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CONGRUENCE OF TWO-DIMENSIONAL SUBSPACES
IN $M_2(K)$ (CHARACTERISTIC $\neq 2$)

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The structure and classification up to isomorphism of a naturally arising class of local rings is determined. Although we are primarily interested in the case of a finite residue field K , our results apply in fact over any field K of characteristic $\neq 2$. The problem is shown to be equivalent to that of classifying two-dimensional subspaces of $M_2(K)$ up to congruence, and it is in these terms that the question is addressed.

1. Introduction.

In investigating the structure of finite local rings one is led to consider such a ring of the form $R = K \oplus J$ in which $K = \mathbf{F}_q$ and the Jacobson radical J is such that $J^3 = 0$ and both J/J^2 and J^2 are two-dimensional over $R/J = K$. Rings with $J^3 = 0$ form a natural object of study, the case $J^2 = 0$ having long been settled [2, 3]. If $J = Kx_1 \oplus Kx_2 \oplus J^2$ and $J^2 = Ky_1 \oplus Ky_2$, then we may write $x_i x_j = \alpha_{ij} y_1 + \beta_{ij} y_2$ ($\alpha_{ij}, \beta_{ij} \in K$) and these four products span J^2 . The ring structure is determined by the pair of (2×2) matrices $A = (\alpha_{ij}), B = (\beta_{ij})$, which are linearly independent over K , and any pair of independent matrices defines such a ring. We wish to determine the number of isomorphism classes of such rings and to find normal forms for the pair of matrices A, B defining them. Chikunji [1] has shown that there are 10 classes for $q = 2$ and, on the basis of computer calculations for $q = 3, 5, 7$, has conjectured that when q is odd the number of classes is $3q + 5$. It is also conjectured that exactly three of these rings are commutative. Our purpose here is *inter alia*, to prove these conjectures.

If (x'_1, x'_2, y'_1, y'_2) is a new basis of J with corresponding matrices A', B' , then x'_1, x'_2 are linear combinations of x_1, x_2, y_1, y_2 . Since $J^3 = 0$, we may assume that the coefficients of y_1, y_2 are zero and write $x'_i = p_{1i}x_1 + p_{2i}x_2$, so that $P = (p_{ij})$ is the transition matrix from the basis (\bar{x}_1, \bar{x}_2) of J/J^2 to the basis (\bar{x}'_1, \bar{x}'_2) . Equally, let $Q = (q_{ij})$ be the transition matrix from the basis (y_1, y_2) of J^2 to (y'_1, y'_2) . If we now calculate $x'_i x'_j$ and compare coefficients of y_i we obtain equations which, in matrix form, are

$$\begin{cases} P^t A P = q_{11} A' + q_{12} B' \\ P^t B P = q_{21} A' + q_{22} B' \end{cases} .$$

Evidently, the problem of classifying our rings up to isomorphism amounts to that of classifying pairs of linearly independent matrices (A, B) under the above relation of *equivalence*, P and Q being arbitrary invertible matrices, and it is to this problem of linear algebra that the paper is devoted. We shall, in fact, solve it over an arbitrary field of characteristic $\neq 2$ and will consider all pairs, independent or otherwise. The approach we take is to first of all deal with pairs of *symmetric* matrices (corresponding to commutative rings) and then to use the fact that a general equivalence class may be represented by the sum of one of the standard symmetric pairs already found with an antisymmetric pair. This is similar to an idea used in [4] for congruence of single matrices.

