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CHARACTERISTIC PROPERTIES OF NEHARI FUNCTIONS

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Let N be the set of all meromorphic functions f defined in the unit disc D that satisfy Nehari's univalence criterion $(1 - |z|^2)^2|Sf(z)| \leq 2$. In this paper we investigate certain properties of the class N. We obtain sharp estimates for the spherical distortion, and also a two-point distortion theorem that actually characterizes the set N. Finally, we study some aspects of the boundary behavior of Nehari functions, and obtain results that indicate how such maps can fail to map D onto a quasidisc.

1. Introduction.

Let f be analytic in the unit disc D and let $Sf = (f''/f')' - (1/2)(f''/f')^2$ be its Schwarzian derivative. In 1949 Nehari showed that if

$$|Sf(z)| \leq \frac{2}{(1 - |z|^2)^2} \tag{1.1}$$

for all $z \in D$ then f is univalent [**11**]. A necessary condition for univalence is obtained by replacing the 2 with a 6 in the numerator of (1.1). It was proved by Kraus in 1932, [**9**], and rediscovered later by Nehari.

Let N be the set of all meromorphic functions satisfying (1.1). This *Nehari Class* was formally introduced and extensively studied in [**5**]. In the present paper we will make use of several results from [**5**] and also earlier papers, and it is our purpose here to investigate further properties of Nehari maps.

We shall consider functions in N normalized in two different ways. In the first one,

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots , \tag{1.2}$$

while in the second, we let

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = 0. \tag{1.3}$$

Both normalizations are achieved by composing f from the left with siutable Möbius transformations. This leaves (1.1) invariant. The second normalization gives rise to the class N_0, and according to a result in [**3**] if f satisfies

(1.1) and (1.3) then it has no poles. In fact, such a function will either be a rotation of the logarithm

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \tag{1.4}$$

or else it will be bounded. The function L has

$$SL(z) = \frac{2}{(1-z^2)^2} \tag{1.5}$$

and plays a very important role and is extremal for many problems in the class N.

There is a classical connection between the Schwarzian and second order linear differential equations. If $Sf = 2p$ and $u = (f')^{-1/2}$ then

$$u'' + pu = 0. \tag{1.6}$$

Conversely, if u_1, u_2 are linearly independent solutions of (1.6) and $f = u_1/u_2$ then $Sf = 2p$.

Much of the work in [**3**] is based on applying comparison theorems for solutions of differential equations to obtain bounds on f and f'. For example, if $f \in N_0$ then $u = (f')^{-1/2}$ satisfies the initial conditions $u(0) = 1, u'(0) = 0$, and it was shown that

$$n(|z|) \le |f(z)| \le L(|z|), \tag{1.7}$$

and

$$n'(|z|) \le |f'(z)| \le L'(|z|), \tag{1.8}$$

where

$$n(z) = \frac{1}{\sqrt{2}}\frac{(1+z)^{\sqrt{2}} - (1-z)^{\sqrt{2}}}{(1+z)^{\sqrt{2}} + (1-z)^{\sqrt{2}}}.$$

The function n belongs to N_0 and has $Sn(z) = -2/(1-z^2)^2$.

The techniques of comparison allow one also to describe the cases of equality: If equality holds in (1.7) or (1.8) at a single $z_0 \neq 0$ then f must be a rotation of the corresponding extremal, n or L.

In Section 2 we shall consider Equation (1.6) but with the dual inital condition, namely, $u(0) = 0, u'(0) = 1$. In terms of the function f this means assuming the normalization (1.2). By considering $g = 1/f$, $f \in N_0$, we will derive in this way sharp upper and lower bounds for the spherical distortion $|f'|/(1+|f|^2)$. We will also obtain a two-point distortion theorem that actually characterizes Nehari functions. This result can be viewed as an analogue of a theorem of Blatter that characterizes the set of all univalent functions in D [**2**].

One of the main results in [**5**] is the fact that, for a function in N, the image domain is a quasidisc as soon as it is a John domain. In other words, linear connectivity comes as a consequence of the John condition. Recall that if f is any univalent function then one of the many characterizations of John domains is that there exits a constant M such that for all $z \in D$

$$\operatorname{diam} f(B(z)) \leq M(1-|z|^2)|f'(z)|, \tag{1.9}$$

where

$$B(z) = \{w : |z| \leq |w| < 1, \ |\arg(w) - \arg(z)| \leq \pi(1-|z|)\}. \tag{1.10}$$

For a detailed exposition of these concepts we refer the reader to [**12**, Chapter 5].

