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CONGRUENCE OF TWO-DIMENSIONAL SUBSPACES IN $M_2(K)$ (CHARACTERISTIC $\neq 2$)

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The structure and classification up to isomorphism of a naturally arising class of local rings is determined. Although we are primarily interested in the case of a finite residue field K, our results apply in fact over any field K of characteristic $\neq 2$. The problem is shown to be equivalent to that of classifying two-dimensional subspaces of $M_2(K)$ up to congruence, and it is in these terms that the question is addressed.

1. Introduction.

In investigating the structure of finite local rings one is led to consider such a ring of the form $R = K \oplus J$ in which $K = \mathbf{F}_q$ and the Jacobson radical J is such that $J^3 = 0$ and both J/J^2 and J^2 are two-dimensional over $R/J = K$. Rings with $J^3 = 0$ form a natural object of study, the case $J^2 = 0$ having long been settled [**2**, **3**]. If $J = Kx_1 \oplus Kx_2 \oplus J^2$ and $J^2 = Ky_1 \oplus Ky_2$, then we may write $x_ix_j = \alpha_{ij}y_1 + \beta_{ij}y_2$ ($\alpha_{ij}, \beta_{ij} \in K$) and these four products span J^2. The ring structure is determined by the pair of (2×2) matrices $A = (\alpha_{ij})$, $B = (\beta_{ij})$, which are linearly independent over K, and any pair of independent matrices defines such a ring.We wish to determine the number of isomorphism classes of such rings and to find normal forms for the pair of matrices A, B defining them. Chikunji [**1**] has shown that there are 10 classes for $q = 2$ and, on the basis of computer calculations for $q = 3, 5, 7$, has conjectured that when q is odd the number of classes is $3q + 5$. It is also conjectured that exactly three of these rings are commutative. Our purpose here is *inter alia*, to prove these conjectures.

If (x_1', x_2', y_1', y_2') is a new basis of J with corresponding matrices A', B', then x_1', x_2' are linear combinations of x_1, x_2, y_1, y_2. Since $J^3 = 0$, we may assume that the coefficients of y_1, y_2 are zero and write $x_i' = p_{1i}x_1 + p_{2i}x_2$, so that $P = (p_{ij})$ is the transition matrix from the basis $(\overline{x}_1, \overline{x}_2)$ of J/J^2 to the basis $(\overline{x}_1', \overline{x}_2')$. Equally, let $Q = (q_{ij})$ be the transition matrix from the basis (y_1, y_2) of J^2 to (y_1', y_2'). If we now calculate $x_i'x_j'$ and compare coefficients of y_i we obtain equations which, in matrix form, are

$$\begin{cases} P^tAP = q_{11}A' + q_{12}B' \\ P^tBP = q_{21}A' + q_{22}B' \end{cases}.$$

Evidently, the problem of classifying our rings up to isomorphism amounts to that of classifying pairs of linearly independent matrices (A, B) under the above relation of *equivalence*, P and Q being arbitrary invertible matrices, and it is to this problem of linear algebra that the paper is devoted. We shall, in fact, solve it over an arbitrary field of characteristic $\neq 2$ and will consider all pairs, independent or otherwise. The approach we take is to first of all deal with pairs of *symmetric* matrices (corresponding to commutative rings) and then to use the fact that a general equivalence class may be represented by the sum of one of the standard symmetric pairs already found with an antisymmetric pair. This is similar to an idea used in [4] for congruence of single matrices.

2. The symmetric case.

We first establish some notation. Let X be the set of all pairs (A, B) of (2×2) matrices over a field K. The group GL_2 acts on the right on X by congruence: $(A, B) \cdot P = (P^tAP, P^tBP)$ and on the left via $Q \cdot (A, B) = (q_{11}A + q_{12}B, q_{21}A + q_{22}B)$, where $Q = (q_{ij})$. These two actions are permutable and define a (left) action of $G = GL_2 \times GL_2$ on X:

$$(P, Q) \cdot (A, B) = Q \cdot (A, B) \cdot P^{-1}.$$

By restriction, G acts on the subset Y consisting of pairs with A, B linearly independent. This amounts to studying the congruence action (via P) of GL_2 on the set $\mathcal{Y}$ of 2-dimensional subspaces of $M_2(K)$, Q just representing a change of basis in a given subspace. In the same way, the whole action of G on X may be reinterpreted as an action of GL_2 on the set $\mathcal{X}$ of subspaces of dimension ≤ 2. Two pairs in the same G-orbit will be called *equivalent*.