2. The symmetric case.

We first establish some notation. Let X be the set of all pairs (A, B) of (2×2) matrices over a field K . The group GL_2 acts on the right on X by congruence: $(A, B) \cdot P = (P^t AP, P^t BP)$ and on the left via $Q \cdot (A, B) = (q_{11}A + q_{12}B, q_{21}A + q_{22}B)$, where $Q = (q_{ij})$. These two actions are permutable and define a (left) action of $G = GL_2 \times GL_2$ on X :

$$(P, Q) \cdot (A, B) = Q \cdot (A, B) \cdot P^{-1}.$$

By restriction, G acts on the subset Y consisting of pairs with A, B linearly independent. This amounts to studying the congruence action (via P) of GL_2 on the set \mathcal{Y} of 2-dimensional subspaces of $M_2(K)$, Q just representing a change of basis in a given subspace. In the same way, the whole action of G on X may be reinterpreted as an action of GL_2 on the set \mathcal{X} of subspaces of dimension ≤ 2 . Two pairs in the same G -orbit will be called *equivalent*.

G also acts by restriction on the set S of pairs with A, B symmetric. *Assuming henceforth that* $\text{char } K \neq 2$, we determine these orbits first. To avoid a plague of parentheses we omit these around ordered pairs of displayed matrices.

Theorem 1. *The following table gives a complete set of representatives for the orbits of G on S , together with their stabilizers:*

Representative	Stabilizing elements (P, Q)
1. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 2ac & ad \end{pmatrix}$
2. $\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} a & \pm\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & \pm 2\delta ac \\ 2ac & \pm(a^2 + \delta c^2) \end{pmatrix}$
3. $\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	All
4. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$
5. $\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}$

In 2) and 5) δ runs through a set of coset representatives of K^{*2} in K^* .

Write $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Q = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$. Before giving the proof it is useful to record that if $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then:

$$P^tAP = \begin{pmatrix} a^2\alpha + ac(\beta + \gamma) + c^2\delta & ab\alpha + ad\beta + bc\gamma + cd\delta \\ ab\alpha + ad\gamma + bc\beta + cd\delta & b^2\alpha + bd(\beta + \gamma) + d^2\delta \end{pmatrix}.$$

In particular we have:

A	$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}$	$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
P^tAP	$\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$	$\begin{pmatrix} a^2 + c^2\delta & ab + cd\delta \\ ab + cd\delta & b^2 + d^2\delta \end{pmatrix}$	$\begin{pmatrix} 2ac & ad + bc \\ ad + bc & 2bd \end{pmatrix}$

Note also that (P, Q) fixes a pair $\Pi = (A, B) \Leftrightarrow \Pi \cdot P = Q \cdot \Pi$.

Proof of Theorem 1. Consider first *independent* pairs (A, B) in S . We claim that any such pair is equivalent to one with $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. To prove this it is enough to show that every 2-dimensional subspace W of the space V of symmetric matrices contains an *isotropic* matrix, in the sense that it is nonsingular and the associated quadratic form represents zero. For all isotropic matrices are congruent to this one. If W equals the space of diagonal matrices, then it contains the isotropic matrix $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. If not, then, since $\dim V = 3$, W is spanned by a diagonal matrix and a non-diagonal matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}$. We may clearly modify the latter so that α or δ equals 0, and then it is isotropic.

So now let (A, B) be independent, with $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. We may take A to be diagonal, $A = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$. Under congruence by $P = B$, if necessary, we may assume that $\alpha \neq 0$, and then, via a suitable Q , that $\alpha = 1$.

We now determine when two pairs $\Pi = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $\Pi' = \begin{pmatrix} 1 & \\ & \delta' \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ are equivalent. This happens when there exist P, Q as above such that $\Pi \cdot P = Q \cdot \Pi'$, or in other words:

$$(1) \quad \begin{pmatrix} a^2 + c^2\delta & ab + cd\delta \\ ab + cd\delta & b^2 + d^2\delta \end{pmatrix}, \begin{pmatrix} 2ac & ad + bc \\ ad + bc & 2bd \end{pmatrix} = \begin{pmatrix} k & l \\ l & k\delta' \end{pmatrix}, \begin{pmatrix} m & n \\ n & m\delta' \end{pmatrix}.$$

Comparing diagonal terms gives

$$(2) \quad \begin{cases} b^2 + d^2\delta = \delta'(a^2 + c^2\delta) \\ bd = \delta'ac \end{cases}.$$

