FREDHOLM PROPERTIES OF TOEPLITZ OPERATORS ON DIRICHLET SPACES

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In this paper, the Fredholm properties of some Toeplitz operators on Dirichlet spaces be discussed, and the essential spectra of Toeplitz operators with symbols in \( C^1 \) or \( H_1^\infty + C^1 \) be computed.

1. Introduction.

Let \( \mathbb{D} \) be the unit disk in the complex plane \( \mathbb{C} \), \( dA = \frac{1}{\pi} dx\,dy \) be the normalized area measure on \( \mathbb{D} \). \( L^{2,1} \) is the space of functions \( u : \mathbb{D} \to \mathbb{C} \), for which the norm

\[
\|u\|_{2,1}^2 = \left[ \int_{\mathbb{D}} \left( \left| \frac{\partial u}{\partial z}(z) \right|^2 + \left| \frac{\partial u}{\partial \bar{z}}(z) \right|^2 \right) dA \right]^{\frac{1}{2}} < \infty.
\]

\( L^{2,1} \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_{2,1} = \left\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right\rangle_{L^2(dA)} + \left\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial \bar{z}} \right\rangle_{L^2(dA)}.
\]

The Dirichlet space, \( \mathcal{D} \), is the subspace of all analytic functions \( g \) in \( L^{2,1} \) with \( g(0) = 0 \). Let \( P \) be the orthogonal projection from \( L^{2,1} \) into \( \mathcal{D} \). \( P \) is an integral operator represented by

\[
P(u)(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z} \frac{\partial K(z, w)}{\partial z} dA,
\]

where \( K(z, w) = \sum_{k=1}^{\infty} \frac{z^k w^k}{k} \) is the reproducing kernel of \( \mathcal{D} \) (see R. Rochberg and Z.J. Wu [5] and Wu [7]). Let \( G \) be a domain in \( \mathbb{C} \), define

\[
C^1(G) = \left\{ u \mid u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in C(G) \right\},
\]

\[
C^1(\bar{G}) = \left\{ u \mid u, \frac{\partial u}{\partial z}, \frac{\partial u}{\partial \bar{z}} \in C(\bar{G}) \right\},
\]

and

\[
H_1^\infty(G) = \{ f \in H(G) | f' \in H_1^\infty(G) \},
\]
where \( C(G) \) (or \( C(\bar{G}) \)) and \( H(G) \) are respectively the set of continuous functions on \( G \) (or \( \bar{G} \)) and the one of analytic functions on \( G \).

In [5] and [7], R. Rochberg and Z.J. Wu define the Toeplitz operator with symbol \( \mu \) as

\[
T_\mu f = \int_D f'(z) \frac{\partial K(z,w)}{\partial z} d\mu \quad (\forall f \in D),
\]

where \( \mu \) is a measure on \( D \). If \( d\mu = \varphi dA, \varphi \in L^\infty(D,dA) \), one can define

\[
T_\varphi f = \int_D f'(z) \frac{\partial K(z,w)}{\partial z} \varphi dA.
\]

We see easily that \( T_\varphi f \neq P(\varphi f) \) in general, in fact, \( P(\varphi f) \) may be undefined for \( \varphi \in L^\infty(D,dA) \). However, it seems to be more natural that the Toeplitz operators be defined as the form \( P(\varphi f) \). In this paper, we try to define the Toeplitz operators with some special symbols as following

**Definition 1.** Suppose \( \varphi \in C^1(\bar{D}) \), the operator

\[
T_\varphi f = P(\varphi f) = \langle \varphi f, K \rangle_{\frac{1}{2}} = \int_D \frac{\partial(\varphi f)}{\partial z} \frac{\partial K(z,w)}{\partial z} dA(z)
\]

is said to be the Toeplitz operator with symbol \( \varphi \).

We will compute the spectra and essential spectra of these operators. For convenience, we use frequently the notation \( \tilde{T}_\varphi \) to denote the Toeplitz operator with symbol \( \varphi \) on Bergman space.

By the way, our results are also true for Toeplitz operators on weighted Dirichlet spaces \( D_\alpha \) (\( \alpha < 1 \)). If \( \alpha = \frac{1}{2} \), then \( D_{\frac{1}{2}} = D \), the usual Dirichlet space (c.f. [5], [7]).

### 2. Toeplitz operators with symbols in \( C^1(\bar{D}) \).

Throughout this paper, we use the symbol “\( \langle \cdot, \cdot \rangle \)” to represent the inner product in \( L^2(\bar{D},dA) \), and “\( \langle \cdot, \cdot \rangle_{\frac{1}{2}} \)” to that in \( D \).

Define the norm in \( C^1(\bar{D}) \) as

\[
\|\varphi\|_* = \max_{z \in \bar{D}} \max \left\{|\varphi|, \left| \frac{\partial \varphi}{\partial z} \right|, \left| \frac{\partial \varphi}{\partial \bar{z}} \right| \right\}, \quad (\forall \varphi \in C^1(\bar{D})).
\]

It is well-known that \( C^1(\bar{D}) \) is a norm-closed algebra relative to \( \|\|_* \).

**Lemma 1.** For any \( \varphi \in C^1(\bar{D}) \), \( H_\varphi f =: (I - P)(\varphi f) \) (\( \forall f \in D \)) is a compact operator from \( D \) to \( D^\perp \).
Proof. For any \( f \in \mathcal{D} \), \( g \in \mathcal{D}^\perp \)

\[
\langle H_\varphi f, g \rangle_{H^2} = \langle \varphi f, g \rangle_{H^2} = \frac{\partial \varphi}{\partial \bar{z}} f \frac{\partial g}{\partial z} + \frac{\partial \varphi}{\partial z} f \frac{\partial g}{\partial \bar{z}} \]

\[
= \frac{\partial \varphi}{\partial \bar{z}} f \frac{\partial g}{\partial z} + \left( \frac{\partial \varphi}{\partial z} f \frac{\partial g}{\partial \bar{z}} + \langle \varphi f, \frac{\partial g}{\partial \bar{z}} \rangle \right).
\]

Note \( g \perp \mathcal{D} \), so \( \frac{\partial g}{\partial \bar{z}} \perp L^2_a \) (\( L^2_a \) is the classical Bergman space on \( \mathbb{D} \)). Let \( P_a \) is the orthogonal projection from \( L^2(\mathbb{D}, dA) \) to \( L^2_a \), then for any \( f \in \mathcal{D} \) and \( g \in \mathcal{D}^\perp \), we have

\[
\langle \varphi f', \frac{\partial g}{\partial z} \rangle_{H^2} = \langle (I - P_a)(\varphi f'), \frac{\partial g}{\partial z} \rangle_{H^2} = \langle \tilde{H}_\varphi f', \frac{\partial g}{\partial z} \rangle_{H^2},
\]

where \( \tilde{H}_\varphi \) is the Hankel operator from \( L^2_a \) to \( L^2_a \) with symbol \( \varphi \). Since \( \varphi \in C^1(\mathbb{D}) \), \( \tilde{H}_\varphi \) is a compact operator. Now assume that \( \{f_k\} \subset \mathcal{D} \) is a sequence which converges weakly 0 and satisfies \( \|f_k\|_D = 1 \), we prove first that \( \|f_k\|_{L^2} \to 0 \) \( (k \to \infty) \). In fact, without loss of generality, assume \( f_k(z) = \sum_{n=1}^{\infty} a_n^{(k)} z^n \), then \( a_n^{(k)} \to 0 \) \( (k \to \infty, \forall n) \) by \( f_k \) weakly to 0. Note \( \|f_k\|_{L^2}^2 = \|f'_k\|_{L^2}^2 = 1 \), so \( \sum_{n=1}^{\infty} |a_n^{(k)}|^2 = 1 \) \( (\forall k) \), thus \( \sum_{n=1}^{\infty} |a_n^{(k)}|^2 \leq 1 \), furthermore, for any \( \epsilon > 0 \), there is a \( N_0 \) which is independent on \( k \) such that \( \sum_{n \geq N_0 + 1} |a_n^{(k)}|^2 \frac{1}{n+1} < \frac{\epsilon}{2} \). Fix such a \( N_0 \), then

\[
\|f_k\|_{L^2}^2 = \sum_{n=1}^{\infty} |a_n^{(k)}|^2 \frac{1}{n+1} \leq \sum_{n=1}^{N_0} |a_n^{(k)}|^2 \frac{1}{n+1} + \sum_{n=N_0+1}^{\infty} \frac{1}{n+1} + \frac{\epsilon}{2},
\]

