $(\mathfrak{sl}(2), \tilde{U}(1))$-submodule $V$ spanned by $\{X^k| k \geq \frac{q}{2}\}$, which is the Harish-Chandra module of the discrete series representation of $\tilde{SL}(2, \mathbb{R})$ with minimal $\tilde{U}(1)$-type $\frac{p+q}{2}$.

The theta-lift of $\mathfrak{h}$ is the irreducible quotient of $P/N'_1$ by $V$ which is a highest weight module with highest weight $p - \frac{q}{2} - 2$.

Further, the theta-lift of $\mathfrak{h}$ is non-unitarizable except in the case (ii) with $p = q = 2$.

Proof. First observe that elements of $B$ transform by the trivial $O(p,q)$ character, simply by checking the action of $\tilde{O}(p) \times \tilde{O}(q)$. Using the formulas (2.3), (2.6), and (2.8), we compute the action of $p' = \text{Span}\{E,F\}$ on the weight vectors in $B$:

$$
E \cdot \bar{1} = pX; \\
E \cdot X^k = \frac{2k+p}{2k+1} X^{k+1}; \\
E \cdot Y^k = (2k-1)(p-2k)Y^{k-1}; \\
F \cdot \bar{1} = qY; \\
F \cdot X^k = (2k-1)(q-2k)X^{k-1}; \\
F \cdot Y^k = \frac{2k+q}{2k+1} Y^{k+1}.
$$

Notice that $E$ annihilates $Y^p$ if $p$ is even, and takes all weight vectors of weight $\frac{p+q}{2} + 2k$ with $2k \neq p$ to weight vectors of weight $\frac{p+q}{2} + 2k + 2$. Similarly, $F$ annihilates $X^q$ if $q$ is even and takes all weight vectors of weight $\frac{p+q}{2} + 2k$ with $2k \neq -q$ to weight vectors of weight $\frac{p+q}{2} + 2k - 2$. The decomposition into submodules and quotients for (i)-(iv) now follows.

The casimir operator of $\mathfrak{sl}(2)$ acts on the Howe quotient by the constant $(\frac{p+q}{2})(\frac{p+q}{2} - 2)$. Non-unitarizability then follows from the descriptions of $\mathfrak{sl}(2)$ modules given in Chapter III of [HT1]. □

For completeness, we shall provide the results for the cases when either $p = 1$ or $q = 1$ or $p = q = 1$. Suppose $\epsilon_1, \epsilon_2 \in \{+, -\}$ and let $\mathfrak{h}_{\epsilon_1, \epsilon_2}$ be the unique character of $O(p,q)$ which restricts to det on $O(p)$ if $\epsilon_1 = -$ and to the trivial character if $\epsilon_1 = +$ and similarly for $O(q)$ and $\epsilon_2$. Recall that all genuine characters of $\tilde{O}(p,q)$ are obtained by twisting these characters by the character $\chi$ defined in the introduction.

**Theorem 2.8.**

(a) $(O(p,1), SL(2, \mathbb{R}))$, $p \geq 2$: 

...
(a)(i) \( \mathbb{I}_{+,+} \in \mathcal{R}(\mathfrak{o}(p, 1), \bar{\mathcal{O}}(p) \times \bar{\mathcal{O}}(1), \omega) \) if \( p \) is even and \( \theta(\mathbb{I}_{+,+}) \) is a lowest weight \((\mathfrak{sl}(2), \bar{U}(1))\) module of lowest weight \(-\frac{p-3}{2}\) and it is not unitarizable unless \( p = 2 \);

(a)(ii) \( \mathbb{I}_{+-} \in \mathcal{R}(\mathfrak{o}(p, 1), \bar{\mathcal{O}}(p) \times \bar{\mathcal{O}}(1), \omega) \) if \( p \) is even and \( \theta(\mathbb{I}_{+-}) \) is a highest weight \((\mathfrak{sl}(2), \bar{U}(1))\) module of highest weight \(\frac{p-3}{2}\) and it is not unitarizable unless \( p = 2 \);

(a)(iii) \( \mathbb{I}_{+-} \in \mathcal{R}(\mathfrak{o}(p, 1), \bar{\mathcal{O}}(p) \times \bar{\mathcal{O}}(1), \omega) \) if \( p \) is odd and \( \theta(\mathbb{I}_{+-}) \) is the non-unitarizable \((\mathfrak{sl}(2), \bar{U}(1))\) principal series representation with infinitesimal character \(\frac{p-1}{2}\), and even \( \bar{U}(1) \)-types if \( \frac{p-1}{2} \) is even, odd \( \bar{U}(1) \)-types otherwise;

(a)(iv) \( \mathbb{I}_{+-} \in \mathcal{R}(\mathfrak{o}(p, 1), \bar{\mathcal{O}}(p) \times \bar{\mathcal{O}}(1), \omega) \) if \( p \) is odd and \( \theta(\mathbb{I}_{+-}) \) is the finite-dimensional \((\mathfrak{sl}(2), \bar{U}(1))\)-module of dimension \(\frac{p-1}{2}\) and it is non-unitarizable unless \( p = 3 \);

(a)(v) \( \mathbb{I} \in \mathcal{R}(\mathfrak{sl}(\bar{U}(1), \omega) \) if \( p \) is odd and \( \theta(\mathbb{I}) \) is the irreducible and unitarizable \((\mathfrak{o}(p, 1), \bar{\mathcal{O}}(p) \times \bar{\mathcal{O}}(1))\) ladder representation with representatives \(\{(x_1 - \sqrt{-1}x_2)^m y^t \mid t = m + \frac{p-1}{2}\}\).

(b) \((O(1, q), SL(2, \mathbb{R}))\), \( q \geq 2 \):

(b)(i) \( \mathbb{I}_{++,} \in \mathcal{R}(\mathfrak{o}(q, 1), \bar{\mathcal{O}}(q) \times \bar{\mathcal{O}}(q), \omega) \) if \( q \) is even and \( \theta(\mathbb{I}_{++,}) \) is a highest weight \((\mathfrak{sl}(2), \bar{U}(1))\) module of highest weight \(\frac{q-3}{2}\) and it is not unitarizable unless \( q = 2 \);

(b)(ii) \( \mathbb{I}_{--} \in \mathcal{R}(\mathfrak{o}(q, 1), \bar{\mathcal{O}}(q) \times \bar{\mathcal{O}}(q), \omega) \) if \( q \) is even and \( \theta(\mathbb{I}_{--}) \) is a lowest weight \((\mathfrak{sl}(2), \bar{U}(1))\) module of lowest weight \(-\frac{q-3}{2}\) and it is not unitarizable unless \( q = 2 \);

(b)(iii) \( \mathbb{I}_{++,} \in \mathcal{R}(\mathfrak{o}(q, 1), \bar{\mathcal{O}}(q) \times \bar{\mathcal{O}}(q), \omega) \) if \( q \) is odd and \( \theta(\mathbb{I}_{++,}) \) is the non-unitarizable \((\mathfrak{sl}(2), \bar{U}(1))\) principal series representation with infinitesimal character \(\frac{q-1}{2}\), and even \( \bar{U}(1) \)-types if \( \frac{q-1}{2} \) is even, odd \( \bar{U}(1) \)-types otherwise;

(b)(iv) \( \mathbb{I}_{--} \in \mathcal{R}(\mathfrak{o}(q, 1), \bar{\mathcal{O}}(q) \times \bar{\mathcal{O}}(q), \omega) \) if \( q \) is odd and \( \theta(\mathbb{I}_{--}) \) is the finite-dimensional \((\mathfrak{sl}(2), \bar{U}(1))\) of dimension \(\frac{q-1}{2}\) and it is non-unitarizable unless \( q = 3 \);

(b)(v) \( \mathbb{I} \in \mathcal{R}(\mathfrak{sl}(\bar{U}(1), \omega) \) if \( q \) is odd and \( \theta(\mathbb{I}) \) is the irreducible and unitarizable \((\mathfrak{o}(q, 1), \bar{\mathcal{O}}(q) \times \bar{\mathcal{O}}(q))\) ladder representation with representatives \(\{x^n(y_1 - \sqrt{-1}y_2)^n \mid s = n + \frac{q-1}{2}\}\).

(c) \((O(1, 1), SL(2, \mathbb{R}))\):

(c)(i) \( \mathbb{I}_{++,} \in \mathcal{R}(\mathfrak{o}(1, 1), \bar{\mathcal{O}}(1) \times \bar{\mathcal{O}}(1), \omega) \) and \( \theta(\mathbb{I}_{++,}) \) is the unitarizable \((\mathfrak{sl}(2), \bar{U}(1))\) principal series representation with infinitesimal character 0;
(c)(ii) \( \mathbb{1}_{-+} \in \mathcal{R}(\mathfrak{o}(1,1), \tilde{O}(1) \times \tilde{O}(1), \omega) \) and \( \theta(\mathbb{1}_{-+}) \) is the unitarizable lowest weight \( (\mathfrak{sl}(2), \tilde{U}(1)) \) module of lowest weight \( 1 \) (i.e., limit of discrete series);

(c)(iii) \( \mathbb{1}_{+-} \in \mathcal{R}(\mathfrak{o}(1,1), \tilde{O}(1) \times \tilde{O}(1), \omega) \) and \( \theta(\mathbb{1}_{+-}) \) is the unitarizable highest weight \( (\mathfrak{sl}(2), \tilde{U}(1)) \) of highest weight \( -1 \) (i.e., limit of discrete series);

(c)(iv) \( \mathbb{1} \in \mathcal{R}(\mathfrak{o}(1,1), \tilde{U}(1), \omega) \) and \( \theta(\mathbb{1}) \) is the non-unitarizable two-dimensional \((\mathfrak{o}(1,1), \tilde{O}(1) \times \tilde{O}(1)) \) module with representatives \( \{1, x\} \).

**Remark.** The cases (a)(iv) and (b)(iv) could be interpreted physically, in terms of the Huygens' Principle. Even the case under Theorem 2.7 could be suitably interpreted in terms of \( O(p,q) \) invariant distributions (see [HT1]).

**Proof.** Basically the computations are similar. The results are obtained by tracking the actions of \( \tilde{O}(p), \tilde{O}(q) \) and \( \mathfrak{sl}(2) \) on the representatives in \( \mathcal{P}/N'_1 \).

\[ \square \]

### 3. The Dual Pairs \((U(p,q),U(1,1))\).

Consider the dual pair \((U(p),U(1,1))\) acting on the \( \tilde{U}(2p) \)-finite vectors of the associated Fock space \( \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p] \) as follows:

(a) Action of \( u(p)C \):
\[
\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}, \quad 1 \leq i, j \leq p.
\]

(b) Action of \( u(1,1)C = \text{Span} \{ h_1, h_2, r^2_1, \Delta_1 \} \), where
\[
h_1 = \sum_{i=1}^{p} z_i \frac{\partial}{\partial z_i}, \quad h_2 = \sum_{i=1}^{p} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}, \quad r^2_1 = \sum_{i=1}^{p} z_i \bar{z}_i, \quad \Delta_1 = \sum_{i=1}^{p} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.
\]

It is easy to see that the duality correspondence is as follows:

\[
\mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p] \big|_{U(p) \times \tilde{U}(1,1)} = \sum_{m=0}^{\infty} \mathcal{H}_{\alpha,\beta}^{(p)} \otimes (\det)^{\frac{\alpha-\beta}{2}} V_{p+\alpha+\beta},
\]

where \( \mathcal{H}_{\alpha,\beta}^{(p)} \) is the irreducible \( U(p) \) module characterized as follows:

\[
\mathcal{H}_{\alpha,\beta}^{(p)} = \{ f \in \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p] \mid h_1 f = \alpha f, h_2 f = \beta f, \Delta_1 f = 0 \};
\]

and \( V_{p+\alpha+\beta} \) is the \( SU(1,1) \) lowest weight module of lowest weight \( p + \alpha + \beta \) spanned by \( \{ (r^2_1)^i z_1^\alpha \bar{z}_p^\beta \mid i = 0, 1, \ldots \} \). Note that the representation of \( SU(1,1) \) is twisted by the \( \frac{\alpha-\beta}{2} \)-power of the determinant character det on \( \tilde{U}(1,1) \). Incidentally, \( \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p] \) is bigraded in degrees in the \( z \) coordinates and \( \bar{z} \) coordinates. The subscript in \( \mathcal{H}_{\alpha,\beta}^{(p)} \) indicates that the module lives in the homogeneous component of degree \( \alpha \) in the \( z \) coordinates.
and degree \( \beta \) in the \( \bar{z} \) coordinates. The \( U(p) \) highest weight vector in \( \mathcal{H}_{\alpha,\beta}^{(p)} \) is \( z_1^\alpha \bar{z}_p^\beta \).

The duality correspondence enables us to write

\[
\mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p]|_{U(p)} = \sum_{i,\alpha,\beta=0}^{\infty} (r_1^2)^i \mathcal{H}_{\alpha,\beta}^{(p)},
\]

where \( (r_1^2)^i \mathcal{H}_{\alpha,\beta}^{(p)} \) are \( U(p) \) modules isomorphic to \( \mathcal{H}_{\alpha,\beta}^{(p)} \). We note the analogue of Lemma 2.1.

**Lemma 3.1.** Let \( \phi \in \mathcal{H}_{\alpha,\beta}^{(p)} \) where \( \alpha + \beta \geq 1 \). If

\[
(z_i \phi)^\sim = z_i \phi - \frac{1}{(p + \alpha + \beta - 1)} r_1^2 \frac{\partial \phi}{\partial z_i}, \quad \text{and} \quad (\bar{z}_i \phi)^\sim = \bar{z}_i \phi - \frac{1}{(p + \alpha + \beta - 1)} r_1^2 \frac{\partial \phi}{\partial \bar{z}_i},
\]

then

\[
z_i \phi = (z_i \phi)^\sim + \frac{1}{(p + \alpha + \beta - 1)} r_1^2 \frac{\partial \phi}{\partial z_i}
\]
gives the projection of \( z_i \phi \) into the \( U(p) \) modules \( \mathcal{H}_{\alpha+1,\beta}^{(p)} \) and \( r_1^2 \mathcal{H}_{\alpha,\beta-1}^{(p)} \), while

\[
\bar{z}_i \phi = (\bar{z}_i \phi)^\sim + \frac{1}{(p + \alpha + \beta - 1)} r_1^2 \frac{\partial \phi}{\partial \bar{z}_i}
\]
gives the projection of \( \bar{z}_i \phi \) into the \( U(p) \) modules \( \mathcal{H}_{\alpha,\beta+1}^{(p)} \) and \( r_1^2 \mathcal{H}_{\alpha-1,\beta}^{(p)} \).

**Proof.** Easy. \( \square \)

For convenience, we shall let

\[
c_{p,\alpha,\beta} = \frac{1}{p + \alpha + \beta - 1}.
\]

We note that when \( \alpha + \beta \geq 1 \), \( c_{p,\alpha,\beta} > 0 \).

Likewise, the dual pair \( (U(q), U(1,1)) \) acting on \( \mathbb{C}[w_1, \ldots, w_q, \bar{w}_1, \ldots, \bar{w}_q] \) gives rise to the following decomposition of the Fock space as \( U(q) \) modules:

\[
\mathbb{C}[w_1, \ldots, w_q, \bar{w}_1, \ldots, \bar{w}_q]|_{U(q)} = \sum_{j,\gamma,\delta=0}^{\infty} (r_2^2)^j \mathcal{H}_{\gamma,\delta}^{(q)},
\]

where \( r_2^2 = \sum_{j=1}^{q} w_j \bar{w}_j \), \( (r_2^2)^j \mathcal{H}_{\gamma,\delta}^{(q)} \) is isomorphic to \( \mathcal{H}_{\gamma,\delta}^{(q)} \), the spherical harmonics of degree \( \gamma,\delta \) (i.e., killed by \( \Delta_2 = \sum_{j=1}^{q} \frac{\partial^2}{\partial w_j \partial \bar{w}_j} \)) in the variables \( w_1, \ldots, w_q, \bar{w}_1, \ldots, \bar{w}_q \).