Squaring these and subtracting 4δ times the second from the first leads to $b^2 - d^2\delta = \pm\delta'(a^2 - c^2\delta)$. According to the sign, there are two cases:

$$(i) \quad \begin{cases} b^2 = \delta'a^2 \\ \delta d^2 = \delta\delta'c^2 \end{cases} \quad \text{or} \quad (ii) \quad \begin{cases} b^2 = \delta\delta'c^2 \\ \delta d^2 = \delta'a^2 \end{cases}.$$

In either case it follows from nonsingularity of P that if $\delta' = 0$, then $b = 0, d \neq 0$ and $\delta = 0$. By symmetry we deduce that $\delta = 0 \Leftrightarrow \delta' = 0$. The stabilizer in this case is given by the single condition $b = 0$, and the form of Q follows from (1).

Assume now that $\delta, \delta' \neq 0$. Case (i) cannot now arise, as is shown by the second equation of (2), the first of (i) and nonsingularity of P . It follows

from (ii) that Π and Π' are equivalent $\Leftrightarrow \delta, \delta'$ are in the same square-class. The form of the stabilizer results at once.

We are left with the *dependent* pairs (A, B) in S . Via Q we may assume that $B = 0$, and then (via P) that $A = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$. If $A \neq 0$, then again (via P) we may assume $\alpha \neq 0$, and finally (via Q) that $\alpha = 1$. This gives the remaining types in the table. As for equivalence, these cannot be equivalent to independent pairs, so we only have to examine equivalence between $\Pi = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ and $\Pi' = \begin{pmatrix} 1 & \\ & \delta' \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$. The condition $\Pi \cdot P = Q \cdot \Pi'$ this time gives

$$(3) \quad \begin{cases} b^2 + d^2\delta = \delta'(a^2 + c^2\delta) \\ ab = -cd\delta \end{cases} .$$

(3) is of exactly the same form as (2): we merely have to interchange a, d and replace δ by $-\delta', \delta'$ by $-\delta$. It follows that Π and Π' are equivalent $\Leftrightarrow \delta, \delta'$ are in the same (possibly zero) square-class. Once more, the form of the stabilizers results immediately. \square

3. The general case.

Consider now an *arbitrary* pair $\Pi = (A, B)$. This decomposes uniquely as the sum $\Pi = \Pi_s + \Pi_a$ of a *symmetric* pair $\Pi_s = (A_s, B_s)$ and an *antisymmetric* pair $\Pi_a = (A_a, B_a)$. One checks at once that this decomposition commutes with the action: $((P, Q) \cdot \Pi)_s = (P, Q) \cdot \Pi_s$ and $((P, Q) \cdot \Pi)_a = (P, Q) \cdot \Pi_a$. In particular:

$$(P, Q) \text{ fixes } \Pi \Leftrightarrow \text{it fixes each of } \Pi_s \text{ and } \Pi_a.$$

Let \mathcal{S} be the set of symmetric representatives in Theorem 1. We now have:

Proposition 1. (i) *Each equivalence class contains a pair $\Sigma + T$, where $\Sigma \in \mathcal{S}$ and T is antisymmetric. Moreover, the class determines Σ uniquely.*
 (ii) *If $\Pi = \Sigma + T$ and $\Pi' = \Sigma + T'$ (similarly), then $(P, Q) \cdot \Pi = \Pi' \Leftrightarrow (P, Q)$ stabilizes Σ and $(P, Q) \cdot T = T'$.*

We also record the following evident lemma. Henceforth let $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Lemma 1. *If $T = (\alpha J, \beta J)$ and $T' = (\alpha' J, \beta' J)$ are antisymmetric pairs and $\Delta = \det P$, then*

$$(4) \quad (P, Q) \cdot T = T' \Leftrightarrow \begin{cases} k\alpha + l\beta = \Delta\alpha' \\ m\alpha + n\beta = \Delta\beta' \end{cases} .$$

Prop. 1 shows that each equivalence class has an underlying type in \mathcal{S} , and each type is a union of equivalence classes. We now analyze these types in turn, keeping the notation established above:

1) $\Sigma = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} : (P, Q) \cdot \Pi = \Pi'$ if and only if (P, Q) is as in line 1 of

the table in Theorem 1 and (4) holds, which amounts to $\begin{cases} a\alpha = d\alpha' \\ 2c\alpha + d\beta = d\beta' \end{cases}$.