Since \( a_n^{(k)} \to 0 \) \( (\forall n, k \to \infty) \), there is a \( K_0 \) such that for any \( k > K_0 \), \( \sum_{n=1}^{N_0} |a_n^{(k)}|^2 \frac{1}{n+1} < \frac{\epsilon}{2} \), thus \( \|f_k\|_{L^2}^2 < \frac{\epsilon}{2} \) \( (k > K_0) \), hence \( \|f_k\|_{L^2} \to 0 \). Note

\[
\|H_\varphi f_k\|_{L^2,1}^2 = \langle H_\varphi f_k, H_\varphi f_k \rangle_{H^2} \]

\[
\leq \left\| \frac{\partial \varphi}{\partial \bar{z}} f_k \right\|_{L^2} \|H_\varphi f_k\|_{L^2,1} + \left\| \tilde{H}_\varphi f_k \right\|_{L^2} \|H_\varphi f_k\|_{L^2,1} + \left\| \tilde{H}_\varphi f'_k \right\|_{L^2} \|H_\varphi f_k\|_{L^2,1},
\]

we know easily that \( \|H_\varphi f_k\|_{L^2,1} \to 0 \). This shows that \( H_\varphi \) is compact. \( \square \)

Proposition 2. For any \( \varphi \in C^1(\mathbb{D}) \), \( T_\varphi \) is a bounded operator on \( \mathcal{D} \).
Proof. For any \( f, g \in \mathcal{D} \),
\[
\langle T_\varphi f, g \rangle_{\frac{1}{2}} = \langle \varphi f, g \rangle_{\frac{1}{2}} = \left\langle \frac{\partial \varphi}{\partial z} f, g \right\rangle = \left\langle \frac{\partial \varphi}{\partial z} f, g \right\rangle + \langle \varphi f', g' \rangle,
\]
hence
\[
\left| \langle T_\varphi f, g \rangle_{\frac{1}{2}} \right| \leq \left\| \frac{\partial \varphi}{\partial z} f \right\|_{L^2} \| g' \|_{L^2} + \| \varphi f' \|_{L^2} \| g' \|_{L^2}
\leq \| \varphi \|_* (\| f \|_{L^2} \| g \|_{\mathcal{D}} + \| f \|_{\mathcal{D}} \| g \|_{\mathcal{D}}).
\]
It is not difficult to check \( \| f \|_{L^2} \leq \| f \|_{\mathcal{D}} \) for any \( f \in \mathcal{D} \) (since \( f(0) = 0 \) for any \( f \in \mathcal{D} \)), hence \( \langle T_\varphi f, g \rangle_{\frac{1}{2}} \leq 2 \| \varphi \|_* \| f \|_{\mathcal{D}} \| g \|_{\mathcal{D}} \). This shows that \( \| T_\varphi \| \leq 2 \| \varphi \|_* \). \( \square \)

Remark. In general, \( \| T_\varphi \| \neq \| \varphi \|_* \) even \( \varphi \in C^1(\mathbb{D}) \cap A(\mathbb{D}) \) (A(\mathbb{D}) be the disk algebra), it may be that \( \| \varphi \|_{C^1} < \| T_\varphi \| < 2 \| \varphi \|_{C^1} \). For instance, if \( \varphi(z) = z \), then \( \| \varphi \|_* = \| \varphi \|_{\infty} = 1 \), but \( \| T_\varphi \| \geq \sqrt{2} \).

Proposition 3. Let \( L^{\infty,1} = \{ f \in L^2 \mathbb{D} | f, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial \bar{z}} \in L^\infty(\mathbb{D}, dA) \} \), then for any \( \varphi \in L^{\infty,1} \) and \( \psi \in C^1(\mathbb{D}) \) \( T_\varphi T_\psi - T_{\varphi \psi} \in \mathcal{K}(\mathcal{D}) \) (the compact operator algebra on \( \mathcal{D} \)).

Proof. For any \( f, g \in \mathcal{D} \), as a direct computing, we see that
\[
\langle (T_\varphi T_\psi - T_{\varphi \psi}) f, g \rangle_{\frac{1}{2}} = \left\langle \frac{\partial \varphi}{\partial z} T_\psi f, \frac{\partial g}{\partial z} \right\rangle - \left\langle \frac{\partial \varphi}{\partial z} (\psi f), \frac{\partial g}{\partial z} \right\rangle + \left\langle \varphi \frac{\partial (\psi f)}{\partial z}, \frac{\partial g}{\partial z} \right\rangle,
\]
set \( g = (T_\varphi T_\psi - T_{\varphi \psi}) f \), we have
\[
\| (T_\varphi T_\psi - T_{\varphi \psi}) f \|_{\frac{1}{2}}^2 \leq \| \varphi \|_* \| T_\psi f \|_{L^2} + \| \psi \|_* \| f \|_{L^2}
+ \| \varphi \|_{L^\infty,1} \| (T_\psi T_\psi - T_{\varphi \psi}) f \|_{\mathcal{D}}.
\]
Note for any sequence \( \{ f_k \} \subset \mathcal{D} \) which converges weakly 0 and satisfies \( \| f_k \|_{\mathcal{D}} = 1 \) (certainly, \( T_\psi f_k \rightharpoonup 0 \) in \( \mathcal{D} \), \( \| T_\psi f_k \|_{L^2} \to 0 \), \( \| f_k \|_{L^2} \to 0 \)), hence
\[
\| (T_\varphi T_\psi - T_{\varphi \psi}) f_k \|_{\mathcal{D}} \to 0
\]
by \( H_\psi \) is compact. Futhermore, \( T_\varphi T_\psi - T_{\varphi \psi} \) is compact. \( \square \)

Lemma 4. For any \( \varphi \in L^{\infty,1} \), \( T_\varphi^* - T_\varphi \in \mathcal{K}(\mathcal{D}) \).

Proof. Similar to the proof of Proposition 2, it is easy to see that \( T_\varphi \) is bounded for any \( \varphi \in L^{\infty,1} \). Now assume \( \varphi \in L^{\infty,1} \), then for any \( f \cdot g \in \mathcal{D} \),
\begin{equation*}
\langle (T^*_\varphi - T\varphi) f, g \rangle \frac{1}{2} = \langle \frac{\partial f}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}} g \rangle - \langle f, \frac{\partial \varphi}{\partial \bar{z}} \rangle. \text{ Thus}
\end{equation*}
\begin{equation*}
|\langle (T^*_\varphi - T\varphi) f, g \rangle\rangle | \leq \left\| \frac{\partial f}{\partial z} \right\|_{L^2} \left\| \frac{\partial \varphi}{\partial \bar{z}} g \right\|_{L^2} + \left\| f \right\|_{L^2} \left\| \frac{\partial \varphi}{\partial \bar{z}} \right\|_{L^2} \leq ||\varphi||_* \left\| g \right\|_{L^2} \left\| f \right\|_{D} + ||\varphi||_* \left\| f \right\|_{L^2} \left\| g \right\|_{D},
\end{equation*}
where \( ||\varphi||_* = \text{esssup}_{z \in \mathbb{D}} \max\{|\varphi(z)|, |\frac{\partial \varphi}{\partial \bar{z}}(z)|, |\frac{\partial^2 \varphi}{\partial z \partial \bar{z}}(z)|\} \). Hence for any \( \{f_k\} \subset D, ||f_k|| = 1, f_k \xrightarrow{w} 0 \), we have
\begin{equation*}
|| (T^*_\varphi - T\varphi) f_k ||_{L^2}^2 \leq ||\varphi||_*(||T^*_\varphi - T\varphi||_{L^2}||f_k||_D + ||\varphi||_*(||f_k||_{L^2}||T^*_\varphi - T\varphi||_{L^2})).
\end{equation*}
Note \( (T^*_\varphi - T\varphi) f_k \xrightarrow{w} 0 \), so \( ||(T^*_\varphi - T\varphi) f_k||_{L^2} \rightarrow 0 \). Consequently, \( ||(T^*_\varphi - T\varphi) f_k||_{D} \rightarrow 0 \). This shows that \( T^*_\varphi - T\varphi \) is compact.
\begin{remark}
It is well-known that \( T^*_\varphi = T\varphi \) for any \( \varphi \in L^\infty(\mathbb{D}, dA) \) on \( L^2_\alpha \), but it is not difficult to check \( T^*_\varphi \not\equiv T\varphi \) in general. For example, if \( \varphi(z) = \sum_{n=1}^{\infty} a_n z^n \), then \( \varphi \in H^\infty \), but \( ||\varphi||_\infty = \infty \), one can check easily that \( T^*_\varphi \) is unbounded, however, \( T\varphi \) is always bounded for every \( \varphi \in H^\infty \), so is \( T^*_\varphi \), hence \( T^*_\varphi - T\varphi \) is unbounded.
\end{remark}