Consider \( P = \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p, w_1, \ldots, w_q, \bar{w}_1, \ldots, \bar{w}_q] \). This is the space of \( \bar{U}(2p + 2q) \)-finite vectors of the associated Fock model for the dual
pair \((U(p, q), U(1, 1))\), and the actions of the complexified Lie algebras of \(U(p, q)\) and \(\tilde{U}(1, 1)\) can be described as follows:

(3.3)

(a) Action of \(u(p, q)_C = u(p)_C \oplus u(q)_C \oplus p\):

(i) Action of \(u(p)_C\):
\[
z_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i}, \quad 1 \leq i, j \leq p;
\]

(ii) Action of \(u(q)_C\):
\[
w_i \frac{\partial}{\partial w_j} - \bar{w}_j \frac{\partial}{\partial \bar{w}_i}, \quad 1 \leq i, j \leq q;
\]

(iii) Action of \(p\):
\[
S_{ij} = z_i w_j - \frac{\partial^2}{\partial z_i \partial w_j}, \quad 1 \leq i \leq p, 1 \leq j \leq q;
\]
\[
T_{ij} = \bar{z}_i \bar{w}_j - \frac{\partial^2}{\partial \bar{z}_i \partial \bar{w}_j}, \quad 1 \leq i \leq p, 1 \leq j \leq q.
\]

(b) Action of \(u(1, 1)_C = \text{Span}\ \{H_1, H_2, E, F\}:

(i) \(H_1 = \sum_{i=1}^{p} z_i \frac{\partial}{\partial z_i} - \sum_{j=1}^{q} w_j \frac{\partial}{\partial w_j} + \frac{p - q}{2}\);

(ii) \(H_2 = -\sum_{i=1}^{p} \bar{z}_i \frac{\partial}{\partial \bar{z}_i} + \sum_{j=1}^{q} \bar{w}_j \frac{\partial}{\partial \bar{w}_j} - \frac{p - q}{2}\);

(iii) \(E = r_1^2 - \Delta_2\),

(iv) \(F = r_2^2 - \Delta_1\).

We note that \(\tilde{U}(p, q)\) is split while \(\tilde{U}(1, 1)\) is split when \(p + q\) is even and non-split otherwise.

Because of the decompositions (3.1) and (3.2), we have the following decomposition of \(P\) as a \(U(p) \times U(q)\) module:

(3.4) \(P|_{U(p) \times U(q)} = \sum_{i,j,\alpha,\beta,\gamma,\delta=0}^{\infty} (r_1^2)^i (r_2^2)^j \mathcal{H}_{\alpha,\beta}^{(p)} \mathcal{H}_{\gamma,\delta}^{(q)}\).

Again, we will take as a “basis” for \(P\), elements of the form

(3.5) \([i, j, \alpha, \beta, \gamma, \delta] = (r_1^2)^i (r_2^2)^j \phi_1 \phi_2\), where \(\phi_1 \in \mathcal{H}_{\alpha,\beta}^{(p)}\) and \(\phi_2 \in \mathcal{H}_{\gamma,\delta}^{(q)}\).

**Lemma 3.2.** The actions of \(u(1, 1)_C\) and \(p \subset u(p, q)_C\) on the basis in (3.5) are as follows:

\[
H_1 \cdot [i, j, \alpha, \beta, \gamma, \delta] = \left(i - j + \alpha - \gamma + \frac{p - q}{2}\right) [i, j, \alpha, \beta, \gamma, \delta];
\]
\[
H_2 \cdot [i, j, \alpha, \beta, \gamma, \delta] = \left(j - i + \delta - \beta - \frac{p - q}{2}\right) [i, j, \alpha, \beta, \gamma, \delta];
\]
\[ E \cdot [i, j, \alpha, \beta, \gamma, \delta] = [i + 1, j, \alpha, \beta, \gamma, \delta] \\
- j(q + \gamma + \delta + j - 1)[i, j - 1, \alpha, \beta, \gamma, \delta]; \]
\[ F \cdot [i, j, \alpha, \beta, \gamma, \delta] = [i, j + 1, \alpha, \beta, \gamma, \delta] \\
- i(p + \alpha + \beta + i - 1)[i - 1, j, \alpha, \beta, \gamma, \delta] \]
\[ S_{kl} \cdot [i, j, \alpha, \beta, \gamma, \delta] = [i, j + 1, \alpha, \beta, \gamma, \delta + 1] \\
+ c_{p,\alpha,\beta}[i + 1, j, \alpha, \beta - 1, \gamma + 1, \delta] \\
+ c_{q,\gamma,\delta}[i, j + 1, \alpha + 1, \beta, \gamma, \delta - 1] \\
+ c_{p,\alpha,\beta}c_{q,\gamma,\delta}[i + 1, j + 1, \alpha, \beta - 1, \gamma, \delta - 1] \\
- ij[i - 1, j - 1, \alpha + 1, \beta, \gamma + 1, \delta] \\
- i(jc_{q,\gamma,\delta} + 1)[i - 1, j, \alpha + 1, \beta, \gamma, \delta - 1] \\
- j(ic_{p,\alpha,\beta} + 1)[i, j - 1, \alpha, \beta - 1, \gamma + 1, \delta] \\
- (ic_{p,\alpha,\beta} + 1)(jc_{q,\gamma,\delta} + 1)[i, j, \alpha - 1, \beta, \gamma - 1, \delta - 1]; \]
\[ T_{kl} \cdot [i, j, \alpha, \beta, \gamma, \delta] = [i, j, \alpha, \beta + 1, \gamma, \delta + 1] \\
+ c_{p,\alpha,\beta}[i + 1, j, \alpha - 1, \beta, \gamma, \delta + 1] \\
+ c_{q,\gamma,\delta}[i, j + 1, \alpha + 1, \beta, \gamma - 1, \delta] \\
+ c_{p,\alpha,\beta}c_{q,\gamma,\delta}[i + 1, j + 1, \alpha - 1, \beta, \gamma - 1, \delta] \\
- ij[i - 1, j - 1, \alpha, \beta + 1, \gamma, \delta + 1] \\
- i(jc_{q,\gamma,\delta} + 1)[i - 1, j, \alpha, \beta + 1, \gamma - 1, \delta] \\
- j(ic_{p,\alpha,\beta} + 1)[i, j - 1, \alpha, \beta - 1, \gamma, \delta + 1] \\
- (ic_{p,\alpha,\beta} + 1)(jc_{q,\gamma,\delta} + 1)[i, j, \alpha - 1, \beta, \gamma - 1, \delta]. \]

**Proof.** Similar to Lemma 2.2. □

For \( \nu \in \frac{1}{2} \mathbb{Z} \), let \( \det^{(\nu)} \) be the \( \nu \)-power of the determinant character on \( \tilde{U}(1, 1) \). Let
\[ N_\nu = \text{Span } \{ H_1 f - \nu f, H_2 f - \nu f, E f, F f \mid f \in \mathcal{P} \}. \]

**Lemma 3.3.** For \( \nu \in \frac{1}{2} \mathbb{Z} \), \( \det^{(\nu)} \in \mathcal{R}(u(1, 1), \tilde{U}(1) \times \tilde{U}(1), \omega) \) if and only if \( \frac{p-1}{2} - \nu \in \mathbb{Z} \). Consider the basis of \( \mathcal{P} \) as in (3.5). If \( \frac{p-1}{2} - \nu \in \mathbb{Z} \), \( \mathcal{P}/N_\nu = \frac{\text{Span of (images of)}}{\{ [0, 0, \alpha, \beta, \gamma, \delta] \mid \alpha - \gamma + \frac{p - q}{2} - \nu = \delta - \beta - \frac{p - q}{2} - \nu = 0 \}}. \]

**Proof.** From Lemma 3.2, we infer that
Theorem 3.4. Assume that $\frac{p-q}{2} - \nu \in \mathbb{Z}$. The theta lift of the representation $\det^\nu$ of $\tilde{U}(1,1)$ is the irreducible and unitarizable $(\mathfrak{u}(p,q), U(p) \times U(q))$ module

$$H_{p,q,\nu} = \text{Span} \left( \text{of the images} \right) \left\{ [0,0,\alpha,\beta,\gamma,\delta] \mid \alpha - \gamma + \frac{p-q}{2} - \nu = 0, \right.$$

$$\delta - \beta - \frac{p-q}{2} - \nu = 0 \left. \right\}.$$

Remark. The representations are also known as ladder representations (see [AFR]), even though their $K$-spectrum has two parameters. They are restrictions of the ladder representation $L_{2p,2q}$ of $O(2p,2q)$:

$$L_{2p,2q}|_{U(p,q)} = \sum_{\nu \in \frac{1}{4}\mathbb{Z}} H_{p,q,\nu}$$

and have Gelfand-Kirillov dimensions $2p + 2q - 4$ (compare with $2p + 2q - 3$ of $L_{2p,2q}$).

Proof. We note the action of $\mathfrak{p} \subset u(p,q)_\mathbb{C}$ using Lemma 3.2:

$$S_{kl} \cdot [0,0,\alpha,\beta,\gamma,\delta] = [0,0,\alpha + 1,\beta,\gamma + 1,\delta] - [0,0,\alpha,\beta - 1,\gamma,\delta - 1] \mod \mathcal{N}_\nu;$$

$$T_{kl} \cdot [0,0,\alpha,\beta,\gamma,\delta] = [0,0,\alpha,\beta + 1,\gamma,\delta + 1] - [0,0,\alpha - 1,\beta,\gamma - 1,\delta] \mod \mathcal{N}_\nu.$$

This shows that $H_{p,q,\nu}$ is irreducible as a $(\mathfrak{u}(p,q), \tilde{U}(p) \times \tilde{U}(q))$ module. Unitarity follows from the observation that they are restrictions of the unitarizable $(\mathfrak{o}(2p,2q), \tilde{O}(2p) \times \tilde{O}(2q))$ module $L_{2p,2q}$.
Now we compute the Howe quotient corresponding to the trivial representation of $U(p, q)$ for the dual pair $(U(p, q), U(1, 1))$. Assume that $p, q \geq 1$. If
\[ \mathcal{P} = \mathbb{C}[z_1, \ldots, z_p, \bar{z}_1, \ldots, \bar{z}_p, w_1, \ldots, w_q, \bar{w}_1, \ldots, \bar{w}_q] \]
as before, let
\[ \mathcal{N}'_\mathbb{H} = \text{Span}\{Xf | f \in \mathcal{P}, X \in u(p, q)_{\mathbb{C}}\}. \]
Then by Proposition 2.3, the Howe quotient is $\mathcal{P}/\mathcal{N}'_\mathbb{H}$. Recall that the action of $u(p, q)_{\mathbb{C}}$ and $u(1, 1)_{\mathbb{C}}$ is given in (3.3).

**Lemma 3.5.** Let $1, Z,$ and $W$ be the elements of $\mathcal{P}/\mathcal{N}'_\mathbb{H}$ given by
\[ \bar{I} = 1 + \mathcal{N}'_\mathbb{H}, \]
\[ Z = z_1 \bar{z}_1 + \mathcal{N}'_\mathbb{H}, \]
\[ W = w_1 \bar{w}_1 + \mathcal{N}'_\mathbb{H}. \]
Then $B = \{\bar{I}, Z^k, W^l | k, l \in \mathbb{Z}_{\geq 1}\}$ is a basis of $\mathcal{P}/\mathcal{N}'_\mathbb{H}$.

**Proof.** For $\lambda = (\lambda_1, \ldots, \lambda_p, \bar{\lambda}_1, \ldots, \bar{\lambda}_p)$ and $\mu = (\mu_1, \ldots, \mu_q, \bar{\mu}_1, \ldots, \bar{\mu}_q)$ with $\lambda_i, \bar{\lambda}_i, \mu_j, \bar{\mu}_j \in \mathbb{Z}_{\geq 0}$, let $z^\lambda w^\mu$ be the monomial $\prod_{i=1}^{\lambda_i} z_i^{\lambda_i} \prod_{j=1}^{\mu_j} w_j^{\mu_j}$. For $1 \leq i, j \leq p$, and arbitrary $z^\lambda w^\mu$, we have that
\[ \left( z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} \right) z_i z_j z^\lambda w^\mu = (\lambda_j + 1) z_i z_i - (\bar{\lambda}_i + 1) z_j z_j ) z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}, \]
so that
\[(3.6a) z_i \bar{z}_i z^\lambda w^\mu \equiv \frac{\lambda_i + 1}{\lambda_j + 1} z_j \bar{z}_j z^\lambda w^\mu \mod \mathcal{N}'_\mathbb{H}.\]

Similarly, we have for $1 \leq i, j \leq q$,
\[(3.6b) w_i \bar{w}_i z^\lambda w^\mu \equiv \frac{\mu_i + 1}{\mu_j + 1} w_j \bar{w}_j z^\lambda w^\mu \mod \mathcal{N}'_\mathbb{H}.\]

If $\bar{\lambda}_i = 0$, then
\[ \left( z_i \frac{\partial}{\partial z_j} - z_j \frac{\partial}{\partial z_i} \right) z_j z^\lambda w^\mu = (\lambda_j + 1) z_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}. \]
This implies that $z_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}$ whenever $\lambda_i = 0$. Similarly, $\bar{z}_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}$ whenever $\lambda_i = 0$, $w_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}$ whenever $\mu_i = 0$, and $\bar{w}_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}$ whenever $\mu_i = 0$. Using (3.6) with $i = j$, we see that if $\lambda_i \neq \bar{\lambda}_i$ for some $i$, then $z_i \bar{z}_i z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}$ (and similarly if $\mu_i \neq \bar{\mu}_i$), so that we have
\[(3.7) \lambda_i \neq \bar{\lambda}_i \text{ for some } i \text{ or } \mu_i \neq \bar{\mu}_i \text{ for some } i \Rightarrow z^\lambda w^\mu \in \mathcal{N}'_\mathbb{H}. \]

For $1 \leq i \leq p$ and $1 \leq j \leq q$, applying the operator $z_i w_j - \frac{\partial^2}{\partial z_i \partial w_j}$ to the monomial $\bar{z}_i \bar{w}_j z^\lambda w^\mu$ yields
\[(3.8) z_i \bar{z}_i w_j \bar{w}_j z^\lambda w^\mu \equiv (\bar{\lambda}_i + 1)(\bar{\mu}_j + 1) z^\lambda w^\mu \mod \mathcal{N}'_\mathbb{H}. \]
Using (3.6), (3.7), and (3.8), we see that every monomial in $\mathcal{P}$ is either in $\mathcal{N}'_\mathbb{H}$ or in one of the cosets listed in $B$. So we have that $B$ spans $\mathcal{P}/\mathcal{N}'_\mathbb{H}$. To
see that the elements of $B$ are linearly independent, we notice that each is a weight vector for $(\mathfrak{u}(1) \oplus \mathfrak{u}(1))_C \subset \mathfrak{u}(1,1)_C$, of the following weights:

$$
\begin{align*}
\tilde{1} & \text{ has weight } \left( \frac{p-q}{2}, -\frac{p-q}{2} \right), \\
Z^k & \text{ has weight } \left( \frac{p-q}{2} + k, -\frac{p-q}{2} - k \right), \\
W^k & \text{ has weight } \left( \frac{p-q}{2} - k, -\frac{p-q}{2} + k \right).
\end{align*}
$$

(3.9)

Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. \qed

**Remark.** The proof of Lemma 3.5 also shows that all $\tilde{U}(1) \times \tilde{U}(1)$-types in $\mathcal{P}/\mathcal{N}'_1$ have multiplicity one.

**Theorem 3.6.** Suppose $p, q \geq 1$. The trivial $(\mathfrak{u}(p,q), U(p) \times U(q))$-module $\mathbb{I}$ belongs to $\mathcal{R}(\mathfrak{u}(p,q), U(p) \times U(q), \omega)$. The Howe quotient $\mathcal{P}/\mathcal{N}'_1$ has two irreducible $(\mathfrak{u}(1,1), \tilde{U}(1) \times \tilde{U}(1))$-submodules $V_1$ and $V_2$ spanned by $\{Z^k|k \geq q\}$ and $\{W^k|k \geq p\}$ respectively, which are discrete series representations with minimal $\tilde{U}(1) \times \tilde{U}(1)$-types $\left( \frac{p-q}{2}, -\frac{p-q}{2} \right)$ and $\left( -\frac{p+q}{2}, \frac{p+q}{2} \right)$ respectively. The theta-lift of $\mathbb{I}$ is the irreducible quotient of $\mathcal{P}/\mathcal{N}'_1$ by $V_1 \oplus V_2$, the $(\mathfrak{u}(1,1), \tilde{U}(1) \times \tilde{U}(1))$-module of dimension $p+q-1$, with $\tilde{U}(1) \times \tilde{U}(1)$-types $\{(k+\frac{p-q}{2}, -k-\frac{p-q}{2})| -q+1 \leq k \leq p-1\}$. The theta-lift of $\mathbb{I}$ is unitarizable only in the case when $p = q = 1$.