If $\alpha = 0$, then $\alpha' = 0$ and $\beta' = \beta$. Thus there is *one orbit for each* $\beta \in K$, corresponding to $T = (0, \beta J)$. The stabilizer for each of these is all of $\text{Stab}(\Sigma)$. If $\alpha \neq 0$, we may take $a = 1, d = \alpha, c = -\beta/2$ to get $\alpha' = 1, \beta' = 0$, resulting in *one more orbit* given by $T = (J, 0)$. The stabilizer is given by the equations $a = d, c = 0$, hence consists of the pairs $(P, Q) = (aI, a^2I)$.

2) $\Sigma = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} : \text{Let } O_{2,\lambda} = \left\{ \begin{pmatrix} x & \mp \lambda y \\ y & \pm x \end{pmatrix} : x^2 + \lambda y^2 = 1 \right\}$ be the orthogonal group of the quadratic form $(1, \lambda)$. The form of (P, Q) shows that $Q/\Delta \in O_{2,-\delta}$ and Equations (4) say that Q/Δ sends (α, β) to (α', β') . Hence these vectors have the same length with respect to the form $(1, -\delta)$, in other words $\alpha^2 - \delta\beta^2 = \alpha'^2 - \delta\beta'^2$.

Conversely, let (α, β) and (α', β') be non-zero vectors satisfying this condition. Then by Witt's Extension Theorem (cf. [4, Prop. 3]) there exists $R = \begin{pmatrix} x & \pm \delta y \\ y & \pm x \end{pmatrix}$ in $O_{2,-\delta}$ (so that $x^2 - \delta y^2 = 1$) sending (α, β) to (α', β') . We can now choose a, c such that $R = Q/\Delta$. Namely, if $x \neq \mp 1$, let $a = \delta^{-1}(1 \pm x), c = \pm \delta^{-1}y$ and if $x = \mp 1$, let $a = 0, c = 1$. Now (4) holds, so Π and Π' are equivalent.

Thus, apart from the symmetric class (given by $T = (0, 0)$), there is *one orbit for each element of K represented (non-trivially) by the form $\alpha^2 - \delta\beta^2$* , corresponding to $T = (\alpha J, \beta J)$.

The stabilizers are easily found from (4), with $\alpha' = \alpha, \beta' = \beta$.

If $P = \begin{pmatrix} a & \delta c \\ c & a \end{pmatrix}$, this condition becomes $\begin{cases} c(c\alpha + a\beta) = 0 \\ c(a\alpha + \delta c\beta) = 0 \end{cases}$, which reduces to $c = 0$, P being nonsingular. Thus $(P, Q) = (aI, a^2I)$.

If $P = \begin{pmatrix} a & -\delta c \\ c & -a \end{pmatrix}$, it amounts to $a\alpha = \delta c\beta$, so that $(a, c) = \mu(\delta\beta, \alpha)$ ($\mu \neq 0$). The only other condition which must be met is that $\Delta \neq 0$, or equivalently $\alpha^2 - \delta\beta^2 \neq 0$. Provided this is so, the stabilizer contains elements of this second type, namely $(P, Q) = \mu \begin{pmatrix} \delta\beta & -\delta\alpha \\ \alpha & -\delta\beta \end{pmatrix}, \delta\mu^2 \begin{pmatrix} \alpha^2 + \delta\beta^2 & -2\delta\alpha\beta \\ 2\alpha\beta & -(\alpha^2 + \delta\beta^2) \end{pmatrix}$. Otherwise such elements do not arise.