\begin{proposition}
If \( \varphi \in C^1(\mathbb{D}) \), then \( T\varphi \) is a compact operator on \( D \) if and only if \( \varphi|_{\partial \mathbb{D}} \equiv 0 \).
\end{proposition}

\begin{proof}
Assume \( \varphi|_{\partial \mathbb{D}} \equiv 0 \), then \( T|_{\partial}{\varphi} \) is a compact operator on \( L^2_\alpha(\mathbb{D}, dA) \), so for any \( \{f_k\} \subset L^2_\alpha, f_k \xrightarrow{w} 0, \|f_k\|_{L^2} = 1 \), we have \( \|T|_{\partial}{\varphi} f_k\|_{L^2} \rightarrow 0 (k \rightarrow \infty) \).
If \( T\varphi \) is not compact on \( D \), then there is a sequence \( \{F_k\} \subset D, \|F_k\|_{D} = 1, F_k \xrightarrow{w} 0 \) such that \( \|T\varphi F_k\|_{D} \rightarrow 0 \), thus \( \|\varphi F_k\|_{L^2} \rightarrow 0 \), furthermore
\begin{equation*}
\left\langle \frac{\partial (\varphi F_k)}{\partial z}, \frac{\partial (\varphi F_k)}{\partial \bar{z}} \right\rangle + \left\langle \frac{\partial (\varphi F_k)}{\partial \bar{z}}, \frac{\partial (\varphi F_k)}{\partial z} \right\rangle \rightarrow 0.
\end{equation*}
Note
\begin{equation*}
\frac{\partial (\varphi F_k)}{\partial z} = \frac{\partial \varphi}{\partial z} F_k + \varphi \frac{\partial F_k}{\partial z}, \frac{\partial (\varphi F_k)}{\partial \bar{z}} = \frac{\partial \varphi}{\partial \bar{z}} F_k,
\end{equation*}
thus
\begin{equation*}
\left\langle \frac{\partial (\varphi F_k)}{\partial z}, \frac{\partial (\varphi F_k)}{\partial \bar{z}} \right\rangle \leq \|\varphi\|^2 \|F_k\|^2_{L^2} \rightarrow 0,
\end{equation*}
and
\begin{equation*}
\left\| \frac{\partial (\varphi F_k)}{\partial z} \right\|_{L^2}^2 = \left\langle \frac{\partial \varphi}{\partial z} F_k, \frac{\partial \varphi}{\partial \bar{z}} F_k \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} F_k, \varphi F'_k \right\rangle + \left\langle \varphi F'_k, \frac{\partial \varphi}{\partial z} F_k \right\rangle + \left\langle \varphi F'_k, \varphi F'_k \right\rangle.
\end{equation*}
Since \( \|F'_k\|_{L^2} = \|F_k\|_{D} = 1 \), and \( F'_k \xrightarrow{w} 0 \) in \( L^2_\alpha \),
\begin{equation*}
\langle \varphi F'_k, \varphi F'_k \rangle = \langle |\varphi|^2 F'_k, F'_k \rangle = \langle T|_{\partial}{\varphi} \rangle F'_k, F'_k \rangle \rightarrow 0,
\end{equation*}
consequently, \( \| \frac{\partial (|\varphi|^2_f)}{\partial z} \|_{L^2}^2 \to 0 \) by \( \left| \langle \frac{\partial \varphi}{\partial z} F_k, \frac{\partial \varphi}{\partial z} F_k \rangle \right| \leq \| \varphi \|_2^2 \| F_k \|_{L^2}^2 \) and \( \left| \langle \frac{\partial \varphi}{\partial z} F_k, \varphi F'_k \rangle \right| \leq \| \varphi \|_2^2 \| F_k \|_{L^2} \| F'_k \|_{L^2} \), and \( \| F_k \|_{L^2} \to 0 \). This contradicts that \( \| \varphi F_k \|_{L^2} \to 0 \). Hence \( T_\varphi \) must be a compact operator on \( \mathcal{D} \).

Conversely, assume that \( T_\varphi \) is compact on \( \mathcal{D} \). We need to prove that \( \varphi|_{\partial \mathcal{D}} = 0 \). Otherwise, \( \widetilde{T}_{|\varphi|} \) is not compact on \( L^2_0(\mathbb{D}) \), thus there is a sequence \( \{ f_k \} \subset L^2_0, \| f_k \|_{L^2} = 1, f_k \rightharpoonup 0 \) such that \( \| \widetilde{T}_{|\varphi|} f_k \|_{L^2} \to 0, \) thus \( \| |\varphi|^2 f_k \|_{L^2} \to 0 \), that is \( \int_D |\varphi|^2 |f_k|^2 dA \to 0 \). But \( T_\varphi \) is compact on \( \mathcal{D} \), so for \( F_k = f_k^2 dA \), \( \| T_\varphi F_k \|_D \to 0 \), further \( \| T_{|\varphi|^2} T_\varphi F_k \|_D \to 0 \). Note \( T_{|\varphi|^2} - T_\varphi \in \mathcal{K} \), so \( \| T_{|\varphi|^2} F_k \|_D \to 0 \) (since \( F_k \rightharpoonup 0 \) in \( \mathcal{D} \) and \( \| F_k \|_D = 1 \), thus \( \langle T_{|\varphi|^2} F_k, F_k \rangle \frac{1}{2} \to 0 \), hence

\[
\left\langle \frac{\partial (|\varphi|^2 F_k)}{\partial z}, F_k \right\rangle \to 0,
\]

that is

\[
\left\langle \frac{\partial (|\varphi|^2 F_k)}{\partial z}, F_k \right\rangle + \langle |\varphi|^2 F_k, F'_k \rangle \to 0.
\]

Note

\[
\langle |\varphi|^2 F'_k, F_k \rangle = \langle |\varphi|^2 f_k, f_k \rangle = \int_D |\varphi|^2 |f_k|^2 dA \to 0,
\]

and

\[
\left\langle \frac{\partial (|\varphi|^2 F_k)}{\partial z}, F_k \right\rangle \leq \| |\varphi|^2 \|_2 \| F_k \|_{L^2} \| f_k \|_{L^2} \to 0,
\]

hence \( \langle T_{|\varphi|^2} F_k, F_k \rangle \frac{1}{2} \to 0 \), this contradiction shows that \( \varphi|_{\partial \mathcal{D}} \equiv 0 \). \( \square \)

Remark. If \( \varphi \notin C^1(\overline{\mathbb{D}}) \), then \( T_\varphi \) may be non-compact even \( \varphi \in C(\overline{\mathbb{D}}) \) and \( \varphi|_{\partial \mathcal{D}} = 0 \). For instance, let \( \varphi = \sqrt{1 - |z|^2} \), then it is not difficult to check \( T_\varphi \) is not a compact operator. In fact, \( T_\varphi \) is also a unbounded operator on \( \mathcal{D} \).