In Section 3 we will derive an estimate for $\operatorname{diam} f(B(z))$ when $f \in N$, which will indicate how a Nehari domain can fail to be a quasidisc. Finally, in Section 4 we will be concerned with other 'quasidisc like' properties of Nehari domains, expressed in terms of f''/f'.

2. Two-point distortion and characterization.

The starting point in this section is a comparison lemma. It is essentially contained in [**3**], and we include here a brief proof for the convenience of the reader.

Lemma 1. *Let $P = P(x) \geq 0$ be continuous for $x \in [0,1)$ and suppose that the solution of*

$$v''(x) + P(x)v(x) = 0, \ v(0) = 0, \ v'(0) = 1 \tag{2.1}$$

is positive in the open interval $(0,1)$. Let w be solution of

$$w''(x) - P(x)w(x) = 0, \ w(0) = 0, \ w'(0) = 1. \tag{2.2}$$

If $p = p(z)$ is analytic in D and $|p(z)| \leq P(|z|)$ then the solution of

$$u'' + pu = 0 \ , \ u(0) = 0 \, , \ u'(0) = 1 \tag{2.3}$$

satisfies

$$v(|z|) \leq |u(z)| \leq w(|z|). \tag{2.4}$$

Proof. We consider u along rays starting from the origin, and without loss of generality, we may take the segment $[0,1)$. Thus let $\varphi(x) = |u(x)|$ for $x \in [0,1)$. At $x = 0$ the right hand derivative of φ exists and equals 1. Whenever $u(x) \neq 0$ then φ is smooth, and it is not difficult to show that

$$\varphi''(x) + |p(x)|\varphi(x) \geq 0.$$

Since the function v is positive in $(0,1)$, it follows from the Sturm comparison theorem that

$$\varphi(x) \geq v(x)$$

for all $x \in [0,1)$. This proves the lower bound in (2.4).

In order to establish the remaining inequality we turn (2.3) into the integral equation

$$u(z) = z - \int_0^z (z-\zeta)p(\zeta)u(\zeta)d\zeta.$$

Since $|p(z)| \le P(|z|)$ it is a consequence of Lemma 8 in [**6**] that

$$|u(z)| \le w(|z|).$$

This finishes the proof. □

Theorem 1. *Let $f \in N_0$. Then*

$$\frac{n'(|z|)}{1+n^2(|z|)} \le \frac{|f'(z)|}{1+|f(z)|^2} \le \frac{L'(|z|)}{1+L^2(|z|)}. \tag{2.5}$$

If equality holds in either inequality at a single $z_0 \neq 0$ then f is a rotation of the corresponding extremal.

Proof. Let $f \in N_0$ and let $g = 1/f$. Then g satisfies (1.1) and (1.2), hence for $u = (g')^{-1/2}$ one has

$$u'' + \left(\frac{1}{2}Sf\right)u = 0,\ u(0) = 0,\ u'(0) = 1.$$

With $P(x) = (1-x^2)^{-2}$ the functions v, w of the lemma are given by

$$v(x) = \sqrt{\frac{L^2}{L'}(x)}$$

and

$$w(x) = \sqrt{\frac{n^2}{n'}(x)}.$$

So the lemma yields

$$\frac{n'}{n^2}(|z|) \le \left|\frac{f'}{f^2}(z)\right| \le \frac{L'}{L^2}(|z|).$$

Hence, using in addition (1.8), we obtain

$$\frac{1+|f(z)|^2}{|f'(z)|} = \frac{1}{|f'(z)|} + \frac{|f^2(z)|}{|f'(z)|} \ge \frac{1}{L'(|z|)} + \frac{L^2(|z|)}{L'(|z|)},$$

and similarly,

$$\frac{1+|f(z)|^2}{|f'(z)|} \le \frac{1}{n'(|z|)} + \frac{n^2(|z|)}{n'(|z|)}.$$

These two inequalities give (2.5).