3) $\Sigma = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Taking $P = I$ and Q arbitrary shows that in addition to the symmetric class there is just *one orbit* with $T \neq 0$. We may, for example, take $T = (J, 0)$. The stabilizer then consists of all pairs $(P, Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \Delta & l \\ 0 & n \end{pmatrix}$.

4) $\Sigma = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Here (4) becomes $\begin{cases} a^2\alpha + l\beta = ad\alpha' \\ n\beta = ad\beta' \end{cases}$, which implies that $\beta = 0 \Leftrightarrow \beta' = 0$. As well as the symmetric class we have the cases:

(i) $\beta' \neq 0$: This is equivalent to the case $(\alpha, \beta) = (0, 1)$ as follows by taking $a = d = 1, l = \alpha' - \alpha, n = \beta'$. So we get *one orbit* corresponding to $T = (0, J)$. The stabilizer consists of the pairs $(P, Q) = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 0 & ad \end{pmatrix}$.

(ii) $\alpha' \neq 0, \beta' = 0$: This is equivalent to $(\alpha, \beta) = (1, 0)$ (take $a = \alpha', d = 1$), and there is again *one orbit*, given by $T = (J, 0)$. The stabilizer consists of the pairs $(P, Q) = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$.

5) $\Sigma = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Now (4) is $\begin{cases} (a^2 + \delta c^2)\alpha + l\beta = \pm(a^2 + \delta c^2)\alpha' \\ n\beta = \pm(a^2 + \delta c^2)\beta' \end{cases}$, leading again to $\beta = 0 \Leftrightarrow \beta' = 0$. Apart from the symmetric class we must consider:

(i) $\beta' \neq 0$: As before, this reduces to *one orbit*, given by $T = (0, J)$. The stabilizer is the set of all $(P, Q) = \begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & 0 \\ 0 & \pm(a^2 + \delta c^2) \end{pmatrix}$.

(ii) $\alpha' \neq 0, \beta' = 0$: It follows that $\alpha = \pm\alpha'$, and thus that the distinct orbits are given by $T = (\alpha J, 0)$, α running over $K^*/\{\pm 1\}$. To calculate the stabilizers we put $\alpha = \alpha', \beta = \beta' = 0$ in the equations above. This forces the sign to be +, and hence the stabilizer in the set of $(P, Q) = \begin{pmatrix} a & -\delta c \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}$.

We collect our results in the next theorem. Since we have dealt already with the symmetric classes in Theorem 1, we confine ourselves to the rest:

Theorem 2. *The following table gives a complete set of representatives for the orbits of G on $X - S$ (the non-symmetric classes), together with their stabilizers:*

Representative	Stabilizing elements (P, Q)
1a. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 1 - \beta & 1 + \beta \end{pmatrix}$ $(\beta \in K^*)$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 2ac & ad \end{pmatrix}$
1b. $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \end{pmatrix}$	$\begin{pmatrix} a & \\ & a \end{pmatrix}, \begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}$
2a. $\begin{pmatrix} 1 & \alpha \\ -\alpha & \delta \end{pmatrix}, \begin{pmatrix} 1 - \beta & 1 + \beta \end{pmatrix}$ in 1-1 correspondence with the values in K represented by $\alpha^2 - \delta\beta^2$, for each $\delta \in K^*/K^{*2}$	$\begin{pmatrix} a & \\ & a \end{pmatrix}, \begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}$ if $\alpha^2 - \delta\beta^2 = 0$ Otherwise, the above pairs plus: $\mu \begin{pmatrix} \delta\beta & -\delta\alpha \\ \alpha & -\delta\beta \end{pmatrix},$ $\delta\mu^2 \begin{pmatrix} \alpha^2 + \delta\beta^2 & -2\delta\alpha\beta \\ 2\alpha\beta & -(\alpha^2 + \delta\beta^2) \end{pmatrix}$
3a. $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \Delta & l \\ 0 & n \end{pmatrix}$
4a. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 0 & ad \end{pmatrix}$
4b. $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$
5a. $\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix},$ $\begin{pmatrix} a^2 + \delta c^2 & 0 \\ 0 & \pm(a^2 + \delta c^2) \end{pmatrix}$
5b. $\begin{pmatrix} 1 & \alpha \\ -\alpha & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ $(\alpha \in K^*/\{\pm 1\})$	$\begin{pmatrix} a & -\delta c \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}$