**Remark.** The situation in this case is very much controlled by the situation for the dual pair $(O(2p,2q), SL(2, \mathbb{R}))$ (see Theorem 2.7(ii)). Again, Lee and Zhu [LZ2] has treated these representations of $U(n,n)$ for the dual pairs $(U(p,q), U(n,n))$ and have explicit information on the structure of the corresponding Howe quotients.

**Proof.** Using the formulas (3.3), (3.6), and (3.8), we compute the action of $p' = \text{Span}\{E,F\}$ on the weight vectors in $B$:

$$
\begin{align*}
E \cdot \tilde{1} & = pZ; \\
E \cdot Z^k & = \frac{k+p}{k+1}Z^{k+1}; \\
E \cdot W^k & = k(p-k)W^{k-1}; \\
F \cdot \tilde{1} & = qW; \\
F \cdot Z^k & = k(q-k)Z^{k-1}; \\
F \cdot W^k & = \frac{k+q}{k+1}W^{k+1}.
\end{align*}
$$

Notice that $E$ annihilates $W^p$, and takes all weight vectors of weight $\left( k + \frac{p-q}{2}, -k - \frac{p-q}{2} \right)$ to weight vectors of weight $\left( k + 1 + \frac{p-q}{2}, -k - 1 - \frac{p-q}{2} \right)$. Similarly, $F$ annihilates $Z^q$ and takes all weight vectors of weight $\left( k + \frac{p-q}{2}, -k - \frac{p-q}{2} \right)$ to weight vectors of weight $\left( k - 1 + \frac{p-q}{2}, -k + 1 - \frac{p-q}{2} \right)$. The result
follows since we know that only the trivial finite-dimensional module is unitarizable.

4. The Dual Pairs $(Sp(p, q), O^*(4))$.

Consider the dual pair $(Sp(p), O^*(4))$ acting on the $\widetilde{U}(4p)$-finite vectors of the associated Fock space $\mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}]$ as follows:

\[(4.1)\]

(a) Action of $\mathfrak{sp}(p)_C$:
\[
\begin{align*}
(i) & \quad z_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i} - z_{p+j} \frac{\partial}{\partial z_{p+i}} + \bar{z}_{p+i} \frac{\partial}{\partial \bar{z}_{p+j}}, & & 1 \leq i, j \leq p; \\
(ii) & \quad \frac{1}{z_i} \frac{\partial}{\partial z_{p+j}} - \frac{1}{\bar{z}_j} \frac{\partial}{\partial \bar{z}_i} + \frac{1}{z_{p+j}} \frac{\partial}{\partial z_{p+i}} - \frac{1}{\bar{z}_{p+i}} \frac{\partial}{\partial \bar{z}_{p+j}}, & & 1 \leq i \leq j \leq p; \\
(iii) & \quad \frac{1}{z_{p+j}} \frac{\partial}{\partial z_{p+i}} - \frac{1}{\bar{z}_{p+i}} \frac{\partial}{\partial \bar{z}_{p+j}} + \frac{1}{z_{p+i}} \frac{\partial}{\partial z_{p+j}} - \frac{1}{\bar{z}_{p+j}} \frac{\partial}{\partial \bar{z}_{p+i}}, & & 1 \leq i \leq j \leq p.
\end{align*}
\]

(b) Action of $\mathfrak{o}^*(4)_C = \text{Span}\{E^z_1 + p, E^z_{22} + p, E^z_{12}, E^z_{21}, r_1^2, \Delta_1\}$:
\[
\begin{align*}
E^z_1 &= \sum_{i=1}^{2p} z_i \frac{\partial}{\partial z_i}, & & E^z_{22} = \sum_{i=1}^{2p} \bar{z}_i \frac{\partial}{\partial \bar{z}_i}, & & E^z_{12} = \sum_{i=1}^{2p} \left( z_{p+i} \frac{\partial}{\partial z_i} - z_i \frac{\partial}{\partial z_{p+i}} \right), \\
E^z_{21} &= \sum_{i=1}^{2p} \left( \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - \bar{z}_{p+i} \frac{\partial}{\partial z_{p+i}} \right), & & r_1^2 = \sum_{i=1}^{2p} z_i \bar{z}_i, & & \Delta_1 = \sum_{i=1}^{2p} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.
\end{align*}
\]

Observe that
\[
\mathfrak{o}^*(4)_C \simeq \text{Span}\{E^z_1 - E^z_{22}, E^z_{12}, E^z_{21}\} \oplus \text{Span}\{E^z_1 + E^z_{22} + 2p, r_1^2, \Delta_1\}
\]
\[
\simeq \mathfrak{su}(2)_C \oplus \mathfrak{sl}(2)_C \implies O^*(4) \simeq (SU(2) \times SL(2, \mathbb{R}))/\{\pm I\}.
\]

Define the spherical harmonics as in the last section:
\[
\mathcal{H}(\mathbb{C}^{2p}) = \{ f \in \mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}] \mid \Delta_1 f = 0 \}.
\]

We have the following decomposition (see [HT2]):
\[
\mathcal{H}(\mathbb{C}^{2p})|_{Sp(p) \times SU(2)} = \sum_{\xi_1, \xi_2 \geq 0} K^{(p)}_{(\xi_1, \xi_2, \xi_2)} \otimes V^{z_{\xi_1}}_1,
\]
where $SU(2)$ is the group with (complexified) Lie algebra $\{E^z_1 - E^z_{22}, E^z_{12}, E^z_{21}\}$, $V^{z_{\xi_1}}_1$ is the irreducible unitary representation of $SU(2)$ of dimension $m+1$, and $K^{(p)}_{(\xi_1, \xi_2, \xi_2)}$ is the $Sp(p)$ module with highest weight $(\xi_1 + \xi_2, \xi_2, 0, \ldots, 0)$ with respect to the Cartan subalgebra spanned by
\[
\left\{ z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} - z_{p+i} \frac{\partial}{\partial z_{p+i}} + \bar{z}_{p+i} \frac{\partial}{\partial \bar{z}_{p+i}} \mid i = 1, \ldots, p \right\}.
\]
The joint $Sp(p) \times SU(2)$ highest weight vector of $K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1}$ is given by

\[(4.2) \quad \gamma_{(\xi_1, \xi_2)} = \begin{vmatrix} z_1 & \bar{z}_{p+1} \\ z_2 & \bar{z}_{p+2} \end{vmatrix}^{\xi_2} \cdot \]

More precisely, there are $\xi_1 + 1$ copies of the $Sp(p)$ representations in $K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1}$, and the $Sp(p)$ highest weight vectors are

\[u_{(p, \xi_1, \xi_2, j)} = \frac{z_j}{\bar{z}_{p+1}} v_{(p, \xi_1+\xi_2-j, \xi_2)} = \begin{vmatrix} z_1 & \bar{z}_{p+1} \\ z_2 & \bar{z}_{p+2} \end{vmatrix}^{\xi_2}, \quad j = 0, 1, \ldots, \xi_1.\]

We shall denote the $Sp(p)$ module with highest weight vector $u_{(p, \xi_1, \xi_2, j)}$ by $K^{(p)}_{(\xi_1+\xi_2, \xi_2), j}$. Thus

\[\mathcal{H}(\mathbb{C}^{2p} |_{Sp(p)}) = \sum_{\xi_1 \geq 0, \xi_2 \geq 0, j=0, \ldots, \xi_1} K^{(p)}_{(\xi_1+\xi_2, \xi_2), j} \]

We also note that $K^{(p)}_{(\xi_1+\xi_2, \xi_2), j} \subset \mathcal{H}^{(2p)}_{\xi_1+\xi_2-j, \xi_2+j}$. In particular, we can extract the decomposition of $\mathcal{H}^{(2p)}_{\alpha, \beta}$ into $Sp(p)$ modules:

\[\mathcal{H}^{(2p)}_{\alpha, \beta} |_{Sp(p)} = K^{(p)}_{(\alpha+\beta, 0), \beta} \oplus K^{(p)}_{(\alpha+\beta-1, 1), \beta-1} \oplus K^{(p)}_{(\alpha+\beta-2, 2), \beta-2} \]

\[\oplus \ldots \oplus K^{(p)}_{(\max(\alpha, \beta), \min(\alpha, \beta)), \beta-\min(\alpha, \beta) \cdot} \]

The duality correspondence is as follows:

\[\mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}] |_{Sp(p) \times O^+(4)} = \sum_{\xi_1, \xi_2 = 0}^{\infty} K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1} \otimes V_{p+\xi_1+2\xi_2},\]

where $V_{p+\xi_1+2\xi_2}$ is the $SL(2, \mathbb{R})$ lowest weight module of lowest weight $p + \xi_1 + 2\xi_2$ spanned by $\{(r^2)_{(\xi_1+\xi_2)}i \gamma_{(\xi_1, \xi_2)} | i = 0, 1, \ldots \}$. The duality correspondence enables us to write

\[(4.3) \quad \mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}] |_{Sp(p) \times SU(2)} = \sum_{i, \xi_1, \xi_2 = 0}^{\infty} (r^2)_{(\xi_1+\xi_2)} i K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1},\]

where $K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1}$ are $Sp(p) \times SU(2)$ modules isomorphic to $K^{(p)}_{(\xi_1+\xi_2, \xi_2)} \otimes V_1^{\xi_1}$. 
Likewise, the dual pair \((Sp(q), O^*(4))\) acting on \(\mathbb{C}[w_1, \ldots, w_{2q}, \bar{w}_1, \ldots, \bar{w}_{2q}]\) gives rise to the following decomposition as \(Sp(q) \times SU(2)\) modules:

\[
\mathbb{C}[w_1, \ldots, w_{2q}, \bar{w}_1, \ldots, \bar{w}_{2q}]|_{Sp(q) \times SU(2)} = \sum_{i, \eta_1, \eta_2=0}^{\infty} (r_2^2)^i \mathcal{K}_{(\eta_1+\eta_2, \eta_2)}^{(p)} \otimes V_1^{\eta_1},
\]

where \((r_2^2)^i \mathcal{K}_{(\eta_1+\eta_2, \eta_2)}^{(p)} \otimes V_1^{\eta_1}\) are \(Sp(q) \times SU(2)\) modules isomorphic to \(\mathcal{K}_{(\eta_1+\eta_2, \eta_2)}^{(q)} \otimes V_1^{\eta_1}\), which are analogously defined as in the \((Sp(p), O^*(4))\) case.

Consider \(\mathcal{P} = \mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}, w_1, \ldots, w_{2q}, \bar{w}_1, \ldots, \bar{w}_{2q}]\). This is the space of \(\bar{U}(4p+4q)\)-finite vectors of the associated Fock model for the dual pair \((Sp(p, q), O^*(4))\), and the actions of the complexified Lie algebras of \(Sp(p, q)\) and \(O^*(4)\) can be described as follows:

**Lemma 4.1.**

(a) **Action of \(sp(p, q)_{\mathbb{C}} = sp(p)_{\mathbb{C}} \oplus sp(q)_{\mathbb{C}} \oplus \mathfrak{p}\):**

(i) **Action of \(sp(p)_{\mathbb{C}}\): as in (4.1);

(ii) **Action of \(sp(q)_{\mathbb{C}}\): similar to (4.1);

(iii) **Action of \(\mathfrak{p}\):**

\[
P_{ij} = z_i w_j - \frac{\partial^2}{\partial z_i \partial w_j} + z_{p+i} \bar{w}_{q+j} - \frac{\partial^2}{\partial z_{p+i} \partial w_{q+j}}, \quad 1 \leq i \leq p, \ 1 \leq j \leq q;
\]

\[
Q_{ij} = z_i w_{q+j} - \frac{\partial^2}{\partial z_i \partial w_{q+j}} - z_{p+i} \bar{w}_j + \frac{\partial^2}{\partial z_{p+i} \partial w_j}, \quad 1 \leq i \leq p, \ 1 \leq j \leq q;
\]

\[
R_{ij} = z_{p+i} w_j - \frac{\partial^2}{\partial z_{p+i} \partial w_j} - z_i \bar{w}_{q+j} + \frac{\partial^2}{\partial z_i \partial w_{q+j}}, \quad 1 \leq i \leq p, \ 1 \leq j \leq q;
\]

\[
S_{ij} = z_{p+i} w_{q+j} - \frac{\partial^2}{\partial z_{p+i} \partial w_{q+j}} + z_i \bar{w}_j - \frac{\partial^2}{\partial z_i \partial w_j}, \quad 1 \leq i \leq p, \ 1 \leq j \leq q.
\]

(b) **Action of \(O^*(4)_{\mathbb{C}} = \text{Span}\{E_{11} + p - q, E_{22} + p - q, E_{12}, E_{21}, E, F\}:**

\[
E_{11} = E_{11}^z - E_{11}^w, \quad E_{22} = E_{22}^z - E_{22}^w, \quad E_{12} = E_{12}^z - E_{21}^w, \quad E_{21} = E_{21}^z - E_{12}^w, \quad E = r_1^2 - \Delta_2, \quad F = r_2^2 - \Delta_1.
\]

**Proof.** Omitted. \(\square\)

We note that \(\bar{S}p(p, q)\) and \(\bar{O}^*(4)\) are both split extensions. Because of the decompositions (4.3) and (4.4), we have the following decomposition of \(\mathcal{P}\) as \(Sp(p) \times Sp(q) \times SU(2)\) modules:

\[
\mathcal{P}|_{Sp(p) \times Sp(q) \times SU(2)} = \sum (r_1^2)^i (r_2^2)^j \mathcal{K}_{(\xi_1, \xi_2)}^{(p)} \otimes \mathcal{K}_{(\eta_1+\eta_2, \eta_2)}^{(q)} \otimes V_1^{\eta_1},
\]
where the sum is over the 7-tuples $(i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu) \in \mathbb{Z}_{\geq 0}$ such that
\[
|\xi_1 - \eta_1| \leq \mu \leq \xi_1 + \eta_1,
\]
\[
\mu \equiv \xi_1 + \eta_1 \mod 2.
\]
This comes about by a direct application of the Clebsch-Gordan formula for the decomposition of a tensor product of two $SU(2)$ modules:
\[
(4.6) \quad V_1^{\xi_1} \otimes V_1^{\eta_1} = V_1^{\xi_1 + \eta_1} \oplus V_1^{\xi_1 + \eta_1 - 2} \oplus V_1^{\xi_1 + \eta_1 - 4} \oplus \ldots \oplus V_1^{[\xi_1 - \eta_1]}.
\]
Observe that $Sp(q)$ acts in a contragredient fashion, so to obtain a set of $Sp(p) \times Sp(q) \times SU(2)$ highest weight vectors in $\mathcal{P}$, we need a little adjustment. Recall that
\[
\gamma(\xi, \eta) = \frac{\xi_1}{z_1} \begin{vmatrix} \xi_2 & \xi_2 & \xi_2 \\ \eta_1 & \eta_2 & \eta_2 \\ 1 & 1 & 1 \end{vmatrix}
\]
is a joint $Sp(p) \times SU(2)$ highest weight vector. In a similar way,
\[
(4.7) \quad \theta_{(\eta_1, \eta_2)} = \frac{\bar{w}_{\eta_1+1}}{w_{\eta_1}} \begin{vmatrix} w_1 & w_{\eta_1+1} & w_{\eta_1+2} \\ \eta_1 & \eta_2 & \eta_2 \\ 1 & 1 & 1 \end{vmatrix}
\]
is a joint $Sp(q) \times SU(2)$ highest weight vector (relative to another choice of positive system for $SU(2)$) satisfying
\[
-E_{11}^\mu \theta_{(\eta_1, \eta_2)} = -\eta_2 \theta_{(\eta_1, \eta_2)};
\]
\[
-E_{22}^\mu \theta_{(\eta_1, \eta_2)} = -(\eta_1 + \eta_2) \theta_{(\eta_1, \eta_2)};
\]
\[
-E_{21}^\mu \theta_{(\eta_1, \eta_2)} = 0.
\]
We will take as a “basis” for $\mathcal{P}$ elements of the form (see (4.2) and (4.7) for the definitions of $\gamma(\xi, \eta)$ and $\theta_{(\eta_1, \eta_2)}$)
\[
(4.8) \quad [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (\frac{q}{2})^i (\frac{q}{2})^j \gamma(\xi_1 - \nu, \xi_2) \theta_{(\eta_1 - \nu, \eta_2)} (z_1 w_1 + \bar{z}_{\eta_1+1} \bar{w}_{\eta_1+1})^\nu,
\]
where $\mu = \xi_1 + \eta_1 - 2\nu$ and $|\xi_1 - \eta_1| \leq \mu \leq \xi_1 + \eta_1$.