Theorem 6. Suppose \( \varphi \in C^1(\overline{\mathbb{D}}) \), then \( \sigma_e(T_\varphi) = \varphi(\partial \mathcal{D}) \).

Proof. Without loss of generality, assume \( 0 \in \varphi(\partial \mathcal{D}) \), thus there is a \( \zeta \in \partial \mathcal{D} \) such that \( \varphi(\zeta) = 0 \). Write \( f_k(z) = (\frac{1+z\zeta}{2})^k \), then \( f_k \rightharpoonup w \) on \( \mathcal{D} \) and \( \| f_k \|_{L^2} \leq \| w \|_{L^2} \), clearly

\[
\| \varphi f_k \|_{L^2}^2 = \left\langle \frac{\partial \varphi}{\partial z} f_k + \varphi f'_k, \frac{\partial \varphi}{\partial z} f_k + \varphi f'_k \right\rangle = \left\langle \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\rangle + \left\langle \varphi f'_k, \frac{\partial \varphi}{\partial z} f_k \right\rangle + \left\langle \frac{\partial \varphi}{\partial z} f_k, \varphi f'_k \right\rangle + \left\langle \varphi f'_k, \varphi f'_k \right\rangle + \left\langle \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\rangle.
\]
\[
\left| \frac{\langle \frac{\partial f}{\partial z} f_k, \frac{\partial f}{\partial z} f_k \rangle + \langle \frac{\partial f}{\partial z} f_k', \frac{\partial f}{\partial z} f_k \rangle + \langle \frac{\partial f}{\partial z} f_k, \varphi f_k' \rangle + \langle \frac{\partial f}{\partial z} f_k, \varphi f_k' \rangle}{\|f_k\|^2_D} \right| \to 0,
\]
and
\[
\frac{\langle \varphi f_k', \varphi f_k' \rangle}{\|f_k\|^2_D} = \int_D |\varphi|^2 |f_k'|^2 dA \int_D |f_k|^2 dA
\]
\[
= \int_D |\varphi|^2 \left( \frac{1+\bar{z} \zeta}{2} \right)^{2(k-1)} |dA \int_D \left( \frac{1+\bar{z} \zeta}{2} \right)^{2(k-1)} |dA
\]
\[
= \int_D |\varphi|^2 f_k^2 dA \int_D f_k^{-1} dA \to |\varphi(\zeta)|^2 = 0
\]
(since \(f_k\) is a peak function at \(\zeta\)). Thus \(\|T_\varphi f_k\|_D \to 0\), this shows that \(T_\varphi\) can not be Fredholm, hence \(0 \in \sigma_e(T_\varphi)\). That is \(\varphi(\partial\mathbb{D}) \subset \sigma_e(T_\varphi)\).

Conversely, if \(0 \notin \varphi(\partial\mathbb{D})\), then \(|\varphi(\zeta)| > \epsilon_0 > 0\) for any \(\zeta \in \partial\mathbb{D}\), thus \(\tilde{T}_\varphi\) is Fredholm on \(L^2_a\), if \(T_\varphi\) is not Fredholm on \(D\), then there is a sequence \(\{F_k\} \subset D\) with \(\|F_k\|_D = 1, F_k \to 0\) such that \(\|T_\varphi F_k\|_D \to 0\), or \(\|T_\varphi^* F_k\|_D \to 0\).

Since \(\tilde{T}_\varphi\) is Fredholm on \(L^2_a\), there is an \(S \in L(L^2_a)\) such that \([T_\varphi S] = [I]\) in the Calkin algebra \(L(L^2_a) / \mathcal{K}(L^2_a)\), assume \(\|T_\varphi^* F_k\|_D \to 0\). (For the case of \(\|T_\varphi F_k\|_D \to 0\), we can complete the proof similarly.) Write \(f_k = F_k'\), then \(\|f_k\|_{L^2} = 1\), and \(f_k \to 0\) in \(L^2_a\), thus \(\langle S^* T_\varphi^* f_k, f_k \rangle \to 1\), that is \(\langle T_\varphi^* f_k, S f_k \rangle \to 1\).

On the other hand,
\[
\langle T_\varphi^* F_k, \int_0^z (S f_k)(w) dw \rangle_{\frac{1}{2}} = \langle F_k, \varphi \int_0^z (S f_k)(w) dw \rangle_{\frac{1}{2}}
\]
\[
= \langle f_k, \frac{\partial f}{\partial z} \int_0^z (S f_k)(w) dw + \varphi S f_k \rangle
\]
\[
= \langle f_k, \frac{\partial f}{\partial z} \int_0^z S f_k dw \rangle + \langle f_k, \varphi S f_k \rangle.
\]
Since \(S f_k \to 0\) in \(L^2_a\), \(\| \int_0^z S f_k dw \|_{L^2_a} \to 0\), thus \(\langle f_k, \frac{\partial f}{\partial z} \int_0^z S f_k(w) dw \rangle \to 0\).

Note \(\langle f_k, \varphi S f_k \rangle = \langle T_\varphi^* f_k, S f_k \rangle \to 1\) \(\neq 0\), this shows that \(\langle T_\varphi^* F_k, \int_0^z (S f_k)(w) dw \rangle_{\frac{1}{2}} \to 1\), thus

\[
\|T_\varphi^* F_k\|_D \|S\|_{L(L^2_a)} \geq \langle T_\varphi^* F_k, \int_0^z S f_k dw \rangle_{\frac{1}{2}} \to 1,
\]
this contradicts that \(\|T_\varphi^* F_k\|_D \to 0\). Hence \(T_\varphi\) must be Fredholm, that is \(0 \notin \sigma_e(T_\varphi)\). This follows the theorem. \(\square\)
Lemma 7. For any $\varphi \in C^1(\mathbb{D})$, $\|T\varphi\|_e = \|[T\varphi]\| = \|\varphi|_{\partial \mathbb{D}}\|_\infty$, where $\|T\varphi\|_e$ denotes the essential norm of $T\varphi$.

Proof. Assume $\zeta \in \partial \mathbb{D}$ such that $|\varphi(\zeta)| = \|\varphi|_{\partial \mathbb{D}}\|_\infty$, set $f_\zeta(z) = \frac{1 + \zeta^2}{2}$, then $f_k(z) = \frac{f_\zeta(z)}{f_\zeta(z)} \to 0$, thus for any $K \in \mathcal{K}$, $\|K f_k\| \to 0$. Note

$$\left\| \langle T\varphi, f_k \rangle, f_k \right\|_\frac{1}{2} = \left\| \left\langle \frac{\partial \varphi}{\partial z} f_k, f_k' \right\rangle + \left\langle \varphi f_k', f_k' \right\rangle \right\| \to |\varphi(\zeta)| = \|\varphi|_{\partial \mathbb{D}}\|_\infty$$

and $|(\langle T\varphi + K \rangle f_k, f_k)_{\frac{1}{2}}| \leq \|T\varphi + K\|$. So $\|\varphi|_{\partial \mathbb{D}}\|_\infty \leq \|T\varphi + K\|$ for any $K \in \mathcal{K}$. Hence $\|\varphi|_{\partial \mathbb{D}}\|_\infty \leq \|T\varphi\|_e$.