Finally, if equality holds in (2.5) at some $z_0 \neq 0$ then it follows already from the case of equality in (1.8) that f must be a rotation of n or L. This finishes the proof. □

If $f \in S$, the class of all univalent function in D with $f(0) = 0, f'(0) = 1$ then, as mentioned in the introduction, one has $|Sf(z)| \le 6(1 - |z|^2)^{-2}$. Again by looking at $g = 1/f$ and $u = (g')^{-1/2}$ we can apply Lemma 1, but now only with the solution w because the corresponding function v has (infinitely many) zeroes [**8**, p. 492]. The function w arises from the Koebe function $k(z) = z/(1 - z)^2$, that is,

$$w(x) = \sqrt{\frac{k^2}{k'}(x)},$$

and we obtain in this fashion the sharp estimate

$$\left|\frac{f'}{f^2}(z)\right| \ge \frac{1 - |z|^2}{|z|^2}.$$

This inequality is equivalent to one established in 1919 by Löwner, namely that for functions g in the class Σ,

$$|g'(\zeta)| \ge 1 - \frac{1}{|\zeta|^2}, \ |\zeta| > 1.$$

It is interesting to note that in our proof we only use the fact that $(1 - |z|^2)^2|Sf(z)| \le 6$, rather than the univalence of f.

The next result characterizes Nehari functions in terms of a two-point distortion property. Let $d_h(z_1, z_2)$ be the hyperbolic distance between points in D.

Theorem 2. *Let f be meromorphic and locally univalent in D. Then*

$$(1 - |z|^2)^2|Sf(z)| \le 2 \tag{2.6}$$

for all $z \in D$ if and only if

$$(1 - |z_1|^2)|f'(z_1)|(1 - |z_2|^2)|f'(z_2)|\, d_h(z_1, z_2)^2 \le |f(z_1) - f(z_2)|^2 \tag{2.7}$$

for all $z_1, z_2 \in D$. Furthermore, equality holds for $z_1 \ne z_2$ if and only if f is of the form $T \circ L \circ \tau$, where T is Möbius and τ an automorphism of D with $\tau(z_1), \tau(z_2) \in (-1, 1)$.

Proof. Suppose first that (2.6) holds. Then

$$g(z) = \frac{(1 - |z_1|^2)f'(z_1)}{f\left(\dfrac{z + z_1}{1 + \bar{z}_1 z}\right) - f(z_1)} = \frac{1}{z} + b_0 + b_1 z + \cdots \tag{2.8}$$

also satisfies (2.6). It follows from Theorem 2 in [**5**] that

$$(1 - |z|^2)d_h(0, z)^2|g'(z)| \le 1 \tag{2.9}$$

for all $z \in D$. This gives

$$(1-|z|^2)\frac{(1-|z_1|^2)^2|f'(z_1)|\left|f'\left(\frac{z+z_1}{1+\bar{z}_1 z}\right)\right|}{|1+\bar{z}_1 z|^2\left|f\left(\frac{z+z_1}{1+\bar{z}_1 z}\right)-f(z_1)\right|^2}d_h(0,z)^2 \le 1.$$

With $z_2 = \dfrac{z+z_1}{1+\bar{z}_1 z}$, the above inequality gives (2.7).

The case of equality in (2.7) for $z_1 \neq z_2$ corresponds to the case of equality in (2.9) for $z \neq 0$. As shown in [**5**] this occurs if and only if g is a rotation of a function of the form $1/L + a$. Hence f is of the form stated.

Let us assume now that (2.7) holds. Then the function g as defined in (2.8) satisfies (2.9). Hence, for $z \in D$ we have

$$(1-|z|^2)\left(|z|+\frac{1}{3}|z|^3+\cdots\right)^2\left|-\frac{1}{z^2}+b_1+\cdots\right| \le 1,$$

which implies that

$$\left(1-\frac{1}{3}|z|^2+O(z^3)\right)(1-\operatorname{Re}\{b_1 z^2\}+O(z^3)) \le 1$$

as $z \to 0$. Therefore

$$\operatorname{Re}\left\{b_1\frac{z^2}{|z|^2}\right\} \le \frac{1}{3}+O(z)$$

as $z \to 0$, which in turn gives that $|b_1| \le 1/3$. Thus

$$(1-|z_1|^2)^2|Sf(z_1)| = |Sg(0)| = 6|b_1| \le 2.$$

Since the point z_1 is arbitrary we conclude that $f \in N$. □

Remarks.