Note that if we set $i = j = 0$, then
\[
E_{11}[0, 0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (\xi_1 + \xi_2 - \eta_2 - \nu)[0, 0, 0, 0, \xi_2, \eta_1, \eta_2, \mu],
\]
\[
E_{22}[0, 0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (\xi_1 - \eta_1 + \eta_2 + \nu)[0, 0, 0, 0, \xi_2, \eta_1, \eta_2, \mu],
\]
\[
E_{12}[0, 0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = 0.
\]
In other words, $[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$ is the set of joint $Sp(p) \times Sp(q) \times SU(2)$ pluriharmonics (up to multiples, of course).

**Lemma 4.2.** The actions of $sl(2)_C \subset \sigma^*(4)$ on the basis in (4.8) are as follows:

(a) $E_{11} \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (2i + \xi_1 + \xi_2 - \eta_2)[i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$;
(b) $E_{22} \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = (-2j + \xi_2 - \eta_1 - \eta_2)[i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$;
(c) $E \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = [i + 1, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]$.
\[ j(2q + \eta_1 + 2\eta_2 + j - 1)[i, j - 1, \xi_1, \xi_2, \eta_1, \eta_2, \mu]; \]
\[ F \cdot [i, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu] = [i, j + 1, \xi_1, \xi_2, \eta_1, \eta_2, \mu] \]
\[ - i(2p + \xi_1 + 2\xi_2 + i - 1)[i - 1, j, \xi_1, \xi_2, \eta_1, \eta_2, \mu]. \]

**Proof.** Similar to Lemma 2.2. \qed

**Proposition 4.3.** Consider the basis of \( \mathcal{P} \) as in (4.8). Then \( V_1^\mu \otimes \mathbb{I} \in \mathcal{R}(\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\}, \omega) \) if and only if \( \mu \) is even. If \( \mu \) is even, the Howe quotient corresponding to \( V_1^\mu \otimes \mathbb{I} \) (i.e., the trivial representation of \( SL(2, \mathbb{R}) \) twisted by a unitary representation of \( SU(2) \)) of \( O^*(4) \) is

\[ \mathcal{P}/\mathcal{N}_\mu = \text{Span of (images of)} \{ [0, 0, \xi_1, \eta_1, \eta_2, \mu] \mid |\xi_1 - \eta_1| \leq \mu \leq \xi_1 + \eta_1, \mu \equiv \xi_1 + \eta_1 \mod 2, \xi_1 + 2\xi_2 + 2p - 2q = \eta_1 + 2\eta_2 \}. \]

In particular,

\[ \mathcal{P}/\mathcal{N}_\mu \simeq L_\mu \otimes (V_1^\mu \otimes \mathbb{I}) \]

is an irreducible \((\mathfrak{sp}(p, q), Sp(p) \times Sp(q)) \times (\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\})\) module, so the theta lift of \( V_1^\mu \otimes \mathbb{I} \) is the irreducible and unitarizable representation \( L_\mu \).

**Remark.** This is not surprising; in fact, the restriction of the ladder representation \( L_{4p,4q} \) of \( O(4p,4q) \) to \( Sp(p, q) \) decomposes as follows:

\[ L_{4p,4q}|_{Sp(p,q) \times SU(2)} = \sum_{\mu=0, \mu \text{ even}}^\infty L_\mu \otimes V_1^\mu. \]

Here \( SU(2) \) is not embedded in \( O(4p,4q) \). It arises from the exponentiated action of the Lie algebra \( \mathfrak{so}(2) = \text{Span} \{E_{11} - E_{22}, E_{12}, E_{21}\} \) (see Lemma 4.1). We shall call the \( L_\mu \) ladder representations. They have Gelfand-Kirillov dimensions \( 4p + 4q - 6 \) (compare with \( 4p + 4q - 3 \) of \( L_{4p,4q} \)).

**Proof.** Let \( \mathcal{P}/\mathcal{N}_\mu \) be the Howe quotient corresponding to the representation \( V_1^\mu \otimes \mathbb{I} \). As an \( SU(2) \) module, \( \mathcal{P} \) is completely reducible, i.e., \( \mathcal{P} = \sum_{\mu \in \mathbb{Z}} \mathcal{P}_\mu \) where \( \mathcal{P}_\mu \) denotes the \( V_1^\mu \)-isotypic component of \( \mathcal{P} \). Thus as an \( SU(2) \) module,

\[ \mathcal{P}/\mathcal{N}_\mu \simeq \mathcal{P}_\mu/(\mathcal{N}_\mu \cap \mathcal{P}_\mu). \]

Let \( \mathcal{N}_\mathbb{I} = \{Xf \mid X \in \mathfrak{sl}(2), f \in \mathcal{P}\} \). Since \( \mathfrak{sl}(2) \) acts trivially, we have \( \mathcal{N}_\mathbb{I} \cap \mathcal{P}_\mu \subset \mathcal{N}_\mu \cap \mathcal{P}_\mu \). But \( \mathcal{N}_\mu = \cap \mathcal{N} \) where \( \mathcal{N} \subset \mathcal{P} \) is such that \( \mathcal{P}/\mathcal{N} \simeq \mathcal{P}_\mu/(\mathcal{N} \cap \mathcal{P}_\mu) \simeq V_1^\mu \otimes \mathbb{I} \) as \( SU(2) \) modules. Thus \( \mathcal{N}_\mu \cap \mathcal{P}_\mu = \cap (\mathcal{N} \cap \mathcal{P}_\mu) \) is the smallest subspace in \( \mathcal{P}_\mu \) such that \( \mathcal{P}_\mu/(\mathcal{N}_\mu \cap \mathcal{P}_\mu) \simeq V_1^\mu \otimes \mathbb{I} \). Hence \( \mathcal{N}_\mathbb{I} \cap \mathcal{P}_\mu = \mathcal{N}_\mu \cap \mathcal{P}_\mu \) and thus

\[ \mathcal{P}/\mathcal{N}_\mu \simeq \mathcal{P}_\mu/(\mathcal{N}_\mathbb{I} \cap \mathcal{P}_\mu) \]

as \((\mathfrak{o}^*(4), (SU(2) \times SO(2))/\{\pm I\})\) modules.
From the above and Lemma 4.2, the \((\mathfrak{sp}(p, q), Sp(p) \times Sp(q))\) module \(L_\mu\) has a multiplicity free \(Sp(p) \times Sp(q)\) spectrum:

\[
L_\mu|_{Sp(p) \times Sp(q)} = \sum \mathcal{K}^{(p)}_{(\xi_1 + \xi_2, \xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1 + \eta_2, \eta_2)},
\]

where the sum runs through non-negative integer tuples \((\xi_1, \xi_2, \eta_1, \eta_2)\) satisfying

\[
\begin{align*}
|\xi_1 - \eta_1| &\leq \mu \leq \xi_1 + \eta_1, \\
\mu &\equiv \xi_1 + \eta_1 \mod 2, \\
\xi_1 + 2\xi_2 + 2\eta_1 - 2\eta_2 &= \eta_1 + 2\eta_2.
\end{align*}
\]

The above relations show that \(\mu\) must be even.

There are several ways to show irreducibility. In the spirit of this paper, we note the transitions from \(\mathcal{K}^{(p)}_{(\xi_1 + \xi_2, \xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1 + \eta_2, \eta_2)}\) to neighbouring \(Sp(p) \times Sp(q)\) types as follows. Let the operators \(P_{11}, P_{12}\) and \(P_{21}\) from \(\mathfrak{p}\) be given as in Lemma 4.1 and

\[
\begin{align*}
X_{21} &= z_2 \frac{\partial}{\partial z_2} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_2} - z_{p+1} \frac{\partial}{\partial z_{p+2}} + \bar{z}_{p+2} \frac{\partial}{\partial \bar{z}_{p+1}}, \\
Y_{21} &= w_2 \frac{\partial}{\partial w_2} - \bar{w}_1 \frac{\partial}{\partial \bar{w}_2} - w_{p+1} \frac{\partial}{\partial w_{p+2}} + \bar{w}_{p+2} \frac{\partial}{\partial \bar{w}_{p+1}}
\end{align*}
\]

be operators coming from \(\mathfrak{sp}(p)\) and \(\mathfrak{sp}(q)\) (see (4.1)). Then the formulae

\[
\begin{align*}
(a) \quad P_{11}[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] &= [0, 0, \xi_1 + 1, \xi_2, \eta_1 + 1, \eta_2, \mu], \\
(b) \quad (P_{11}X_{21} - \xi_1 P_{21})[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] &= \left(\frac{\eta_1 - \xi_1 - \mu}{2}\right) [0, 0, \xi_1 - 1, \xi_2 + 1, \eta_1 + 1, \eta_2, \mu], \\
(c) \quad (P_{11}Y_{21} - \eta_1 P_{12})[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] &= \left(\frac{\eta_1 + \mu - \xi_1}{2}\right) [0, 0, \xi_1 + 1, \xi_2, \eta_1 - 1, \eta_2 + 1, \mu], \\
(d) \quad (P_{11}X_{21}Y_{21} - \eta_1 P_{12}Y_{21} - \xi_1 P_{12}X_{21} + \xi_1 \eta_1 P_{12}P_{21})[0, 0, \xi_1, \xi_2, \eta_1, \eta_2, \mu] &= \left(\frac{\xi_1 + \eta_1 - \mu}{2} - \frac{(\eta_1 - \xi_1 - \mu)(\xi_1 - \eta_1 - \mu)}{4}\right) \cdot [0, 0, \xi_1 - 1, \xi_2 + 1, \eta_1 - 1, \eta_2 + 1, \mu],
\end{align*}
\]

describes the transitions (of \(Sp(p) \times Sp(q)\)-types) \(\mathcal{K}^{(p)}_{(\xi_1 + \xi_2, \xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1 + \eta_2, \eta_2)}\) to \(\mathcal{K}^{(p)}_{(\xi_1 + \xi_2 + 1, \xi_2)} \otimes \mathcal{K}^{(q)}_{(\eta_1 + \eta_2 + 1, \eta_2)}\), and to \(\mathcal{K}^{(p)}_{(\xi_1 + 2\xi_2 - 2, \xi_2 + 1)} \otimes \mathcal{K}^{(q)}_{(\eta_1 + \eta_2, \eta_2)}\) respectively. Noting that the lowest joint harmonic (see [Ho2]) has trivial \(Sp(p) \times Sp(q)\) type, these transitions are enough to show that the \((\mathfrak{sp}(p, q), Sp(p) \times Sp(q))\) module \(L_\mu\)
is irreducible. Unitarity follows from the unitarizability of the \((\mathfrak{o}(4p, 4q), \widehat{O}(4p) \times \widehat{O}(4q))\) module \(L_{4p, 4q}\).

Now we compute the Howe quotient corresponding to the trivial representation of \(Sp(p, q)\) for the dual pair \((Sp(p, q), O^*(4))\). Assume that \(p, q \geq 1\).

If
\[
\mathcal{P} = \mathbb{C}[z_1, \ldots, z_{2p}, \bar{z}_1, \ldots, \bar{z}_{2p}, w_1, \ldots, w_{2q}, \bar{w}_1, \ldots, \bar{w}_{2q}]
\]
as before, let
\[
\mathcal{N}'_{\mathbb{H}} = \text{Span}\{Xf | f \in \mathcal{P}, X \in \mathfrak{sp}(p, q)_{\mathbb{C}}\}.
\]
Then by Proposition 2.3, the Howe quotient is \(\mathcal{P}/\mathcal{N}'_{\mathbb{H}}\). Recall that the action of \(\mathfrak{sp}(p, q)_{\mathbb{C}}\) and \(\mathfrak{o}^*(4)_{\mathbb{C}} \equiv (\mathfrak{su}(2) \oplus \mathfrak{s}(2))_{\mathbb{C}}\) is given in Lemma 4.1.

**Lemma 4.4.** Let \(\bar{1}, Z, \) and \(W\) be the elements of \(\mathcal{P}/\mathcal{N}'_{\mathbb{H}}\) given by
\[
\bar{1} = 1 + \mathcal{N}'_{\mathbb{H}},
\]
\[
Z = z_1 \bar{z}_1 + \mathcal{N}'_{\mathbb{H}},
\]
\[
W = w_1 \bar{w}_1 + \mathcal{N}'_{\mathbb{H}}.
\]
Then \(B = \{\bar{1}, Z^k, W^l | k, l \in \mathbb{Z}_{\geq 1}\}\) is a basis of \(\mathcal{P}/\mathcal{N}'_{\mathbb{H}}\).

**Proof.** For \(\lambda = (\lambda_1, \ldots, \lambda_2p, \bar{\lambda}_1, \ldots, \bar{\lambda}_2p)\) and \(\mu = (\mu_1, \ldots, \mu_2q, \bar{\mu}_1, \ldots, \bar{\mu}_2q)\) with \(\lambda_i, \bar{\lambda}_i, \mu_j, \bar{\mu}_j \in \mathbb{Z}_{\geq 0}\), let \(z^\lambda w^\mu\) be the monomial \(\prod_{i=1}^{2p} z_i^{\lambda_i} \bar{z}_i^{\bar{\lambda}_i} \prod_{i=1}^{2q} w_i^{\mu_i} \bar{w}_i^{\bar{\mu}_i}\).

For \(1 \leq i \leq p\), and arbitrary \(z^\lambda w^\mu\), and using (4.1)(a)(ii) with \(i = j\), we have that
\[
\left(\frac{\partial}{\partial z_{p+i}} - \frac{\partial}{\partial \bar{z}_{p+i}}\right) \bar{z}_{p+i} z^\lambda w^\mu = \left((\lambda_{p+i} + 1)z_i \bar{z}_i - (\bar{\lambda}_i + 1)z_{p+i} \bar{z}_{p+i}\right) z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{H}},
\]
so that
\[
(4.9a) \Rightarrow z_i \bar{z}_i z^\lambda w^\mu \equiv \frac{\lambda_i + 1}{\lambda_{p+i} + 1} z_{p+i} \bar{z}_{p+i} z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{H}}.
\]
Similarly, we have for \(1 \leq i \leq q\),
\[
(4.9b) \Rightarrow w_i \bar{w}_i z^\lambda w^\mu \equiv \frac{\mu_i + 1}{\mu_{q+i} + 1} w_{q+i} \bar{w}_{q+i} z^\lambda w^\mu \mod \mathcal{N}'_{\mathbb{H}}.
\]
If \(\lambda_i = 0\), then using (4.1)(a)(iii) with \(i = j\) we get
\[
\left(\frac{\partial}{\partial z_i} - \frac{\partial}{\partial \bar{z}_i}\right) \bar{z}_{p+i} z^\lambda w^\mu = -\left((\lambda_{p+i} + 1)z_i \bar{z}_i\right) z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{H}}.
\]
This implies that \(z_i \bar{z}_{p+i} z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{H}}\) whenever \(\lambda_i = 0\). Analogous statements hold for the cases \(\bar{\lambda}_i = 0\), \(\lambda_{p+i} = 0\), \(\bar{\lambda}_{p+i} = 0\), and for \(1 \leq i \leq 2q\), \(\mu_i = 0\), and \(\bar{\mu}_i = 0\). Applying (4.9) repeatedly if necessary yields
\[
(4.10) \begin{cases} 
\lambda_i \neq \bar{\lambda}_i \text{ for some } i \leq 2p \\
\text{or} \\
\mu_i \neq \bar{\mu}_i \text{ for some } i \leq 2q
\end{cases} \Rightarrow z^\lambda w^\mu \in \mathcal{N}'_{\mathbb{H}}.
\]
Now suppose \( z^\lambda w^\mu \) satisfies
\[
\lambda_i = \bar{\lambda}_i \text{ for } 1 \leq i \leq 2p \text{ and } \mu_i = \bar{\mu}_i \text{ for } 1 \leq i \leq 2q.
\]
Let \( 1 \leq i, j \leq p \). Then
\[
\left( z_i \frac{\partial}{\partial z_j} - \tilde{z}_j \frac{\partial}{\partial \tilde{z}_i} - z_{p+j} \frac{\partial}{\partial z_{p+i}} + \tilde{z}_{p+i} \frac{\partial}{\partial \tilde{z}_{p+j}} \right) \tilde{z}_i \bar{w}_j z^\lambda w^\mu \\
= (\lambda_j + 1)z_i \bar{z}_j z^\lambda w^\mu - (\bar{\lambda}_i + 1)\bar{z}_j \bar{z}_j z^\lambda w^\mu - M_1 + M_2,
\]
where \( M_1 \) and \( M_2 \) are in \( N'_1 \) by (4.10), so we have that
\[
(4.12a) \quad \tilde{z}_i \bar{z}_i z^\lambda w^\mu \equiv \frac{\lambda_i + 1}{\mu_i + 1} \tilde{z}_j \bar{z}_j z^\lambda w^\mu \mod N'_1.
\]
Similarly,
\[
(4.12b) \quad z_{p+i} \bar{z}_{p+i+1} z^\lambda w^\mu \equiv \frac{\lambda_{p+i} + 1}{\mu_{p+i} + 1} z_{p+j} \bar{z}_{p+j} z^\lambda w^\mu \mod N'_1,
\]
and for \( 1 \leq i, j \leq q \),
\[
(4.12c) \quad w_i \bar{w}_i z^\lambda w^\mu \equiv \frac{\lambda_j + 1}{\mu_j + 1} w_{j} \bar{w}_j z^\lambda w^\mu \mod N'_1,
\]
and
\[
(4.12d) \quad w_{q+i} \bar{w}_{q+i} z^\lambda w^\mu \equiv \frac{\lambda_{q+i} + 1}{\mu_{q+i} + 1} w_{q+j} \bar{w}_{q+j} z^\lambda w^\mu \mod N'_1.
\]
Now suppose again that \( z^\lambda w^\mu \) satisfies (4.11), and that \( 1 \leq i \leq p \) and \( 1 \leq j \leq q \). Then
\[
\left( z_i w_j - \frac{\partial^2}{\partial z_i \partial w_j} + \tilde{z}_{p+i} \bar{w}_{q+j} - \frac{\partial^2}{\partial \tilde{z}_{p+i} \partial \bar{w}_{q+j}} \right) \tilde{z}_i \bar{w}_j z^\lambda w^\mu \\
= z_i \tilde{z}_i w_j \bar{w}_j z^\lambda w^\mu - (\lambda_i + 1)(\mu_j + 1)z^\lambda w^\mu + M_3 - M_4,
\]
where \( M_3 \) and \( M_4 \) are in \( N'_1 \) by (4.10). Consequently,
\[
(4.13a) \quad \tilde{z}_i \bar{w}_j \bar{z}_j z^\lambda w^\mu \equiv (\lambda_i + 1)(\mu_j + 1)z^\lambda w^\mu \mod N'_1.
\]
Similarly,
\[
(4.13b) \quad z_{p+i} \tilde{z}_{p+i} w_{q+j} \bar{w}_{q+j} z^\lambda w^\mu \equiv (\lambda_{p+i} + 1)(\mu_{q+j} + 1)z^\lambda w^\mu \mod N'_1.
\]
Notice that (4.12) and (4.13) also hold if \( z^\lambda w^\mu \in N'_1 \).