G also acts by restriction on the set S of pairs with A, B symmetric. *Assuming henceforth that* char $K \neq 2$, we determine these orbits first. To avoid a plague of parentheses we omit these around ordered pairs of displayed matrices.

Theorem 1. *The following table gives a complete set of representatives for the orbits of G on S, together with their stabilizers:*

Representative	Stabilizing elements (P, Q)
1. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 2ac & ad \end{pmatrix}$
2. $\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} a & \pm\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2+\delta c^2 & \pm 2\delta ac \\ 2ac & \pm(a^2+\delta c^2) \end{pmatrix}$
3. $\begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	All
4. $\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$
5. $\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2+\delta c^2 & l \\ 0 & n \end{pmatrix}$

In 2) and 5) δ runs through a set of coset representatives of K^{*2} in K^*.

Write $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $Q = \begin{pmatrix} k & l \\ m & n \end{pmatrix}$. Before giving the proof it is useful to record that if $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then:

$$P^t A P = \begin{pmatrix} a^2\alpha + ac(\beta+\gamma) + c^2\delta & ab\alpha + ad\beta + bc\gamma + cd\delta \\ ab\alpha + ad\gamma + bc\beta + cd\delta & b^2\alpha + bd(\beta+\gamma) + d^2\delta \end{pmatrix}.$$

In particular we have:

A	$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}$	$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$
$P^t A P$	$\begin{pmatrix} a^2 & ab \\ ab & b^2 \end{pmatrix}$	$\begin{pmatrix} a^2+c^2\delta & ab+cd\delta \\ ab+cd\delta & b^2+d^2\delta \end{pmatrix}$	$\begin{pmatrix} 2ac & ad+bc \\ ad+bc & 2bd \end{pmatrix}$

Note also that (P, Q) fixes a pair $\Pi = (A, B) \Leftrightarrow \Pi \cdot P = Q \cdot \Pi$.

Proof of Theorem 1. Consider first *independent* pairs (A, B) in S. We claim that any such pair in equivalent to one with $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. To prove this it is enough to show that every 2-dimensional subspace W of the space V of symmetric matrices contains an *isotropic* matrix, in the sense that it is nonsingular and the associated quadratic form represents zero. For all isotropic matrices are congruent to this one. If W equals the space of diagonal matrices, then it contains the isotropic matrix $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. If not, then, since $\dim V = 3$, W is spanned by a diagonal matrix and a non-diagonal matrix $\begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}$. We may clearly modify the latter so that α or δ equals 0, and then it is isotropic.

So now let (A, B) be independent, with $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$. We may take A to be diagonal, $A = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$. Under congruence by $P = B$, if necessary, we may assume that $\alpha \neq 0$, and then, via a suitable Q, that $\alpha = 1$.

We now determine when two pairs $\Pi = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ and $\Pi' = \begin{pmatrix} 1 & \\ & \delta' \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ are equivalent. This happens when there exist P, Q as above such that $\Pi \cdot P = Q \cdot \Pi'$, or in other words:

(1)
$$\begin{pmatrix} a^2 + c^2\delta & ab + cd\delta \\ ab + cd\delta & b^2 + d^2\delta \end{pmatrix}, \begin{pmatrix} 2ac & ad + bc \\ ad + bc & 2bd \end{pmatrix} = \begin{pmatrix} k & l \\ l & k\delta' \end{pmatrix}, \begin{pmatrix} m & n \\ n & m\delta' \end{pmatrix}.$$

Comparing diagonal terms gives

$$\begin{cases} b^2 + d^2\delta = \delta'(a^2 + c^2\delta) \\ bd = \delta' ac \end{cases} . \tag{2}$$

Squaring these and subtracting 4δ times the second from the first leads to $b^2 - d^2\delta = \pm\delta'(a^2 - c^2\delta)$. According to the sign, there are two cases:

$$\text{(i)} \quad \begin{cases} b^2 = \delta' a^2 \\ \delta d^2 = \delta\delta' c^2 \end{cases} \qquad \text{or} \quad \text{(ii)} \quad \begin{cases} b^2 = \delta\delta' c^2 \\ \delta d^2 = \delta' a^2 \end{cases} .$$

In either case it follows from nonsingularity of P that if $\delta' = 0$, then $b = 0$, $d \neq 0$ and $\delta = 0$. By symmetry we deduce that $\delta = 0 \Leftrightarrow \delta' = 0$. The stabilizer in this case is given by the single condition $b = 0$, and the form of Q follows from (1).