1. If $\Omega = f(D)$ is the image domain with Poincaré metric $\lambda(w)|dw|$ and hyperbolic distance δ_h, then (2.7) can be rewritten as

$$\delta_h(w_1,w_2) \le \sqrt{\lambda(w_1)\lambda(w_2)}\,|w_1-w_2|$$

for points $w_1, w_2 \in \Omega$.

2. Theorem 2 resembles a result of Blatter, according to which an analytic function in the unit disc is univalent if and only if

$$\begin{aligned}&|f(z_1)-f(z_2)|^2\\ &\ge \frac{1}{8}\frac{\sinh^2(2d_h(z_1,z_2))}{\cosh(4d_h(z_1,z_2))}\{(1-|z_1|^2)^2|f'(z_1)|^2+(1-|z_2|^2)^2|f'(z_2)|^2\}.\end{aligned}$$

Here the cases of equality for $z_1 \neq z_2$ only happen when f is of the form $ak \circ \tau + b$ with $\tau(D) = D$, $\tau(z_1), \tau(z_2) \in (-1,1)$, and k the Koebe function.

3. A diameter bound.

As was pointed out in the proof of Theorem 2, if $f \in N$ is normalized so that $f(z) = 1/z + b_0 + b_1 z + \cdots$ then

$$(1 - |z|^2)L^2(|z|)|f'(z)| \le 1.$$

Lemma 2. *Let $f(z) = 1/z + b_0 + b_1 z + \cdots$ belong to N. Then for $|\zeta| = 1$*

$$q(r) = (1 - r^2)L^2(r)|f'(r\zeta)| \tag{3.1}$$

is decreasing for $r \in [0, 1)$.

Proof. Let $|\zeta| = 1$ and for $r \in [0, 1)$ define

$$u(r) = \frac{1}{\sqrt{(1 - r^2)|f'(r\zeta)|}}.$$

It was shown in [**5**] that

$$\frac{d}{dr}[(1 - r^2)u'(r)] \ge 0. \tag{3.2}$$

Also, $u(0) = 0$ and $u'(0) = 1$. Let

$$v(r) = (1 - r^2)u'(r)L(r) - u(r).$$

A simple calculation shows that

$$v'(r) = L(r)\frac{d}{dr}[(1 - r^2)u'(r)],$$

hence v is increasing. Since $v(0) = 0$ we conclude that $v(r) \ge 0$ for $r \in [0, 1)$, and therefore

$$\frac{d}{dr}\left(\frac{u(r)}{L(r)}\right) = \frac{v(r)}{(1 - r^2)L^2(r)} \ge 0.$$

Since $q(r) = (L(r)/u(r))^2$ the lemma follows. □

Theorem 3. *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots$$

belong to N. Then

$$\operatorname{diam} f(B(z)) \le K(1 - |z|^2)|f'(z)|L(|z|), \tag{3.3}$$

where K is an absolute constant.

Proof. Let $z = re^{it}$ and $w = \rho e^{i\theta} \in B(z)$. We write $q = (1 - |z|^2)|f'(z)|$. The classical distortion theorems for univalent functions imply that

$$|f(re^{i\theta}) - f(re^{it})| \le K_1 q \tag{3.4}$$

and

$$(1 - r^2)|f'(re^{i\theta})| \le K_2 q, \tag{3.5}$$

where K_1, K_2 are absolute constants. It follows now from Lemma 2 and Equation (3.5) that

$$(1-\rho^2)|f'(\rho e^{i\theta})|L^2(\rho) \le (1-r^2)|f'(re^{i\theta})|L^2(r) \le K_2 q L^2(r). \tag{3.6}$$

Hence

$$|f(\rho e^{i\theta}) - f(re^{i\theta})| \le \int_r^\rho |f'(se^{i\theta})|ds \le K_2 q \int_r^\rho \frac{L^2(r)}{(1-s^2)L^2(s)} ds$$
$$\le K_2 q L^2(r) \int_r^1 \frac{L'(s)}{L^2(s)} ds = K_2 q L(r).$$