Using (4.9), (4.10), (4.12), and (4.13), we see that every monomial in \( \mathcal{P} \)
is either in \( N'_1 \) or in one of the cosets listed in \( B \). So we have that \( B \) spans \( \mathcal{P}/N'_1 \). To see that the elements of \( B \) are linearly independent, we notice that each is a weight vector for \( u(1)_C \subset \mathfrak{sl}(2)_C \), of the following weights:
\[
\bar{1} \text{ has weight } 2(p-q), \\
Z^k \text{ has weight } 2(p-q) + 2k, \\
W^k \text{ has weight } 2(p-q) - 2k.
\]
Since all weights are distinct, the vectors must be linearly independent, and the lemma is proved. \(\Box\)
Remark. The proof of Lemma 3.5 also shows that all $U(1)$-types in $P/N_1'$ have multiplicity one.

Theorem 4.5. Suppose $p,q \geq 1$. The trivial $(sp(p,q),Sp(p) \times Sp(q))$-module $1$ belongs to $\mathcal{R}(sp(p,q),Sp(p) \times Sp(q),\omega)$. Since $O^*(4)$ is a quotient of $SU(2) \times SL(2,\mathbb{R})$, we may regard the Howe quotient $P/N_1'$ as an $(su(2) \oplus sl(2),SU(2) \times U(1))$-module. This module is of the form $1 \otimes V$. The $(sl(2),U(1))$-module $V$ has two irreducible submodules $V_1$ and $V_2$ spanned by $\{Z^k| k \geq 2q\}$ and $\{W^k| k \geq 2p\}$ respectively, which are discrete series representations with minimal $U(1)$-types $2p$ and $-2q$ respectively. The quotient of $V$ by $V_1 \oplus V_2$ is irreducible and of dimension $2(p+q)-1$. If $\sigma_{p,q}$ is the unique irreducible $(sl(2),U(1))$-module of dimension $2(p+q)-1$, then the theta-lift of $1$ is $1 \otimes \sigma_{p,q}$, and it is not unitarizable.

Remark. The situation in this case is again controlled by the situation for the dual pair $(U(2p,2q),U(1,1))$ (see Theorem 3.6) which is in turn controlled by the situation in $(O(4p,4q),SL(2,\mathbb{R}))$ (see Theorem 2.7(ii)).

Proof. Using the formulas of Lemma 4.1, it is easy to confirm that $su(2)$ (and hence $SU(2)$) acts trivially on $P/N_1'$. Using (4.9), (4.12), and (4.13), we compute the action of $p' = \text{Span } \{E,F\} \subset sl(2)_{\mathbb{C}}$ (see Lemma 4.1) on the weight vectors in $B$:

$$E \cdot \bar{1} = 2pZ;$$
$$E \cdot Z^k = \frac{k+2p}{k+1}Z^{k+1};$$
$$E \cdot W^k = k(2p-k)W^{k-1};$$
$$F \cdot \bar{1} = 2qW;$$
$$F \cdot Z^k = k(2q-k)Z^{k-1};$$
$$F \cdot W^k = \frac{k+2q}{k+1}W^{k+1}.$$ 

Notice that $E$ annihilates $W^{2p}$, and takes all weight vectors of $U(1)$-weight $2(p-q) + 2k$ to weight vectors of $U(1)$-weight $2(p-q) + 2k + 2$. Similarly, $F$ annihilates $Z^{2q}$ and takes all weight vectors of $U(1)$-weight $2(p-q) + 2k$ to weight vectors of $U(1)$-weight $2(p-q) + 2k - 2$. The result follows.

References


ON THE DUAL PAIRS \((O(p,q), SL(2, \mathbb{R}))\)


Received September 22, 1997 and revised March 3, 1998. Part of the ideas in this paper was developed when the second author was visiting the Department of Mathematics at the University of California at Berkeley. The second author wishes to thank the Department and Joseph Wolf for their hospitality. We also thank Jeff Adams and Chen-Bo Zhu for helpful conversations and the referee for his/her comments and suggestions.

NATIONAL UNIVERSITY OF SINGAPORE
KENT RIDGE, SINGAPORE 119260

E-mail address: mattanec@nus.edu.sg
Weiping Zhang

In memory of Professor Weishu Shi

We construct skew-adjoint operators associated to nowhere zero vector fields on manifolds with vanishing Euler number. The mod 2 indices of these operators provide potentially new invariants for such manifolds. An odd index theorem for corresponding Toeplitz operators is established. This last result may be viewed as an odd dimensional analogue of the Gauss-Bonnet-Chern theorem.

0. Introduction.

It is well-known that the classical Gauss-Bonnet-Chern theorem [C] can be interpreted as an index theorem for de Rham-Hodge operators (cf. [BGV] and [LM]) and that it is nontrivial only for even dimensional manifolds.

This paper arose from an attempt to search an odd dimensional analogue of this index theorem.

Recall that the Gauss-Bonnet-Chern theorem is closely related to the famous Poincaré-Hopf index formula expressing the Euler number as the sum of indices of singularities of a tangent vector field.

Now for odd dimensional manifolds, or more generally for manifolds with vanishing Euler number, we take the advantage that another result of Hopf asserts that there always exist nowhere zero vector fields (see e.g. Steenrod [S]).

Thus let $M$ be an odd dimensional oriented closed manifold and let $V$ be a nowhere zero vector field on $M$. Let $\gamma$ be the one dimensional oriented vector bundle over $M$ generated by $V$. Then $TM/\gamma$ carries a canonically induced orientation. Let $e(TM/\gamma)$ be the Euler class of $TM/\gamma$.

For any integral element $\omega$ in $H^1(M, \mathbb{Q})$, we will take $\langle \omega e(TM/\gamma), [M] \rangle$ as our substitute for the Euler class appeared in the Gauss-Bonnet-Chern theorem. The main result of this paper gives an analytic formula for this number as the index of certain elliptic Toeplitz operators.

In fact, a general odd index theory has already been developed by Baum-Douglas [BD] who pointed out that the associated index can be computed...
from the original Atiyah-Singer index theorem [AS1]. Thus, our result provides a new example for this theory.

Although our index formula can be deduced from the Atiyah-Singer index theorem in its general form, here we will instead develop a purely analytic approach by emphasizing the heat kernel aspects of the index theory. See Section 3 for more details.

Of particular interests is that the operators we construct in Section 2 are in its real form skew-adjoint. Thus, according to Atiyah and Singer [AS2], they would provide interesting mod 2 invariants for manifolds with vanishing Euler number. We will however leave the possible systematic study to elsewhere.

It might also be interesting to note that on the topological side, an odd analogue of the Poincaré-Hopf index theorem has been proved by Geoghegan and Nicas [GN, Theorem 3.1].

This paper is organized as follows. In Section 1, we recall some algebraic preliminaries. In Section 2, we construct the skew-adjoint operators and study its basic properties. In Section 3, we prove an odd index theorem associated to the operators constructed in Section 2. Finally in Section 4 we apply the results of Section 3 to give an analytic interpretation of the above mentioned invariants $\langle \omega(e(TM/\gamma), [M]) \rangle$.

Acknowledgements. The author would like to thank Mai Zhou for stimulating discussions.

1. Algebraic preliminaries.

In this section, we describe the basic algebraic data.

Let $E$ be an oriented Euclidean vector space. If $e \in E$, let $e^* \in E^*$ corresponds to $e$ by the scalar product of $E$. If $e \in E$, let $c(e), \hat{c}(e)$ be the operators acting on the exterior algebra $\wedge(E^*)$,

\begin{align*}
  c(e) &= e^* \wedge -i_e, \\
  \hat{c}(e) &= e^* \wedge +i_e,
\end{align*}

where $e^* \wedge$ and $i_e$ are the standard notation for exterior and inner multiplications. If $e, e' \in E$, the following identities hold,

\begin{align*}
  c(e)c(e') + c(e')c(e) &= -2\langle e, e' \rangle, \\
  \hat{c}(e)\hat{c}(e') + \hat{c}(e')\hat{c}(e) &= 2\langle e, e' \rangle, \\
  c(e)\hat{c}(e') + \hat{c}(e')c(e) &= 0.
\end{align*}

If we view $\wedge(E^*) = \wedge^{\text{even}}(E^*) \oplus \wedge^{\text{odd}}(E^*)$ as a $\mathbb{Z}_2$-graded space, then clearly, $c(e), \hat{c}(e)$ are odd elements of $\text{End}(\wedge(E^*))$. Also, $\text{End}(\wedge(E^*))$ is generated as an algebra by 1 and the $c(e), \hat{c}(e)$'s.

Let $e_1, ..., e_n$ be an orthonormal base of $E$. 

Proposition 1.1. Among the monomials in the $c(e_i), \dot{c}(e_i)$’s, only $c(e_1)\dot{c}(e_1)...c(e_n)\dot{c}(e_n)$ has a nonzero supertrace. Moreover,

$$\text{Tr}_s[c(e_1)\dot{c}(e_1)...c(e_n)\dot{c}(e_n)] = (-2)^n.$$  

(1.3)

For a proof of this Proposition, see [BZ, Section 4d].

We now assume $n = 2m + 1$.

Let $\dot{c}^{\text{odd}}(E)$ be the algebra generated by monomials of the type

$$c_{I,J} = c(e_{i_1})...c(e_{i_k})\dot{c}(e_{j_1})...\dot{c}(e_{j_l}),$$

(1.4)

where both $k$ and $l$ are odd integers. Then $\dot{c}^{\text{odd}}(E)$ preserves $\wedge^{\text{even}}(E^*)$ and $\wedge^{\text{odd}}(E^*)$. We will view $\dot{c}^{\text{odd}}(E)$ as a subalgebra of $\text{End}(\wedge^{\text{even}}(E^*))$.

Proposition 1.2. Among the monomials of the type (1.4), only $c(e_1)\dot{c}(e_1)...c(e_n)\dot{c}(e_n)$ has a nonzero trace on $\wedge^{\text{even}}(E^*)$. Moreover,

$$\text{Tr}[c(e_1)\dot{c}(e_1)...c(e_n)\dot{c}(e_n)] = -2^{2m}.$$  

(1.5)

Proof. We consider an element of the form (1.4). We assume that no two of $j_l$’s are equal to each other. As $l$ is odd, the set $\{j_1,...,j_l\}$ is not empty. We can assume without loss of generality that $j_1 = 1$.

Let $E'$ be the subspace of $E$ generated by the $e_i$’s with $i > 1$. Then one clearly has

$$\wedge^{\text{even}}(E^*) = \wedge^{\text{even}}(E'^*) \oplus \lambda^1([e_1]^*) \otimes \wedge^{\text{odd}}(E'^*).$$  

(1.6)

Now there are two possibilities.

(i) $1$ does not appear in the set $i_1,...,i_k$. Then one clearly has $\text{Tr}[c_{I,J}] = 0$.

(ii) We can assume without loss of generality that $i_1 = 1$ and that none of $i_s$, $s > 1$ is equal to $1$. Then

$$c_{I,J} = (-1)^{k-1}c(e_1)\dot{c}(e_1)c(e_{i_2})...c(e_{i_k})\dot{c}(e_{j_2})...\dot{c}(e_{j_l}).$$

And one verifies directly that

$$\text{Tr}[c_{I,J}] = (-1)^k\text{Tr}_s[c(e_{i_2})...c(e_{i_k})\dot{c}(e_{j_2})...\dot{c}(e_{j_l}) |_{\wedge(E'^*)}],$$

(1.7)

reducing the problem to Proposition 1.1. \qed

2. Skew-adjoint operators associated to nowhere zero vector fields.

Let $M$ be a compact oriented manifold. We make the assumption that the Euler number of $M$ vanishes.

By a classical result of Hopf, there exists a nowhere zero vector field on $M$. That is, a vector field $V$ on $M$ such that $V(x) \neq 0$ for all $x \in M$. Without loss of generality we will always assume $V$ is smooth.

If $V_1, V_2$ are two nowhere zero vector fields on $M$, we say $V_1$ is homotopic to $V_2$ if there is a smooth family of nowhere zero vector fields $V(t)$, $0 \leq t \leq 1$ with $V(0) = V_1, V(1) = V_2$. 


We now fix a nowhere zero vector field $V$ on $M$.

Let $g^{TM}$ be a metric on $M$. We make the assumption that

\begin{equation}
\|V\|_{g^{TM}} = 1.
\end{equation}

In fact, for any metric $g^{TM}$, we can find a positive function on $M$ so that $fV$ verifies (2.1).

Let $d$ as usual be the exterior derivation acting on $\Gamma(\wedge(T^*M))$. Let $*$ be the Hodge star operator of $g^{TM}$.

Let $\langle \ , \ \rangle$ be the inner product on $\Gamma(\wedge(T^*M))$ defined by

\begin{equation}
\langle \alpha, \beta \rangle = \int_{M} \alpha \wedge * \beta, \quad \alpha, \beta \in \Gamma(\wedge(T^*M)).
\end{equation}

Let $\delta = d^*$ be the formal adjoint of $d$ with respect to the inner product (2.2).

The Clifford actions $c$, $\hat{c}$ in Section 1 can now be defined for $\wedge(T^*M)$ in the same way. Thus $\hat{c}(V)$ acts on $\wedge(T^*M)$ and interchanges $\wedge^{\text{even}}$ and $\wedge^{\text{odd}}$.

**Definition 2.1.** The operator $D_V$ is the operator acting on $\Gamma(\wedge^{\text{even}}(T^*M))$ defined by

\begin{equation}
D_V = \frac{1}{2}(\hat{c}(V)(d + \delta) - (d + \delta)\hat{c}(V)).
\end{equation}

Since the Clifford actions $c$ and $\hat{c}$ anticommute with each other, one verifies easily that $D_V$ is a (real) skew-adjoint elliptic first order differential operator. To be more precise, if $e_1, \ldots, e_n$ is an orthonormal base of $TM$ and $\nabla^{TM}$ is the Levi-Civita connection of $g^{TM}$, then one has the following formula for $D_V$.