Conversely, by Lemma 1.2 in C.K. Fong [3], there is an orthogonal sequence $\{f_k\} \subset \mathcal{D}$ such that $\|T\varphi f_k\| \to \|T\varphi\|_e$, so for any $\epsilon > 0$, there is a $k_0$ such that for any $k > k_0$, $\|T\varphi f_k\| > \|T\varphi\|_e - \epsilon$. Since $\varphi \in C^1(\mathbb{D})$, there is a $0 < r < 1$ such that $|\varphi(z)| < \|\varphi|_{\partial \mathbb{D}}\|_\infty + \epsilon$ for any $|z| > r$. Note

$$\|T\varphi f\|_D^2 \leq \|\varphi f\|_{L^2,1}^2 = \langle \varphi f_k, \varphi f_k \rangle_{\frac{1}{2}}$$

$$= \langle \varphi f_k', \varphi f_k' \rangle + \left\| \frac{\partial \varphi}{\partial z} f_k, \varphi f_k' \right\|$$

$$+ \left\| \varphi f_k', \frac{\partial \varphi}{\partial z} f_k \right\| + \left\| \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\|$$

clearly,

$$\left\| \frac{\partial \varphi}{\partial z} f_k, \varphi f_k' \right\| + \left\| \varphi f_k', \frac{\partial \varphi}{\partial z} f_k \right\| + \left\| \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\| + \left\| \frac{\partial \varphi}{\partial z} f_k, \frac{\partial \varphi}{\partial z} f_k \right\| \to 0,$$

so there is a $k_1$ such that for any $k > k_1$, $\|T\varphi f_k\|^2 \leq |\langle \varphi f_k', \varphi f_k' \rangle| + \epsilon$. Since $f_k \to 0$, we know that $f_k$ uniformly $0$ on any compact subset of $\mathbb{D}$, so for any $0 < t < 1$, $\int_{|z| \leq t} |f_k|^2 dA \to 0$ ($k \to \infty$), assume $f_k(z) = \sum_{n=1}^\infty a_n^{(k)} z^n$, thus

$$\sum_{n=1}^\infty a_n^{(k)} \frac{t^{2(n+1)}}{n+1} = \int_{|z| \leq t} |f_k|^2 dA \to 0.$$

Without loss of generality, assume $t > r$, thus

$$\int_{|z| \leq r} |f_k'|^2 dA = \sum_{n=1}^\infty |a_n^{(k)}|^2 n r^{2n} = \sum_{n=1}^\infty |a_n^{(k)}|^2 \frac{t^{2(n+1)}}{n+1} n (n+1) t^{-2} \left( \frac{r}{t} \right)^{2n},$$
since \((\xi^2)^2n(n+1)t^{-2} \to 0 \ (n \to \infty)\), it is clearly that \(\int_{\{|z| \leq r\}} |f_k'|^2dA \to 0\), so there is a \(k_2\) such that for any \(k > k_2\), \(\int_{\{|z| \leq r\}} |f_k'|^2dA < \epsilon\), consequently,

\[
|\langle \varphi f_k', \varphi f_k' \rangle| = \int_{D} |\varphi|^2 |f_k'|^2dA
= \int_{\{|z| \leq r\}} |\varphi|^2 |f_k'|^2dA + \int_{\{|z| > r\}} |\varphi|^2 |f_k'|^2dA
\leq \|\varphi\|_2^2 \int_{\{|z| \leq r\}} |f_k'|^2dA + (\|\varphi\|_{\partial D} \| + \epsilon)^2 \int_{\{|z| > r\}} |f_k'|^2dA
\leq \|\varphi\|_2^2 \int_{\{|z| \leq r\}} |f_k'|^2dA + (\|\varphi\|_{\partial D} \| + \epsilon)^2
< \|\varphi\|_2^2 \epsilon + (\|\varphi\|_{\partial D} \| + \epsilon)^2.
\]

Furthermore, for any \(k > \max\{k_0, k_1, k_2\}\)

\[
\|T_{\varphi} f_k\|^2 \leq \|\varphi\|_2^2 \int_{\{|z| \leq r\}} |f_k'|^2dA + (\|\varphi\|_{\partial D} \| + \epsilon)^2 + \epsilon
\leq (\|\varphi\|_2^2 + 1) \epsilon + (\|\varphi\|_{\partial D} \| + \epsilon)^2.
\]

Hence \(\lim_{k \to \infty} \|T_{\varphi} f_k\| \leq \|\varphi\|_{\partial D} \| \) by the arbitrariness of \(\epsilon\). This shows that \(\|T_{\varphi}\| \leq \|\varphi\|_{\partial D} \| \). We are done. \(\square\)

**Theorem 8.** Let \(\mathcal{I} = \mathcal{I}(C^1)\) be the \(C^*\)-algebra generated by \(\{T_{\varphi} | \varphi \in C^1(\mathbb{D})\}\), then the commutator ideal \(\mathcal{C}(C^1)\) of \(\mathcal{I}\) equals \(\mathcal{K}(D)\), and \(\frac{\mathcal{I}}{\mathcal{K}} \cong C(\partial \mathbb{D})\). Consequently, following short sequence

\[
(*) \quad 0 \to \mathcal{K} \to \mathcal{I} \to C(\partial \mathbb{D}) \to 0
\]

is exact.

**Proof.** By Proposition 3, we know that \(C(C^1) \subset \mathcal{K}\), since \(\mathcal{K}\) is minimal, \(\mathcal{C}(C^1) = \mathcal{K}\). By Lemma 4, \([T_{\varphi}'] = [T_{\varphi}']\) in \(\mathcal{K}\). Define \(\xi : \{[T_{\varphi}] | \varphi \in C^1(\mathbb{D})\} \to C(\partial \mathbb{D})\) as \(\xi([T_{\varphi}]) = \varphi_{|\partial \mathbb{D}}\), it is easy to see that \(\xi\) is well-defined, and one-to-one by Proposition 5. By Proposition 3, Lemma 4 and Lemma 7, we see that \(\xi\) is an isometric *-homomorphism. Hence \(\xi\) can be extended to \(\frac{\mathcal{I}}{\mathcal{K}}\), in fact, for any \([T] \in \mathcal{K}\), there is a sequence \([T_{\varphi_k}]\) such that \(\|[T_{\varphi_k}] - [T]\| \to 0\), thus \(\|(\varphi_k - \varphi_j)_{|\partial \mathbb{D}}\|_\infty = \|[T_{\varphi_k - \varphi_j}]\| = \|[T_{\varphi_k}] - [T_{\varphi_j}]\| \to 0\). Hence there is a \(\varphi \in C(\partial \mathbb{D})\) such that \(\|\varphi_k - \varphi\|_\infty \to 0\). Let \(\xi([T]) = \varphi\), then \(\xi\) is well-defined on \(\mathcal{K}\), and \(\|\xi([T])\|_\infty = \|\varphi\|_\infty = \lim_{k \to \infty} \|\varphi_k - \varphi\|_\infty = \lim_{k \to \infty} \|[T_{\varphi_k}]\| = \|[T]\|\), so \(\xi\) is an isometry from \(\mathcal{K}\) into \(C(\partial \mathbb{D})\). For any \(\varphi \in C(\partial \mathbb{D})\), there is a polynomial sequence \(\{p_k\} \subset C^1(\mathbb{D})\) such that \(\|p_k - \varphi\|_\infty \to 0\), thus \(\|(p_k - p_j)_{|\partial \mathbb{D}}\|_\infty \to 0\). Furthermore, \(\|[T_{p_k}] - [T_{p_j}]\| \to 0\), so there is a \([T] \in \mathcal{K}\) such that \(\|[T_{p_k}] - [T]\| \to 0\), hence \(\xi([T]) = \varphi\), i.e. \(\xi\) is a surjection onto \(C(\partial \mathbb{D})\). This shows that \(\xi\) is a *-isomorphism between \(\mathcal{K}\) and \(C(\partial \mathbb{D})\). The proof is thus completed. \(\square\)
Theorem 9. \( \| \cdot \|_* \) is a norm on \( H_1^\infty + C^1(\mathbb{D}) \).