Assume now that $\delta, \delta' \neq 0$. Case (i) cannot now arise, as is shown by the second equation of (2), the first of (i) and nonsingularity of P. It follows from (ii) that Π and Π' are equivalent $\Leftrightarrow \delta, \delta'$ are in the same square-class. The form of the stabilizer results at once.

We are left with the *dependent* pairs (A, B) in S. Via Q we may assume that $B = 0$, and then (via P) that $A = \begin{pmatrix} \alpha & \\ & \delta \end{pmatrix}$. If $A \neq 0$, then again (via P) we may assume $\alpha \neq 0$, and finally (via Q) that $\alpha = 1$. This gives the remaining types in the table. As for equivalence, these cannot be equivalent to independent pairs, so we only have to examine equivalence between $\Pi = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ and $\Pi' = \begin{pmatrix} 1 & \\ & \delta' \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$. The condition $\Pi \cdot P = Q \cdot \Pi'$ this time gives

$$(3) \qquad \begin{cases} b^2 + d^2\delta = \delta'(a^2 + c^2\delta) \\ ab = -cd\delta \end{cases}.$$

(3) is of exactly the same form as (2): we merely have to interchange a, d and replace δ by $-\delta'$, δ' by $-\delta$. It follows that Π and Π' are equivalent $\Leftrightarrow \delta, \delta'$ are in the same (possibly zero) square-class. Once more, the form of the stabilizers results immediately. □

3. The general case.

Consider now an *arbitrary* pair $\Pi = (A, B)$. This decomposes uniquely as the sum $\Pi = \Pi_s + \Pi_a$ of a *symmetric* pair $\Pi_s = (A_s, B_s)$ and an *antisymmetric* pair $\Pi_a = (A_a, B_a)$. One checks at once that this decomposition commutes with the action: $((P, Q) \cdot \Pi)_s = (P, Q) \cdot \Pi_s$ and $((P, Q) \cdot \Pi)_a = (P, Q) \cdot \Pi_a$. In particular:

$$(P, Q) \text{ fixes } \Pi \Leftrightarrow \text{ it fixes each of } \Pi_s \text{ and } \Pi_a.$$

Let $\mathcal{S}$ be the set of symmetric representatives in Theorem 1. We now have:

Proposition 1. (i) *Each equivalence class contains a pair* $\Sigma + \mathrm{T}$, *where* $\Sigma \in \mathcal{S}$ *and* T *is antisymmetric. Moreover, the class determines* Σ *uniquely.* (ii) *If* $\Pi = \Sigma + \mathrm{T}$ *and* $\Pi' = \Sigma + \mathrm{T}'$ (*similarly*), *then* $(P, Q) \cdot \Pi = \Pi' \Leftrightarrow (P, Q)$ *stabilizes* Σ *and* $(P, Q) \cdot \mathrm{T} = \mathrm{T}'$.

We also record the following evident lemma. Henceforth let $J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$.

Lemma 1. *If* $\mathrm{T} = (\alpha \mathrm{J}, \beta \mathrm{J})$ *and* $\mathrm{T}' = (\alpha' \mathrm{J}, \beta' \mathrm{J})$ *are antisymmetric pairs and* $\Delta = \det P$, *then*

$$(4) \qquad (P, Q) \cdot \mathrm{T} = \mathrm{T}' \Leftrightarrow \begin{cases} k\alpha + l\beta = \Delta\alpha' \\ m\alpha + n\beta = \Delta\beta' \end{cases}.$$

Prop. 1 shows that each equivalence class has an underlying type in $\mathcal{S}$, and each type is a union of equivalence classes. We now analyze these types in turn, keeping the notation established above:

1) $\Sigma = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$: $(P,Q)\cdot\Pi = \Pi'$ if and only if (P,Q) is as in line 1 of the table in Theorem 1 and (4) holds, which amounts to $\begin{cases} a\alpha = d\alpha' \\ 2c\alpha + d\beta = d\beta'. \end{cases}$

If $\alpha = 0$, then $\alpha' = 0$ and $\beta' = \beta$. Thus there is *one orbit for each* $\beta \in K$, corresponding to $\mathrm{T} = (0, \beta\mathrm{J})$. The stabilizer for each of these is all of $\mathrm{Stab}(\Sigma)$. If $\alpha \neq 0$, we may take $a = 1$, $d = \alpha$, $c = -\beta/2$ to get $\alpha' = 1$, $\beta' = 0$, resulting in *one more orbit* given by $\mathrm{T} = (\mathrm{J}, 0)$. The stabilizer is given by the equations $a = d$, $c = 0$, hence consists of the pairs $(P,Q) = (aI, a^2I)$.