**Proposition 2.2.** The following identity holds,

\begin{equation}
D_V = \hat{c}(V)(d + \delta) - \frac{1}{2} \sum_{i} c(e_i)\hat{c}(\nabla_{e_i}^{TM} V).
\end{equation}

**Proof.** Clearly,

\begin{equation}
d + \delta = \sum_{i} c(e_i)\nabla_{e_i}^{\wedge^{\text{even}}(T^*M)},
\end{equation}

where $\nabla^{\wedge^{\text{even}}(T^*M)}$ is the Euclidean connection on $\wedge^{\text{even}}(T^*M)$ induced canonically by $\nabla^{TM}$.

From (2.3) and (2.5), one deduces that

\[
D_V = \hat{c}(V)(d + \delta) - \frac{1}{2} (\hat{c}(V)(d + \delta) + (d + \delta)\hat{c}(V))
\]

\[
= \hat{c}(V)(d + \delta) - \frac{1}{2} \sum_{i} c(e_i)\hat{c}(\nabla_{e_i}^{TM} V).
\]

$\square$
Now according to Atiyah and Singer [AS2], for any real skew-adjoint elliptic operator $D$, the dimension of the kernel of $D$ is a mod 2 homotopy invariant (cf. also Lawson-Michelsohn [LM] for an exposition). This is the so called mod 2 index of $D$.

**Definition 2.3.** Let $\alpha(V)$ be the mod 2 index of $D_V$:

$$\alpha(V) \equiv \dim \ker D_V \pmod{2}.$$  \hfill (2.6)

Clearly, $\alpha(V)$ depends only on the homotopy class of $V$. The Atiyah-Singer mod 2 index theorem [AS2] provides a purely topological formula for $\alpha(V)$.

If we denote by $\text{NZ}(M)$ the set of homotopy classes of nowhere zero vector fields on $M$, then $\alpha$ defines a map

$$\alpha : \text{NZ}(M) \longrightarrow \mathbb{Z}_2.$$  \hfill (2.7)

**Example 2.4.** Take $M = S^1 \times Y$ and assume $M$ has a product metric. Let $V$ be the unit vector field on $S^1$. Then $V$ lifts to a unit vector field on $M$ in an obvious way. One verifies easily that $\alpha(V) \equiv \chi(Y) \pmod{2}$.

From Example 2.4, we see that the map $\alpha$ is nontrivial for manifolds of odd dimensions $4q + 1$. It would be very interesting if this map is also nontrivial in other dimensions. In particular, they might provide new invariants for 3-manifolds, as well as for 4-manifolds with vanishing Euler number. Anyway we will leave the possible systematic study to elsewhere.

Here instead, we prove the following result.

**Theorem 2.5.** If $\dim M = 4q + 1$, then the map $\alpha$ in (2.7) is a constant map with value the Kervaire semi-characteristic.

**Proof.** Recall that the Kervaire semi-characteristic is defined by

$$k(M) = \sum_{i=0}^{2q} \dim H^{2i}(M, \mathbb{R}) \pmod{2}.$$  \hfill (2.8)

As $\dim M = 4q + 1$, $k(M)$ has the following mod 2 index interpretation (cf. Atiyah-Singer [AS2]).

Let $D_R$ be the operator defined by

$$D_R = \hat{c}(e_1)\ldots\hat{c}(e_{4q+1}) (d + \delta) : \Gamma(\wedge^{\text{even}}(T^*M)) \longrightarrow \Gamma(\wedge^{\text{even}}(T^*M)).$$  \hfill (2.9)

Then one verifies easily that $D_R$ is a real skew-adjoint elliptic operator with

$$\dim \ker D_R \equiv k(M) \pmod{2}.$$  \hfill (2.10)

Now for any unit vector field $V$ of $(TM, g^{TM})$, set

$$D'_R = D_R - \frac{1}{2} \hat{c}(V)\hat{c}(e_1)\ldots\hat{c}(e_{4q+1}) \sum_{i} c(e_i)\hat{c}(\nabla_{e_i}^{TM} V).$$  \hfill (2.11)
Also one verifies that
\[(2.12) \quad \langle \nabla^{TM} V, V \rangle = 0.\]

From (2.12), one finds that the elliptic operator \(D_R'\) is also real skew-adjoint. Furthermore, one has the following family of real skew-adjoint elliptic operators,
\[(2.13) \quad D_R(u) = (1 - u)D_R + uD_R', \quad 0 \leq u \leq 1.\]

By the homotopy invariance of the mod 2 indices ([AS2]), and by (2.4), (2.9), (2.11) and (2.12), one then has
\[(2.14) \quad \dim \ker D_R \equiv \dim \ker D_R' \pmod{2} = \dim \ker D_V.\]

From (2.14), (2.10), one deduces that
\[(2.15) \quad \dim \ker D_V \equiv k(X) \pmod{2}.\]

The proof of Theorem 2.5 is completed. \(\square\)

**Remark 2.6.** The above argument does not work for dimensions \(4q - 1\). Thus it remains an interesting question that whether the map \(\alpha\) would still be a constant map for 3-manifolds.

### 3. An odd index theorem for nowhere zero vector fields.

In this Section, we prove the main result of this paper, which is an odd index theorem for the operators \(D_V\) constructed in Section 2. We assume in this section that \(M\) is of odd dimension.

Recall that the odd index theory for self-adjoint elliptic operators has been developed by Baum and Douglas in [BD]. The index theorem we will prove is for the operator \(\sqrt{-1}D_V\) and can in fact as in [BD] be obtained by an application of the Atiyah-Singer index theorem [AS1] for elliptic pseudodifferential operators. However, here we prefer to give a direct geometric proof of our result. This proof consists of two steps. The first step is to use a theorem of Booss and Wojciechowski [BW], reducing the problem to a computation of a spectral flow. This spectral flow is then evaluated in the second step by heat equation methods.

This Section is organized as follows. In a), we state the main result of this Section. In b), we reduce the problem to computations of spectral flows. In c), we give an expression of the spectral flow in terms of heat kernels, and in d) we state a result computing the asymptotic expansion constant appearing in the formula in c). In e), we prove some Lichnerowicz type formulas. These formulas will be used in f) to give the proof of the result stated in d).
a). An index theorem for certain Toeplitz operators.

Recall that $V$ is a unit vector field on an oriented compact odd dimensional Riemannian manifold $M$. We consider the operator $D_V = \frac{1}{2}(\hat{c}(V)(d + \delta) - (d + \delta)c(V))$ constructed in Section 2.

In order to apply the ideas of odd index theory for self-adjoint operators, we should complexify our geometric data. So in this Section we consider the exterior algebra bundle with complex coefficients $\bigwedge^\C(T^*M) = \bigwedge(T^*M) \otimes \C$. Without confusion we still denote it by $\bigwedge^\C(T^*M)$. The inner product $\langle \ , \ \rangle$ now should be modified accordingly,

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge *\beta, \ \alpha, \beta \in \bigwedge(T^*M).$$

**Definition 3.1.** Let $\tilde{D}_V$ be the operator $\sqrt{-1}D_V$ acting on $\Gamma(\bigwedge^{\text{even}}(T^*M))$.

Clearly, $\tilde{D}_V$ is a self-adjoint first order elliptic differential operator. We use the same symbol to denote its closed extension on $L^2(\bigwedge^{\text{even}}(T^*M))$.

Let $L^2_+(\bigwedge^{\text{even}}(T^*M))$ be the direct sum of eigenspaces of $\tilde{D}_V$ associated to nonnegative eigenvalues. Denote by $P_+$ the orthogonal projection operator from $L^2(\bigwedge^{\text{even}}(T^*M))$ to $L^2_+(\bigwedge^{\text{even}}(T^*M))$.

Let $\mathbf{C}^N$ be a trivial complex vector bundle over $M$ carrying the trivial metric and connection. Then $\tilde{D}_V$ extends trivially as an operator acting on $\Gamma(\bigwedge^{\text{even}}(T^*M) \otimes \mathbf{C}^N)$.

Let $g : M \to U(N)$ be a smooth map from $M$ to the unitary group $U(N)$. Then $g$ extends to an action on $\bigwedge^{\text{even}}(T^*M) \otimes \mathbf{C}^N$ as $\text{Id}_{\bigwedge^{\text{even}}(T^*M)} \otimes g$. We still note this action by $g$.

**Definition 3.2.** Let $T_{V,g}$ be the operator

$$\tilde{T}_{V,g} = (P_+ \otimes \text{Id}_{\mathbf{C}^N})g : L^2_+(\bigwedge^{\text{even}}(T^*M) \otimes \mathbf{C}^N) \to L^2_+(\bigwedge^{\text{even}}(T^*M) \otimes \mathbf{C}^N).$$

This is the so-called Toeplitz operator associated to $\tilde{D}_V$ and $g$.

Since $g$ is invertible, one verifies that $T_{V,g}$ is a bounded Fredholm operator between Hilbert spaces.

Let $\gamma$ be the one dimensional oriented vector bundle over $M$ generated by $V$. Let $E$ be the subbundle of $TM$ orthogonal to $\gamma$. Then $E$ carries a canonically induced orientation so that $o(\gamma, E) = o(TM)$. Take a metric on $E$ and let $\nabla$ be a Euclidean connection for this metric on $E$. Let $R$ be the curvature of $\nabla$.

The main result of this Section can be stated as follows.

**Theorem 3.3.** The following identity holds,

$$\text{ind}(T_{V,g}) = \frac{1}{2\pi\sqrt{-1}} \int_M \text{Tr}[g^{-1}dg] \text{Pf} \left( \frac{R}{2\pi} \right).$$
Theorem 3.3 can be proved by an application of the Atiyah-Singer index theorem [AS1] as in Baum-Douglas [BD]. In the rest of this Section, we will give a geometric proof of (3.3).

b). Toeplitz operators and spectral flow.

Note that by now $\tilde{D}_V$ is a self-adjoint elliptic operator acting on $\Gamma(\wedge^{\text{even}}(T^*M) \otimes \mathbb{C}^N)$.

Since $g$ is unitary, the operator (3.4) $\tilde{D}_{V,g} = g^{-1} \tilde{D}_V g$ is also a self-adjoint operator on $\Gamma(\wedge^{\text{even}}(T^*M) \otimes \mathbb{C}^N)$.

Let $\tilde{D}_V(u)$ be the family of self-adjoint operators (3.5) $\tilde{D}_V(u) = (1-u)\tilde{D}_V + u\tilde{D}_{V,g}, \ 0 \leq u \leq 1$.

Let $\text{sf}\{\tilde{D}_V(u), \ 0 \leq u \leq 1\}$ be the spectral flow of the family $\{\tilde{D}_V(u)\}_{0 \leq u \leq 1}$ in the sense of Atiyah, Patodi and Singer [APS2]. Then one has:

**Theorem 3.4.** The following identity holds, (3.6) $\text{ind}(T_{V,g}) = -\text{sf}\{\tilde{D}_V(u), \ 0 \leq u \leq 1\}$.

**Proof.** Formula (3.6) is a corollary of a general result of Booss and Wojciechowski (cf. [BW, Theorem 17.17]). □

c). $\eta$ invariants and spectral flow

For a self-adjoint first order elliptic differential operator $D$, we will adopt the standard notation $\eta(D)$ for the $\eta$ invariant of $D$ in the sense of Atiyah, Patodi and Singer [APS1]. Let $\bar{\eta}(D)$ be the reduced $\eta$ invariant of $D$, also defined in [APS1]:

(3.7) $\bar{\eta}(D) = \frac{1}{2}(\dim \ker(D) + \eta(D))$.

Let $\tilde{D}_V(u), \ 0 \leq u \leq 1$, be our family of first order elliptic self-adjoint operators. Clearly, for any $u \in [0,1]$, (3.8) $\frac{\partial}{\partial u} \tilde{D}_V(u) = \tilde{D}_{V,g} - \tilde{D}_V$ is a bounded operator.

By the standard results for heat kernel asymptotics, one has the following asymptotic expansion as $t \to 0^+$,

(3.9) $\text{Tr} \left[ \frac{\partial}{\partial u} \tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2) \right] = \frac{c_{-\frac{n}{2}}}{t^\frac{n}{2}} + \cdots + \frac{c_{-\frac{1}{2}}}{t^\frac{1}{2}} + O(t^\frac{1}{2})$,

where $n = \dim M$ is the dimension of $M$ and $c_{-\frac{n}{2}},...,c_{-\frac{1}{2}}$ are smooth functions of $u \in [0,1]$.
The main result of this subsection can be stated as follows.

**Proposition 3.5.** The following identity holds,

\[
\text{sf}\{\tilde{D}_V(u), 0 \leq u \leq 1\} = \int_0^1 \frac{c_{-\frac{1}{2}}}{\sqrt{\pi}} du.
\]

In fact, we will prove the following more general result.

**Proposition 3.6.** For any \(s \in [0,1]\), one has

\[
\text{sf}\{\tilde{D}_V(u), 0 \leq u \leq s\} = \int_0^s \frac{c_{-\frac{1}{2}}}{\sqrt{\pi}} du + \bar{\eta}(\tilde{D}_V(s)) - \bar{\eta}(\tilde{D}_V) + \bar{\eta}(\tilde{D}_V(s)).
\]

**Proof.** First of all, (3.11) holds for \(s = 0\). Let \(I\) be the subset of \([0,1]\) such that for any \(s \in I\), (3.11) holds. Then \(I\) is not empty.

Take \(u_0 \in I\). Let \(\varepsilon_0 > 0\) be sufficiently small so that

\[
\text{Spec}(\tilde{D}_V(u_0)) \bigcap [-\varepsilon_0, \varepsilon_0] = \{0\}.
\]

Since the eigenvalues of \(\tilde{D}_V(u)\) are continuous functions of \(u\), there is a sufficiently small neighborhood \(U\) of \(u_0\) in \([0,1]\) such that for any \(u \in U\), \(\pm \varepsilon_0\) are not eigenvalues of \(\tilde{D}_V(u)\).

Take any \(u \in U\). Let \(E_u^+(\varepsilon_0)\) (resp. \(E_u^-(\varepsilon_0)\)) be the direct sum of eigenspaces of \(\tilde{D}_V(u)\) associated to eigenvalues greater than \(\varepsilon_0\) (resp. less than \(-\varepsilon_0\)). Let \(P_u^+\) (resp. \(P_u^-\)) be the orthogonal projection operator from \(L^2(\wedge_{\text{even}}(T^*M) \otimes \mathbb{C}^N)\) onto \(E_u^+(\varepsilon_0)\) (resp. \(E_u^-(\varepsilon_0)\)).

Set \(P = P_u^+ + P_u^-\).

Let \(s \in \mathbb{C}\) be such that \(\text{Re}(s) \gg 0\). Set

\[
\eta_{\varepsilon_0}(\tilde{D}_V(u), s) = \frac{1}{\Gamma\left(s + \frac{1}{2}\right)} \int_0^\infty t^{-\frac{s+1}{2}} \text{Tr}[P\tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2)P] dt.
\]

This function extends to a meromorphic function on \(\mathbb{C}\), which is holomorphic at \(s = 0\) ([APS1, APS2]).