**Proof.** Our proof is similar to that of Rudin [6]. Assume \( \varphi \in cl(H_1^\infty + C^1(\mathbb{D})) \), to prove \( \varphi \in H_1^\infty + C^1(\mathbb{D}) \), we first prove that for any \( \psi \in H_1^\infty + C^1(\mathbb{D}) \), there are \( \psi_1 \in H_1^\infty \) and \( \psi_2 \in C^1(\mathbb{D}) \) such that \( \psi = \psi_1 + \psi_2 \) and \( \| \psi_1 \|_* \leq 3\| \psi \|_* \), \( \| \psi_2 \|_* \leq 2\| \psi \|_* \). In fact, if \( \psi = \tilde{\psi}_1 + \tilde{\psi}_2 \in H_1^\infty + C^1(\mathbb{D}) \), \( \tilde{\psi}_1 \in H_1^\infty \), then write \( \tilde{\psi}_1(z) = \tilde{\psi}_1(rz) \), clearly, \( \| \tilde{\psi}_1 - \tilde{\psi}_2 \|\rightarrow 0 \) \( (r \rightarrow 1^-) \), and \( \frac{\partial \tilde{\psi}_1}{\partial z}(z) = r\frac{\partial \tilde{\psi}_1}{\partial z}(rz) \). Fix a \( r_0 \) such that \( \| \tilde{\psi}_1 - \tilde{\psi}_2 \|_* \leq \| \psi \|_* \). Set

\[
\psi_2 = \tilde{\psi}_2 - \tilde{\psi}_2 - \psi_{r_0}, \quad \psi_1 = \tilde{\psi}_1 - \tilde{\psi}_2 - \psi_{r_0},
\]

then \( \psi = \psi_1 + \psi_2 \), and \( \| \psi_1 \|_* \leq 3\| \psi \|_* \), \( \| \psi_2 \|_* \leq 2\| \psi \|_* \).

Since \( \varphi \in cl(H_1^\infty + C^1(\mathbb{D})) \), there are \( \varphi_i \in H_1^\infty + C^1(\mathbb{D}) \), such that \( \| \varphi_i \|_* \leq 2^{-i} \) \( (i \geq 2) \), and \( \varphi = \sum_{i=1}^\infty \varphi_i \) (in fact, there is a sequence \( \{p_k\} \subset H_1^\infty + C^1(\mathbb{D}) \) such that \( \| p_k - \varphi \|_* \rightarrow 0 \), there is a subsequence \( \{p_{k_i}\} \) such that \( \| p_{k_{i+1}} - p_{k_i} \|_* < \frac{1}{2^i} \) \( (i \geq 2) \). For each \( \varphi_i \), there is a \( \varphi_i^{(1)} \in H_1^\infty \), \( \varphi_i^{(2)} \in C^1(\mathbb{D}) \) such that \( \| \varphi_i^{(1)} \|_* \leq 3\| \varphi_i \|_* \), \( \| \varphi_i^{(2)} \|_* \leq 2\| \varphi_i \|_* \), \( \varphi_i = \varphi_i^{(1)} + \varphi_i^{(2)} \), thus \( \varphi_i^{(1)} = \sum_{i=1}^\infty \varphi_i^{(1)} \in H_1^\infty \) (since \( H_1^\infty \) is closed relative to \( \| \cdot \|_* \)), \( \varphi_i^{(2)} = \sum_{i=1}^\infty \varphi_i^{(2)} \in C^1(\mathbb{D}) \), and \( \varphi = \varphi^{(1)} + \varphi^{(2)} \). That is \( \varphi \in H_1^\infty + C^1(\mathbb{D}) \).

**Proposition 10.** If \( \varphi \in H_1^\infty + C^1(\mathbb{D}) \) satisfies \( \varphi|_{\partial \mathbb{D}} = 0 \), then \( T_\varphi \) is a compact operator on \( D \), where \( \varphi|_{\partial \mathbb{D}} \) denotes the radial boundary values of \( \varphi \).

**Proof.** Suppose \( \varphi = \varphi_1 + \varphi_2 \), \( \varphi_1 \in H_1^\infty \), \( \varphi_2 \in C^1(\mathbb{D}) \), since \( \varphi|_{\partial \mathbb{D}} = 0 \), \( \varphi_1|_{\partial \mathbb{D}} = -\varphi_2|_{\partial \mathbb{D}} \), thus \( \varphi_1|_{\partial \mathbb{D}} \in C(\partial \mathbb{D}) \). Further \( \varphi_1(z) = P[\varphi_1|_{\partial \mathbb{D}}] \in H_1^\infty \cap A(\mathbb{D}) \) (where \( P[\varphi|_{\partial \mathbb{D}}] \) denotes the Poisson integral of \( \varphi|_{\partial \mathbb{D}} \)), consequently, \( \varphi \in C(\mathbb{D}) \cap (H_1^\infty + C^1(\mathbb{D})) \).
Suppose \( \{f_k\} \subset D, \|f_k\|_D = 1, f_k \overset{w}{\to} 0, \) then
\[
\|T_\varphi f_k\|_D^2 \leq \|\varphi f_k\|_{2,1}^2 = \langle \varphi f_k, \varphi f_k \rangle^\frac{1}{2}
\]
\[
= \left\langle \frac{\partial \varphi}{\partial z} f_k + f_k' \varphi, \frac{\partial \varphi}{\partial z} f_k + f_k' \varphi \right\rangle + \left\langle \frac{\partial \varphi}{\partial \bar{z}} f_k, \frac{\partial \varphi}{\partial \bar{z}} f_k \right\rangle
\]
\[
= \left\| \frac{\partial \varphi}{\partial z} f_k \right\|_{L^2}^2 + 2 \text{Re} \int_D f_k' \overline{\frac{\partial \varphi}{\partial z} f_k} dA + \|\varphi f_k'\|_{L^2}^2 + \left\| \frac{\partial \varphi}{\partial \bar{z}} f_k \right\|_{L^2}^2
\]
\[
\leq \|\varphi\|^2 \|f_k\|^2 + 2\|\varphi\|\|f_k\|_D \|f_k\|_{L^2} + \|\varphi\|^2 \|f_k\|^2 \to 0.
\]
Since \( \|f_k\|_D = 1 \) and \( f_k \overset{w}{\to} 0 \) in \( D \), \( \|f_k\|_{L^2} \to 0 \), so
\[
\|\varphi\|^2 \|f_k\|^2 + 2\|\varphi\|\|f_k\|_D \|f_k\|_{L^2} + \|\varphi\|^2 \|f_k\|^2 \to 0.
\]
Note \( \varphi \in C(\overline{D}) \), and \( \varphi|_{\partial D} = 0 \), hence for any \( \epsilon > 0 \), there is a \( 0 < \delta < 1 \) such that \( |\varphi(z)| < \epsilon \) for any \( |z| > \delta \), thus
\[
\int_{\overline{D}} |\varphi|^2 |f_k|^2 dA \leq \epsilon^2 \int_{\{|z| > \delta\}} |f_k|^2 dA + \int_{\{|z| \leq \delta\}} |\varphi|^2 |f_k|^2 dA
\]
\[
\leq \epsilon^2 + \|\varphi\|^2 \int_{\{|z| \leq \delta\}} |f_k|^2 dA.
\]
By \( f_k \overset{w}{\to} 0 \), we know that \( \int_{\{|z| \leq \delta\}} |f_k'|^2 dA \to 0 \), hence \( \int_{\overline{D}} |\varphi|^2 |f_k'|^2 dA \to 0 \) \((k \to \infty)\), furthermore \( \|T_\varphi f_k\|_D^2 \to 0 \), this shows that \( T_\varphi \) is compact on \( D \).

**Proposition 11.** If \( \varphi \in H_1^\infty(\mathbb{D}) \), then on Dirichlet space \( D \), \( \sigma(T_\varphi) = \overline{\varphi(\mathbb{D})} \).

**Proof.** Note for any \( f \in D \),
\[
\langle f, T_\varphi K \rangle^\frac{1}{2} = \langle T_\varphi f, K \rangle^\frac{1}{2} = \langle \varphi f, K \rangle^\frac{1}{2} = \varphi(z) f(z),
\]
so \( T_\varphi K(z, w) = \overline{\varphi(z)} K(z, w) \), this shows that \( \overline{\varphi(\mathbb{D})} \subset \sigma(T_\varphi) \).

Conversely, if \( 0 \not\in \overline{\varphi(\mathbb{D})} \), then there is a \( \delta > 0 \) such that \( |\varphi(z)| > \delta > 0 \) \((\forall z \in \mathbb{D})\), thus \( \varphi^{-1} \in H_1^\infty \). Note
t\[
\left| \frac{\partial (\varphi^{-1})}{\partial z} \right| = \left| -\frac{\varphi'}{\varphi^2} \right| \leq \frac{1}{\delta^2} |\varphi'| \leq \frac{1}{\delta^2} \|\varphi\|_* < \infty,
\]
we have \( \varphi^{-1} \in H_1^\infty \), so \( T_{\varphi^{-1}} T_\varphi T_{\varphi^{-1}} = I \), that is \( T_\varphi \) is invertible, hence \( \sigma(T_\varphi) \subset \overline{\varphi(\mathbb{D})} \). The proof is thus complete.