This inequality together with (3.4) implies (3.3). □

In Theorem 3 we have used the stated normalization on f in order to make $\partial f(D)$ bounded. If now $f \in N_0$ then, as pointed out in the introduction, the image Ω will either be a parallel strip or else will be bounded. In the latter, it is clear that (3.3) will still hold with K replaced by some constant M depending on f. From the results in [**5**], such a domain will be a quasidisc if for some constant M the stronger estimate holds:

$$\mathrm{diam} f(B(z)) \le M(1-|z|^2)|f'(z)|,$$

that is, Equation (3.3) without the logarithm.

4. Boundary behavior and exceptional points.

It was shown in [**7**] that all Nehari functions admit a (spherically) continuous extension to the closed disc. In this section we shall be interested in studying the behavior of

$$(1-r^2)\mathrm{Re}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\}$$

as $r \to 1$. According to Theorem 4 in [**5**], a Nehari domain is a John domain (hence a quasidisc) if and only if the corresponding function f normalized to be in N_0 satisfies

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\left\{z\frac{f''}{f'}(z)\right\} < 2. \tag{4.1}$$

Recall also that if $f \in N_0$ then in any case

$$(1-|z|^2)\left|\frac{f''}{f'}(z)\right| \le 2. \tag{4.2}$$

See, e.g., [**4**].

The following lemma is of a general nature:

Lemma 3. *Let $h(z)$ be analytic in D and suppose that for some $0 < \alpha < \infty$, $M < \infty$*

$$(1-|z|)^\alpha |g(z)| \le M. \tag{4.3}$$

Then there exist at most countably many points ζ, $|\zeta| = 1$, such that

$$\lim_{r\to 1}(1-r)^\alpha g(r\zeta) =: b(\zeta) \neq 0 \quad \textit{exists.} \tag{4.4}$$

Proof. Let $|\zeta| = 1$ be such that (4.4) holds. For $r \in (0,1)$ and $z \in D$ let

$$f(z,r) = \left(1 - \frac{z+r}{1+rz}\right)^\alpha g\left(\frac{z+r}{1+rz}\zeta\right). \tag{4.5}$$

Hence by (4.3)

$$|f(z,r)| \le 2^\alpha M \left|1 - \frac{z+r}{1+rz}\right|^\alpha \left(1 - \left|\frac{z+r}{1+rz}\right|^2\right)^{-\alpha}$$

$$= \frac{2^\alpha M|1+rz|^\alpha |1-z|^\alpha}{(1+r)^\alpha (1-|z|^2)^\alpha} \le 4^\alpha M \left(\frac{|1-z|}{1-|z|}\right)^\alpha. \tag{4.6}$$

Therefore as $r \to 1$, $f(z,r)$ is locally uniformly bounded in z. Also, by (4.4) and (4.5) we have

$$f(z,r) = (1-\bar{\zeta}w)^\alpha g(w) \to b(\zeta) \tag{4.7}$$

as $r \to 1$, where $w = \zeta \dfrac{z+r}{1+rz}$. From the theorem of Montel we conclude that (4.7) holds locally uniformly in z.
Let

$$\varphi(z) = (1-|z|^2)^\alpha |g(z)|.$$

If ζ is such that (4.4) holds, then according to (4.5)

$$\varphi\left(\frac{z+r}{1+rz}\zeta\right) = \frac{(1-r^2)^\alpha (1-|z|^2)^\alpha}{|1+rz|^{2\alpha}} \left|g\left(\frac{z+r}{1+rz}\zeta\right)\right| \to \left(\frac{1-|z|^2}{|1-z|}\right)^\alpha |b(\zeta)|$$

as $r \to 1$. The convergence is locally uniform in z. Hence as $z \to \zeta$ radially

$$\varphi(z) \to 2^\alpha |b(\zeta)| \neq 0,$$

but we can also find a curve γ ending at ζ along which the function φ tends to 0. The Ambiguous Point Theorem of Bagemihl, [**1**], implies that the number of points ζ for which this can happen is at most countable. This finishes the proof.