Since \(P^2 = P\), one has

\[
\left(\frac{\partial}{\partial u} P\right) P + P\frac{\partial}{\partial u} P = \frac{\partial}{\partial u} P.
\]

From (3.14), one gets

\[
P\left(\frac{\partial}{\partial u} P\right) P = 0.
\]

From (3.13)-(3.15) and by proceeding as in Bismut-Freed [BF, (2.31)], one finds that

\[
\Gamma\left(s + \frac{1}{2}\right) \frac{\partial}{\partial u} \eta_{\varepsilon_0}(\tilde{D}_V(u), s)
\]
\[ = -s \int_0^{\infty} t^{\frac{s-1}{2}} \left\{ \text{Tr} \left[ P \frac{\partial}{\partial u} \tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2)P \right] \right\} dt. \]

On the other hand, since \( \frac{\partial}{\partial a}\tilde{D}_V(u) \) is a bounded operator, from (3.9) one verifies easily that as \( t \to 0^+ \), one has the following asymptotic expansion,

\[
(3.17) \quad \text{Tr} \left[ P \frac{\partial}{\partial u} \tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2)P \right] = \frac{c-\frac{u}{2}}{t^2} + \cdots + \frac{c-\frac{1}{2}}{t^{\frac{1}{2}}} + O(1).
\]

From (3.16), (3.17), one gets

\[
(3.18) \quad \frac{\partial}{\partial u} \eta_{e_0}(\tilde{D}_V(u), 0) = -\frac{2c-\frac{1}{2}}{\sqrt{\pi}}.
\]

Now let \( k_u \) be the number of negative eigenvalues of \( \tilde{D}_V(u) \) in \((-e_0, 0)\). Then by the definition of spectral flow, one has

\[
(3.19) \quad \text{sf}\{\tilde{D}_V(\alpha) : \alpha \text{ from } u_0 \text{ to } u\} = -k_u.
\]

Also, by (3.18) one has,

\[
(3.20) \quad \bar{\eta}(\tilde{D}_V(u)) - \bar{\eta}(\tilde{D}_V(u_0)) = \frac{1}{2} \left( \eta_{e_0}(\tilde{D}_V(u), 0) - \eta_{e_0}(\tilde{D}_V(u_0), 0) \right) - k_u
\]

\[ = -\int_{u_0}^{u} \frac{c-\frac{1}{2}}{\sqrt{\pi}} ds - k_u. \]

Since (3.11) holds for \( u_0 \), by (3.19), (3.20) and by the additivity of spectral flow, it also holds for \( u \in U \).

Thus the set \( I \) is open in \([0,1]\). A similar argument shows that \( I \) is also closed in \([0,1]\). Thus \( I = [0,1] \).

Proposition 3.6 is then proved.

**Proof of Proposition 3.5.** Since \( g^{-1}\tilde{D}_Vg \) is conjugate to \( \tilde{D}_V \), they have the same spectrum. Thus, one has

\[
(3.21) \quad \bar{\eta}(g^{-1}\tilde{D}_Vg) = \bar{\eta}(\tilde{D}_V).
\]

Proposition 3.5 follows from (3.21) by setting \( s = 1 \) in (3.11).

**Remark 3.7.** Proposition 3.6 may be seen as a refinement of [BF, Proposition 2.8] in our situation. This kind of refinements is in fact well-known. We include a proof here just for the sake of completeness.

**Remark 3.8.** By Theorem 3.4 and Proposition 3.5, in order to prove Theorem 3.3, we need only to evaluate \( c-\frac{1}{2} \) for each \( u \in [0,1] \). This is the subject of the following subsections.
d). An asymptotic result involving the heat kernel of $\tilde{D}_V(u)^2$.

Recall that $E$ is the oriented subbundle of $TM$ such that $E$ is orthogonal to the oriented line bundle $\gamma$ generated by $V$ and that $(\gamma, E)$ has the same orientation as $TM$.

Let
\begin{equation}
TM = \gamma \oplus E,
\end{equation}
\begin{equation}
g^{TM} = g^{\gamma} + g^{E}
\end{equation}
be the corresponding orthogonal decomposition of the metric $g^{TM}$. Let $R^{TM}$ be the curvature of the Levi-Civita connection $\nabla^{TM}$. Let $P^E$ (resp. $P^{\gamma}$) be the orthogonal projection from $TM$ to $E$ (resp. $\gamma$). Then
\begin{equation}
\nabla^{\gamma} = P^{\gamma} \nabla^{TM} P^{\gamma},
\end{equation}
\begin{equation}
\nabla^{E} = P^{E} \nabla^{TM} P^{E}
\end{equation}
are Euclidean connections of $g^{\gamma}$, $g^{E}$ respectively.

Let $R^{E} = (\nabla^{E})^2$ be the curvature of $\nabla^{E}$.

For any $u \in [0, 1]$, let $Q^u_t(x, y)$ be the kernel of $\frac{\partial}{\partial u} \tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2)$. Let $dv_M$ be the volume element on $TM$ with respect to $g^{TM}$.

The main result of this subsection can be stated as follows.

**Theorem 3.9.** For any $u \in [0, 1]$, $x \in M$, the following identity holds,
\begin{equation}
\lim_{t \to 0} \sqrt{t} \text{Tr}[Q^u_t(x, x)]dv_M(x) = \frac{\sqrt{-1}}{2\sqrt{\pi}} \text{Tr}[g^{-1}dg] \text{Pf} \left( \frac{R^E}{2\pi} \right).
\end{equation}

By Remark 3.8, Theorem 3.3 is a consequence of Theorem 3.9 and the Chern-Weil theorem (cf. [BGV]) for Euler forms. The rest of this section is devoted to the proof of (3.24).

e). A Lichnerowicz formula for $\tilde{D}_V(u)^2$.

We write $n = 2m + 1$.

Let $e_0, \ldots, e_{2m}$ be an orthonormal base of $TM$.

**Proposition 3.10.** The following identity holds,
\begin{equation}
\tilde{D}_V(u)^2 = \tilde{D}_V^2 + u((d + \delta)c(g^{-1}dg) + c(g^{-1}dg)(d + \delta))
+ u^2 \left( \sum_{i=0}^{2m} c(e_i) c(g^{-1}dg) \tilde{c}(V) \tilde{c}(\nabla^{TM}_e V)
+ \sum_{i=0}^{2m} c(g^{-1}dg) c(e_i) \tilde{c}(V) \tilde{c}(\nabla^{TM}_e V)
+ u^2 (c(g^{-1}dg))^2.\right)
\end{equation}
Proof. By (2.4), Definition 3.1, one has
\[ \tilde{D}_V = \sqrt{-1}\hat{c}(V)(d + \delta) + \frac{1}{2\sqrt{-1}} \sum_{0}^{2m} c(e_i)\hat{c}(\nabla_{e_i}^TM V). \] (3.26)

Also,
\[ g^{-1}\tilde{D}_V g = \tilde{D}_V + g^{-1}[\tilde{D}_V, g] = \tilde{D}_V + \sqrt{-1}\hat{c}(V)c(g^{-1}dg). \] (3.27)

From (3.5) and (3.27), one has
\[ \tilde{D}_V(u) = \tilde{D}_V + u\sqrt{-1}\hat{c}(V)c(g^{-1}dg). \] (3.28)

Thus,
\[ \tilde{D}_V(u)^2 = \tilde{D}_V^2 + u\sqrt{-1}\tilde{D}_V\hat{c}(V)c(g^{-1}dg) + u\sqrt{-1}\hat{c}(V)c(g^{-1}dg)\tilde{D}_V - u^2\hat{c}(V)c(g^{-1}dg)\hat{c}(V)c(g^{-1}dg). \] (3.29)

(3.25) follows from (3.29), (3.26) and (1.2). \qed

Let \( \Delta \) be the standard notation for the Bochner Laplacian
\[ \Delta = \sum_{0}^{2m} \left( \nabla_{e_i}^{even}(T^*M)^{2} - \nabla_{\nabla_{e_i}^TM e_i}^{even}(T^*M) \right). \] (3.30)

Let \( K \) be the scalar curvature of \( g^{TM} \).
We also make the assumption that \( e_0 = V \). Then \( e_1, ..., e_{2m} \) is an orthonormal base of \( E \).

**Theorem 3.11.** The following identity holds,
\[ \tilde{D}_V^2 = -\Delta + \frac{K}{4} \]
\[ + \frac{1}{8} \sum_{0 \leq i,j \leq 2m} \sum_{1 \leq k,l \leq 2m} \langle R(E)(e_i, e_j)e_k, e_l \rangle c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_l) \]
\[ - \frac{1}{2}\hat{c}(V)\hat{c}(\Delta V) - \sum_{0}^{2m} \hat{c}(V)\hat{c}(\nabla_{e_i}^TM V)\nabla_{e_i}^TM - \frac{1}{4} \sum_{0}^{2m} ||S(e_i)V||^2. \] (3.31)

Proof. By (3.26), one has
\[ \tilde{D}_V^2 = -(\hat{c}(V)(d + \delta))^2 + \frac{1}{2} \left( \hat{c}(V)(d + \delta) \sum_{0}^{2m} c(e_i)\hat{c}(\nabla_{e_i}^TM V) \right) \]
\[ + \frac{1}{8} \sum_{0 \leq i,j \leq 2m} \sum_{1 \leq k,l \leq 2m} \langle R(E)(e_i, e_j)e_k, e_l \rangle c(e_i)c(e_j)\hat{c}(e_k)\hat{c}(e_l) \]
\[ - \frac{1}{2}\hat{c}(V)\hat{c}(\Delta V) - \sum_{0}^{2m} \hat{c}(V)\hat{c}(\nabla_{e_i}^TM V)\nabla_{e_i}^TM - \frac{1}{4} \sum_{0}^{2m} ||S(e_i)V||^2. \] (3.32)
ANALYTIC AND TOPOLOGICAL INVARIANTS...

\[ + \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V) \hat{c}(V)(d + \delta) - \frac{1}{4} \left( \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V) \right)^2. \]

Now

\[
(\hat{c}(V)(d + \delta))^2
= - (d + \delta)^2 + \hat{c}(V)((d + \delta)\hat{c}(V) + \hat{c}(V)(d + \delta))(d + \delta)
= - (d + \delta)^2 + \hat{c}(V) \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V)(d + \delta).
\]

By (3.32), (3.33), one deduces

\[
\tilde{D}^2 = (d + \delta)^2 + \frac{1}{2} \left\{ \hat{c}(V)(d + \delta) \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V) \right. \\
- \hat{c}(V) \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V)(d + \delta) \left. \right\} \\
- \frac{1}{4} \left( \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V) \right)^2.
\]

The middle term in the right hand side of (3.34) can be evaluated as follows,

\[
\hat{c}(V)(d + \delta) \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V) - \hat{c}(V) \sum_{0}^{2m} c(e_i) \hat{c}(\nabla^T e_i V)(d + \delta)
= \hat{c}(V) \sum_{0}^{2m} \{(d + \delta)c(e_i) + c(e_i)(d + \delta)\} \hat{c}(\nabla^T e_i V) \\
- \hat{c}(V) \sum_{0}^{2m} c(e_i) \{\hat{c}(\nabla^T e_i V)(d + \delta) + (d + \delta)\hat{c}(\nabla^T e_i V)\}
= \hat{c}(V) \sum_{0}^{2m} (-2\nabla^T e_i) \hat{c}(\nabla^T e_i V) + \hat{c}(V) \sum_{i,j} c(e_j) c(\nabla^T e_j e_i) \hat{c}(\nabla^T e_i V) \\
- \hat{c}(V) \sum_{i,j} c(e_i) c(e_j) \hat{c}(\nabla^T e_j \nabla^T e_i V)
= \hat{c}(V) \left\{ -2 \sum_{i} \hat{c}(\nabla^T e_i V) \nabla^T e_i - \hat{c}(\Delta V) \right\} \\
+ \frac{1}{2} \sum_{i,j} c(e_i) c(e_j) \hat{c}(V) \hat{c}(R^T (e_i, e_j)V).
Let \( S \) be defined by
\[
S = \nabla^{TM} - \nabla^{E} - \nabla^{\gamma}.
\]
Then \( S \) is a one form taking values in skew-symmetric endomorphisms interchanging \( \gamma \) and \( E \).

Recall that \( e_0 = V \). We have
\[
- \left( \sum_{i,j} c(e_i) \hat{c}(\nabla^{TM}_{e_i} V) \right)^2
= \sum_{i,j} c(e_i) c(e_j) \hat{c}(S(e_i) V) \hat{c}(S(e_j) V)
= \frac{1}{2} \sum_{i,j,k,l} \left( \langle S(e_i) V, e_k \rangle \langle S(e_j) V, e_l \rangle \right. \\
- \langle S(e_j) V, e_k \rangle \langle S(e_i) V, e_l \rangle) \hat{c}(e_i) \hat{c}(e_j) \hat{c}(e_k) \hat{c}(e_l)
- \sum_{i,j,k,l} \|S(e_i) V\|^2.
\]

Also, from (3.36), one deduces that for \( 1 \leq k, l \leq 2m \),
\[
\langle R^{TM}(e_i, e_j) e_t, e_k \rangle
= \langle R^{E}(e_i, e_j) e_t, e_k \rangle + \langle S(e_i) S(e_j) e_l - S(e_j) S(e_i) e_l, e_k \rangle
= \langle R^{E}(e_i, e_j) e_l, e_k \rangle + \langle S(e_i) e_l, V \rangle \langle S(e_j) e_k, V \rangle - \langle S(e_j) e_l, V \rangle \langle S(e_i) e_k, V \rangle.
\]

On the other hand, by using an obvious extension of the Lichnerowicz formula [L] for \((d + \delta)^2\) (cf. [BZ, Section 4e]) and \([BGV]\), one has
\[
(d + \delta)^2 = -\Delta + \frac{K}{4} + \frac{1}{8} \sum_{i,j,k,l} \langle R^{TM}(e_i, e_j) e_t, e_k \rangle c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l).
\]

(3.31) follows from (3.34), (3.35) and (3.37)-(3.39).

Proposition 3.10 and Theorem 3.11 together gives the following crucial Lichnerowicz type formula.

**Theorem 3.12.** For any \( u \in [0, 1] \), the following identity holds,
\[
\tilde{D}_u V (u)^2 = -\Delta + \frac{K}{4} + \frac{1}{8} \sum_{0 \leq i,j \leq 2m} \sum_{1 \leq k,l \leq 2m} \langle R^{E}(e_i, e_j) e_t, e_k \rangle c(e_i) c(e_j) \hat{c}(e_k) \hat{c}(e_l).
\]
ANALYTIC AND TOPOLOGICAL INVARIANTS... 393

\[- \frac{1}{2} \hat{c}(V) \hat{c}(\Delta V) - \sum_{0}^{2m} \hat{c}(V) \hat{c}(\nabla_{e_{i}}^{T M} V) \nabla_{e_{i}}^{T M} \]

\[+ u((d + \delta) c(g^{-1} dg) + c(g^{-1} dg)(d + \delta)) \]

\[+ \sum_{0}^{2m} c(e_{i}) \hat{c}(V) \hat{c}(\nabla_{e_{i}}^{T M} V) \]

\[+ u \sum_{0}^{2m} c(g^{-1} dg) c(e_{i}) \hat{c}(V) \hat{c}(\nabla_{e_{i}}^{T M} V) \]

\[+ u^{2}(c(g^{-1} dg))^{2} - \frac{1}{4} \sum_{0}^{2m} \|S(e_{i})V\|^{2}. \]

f). A proof of Theorem 3.9.

We will use the method of Bismut and Zhang [BZ, Section 4] to prove Theorem 3.9. In particular, we adopt the notation used there. For example, for any \( e \in TM \) and \( e^{*} \in T^{*}M \) its corresponding element via \( g^{T M} \), we will write as in [BZ, Section 4] that

\[(3.41) \quad \hat{c}(e) = \hat{e}^{*} \wedge + i \hat{e}. \]

For any \( t > 0, x \in M, e \in TM \) and \( e^{*} \in T^{*}M \) its correspondent, set

\[(3.42) \quad c_{t}(e) = \frac{e^{*}}{t^{\frac{1}{2}}} \wedge - l + \frac{1}{i} \hat{e}, \quad \hat{c}_{t}(e) = \frac{\hat{e}^{*}}{t^{\frac{1}{2}}} \wedge + l + \frac{1}{i} \hat{e}. \]

Then for any \( e, e' \in TM \), one has

\[(3.43) \quad c_{t}(e) c_{t}(e') + c_{t}(e') c_{t}(e) = -2(e, e'), \]

\[\hat{c}_{t}(e) \hat{c}_{t}(e') + \hat{c}_{t}(e') \hat{c}_{t}(e) = 2(e, e'), \]

\[c_{t}(e) \hat{c}_{t}(e') + \hat{c}_{t}(e') c_{t}(e) = 0. \]

Furthermore, for any \( \alpha \in \text{End}(\wedge^{\text{even}}(T^{*}M)) \), if we denote by \( \alpha_{t} \) the element obtained from \( \alpha \) by replacing \( c(e) \) by \( c_{t}(e) \) and \( \hat{c}(e) \) by \( \hat{c}_{t}(e) \), then we have:

**Proposition 3.13.** If \( \alpha \in \text{End}(\wedge^{\text{even}}(T^{*}M)) \), then for any \( t > 0 \),

\[(3.44) \quad \text{Tr}[\alpha] = (-1)^{m+1} 2^{2m} t^{m+\frac{1}{2}} \{\alpha_{t}\}^{\text{max}}, \]

where \( \{\alpha_{t}\}^{\text{max}} \) denotes the coefficient of the monomial \( e^{0} \wedge ... \wedge e^{2m} \wedge \hat{e}^{0} \wedge ... \hat{e}^{2m} \) in the expansion of \( \alpha_{t} \).