**Theorem 12.** If \( \varphi = \varphi_1 + \varphi_2 \in H_1^\infty + C^1(\overline{\mathbb{D}}) \), where \( \varphi_1 \in H_1^\infty, \varphi_2 \in C^1(\overline{\mathbb{D}}) \), then on Dirichlet space \( D \),
\[
\sigma_e(T_\varphi) = \cap_{0<\delta<1} \{ \varphi(z) | z > \delta \} = \varphi(\partial \mathbb{D}).
\]
Proof. Without loss of generality, assume $0 \in \cap_{0<\delta<1}\{\varphi(z)||z|>\delta\}$, then there is a sequence $\{z_k\} \subset \mathbb{D}$ with $z_n \to \zeta \in \partial \mathbb{D}$, such that $\varphi(z_n) \to 0$, thus $T_\varphi$ is not Fredholm on $L^2_a$ (c.f. G.F. Cao [1] or MacDonald [4]), hence there is a sequence $\{f_k\} \subset L^2_a$ with $\|f_k\|_{L^2} = 1$, $f_k \overset{w}{\to} 0$ such that $\|T_\varphi f_k\|_{L^2} \to 0$ or $\|T_\varphi^* f_k\|_{L^2} \to 0$. If $T_\varphi$ is Fredholm on $\mathcal{D}$, then there is a bounded operator $S$ on $\mathcal{D}$ such that $ST_\varphi - I, T_\varphi S - I \in K(\mathcal{D})$, further $S^*T_\varphi - I, T_\varphi^*S^* - I \in K(\mathcal{D})$. Write $F_k(z) = \int_0^z f_k(w)dw$, then $F_k \in \mathcal{D}$ and $\|F_k\|_{\mathcal{D}} = \|f_k\|_{L^2} = 1$, so $\|F_k\|_{L^2} \to 0$. In addition, it is clearly that $F_k \overset{w}{\to} 0$ in $\mathcal{D}$. Thus

$$
\lim_{k \to \infty} \langle ST_\varphi F_k, F_k \rangle = \lim_{k \to \infty} \langle T_\varphi SF_k, F_k \rangle = \lim_{k \to \infty} \langle T_\varphi^* S^* F_k, F_k \rangle = \lim_{k \to \infty} \langle S^* T_\varphi^* F_k, F_k \rangle = 1.
$$

Hence, without loss of generality, we can assume that $\|T_\varphi^* F_k\|_{L^2} \to 0$ (similarly for the case of $\|T_\varphi F_k\|_{L^2} \to 0$). Set $G_k = SF_k$, then $\|G_k\|_{\mathcal{D}} \leq \|S\|||F_k||_{\mathcal{D}} = ||S||$, and $G_k \overset{w}{\to} 0$, so $\|G_k\|_{L^2} \to 0$, and

$$
\left| \langle T_\varphi SF_k, F_k \rangle \right| = \left| \langle \varphi(SF_k)', f_k \rangle + \left( \frac{\partial \varphi}{\partial z} SF_k \right) f_k \right| \\
\leq \left| \langle \varphi(SF_k)', T_\varphi^* f_k \rangle + \|\varphi\|_s ||SF_k||_{L^2} ||f_k||_{L^2} \right| \leq \|SF_k\|_{L^2} \left| ||T_\varphi^* f_k||_{L^2} + ||\varphi||_s ||G_k||_{L^2} ||f_k||_{L^2} \right| = \|G_k\|_{\mathcal{D}} \left| ||T_\varphi^* f_k||_{L^2} + ||\varphi||_s ||G_k||_{L^2} ||f_k||_{L^2} \right| \to 0.
$$

This contradicts that $\langle T_\varphi SF_k, F_k \rangle \to 1$. hence $T_\varphi$ must be non-Fredholm.

That is $0 \in \sigma_0(T_\varphi)$, consequently $\cap_{0<\delta<1}\{\varphi(z)||z|>\delta\} \subset \sigma_0(T_\varphi)$.

Conversely, assume $0 \notin \cap_{0<\delta<1}\{\varphi(z)||z|>\delta\}$, thus there are $\epsilon, \delta > 0$ such that $\varphi(z) > \epsilon$ for any $|z| > \delta$, we prove that $T_\varphi$ is Fredholm on $\mathcal{D}$. Otherwise, there is a sequence $\{F_k\} \subset \mathcal{D}$ with $\|F_k\|_{\mathcal{D}} = 1, F_k \overset{w}{\to} 0$, such that $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$ or $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$. Similar to above proof, we can assume $\|T_\varphi^* F_k\|_{\mathcal{D}} \to 0$ (if $\|T_\varphi F_k\|_{\mathcal{D}} \to 0$, the proof will be simpler). It is well-known that $\tilde{T}_\varphi$ is Fredholm on $L^2_a$ if $\varphi(z) > \epsilon$ for any $|z| > \delta$, hence there is a $S \in \mathcal{L}(L^2_a)$ such that $S\tilde{T}_\varphi - I, S\tilde{T}_\varphi S - I \in K(L^2_a)$, further, $S^*\tilde{T}_\varphi^* - I, \tilde{T}_\varphi^* S^* - I \in K(L^2_a)$. Set $f_k = F_k, g_k = SF_k$, then $f_k \overset{w}{\to} 0, g_k \overset{w}{\to} 0$ in $L^2_a$ and $\|f_k\|_{L^2_a} = \|F_k\|_{\mathcal{D}} = 1$, $\|g_k\|_{L^2_a} \leq \|S\|||f_k||_{L^2_a} = ||S||$, so $\langle S^* T_\varphi^* f_k, f_k \rangle \to 0$. Therefore, $\tilde{T}_\varphi^* f_k \overset{w}{\to} 0$ in $L^2_a$, hence $T_\varphi$ is Fredholm on $\mathcal{D}$. This contradicts that $\langle T_\varphi SF_k, F_k \rangle \to 1$. Hence $T_\varphi$ must be non-Fredholm.
1, that is $\langle \hat{T}_\varphi f_k, S f_k \rangle \to 1$. Note

$$\left\langle T^*_\varphi F_k, \int_0^z S f_k(w) dw \right\rangle_{L^2} = \left\langle F_k, \varphi \int_0^z S f_k dw \right\rangle_{L^2}$$

$$= \left\langle f_k, \left( \frac{\partial \varphi}{\partial z} \int_0^z S f_k dw + \varphi S f_k \right) \right\rangle_{L^2}$$

$$= \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z g_k dw \right\rangle + \langle f_k, \varphi g_k \rangle$$

and $g_k \to 0$, so $\| \int_0^z g_k dw \|_{L^2} \to 0$, and

$$\left| \left\langle f_k, \frac{\partial \varphi}{\partial z} \int_0^z g_k dw \right\rangle \right| \leq \|f_k\|_{L^2} \|\varphi\| \| \int_0^z g_k dw \|_{L^2} \to 0,$$

but $\langle f_k, \varphi g_k \rangle = \langle \hat{T}_\varphi f_k, S f_k \rangle \to 1 \neq 0$, hence $\langle T^*_\varphi F_k, \int_0^z g_k dw \rangle_{L^2} \to 1 \neq 0$, this contradicts that $\|T^*_\varphi F_k\| \to 0$. It shows that $T_\varphi$ must be Fredholm on $D$. That is $0 \notin \sigma_0(T_\varphi)$. Note the functions with derivatives in $C^1(z)$, hence $H^\infty + C^1(\overline{D}) \subset C(\overline{D})$, consequently, $\bigcap_{0 < \delta < 1} \{ \varphi(z) | z | > \delta \} = \varphi(\partial D)$. We are done. \hfill \Box

**Remark.** If $\varphi \in H^\infty$, then $T_\varphi$ is always bounded on $D$, in fact, for any $f, g \in D$,

$$\left| \langle T_\varphi f, g \rangle_{L^2} \right| = \left| \langle \hat{T}_\varphi f', g' \rangle \right| = \left| \langle \hat{T}_\varphi f', \hat{g}' \rangle \right| \leq \left\| T_\varphi \right\| \left\| f' \right\|_{L^2} \left\| g' \right\|_{L^2} \leq \|\varphi\| \|f\| \|g\|,$$

hence $\|T_\varphi f\| \leq \|\varphi\| \|f\|$. However, for $\varphi \in H^\infty$, $T_\varphi$ may be unbounded on $D$. Let

$$L^{1,1}_1 = \left\{ f \in L^{2,1}, f, \frac{\partial f}{\partial z} \in L^\infty(\mathbb{D}, dA) \right\},$$

then $L^{1,1}_1 \cup \overline{H^\infty}$ is perhaps the most suitable symbol space of Toeplitz operators which are bounded. Also, we can prove that if $\varphi \in L^{1,1}_1$ has a compact support set, then $T_\varphi$ is compact on $D$.