By inspection from Theorems 1 and 2 we also have:

Corollary 1. *The orbits of G on Y (the linearly independent classes) are given by lines 1, 2, 1a, 1b, 2a, 4a and 5a.*

4. Finite Fields.

We now specialize the foregoing to the finite field $K = \mathbf{F}_q$ (q odd). In this case $|G| = q^2(q-1)^2(q^2-1)^2$, $|X| = q^8$, $|Y| = q(q^3-1)(q^4-1)$ and $|S| = q^6$. There are two square-classes in K^* , represented by 1 and a fixed non-square ε . Over \mathbf{F}_q quadratic forms of rank ≥ 2 are universal (cf. [5] for example), so that for each of $\delta = 1, \varepsilon$ the form $\alpha^2 - \delta\beta^2$ takes all values in K^* . In addition, when $\delta = 1$ it represents 0, but not when $\delta = \varepsilon$. Let χ denote the quadratic character of K .

From the previous results we can now easily determine the number of equivalence classes and their sizes:

Theorem 3. *The following table gives, for each type of representative, the sizes of the stabilizer and equivalence class and the number of classes:*

Rep.	Stabilizer	Class	Number of classes
1	$q(q-1)^2$	$q(q^2-1)^2$	1
2	$2(q-1)(q-\chi(\delta))$	$\frac{1}{2}q^2(q-1)(q^2-1)(q+\chi(\delta))$	2
3	$ G $	1	1
4	$q^2(q-1)^3$	$(q+1)(q^2-1)$	1
5	$2q(q-1)^2(q-\chi(-\delta))$	$\frac{1}{2}q(q^2-1)(q+\chi(-\delta))$	2
1a	$q(q-1)^2$	$q(q^2-1)^2$	$q-1$
1b	$q-1$	$q^2(q-1)(q^2-1)^2$	1

2a	$\begin{cases} q-1 & \text{if } \alpha^2 - \delta\beta^2 = 0 \\ 2(q-1) & \text{if not} \end{cases}$	$\begin{cases} q^2(q-1)(q^2-1)^2 \\ \frac{1}{2}q^2(q-1)(q^2-1)^2 \end{cases}$	$\begin{cases} 1 \\ 2(q-1) \end{cases}$
3a	$q^2(q-1)^2(q^2-1)$	q^2-1	1
4a	$q(q-1)^2$	$q(q^2-1)^2$	1
4b	$q^2(q-1)^2$	$(q^2-1)^2$	1
5a	$2(q-1)(q-\chi(-\delta))$	$\frac{1}{2}q^2(q-1)(q^2-1)(q+\chi(-\delta))$	2
5b	$q(q-1)^2(q-\chi(-\delta))$	$q(q^2-1)(q+\chi(-\delta))$	$q-1$

In all there are $4q + 10$ classes, of which 7 are symmetric. For the linearly independent pairs, the number of classes is $3q + 5$, and 3 of these are symmetric.

Proof. It is only necessary to observe, for lines 2, 5, 5a and 5b, that if $\xi \in K^*$ then the number of solutions of $\alpha^2 - \xi\beta^2 \neq 0$ is $(q-1)(q-\chi(\xi))$. Note also in line 5b that there are $\frac{1}{2}(q-1)$ classes for each of $\delta = 1, \varepsilon$. \square

As a check on the arithmetic, one readily verifies that the sum of all the class sizes is $q^8 = |X|$. For the symmetric classes the sum is $q^6 = |S|$.

Corollary 2. *For the finite local rings of the Introduction, there are $3q + 5$ isomorphism classes (q odd). Of these, 3 are commutative.*

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