2) $\Sigma = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$: Let $O_{2,\lambda} = \left\{ \begin{pmatrix} x & \mp\lambda y \\ y & \pm x \end{pmatrix} : x^2 + \lambda y^2 = 1 \right\}$ be the orthogonal group of the quadratic form $(1, \lambda)$. The form of (P,Q) shows that $Q/\Delta \in O_{2,-\delta}$ and Equations (4) say that Q/Δ sends (α, β) to (α', β'). Hence these vectors have the same length with respect to the form $(1, -\delta)$, in other words $\alpha^2 - \delta\beta^2 = \alpha'^2 - \delta\beta'^2$.

Conversely, let (α, β) and (α', β') be non-zero vectors satisfying this condition. Then by Witt's Extension Theorem (cf. [**4**, Prop. 3]) there exists $R = \begin{pmatrix} x & \pm\delta y \\ y & \pm x \end{pmatrix}$ in $O_{2,-\delta}$ (so that $x^2 - \delta y^2 = 1$) sending (α, β) to (α', β'). We can now choose a, c such that $R = Q/\Delta$. Namely, if $x \neq \mp 1$, let $a = \delta^{-1}(1 \pm x)$, $c = \pm\delta^{-1}y$ and if $x = \mp 1$, let $a = 0$, $c = 1$. Now (4) holds, so Π and Π' are equivalent.

Thus, apart from the symmetric class (given by $\mathrm{T} = (0,0)$), there is *one orbit for each element of* K *represented* (*non-trivially*) *by the form* $\alpha^2 - \delta\beta^2$, corresponding to $\mathrm{T} = (\alpha\mathrm{J}, \beta\mathrm{J})$.

The stabilizers are easily found from (4), with $\alpha' = \alpha$ $\beta' = \beta$.

If $P = \begin{pmatrix} a & \delta c \\ c & a \end{pmatrix}$, this condition becomes $\begin{cases} c(c\alpha + a\beta) = 0 \\ c(a\alpha + \delta c\beta) = 0 \end{cases}$, which reduces to $c = 0$, P being nonsingular. Thus $(P,Q) = (aI, a^2I)$.

If $P = \begin{pmatrix} a & -\delta c \\ c & -a \end{pmatrix}$, it amounts to $a\alpha = \delta c\beta$, so that $(a,c) = \mu(\delta\beta, \alpha)$ $(\mu \neq 0)$. The only other condition which must be met is that $\Delta \neq 0$, or equivalently $\alpha^2 - \delta\beta^2 \neq 0$. Provided this is so, the stabilizer contains elements of this second type, namely $(P,Q) = \mu\begin{pmatrix} \delta\beta & -\delta\alpha \\ \alpha & -\delta\beta \end{pmatrix}, \delta\mu^2\begin{pmatrix} \alpha^2 + \delta\beta^2 & -2\delta\alpha\beta \\ 2\alpha\beta & -(\alpha^2 + \delta\beta^2) \end{pmatrix}$. Otherwise such elements do not arise.

3) $\Sigma = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Taking $P = I$ and Q arbitrary shows that in addition to the symmetric class there is just *one orbit* with $\mathrm{T} \neq 0$. We may, for example, take $\mathrm{T} = (\mathrm{J}, 0)$. The stabilizer then consists of all pairs $(P, Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \Delta & l \\ 0 & n \end{pmatrix}$.

4) $\Sigma = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Here (4) becomes $\begin{cases} a^2\alpha + l\beta = ad\alpha' \\ n\beta = ad\beta' \end{cases}$, which implies that $\beta = 0 \Leftrightarrow \beta' = 0$. As well as the symmetric class we have the cases:

(i) $\beta' \neq 0$: This is equivalent to the case $(\alpha, \beta) = (0, 1)$ as follows by taking $a = d = 1$, $l = \alpha' - \alpha$, $n = \beta'$. So we get *one orbit* corresponding to $\mathrm{T} = (0, \mathrm{J})$. The stabilizer consists of the pairs $(P, Q) = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 0 & ad \end{pmatrix}$.