**4. An index formula of Toeplitz operators.**

The classical index formula shows that for $\varphi \in C(\partial \mathbb{D})$, if $|\varphi| > \epsilon > 0$, then $T_\varphi$ is a Fredholm operator on Hardy space $H^2(\partial \mathbb{D})$, and $\text{Ind} T_\varphi = -\text{wind} \varphi$. If $\varphi \in C(\overline{\mathbb{D}})$ with $|\varphi(\partial \mathbb{D})| > \epsilon > 0$, then $\text{Ind} T_\varphi = -\text{wind} \varphi |_{\partial \mathbb{D}}$ on Bergman space $L^2_2(\mathbb{D})$. The proof of these index formulas is relative to the topology homotopy of symbol functions, it can not be directly extended to the case of Dirichlet space since we do not know whether there is a $C^1$-function $H_t$ which is continuous with respect to $t \in [0, 1]$ such that $H_0 = \varphi, H_1 = \psi$ if $\varphi, \psi \in C^1(\overline{\mathbb{D}})$ and $\text{wind} \varphi |_{\partial \mathbb{D}} = \text{wind} \psi |_{\partial \mathbb{D}}$. In this section, we use the
short exact sequence (\(\ast\)) to prove an analogy of the above index formulas for \(C^1\)-symbols.

**Theorem 13.** Suppose \(\varphi \in C^1(\mathbb{D})\) such that \(T_\varphi\) is Fredholm on \(\mathcal{D}\), then

\[
\text{Ind}T_\varphi = -\text{wind}\varphi|_{\partial\mathbb{D}}.
\]

**Proof.** By Theorem 8, if \(\varphi, \psi \in C^1(\mathbb{D})\) such that \(T_\varphi\) and \(T_\psi\) are Fredholm on \(\mathcal{D}\) and \(\text{wind}\varphi|_{\partial\mathbb{D}} = \text{wind}\psi|_{\partial\mathbb{D}}\), then there is a \(H_t \in C([0, 1] \times \partial\mathbb{D})\) such that \(H_t \in GC(\partial\mathbb{D})\) (the set of invertible elements in \(C(\partial\mathbb{D})\)) for each \(t \in [0, 1]\) and \(H_0 = \varphi|_{\partial\mathbb{D}}, H_1 = \psi|_{\partial\mathbb{D}}\). Note \(\xi\) is an isometry isomorphism, so \(\xi^{-1}(H_t)\) is continuous with respect to \(t\). On the other hand, \(H_t^{-1}\) is also continuous on \([0, 1] \times \partial\mathbb{D}\), and \(\xi^{-1}(H_t)\xi^{-1}(H_t^{-1}) = \xi^{-1}(H_t^{-1})\xi^{-1}(H_t) = \xi^{-1}(H_tH_t^{-1}) = [I]\). Hence, \(\xi^{-1}(H_t)\) is invertible in \(D/\mathcal{K}\). Furthermore, it is easy to see that \(\text{Ind}T_\varphi = \text{Ind}T_\psi\).

Now suppose \(\varphi \in C^1(\mathbb{D})\) such that \(T_\varphi\) is Fredholm on \(\mathcal{D}\) and \(\text{wind}\varphi|_{\partial\mathbb{D}} = k\), note \(T_z\) is Fredholm on \(\mathcal{D}\) with \(\text{Ind}T_z = -1 = -\text{wind}z|_{\partial\mathbb{D}}\), let

\[
\tilde{z}^k = \begin{cases}
  z^k, & \text{if } k > 0,
  
  \bar{z}^{-k}, & \text{otherwise};
\end{cases}
\]

we see that \(\text{wind}\varphi|_{\partial\mathbb{D}} = \text{wind}\tilde{z}^k|_{\partial\mathbb{D}}\), thus \(\text{Ind}T_\varphi = \text{Ind}T_{\tilde{z}^k}\) by above proof, consequently, \(\text{Ind}T_\varphi = -k = -\text{wind}\varphi|_{\partial\mathbb{D}}\). The proof is thus complete. \(\square\)

**Proposition 14.** Suppose \(\varphi \in H^\infty\) such that \(T_\varphi\) is Fredholm on \(\mathcal{D}\), then

\[
\text{Ind}T_\varphi = -\lim_{r \to 1-} \text{wind}\varphi_r|_{\partial\mathbb{D}},
\]

where \(\varphi_r(z) = \varphi(rz)\).

**Proof.** If \(f \in \text{Ker}T_\varphi\), then for any \(g \in \mathbb{D}\),

\[
0 = \langle T_\varphi f, g \rangle_{\mathbb{D}} = \langle \tilde{T}_\varphi f', g' \rangle_{\mathbb{D}} = \langle \tilde{T}_\varphi, f', g' \rangle,
\]

since \(\{g'| g \in \mathcal{D}\} = L^2_\mathbb{D}, \tilde{T}_\varphi f' = 0\), hence \(\{f'| f \in \text{Ker}T_\varphi\} \subset \text{Ker}\tilde{T}_\varphi\). Consequently, assume \(T_\varphi f = 0, f \in L^2_\mathbb{D}\), set \(F(z) = \int_0^z f dw\), then \(F \in \mathcal{D}\), and for any \(G \in \mathcal{D}\),

\[
\langle T_\varphi F, G \rangle_{\mathbb{D}} = \langle \tilde{T}_\varphi F', G' \rangle = \langle \tilde{T}_\varphi f', G' \rangle = 0,
\]

so \(T_\varphi F = 0\), further \(\{\int_0^z f(w) dw | f \in \text{Ker}\tilde{T}_\varphi\} \subset \text{Ker}T_\varphi\). This shows that

\[
\dim \text{Ker}T_\varphi = \dim \text{Ker}\tilde{T}_\varphi.
\]

Now assume \(f \in \mathcal{D}\) such that \(T_\varphi^* f = 0\), then for any \(g \in \mathcal{D}\),

\[
0 = \langle T_\varphi^* f, g \rangle_{\mathbb{D}} = \langle f, \varphi g \rangle_{\mathbb{D}} = \langle f', \varphi g' \rangle = \langle \tilde{T}_\varphi f', g' \rangle,
\]
hence \( \tilde{T}_\varphi f' = 0 \), that is \( \{ f' | f \in \text{Ker} T^*_\varphi \} \subset \text{Ker} \tilde{T}_\varphi \).

Similarly to above proof, we have also \( \{ \int_0^z f(w) dw | f \in \text{Ker} \tilde{T}_\varphi \} \subset \text{Ker} T^*_\varphi \).

Hence \( \dim \text{Ker} T^*_\varphi = \dim \text{Ker} \tilde{T}_\varphi \), consequently,

\[
\text{Ind} T^*_\varphi = -\text{Ind} \tilde{T}_\varphi = \lim_{r \to 1^-} \text{wind} \varphi_r |_{\partial D} = - \lim_{r \to 1^-} \text{wind} \tilde{\varphi}_r |_{\partial D}.
\]

\[\Box\]

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References


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SICHUAN UNIVERSITY
CHENGDU 610064
CHINA
E-mail address: nic2601@scuu.edu.cn