**Proof.** Equation (3.44) follows from (1.5). \( \square \)
Now we proceed as in Bismut-Zhang [BZ, Section 4] and Getzler [G].

Let \( a > 0 \) be the injectivity radius of \((M, g^T_M)\). Take \( \varepsilon \) such that \( 0 < \varepsilon \leq a/2 \). Take \( x \in M \). Let \( e_0, ..., e_{2m} \) be an orthonormal base of \( T_x M \). We make the assumption that \( e_0 = V(x) \). We identify the open ball \( B^T_{x,M}(0, \varepsilon) \) in \( T_x M \) with the open ball \( B^M(x, \varepsilon) \) in \( M \) using geodesic coordinates. Then \( y \in T_x M, \ |y| \leq \varepsilon \) represents an element of \( B^M(x, \varepsilon) \). For \( y \in T_x M, \ |y| \leq \varepsilon \), we identify \( T_y M, \mathbf{C}^N_y \) to \( T_x M, \mathbf{C}^N_x \) by parallel transports along the geodesic \( t \in [0, 1] \to ty \) with respect to \( \nabla^T_M \) and the trivial connection on \( \mathbf{C}^N \).

Let \( \Gamma^T_M \) be the connection form for \( \nabla^T_M \) in the considered trivialization of \( TM \). By [ABP, Proposition 4.7], we know that

\[
\Gamma^T_M \big|_{y} = \frac{1}{2} g^T_M(y, \cdot) + O(|y|^2).
\]

The induced connection form \( \Gamma^\text{even}(T^*_M) \) on \( \wedge^\text{even}(T^*_M) \) is given by

\[
\Gamma^\text{even}(T^*_M) \big|_{y} = \frac{1}{8} \sum_{i,j} ((R^T_M(y, \cdot)e_i, e_j) + O(|y|^2))(c(e_i)c(e_j) - \hat{c}(e_i)\hat{c}(e_j)).
\]

The operator \( \tilde{D}_V(u)^2 \) now acts on smooth sections of \( \wedge^\text{even}(T^*_M) \otimes \mathbf{C}^N \) over \( B^T_{x,M}(0, \varepsilon) \).

If \( h \) is a smooth section of \( \wedge^\text{even}(T^*_M) \otimes \mathbf{C}^N \) over \( T_x M \), set

\[
T_t h(y) = h \left( \frac{y}{\sqrt{t}} \right).
\]

Let \( K_t(u) \) be the operator

\[
K_t(u) = T^{-1}_t \tilde{D}(u)^2 T_t.
\]

Then \( K_t(u) \) is a differential operator with coefficients in the algebra spanned by the elements of \( \text{End}(\wedge^\text{even}(T^*_M)) \) and \( \text{End}(\mathbf{C}^N) \).

Let \( L_t(u) \) be the operator obtained from \( K_t(u) \) by replacing the Clifford variables \( c(e_i), \hat{c}(e_i) \) by \( c_t(e_i), \hat{c}_t(e_i) \) defined in (3.42). Let \( \nabla^T_x M \) be the flat connection on \( T_x M \). Let \( \Delta^T_x M \) be the flat Laplacian over \( T_x M \) for the metric \( g^T_x M \). Using (3.40), (3.46), one concludes easily that as \( t \to 0 \), the coefficients of \( L_t(u) \) converge uniformly over compact sets together with their derivatives to the coefficients of the operator \( L_0(u) \) given by

\[
\begin{split}
L_0(u) = -\Delta^T_x M - \tilde{e}^0 \nabla^T_{\tilde{e}_i} \tilde{V}(0) \nabla^T_{\tilde{e}_i} M \\
+ \frac{1}{8} \sum_{0 \leq i,j \leq 2m} \sum_{1 \leq k,l \leq 2m} (R^T(e_i, e_j)e_k, e_l)\tilde{c}_i\tilde{c}_j\tilde{c}_k\tilde{c}_l.
\end{split}
\]

On the other hand, from (3.28), one has

\[
\frac{\partial}{\partial u} \tilde{D}_V(u) = \sqrt{-1} \tilde{c}(V)c(g^{-1}dg).
\]
Let \( C_t(u) \) be the operator obtained from \( \frac{\partial}{\partial u} \tilde{D}_V(u) \) by replacing \( c(e_i), \hat{c}(e_i) \) by \( ct(e_i), \hat{ct}(e_i) \) respectively.

Set

\[
\omega = g^{-1}dg.
\]

Using (3.50), (3.51), we find that

\[
\lim_{t \to 0} \sqrt{t}C_t(u) = \sqrt{-1}\hat{e}^0 \sum_{0 \leq i \leq 2m} \omega(e_i)e^i.
\]

Recall that \( Q^t(x, y) \) is the kernel of \( \frac{\partial}{\partial u} \tilde{D}_V(u) \exp(-t\tilde{D}_V(u)^2) \).

Let \( dv_M \) be the volume element on \( TM \) with respect to the metric \( g^{TM} \).

By using Proposition 3.13, Equations (3.49), (3.52), and also the trivial relation \( \hat{e}^{0,2} = 0 \), and by proceeding as in Getzler [G], we see that as \( t \to 0 \),

\[
\sqrt{t} \text{Tr}[Q^t_t(x, x)]dv_M(x) \to \frac{\sqrt{-1}}{2\sqrt{\pi}} \text{Tr}[g^{-1}dg] \text{Pf} \left( \frac{R^E}{2\pi} \right), \text{ uniformly on } M.
\]

This completes the proof of Theorem 3.9. By Remark 3.8, Theorem 3.3 is thus also proved. \( \square \)

**Remark 3.14.** The local index techniques of this section can also be used to prove regularity results for the \( \eta \) functions of \( \tilde{D}_V(u) \)'s.

### 4. Invariants associated to nowhere zero vector fields.

In this Section, we apply the odd index Theorem 3.3 of the last section to get invariants on odd dimensional manifolds.

Let \( M \) as before be an odd dimensional oriented compact manifold.

Let \( V \) be a nowhere zero vector field on \( M \). Let \( \gamma \) be the one dimensional oriented vector bundle generated by \( V \). Then \( E = TM/\gamma \) is an even dimensional vector bundle over \( M \), carrying a canonically induced orientation.

Let \( e(E) \) be the Euler class of \( E \).

**Definition 4.1.** Let \( \alpha_V \) be the homomorphism

\[
\alpha_V : H^1(M, \mathbb{Z}) \to \mathbb{Z}
\]

defined by

\[
\alpha_V : \omega \to \langle \omega e(E), [M] \rangle,
\]

if \( \omega \) is not a torsion class. If \( \omega \) is a torsion class, we set \( \alpha_V(\omega) = 0 \).

By using Theorem 3.3, we can give an analytic formula for the map \( \alpha_V \) as follows.

Let \( \omega \in H^1(M, \mathbb{Z}) \). We assume \( \omega \) is not a torsion class. Let \( \tilde{\omega} \in \wedge^1(T^*M) \) be any de Rham representative of \( \omega \).
We fix a point \( p \) on \( M \). For any \( x \in M \), let \( \gamma_{px} \) be a pass connecting \( p \) and \( x \).

Let \( \rho_{\tilde{\omega}}(x) : M \to U(1) \) be the function on \( M \) defined by

\[
(4.3) \quad \rho_{\tilde{\omega}}(x) = \exp \left\{ 2\pi \sqrt{-1} \int_{p}^{x} \tilde{\omega} \right\},
\]

where the integral is along the pass \( \gamma_{px} \).

Since \([\tilde{\omega}] \in H^1(M, \mathbb{Z})\), \( \rho_{\tilde{\omega}} \) is well defined. Also, one finds

\[
(4.4) \quad \rho_{\tilde{\omega}}^{-1} d\rho_{\tilde{\omega}} = 2\pi \sqrt{-1} \tilde{\omega}.
\]

Let \( g^{TM} \) be a metric on \( TM \) so that \( \|V\|_{g^{TM}} = 1 \). Let \( T_{V,\rho_{\tilde{\omega}}} \) be the Toeplitz operator defined for \( V, \rho_{\tilde{\omega}} \) as in Section 3a). Then from Theorem 3.3 one deduces easily the following result.

**Theorem 4.2.** The following identity holds,

\[
(4.5) \quad \text{ind}(T_{V,\rho_{\tilde{\omega}}}) = \alpha_V(\omega).
\]

**Example 4.3.** The simplest example is by taking \( M = S^1 \times X \) for some compact connected oriented even dimensional manifold \( X \), and take \( V \) to be the canonical vector field lifted from that of \( S^1 \) and \( \omega \) the canonical generator of \( H^1(S^1, \mathbb{Z}) \). Then one has clearly \( \alpha_V(\omega) = \chi(X) \).

As a simple application to problems involving vector fields, we prove the following result.

**Theorem 4.4.** If the homomorphism \( \alpha_V \) is not identically zero, then \( V \) cannot be deformed through nowhere zero vector fields to \(-V\).

**Proof.** Since \( \alpha_V \) is not a zero map, there exists \( \omega \in H^1(M, \mathbb{Z}) \) such that

\[
(4.6) \quad \alpha_V(\omega) = \text{ind}T_{V,\omega} \neq 0.
\]

Now if \( V \) is homotopic to \(-V\), then by the homotopy invariance of the analytic index, one would have

\[
(4.7) \quad \text{ind}(T_{V,\omega}) = \text{ind}(T_{-V,\omega}) = -\text{ind}(T_{V,\omega}).
\]

This contradicts with (4.6). \( \square \)

**Remark 4.5.** In some cases, Theorem 4.4 gives refined obstructions for a vector field \( V \) to be able to homotopic to \(-V\). For example, if \( M = S^1 \times X \), \( \dim M = 4q + 1 \), and \( V \) is the vector field considered in Example 4.3, then Theorem 2.5 only asserts \( \chi(X) \equiv 0 \pmod{2} \) (cf. \([A, \text{Theorem 1.3}]\)), while Theorem 4.4 asserts that \( \chi(X) \) vanishes exactly.
Remark 4.6. Of course, the map $\alpha_V$ can be nontrivial only on nonsimply connected manifolds. For simply connected manifolds, one might expect that the mod 2 indices discussed in Section 2 would provide meaningful invariants.

Remark 4.7. In some sense one may regard Theorem 4.2 as an odd dimensional analogue of the Gauss-Bonnet-Chern theorem. But in line of the above remark, one might also take the mod 2 index theorem [AS2] for the skew-adjoint operator $D_V$ constructed in Section 2 as another candidate.

References


Received August 25, 1997. The author was partially supported by Chinese National Science Foundation.

Nankai Institute of Mathematics
Tianjin 300071
People’s Republic of China
E-mail address: weiping@sun.nankai.edu.cn
<table>
<thead>
<tr>
<th>Article</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gene Abrams, Jeremy Haefner and Angel del Río</td>
<td>The isomorphism problem for incidence rings</td>
<td>201</td>
</tr>
<tr>
<td>Frauke M. Bleher</td>
<td>Finite groups of Lie type of small rank</td>
<td>215</td>
</tr>
<tr>
<td>Mats Boij</td>
<td>Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function</td>
<td>1</td>
</tr>
<tr>
<td>B.L.J. Braaksma, G.K. Immink and Y. Sibuya</td>
<td>The Stokes phenomenon in exact asymptotics</td>
<td>13</td>
</tr>
<tr>
<td>B. Brenken</td>
<td>Hecke algebras and semigroup crossed product C*-algebras</td>
<td>241</td>
</tr>
<tr>
<td>Jean-Noël Corvellec</td>
<td>Quantitative deformation theorems and critical point theory</td>
<td>263</td>
</tr>
<tr>
<td>Angel del Río with Gene Abrams and Jeremy Haefner</td>
<td></td>
<td>201</td>
</tr>
<tr>
<td>Terry Fuller</td>
<td>Generalized nuclei of complex surfaces</td>
<td>281</td>
</tr>
<tr>
<td>Gayatri Garimella</td>
<td>Weak Paley–Wiener property for completely solvable Lie groups</td>
<td>51</td>
</tr>
<tr>
<td>Jeremy Haefner with Gene Abrams and Angel del Río</td>
<td></td>
<td>201</td>
</tr>
<tr>
<td>Dieter Happel and Birge Huisgen-Zimmermann</td>
<td>Viewing finite dimensional representations through infinite dimensional ones</td>
<td>65</td>
</tr>
<tr>
<td>Kin Ming Hui</td>
<td>Singular limit of solutions of the equation ( u_t = \Delta \left( \frac{u^m}{m} \right) ) as ( m \to 0 )</td>
<td>297</td>
</tr>
<tr>
<td>Birge Huisgen-Zimmermann with Dieter Happel</td>
<td></td>
<td>65</td>
</tr>
<tr>
<td>Scott K. Imig</td>
<td>A construction of logarithmic 1-forms on hyperplane arrangements</td>
<td>91</td>
</tr>
<tr>
<td>G.K. Immink with B.L.J. Braaksma and Y. Sibuya</td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>Thomas Michael Keller</td>
<td>Orbit sizes and character degrees</td>
<td>317</td>
</tr>
<tr>
<td>Yiming Long</td>
<td>Bott formula of the Maslov-type index theory</td>
<td>113</td>
</tr>
<tr>
<td>Alexandru Nica, Dimitri Shlyakhtenko and Roland Speicher</td>
<td>Maximality of the microstates free entropy for R-diagonal elements</td>
<td>333</td>
</tr>
<tr>
<td>Masahiro Ohno</td>
<td>On degenerate secant varieties whose Gauss maps have the largest images</td>
<td>151</td>
</tr>
<tr>
<td>Annegret Paul and Eng-Chye Tan</td>
<td>On the dual pairs ((O(p,q), SL(2, \mathbb{R})), (U(p,q), U(1,1))) and ((Sp(p,q), O^*(4)))</td>
<td>349</td>
</tr>
<tr>
<td>Name</td>
<td>Title</td>
<td>Page</td>
</tr>
<tr>
<td>-----------------------</td>
<td>----------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>Leonard R. Rubin</td>
<td>Limit theorem for inverse sequences of metric spaces in extension theory</td>
<td>177</td>
</tr>
<tr>
<td>Philip J. Schapiro</td>
<td></td>
<td>177</td>
</tr>
<tr>
<td>Dimitri Shlyakhtenko</td>
<td>with Alexandru Nica and Roland Speicher</td>
<td>333</td>
</tr>
<tr>
<td>Y. Sibuya</td>
<td>with B.L.J. Braaksma and G.K. Immink</td>
<td>13</td>
</tr>
<tr>
<td>Roland Speicher</td>
<td>with Alexandru Nica and Dimitri Shlyakhtenko</td>
<td>333</td>
</tr>
<tr>
<td>Eng-Chye Tan</td>
<td>with Annegret Paul</td>
<td>349</td>
</tr>
<tr>
<td>Hiroyuki Terakawa</td>
<td>The d-very ampleness on a projective surface in positive characteristic</td>
<td>187</td>
</tr>
<tr>
<td>Weiping Zhang</td>
<td>Analytic and topological invariants associated to nowhere zero vector fields</td>
<td>379</td>
</tr>
</tbody>
</table>
Guidelines for Authors

Authors may submit manuscripts at pjm.math.berkeley.edu/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LATEX, but papers in other varieties of TEX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LATEX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTEX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text (“the curve looks like this:”). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as “Place Figure 1 here”. The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal’s preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a Web site in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.
The isomorphism problem for incidence rings 201
Gene Abrams, Jeremy Haefner and Angel del Río

Finite groups of Lie type of small rank 215
Frauke M. Bleher

Hecke algebras and semigroup crossed product C*-algebras 241
B. Brenken

Quantitative deformation theorems and critical point theory 263
Jean-Noël Corvellec

Generalized nuclei of complex surfaces 281
Terry Fuller

Singular limit of solutions of the equation \( u_t = \Delta \left( \frac{u}{m} \right) \) as \( m \to 0 \) 297
Kin Ming Hui

Orbit sizes and character degrees 317
Thomas Michael Keller

Maximality of the microstates free entropy for \( R \)-diagonal elements 333
Alexandru Nica, Dimitri Shlyakhtenko and Roland Speicher

On the dual pairs \((O(p,q), SL(2,R)), (U(p,q), U(1,1))\) and \((Sp(p,q), O^*(4))\) 349
Annegret Paul and Eng-Chye Tan

Analytic and topological invariants associated to nowhere zero vector fields 379
Weiping Zhang