Theorem 4. *Let $f \in N_0$. Then*

$$\liminf_{r\to 1}(1-r^2)\mathrm{Re}\left\{\zeta \frac{f''}{f'}(r\zeta)\right\} < 2, \tag{4.8}$$

except possibly for countably many points ζ.

Proof. Observe first that in light of (4.2), the quantity on the left hand side of (4.8) is bounded by 2. Suppose ζ is such that we have equality in (4.8). Then

$$\lim_{r\to 1}(1-r^2)\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} = 2.$$

Using (4.2) again we conclude that

$$\lim_{r\to 1}(1-r^2)\mathrm{Im}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\} = 0,$$

hence

$$\lim_{r\to 1}(1-r^2)\frac{f''}{f'}(r\zeta) = 2.$$

But according to Lemma 3 applied to $g = f''/f'$ and $\alpha = 1$, the last equation can only happen for countably many points ζ.

Lemma 4. *Let*

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots$$

be in N. *Then for* $|\zeta| = 1$ *and* $r \in (0,1)$

$$|f(\zeta) - f(r\zeta)| \le \int_r^1 |f'(s\zeta)|ds \le \frac{(1-r^2)|f'(r\zeta)|}{r - \dfrac{1-r^2}{2}\mathrm{Re}\left\{\zeta\dfrac{f''}{f'}(r\zeta)\right\}}. \tag{4.9}$$

Remark. The right-hand side of (4.9) is rather similar to the extension operator considered in [**4**].

Proof. Let $|\zeta| = 1$ and let again

$$u(r) = \frac{1}{\sqrt{(1-r^2)|f'(r\zeta)|}}.$$

Then $((1-r^2)u'(r))' \ge 0$, hence for $0 < r < s < 1$

$$(1-s^2)u'(s) \ge (1-r^2)u'(r),$$

and therefore

$$u(s) - u(r) \ge (1-r^2)u'(r)(L(s) - L(r)).$$

Thus

$$(1-s^2)|f'(s\zeta)| = \frac{1}{u^2(s)} \le \frac{1}{u^2(r)\left[1 + (1-r^2)\dfrac{u'}{u}(r)(L(s)-L(r))\right]^2},$$

or

$$|f'(s\zeta)| \le \frac{(1-r^2)|f'(r\zeta)|L'(s)}{\left[1+(1-r^2)\frac{u'}{u}(r)(L(s)-L(r))\right]^2}$$

$$= \frac{|f'(r\zeta)|}{(u'/u)(r)} \frac{d}{ds}\left[\frac{-1}{1+(1-r^2)\frac{u'}{u}(r)(L(s)-L(r))}\right].$$

This implies

$$\int_r^1 |f'(s\zeta)|ds \le \frac{|f'(r\zeta)|}{(u'/u)(r)},$$

which is equivalent to (4.9) since

$$(1-r^2)\frac{u'}{u}(r) = r - \frac{1-r^2}{2}\mathrm{Re}\left\{\zeta\frac{f''}{f'}(r\zeta)\right\}.$$

□

Theorem 5. *Let $f \in N$. Then*

$$\liminf_{r\to 1} \frac{|f(\zeta)-f(r\zeta)|}{(1-r^2)|f'(r\zeta)|} < \infty \tag{4.10}$$

with the exception of at most countably many points $\zeta \in \partial D$.

Proof. Without loss of generality we may assume that

$$f(z) = \frac{1}{z} + b_0 + b_1 z + \cdots ,$$

since such a normalization can affect condition (4.10) at most at one boundary point. Also, we may take $b_0 = 0$. Then (4.10) follows directly from (4.8) and (4.9) provided we can show that (4.8) still holds when $f(z) = 1/z + \cdots$. Let $g = 1/f$. Then $g \in N_0$, hence it is either a rotation of L or else it is bounded. In the latter, it is easy to see that (4.8) for g implies (4.8) for f, while if $g = 1/L$ then (4.8) can be verified directly.

To conclude, we remark that if $f(D)$ is a bounded quasidisc then

$$\limsup_{r\to 1} \frac{|f(\zeta)-f(r\zeta)|}{(1-r^2)|f'(r\zeta)|} < \infty \tag{4.11}$$

for all ζ. It is natural to ask whether a stronger form of Theorem 5 is true, where (4.11) holds with at most countably many exceptions. □

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