(ii) $\alpha' \neq 0$, $\beta' = 0$: This is equivalent to $(\alpha, \beta) = (1, 0)$ (take $a = \alpha'$, $d = 1$), and there is again *one orbit*, given by $\mathrm{T} = (\mathrm{J}, 0)$. The stabilizer consists of the pairs $(P, Q) = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$.

5) $\Sigma = \begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$: Now (4) is $\begin{cases} (a^2 + \delta c^2)\alpha + l\beta = \pm(a^2 + \delta c^2)\alpha' \\ n\beta = \pm(a^2 + \delta c^2)\beta' \end{cases}$, leading again to $\beta = 0 \Leftrightarrow \beta' = 0$. Apart from the symmetric class we must consider:

(i) $\beta' \neq 0$: As before, this reduces to *one orbit*, given by $\mathrm{T} = (0, \mathrm{J})$. The stabilizer is the set of all $(P, Q) = \begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & 0 \\ 0 & \pm(a^2 + \delta c^2) \end{pmatrix}$.

(ii) $\alpha' \neq 0$, $\beta' = 0$: It follows that $\alpha = \pm\alpha'$, and thus that the distinct orbits are given by $\mathrm{T} = (\alpha\mathrm{J}, 0)$, α running over $K^*/\{\pm 1\}$. To calculate the stabilizers we put $\alpha = \alpha'$, $\beta = \beta' = 0$ in the equations above. This forces the sign to be $+$, and hence the stabilizer in the set of $(P, Q) = \begin{pmatrix} a & -\delta c \\ c & a \end{pmatrix}$, $\begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}$.

We collect our results in the next theorem. Since we have dealt already with the symmetric classes in Theorem 1, we confine ourselves to the rest:

Theorem 2. *The following table gives a complete set of representatives for the orbits of G on $X - S$ (the* non-symmetric *classes), together with their stabilizers:*

	Representative	Stabilizing elements (P, Q)
1a.	$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1+\beta \\ 1-\beta & \end{pmatrix}$ $(\beta \in K^*)$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 2ac & ad \end{pmatrix}$
1b.	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$	$\begin{pmatrix} a & \\ & a \end{pmatrix}, \begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}$
2a.	$\begin{pmatrix} 1 & \alpha \\ -\alpha & \delta \end{pmatrix}, \quad \begin{pmatrix} & 1+\beta \\ 1-\beta & \end{pmatrix}$ in 1-1 correspondence with the values in K represented by $\alpha^2 - \delta\beta^2$, for each $\delta \in K^*/K^{*2}$	$\begin{pmatrix} a & \\ & a \end{pmatrix}, \begin{pmatrix} a^2 & \\ & a^2 \end{pmatrix}$ if $\alpha^2 - \delta\beta^2 = 0$ Otherwise, the above pairs plus: $\mu \begin{pmatrix} \delta\beta & -\delta\alpha \\ \alpha & -\delta\beta \end{pmatrix},$ $\delta\mu^2 \begin{pmatrix} \alpha^2 + \delta\beta^2 & -2\delta\alpha\beta \\ 2\alpha\beta & -(\alpha^2 + \delta\beta^2) \end{pmatrix}$
3a.	$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \Delta & l \\ 0 & n \end{pmatrix}$
4a.	$\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 0 & ad \end{pmatrix}$
4b.	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}$
5a.	$\begin{pmatrix} 1 & \\ & \delta \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} a & \mp\delta c \\ c & \pm a \end{pmatrix},$ $\begin{pmatrix} a^2 + \delta c^2 & 0 \\ 0 & \pm(a^2 + \delta c^2) \end{pmatrix}$
5b.	$\begin{pmatrix} 1 & \alpha \\ -\alpha & \delta \end{pmatrix}, \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}$ $(\alpha \in K^*/\{\pm 1\})$	$\begin{pmatrix} a & -\delta c \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}$

By inspection from Theorems 1 and 2 we also have:

Corollary 1. *The orbits of* G *on* Y (*the* linearly independent *classes*) *are given by lines* 1, 2, 1a, 1b, 2a, 4a *and* 5a.

4. Finite Fields.

We now specialize the foregoing to the finite field $K = \mathbf{F}_q$ (q odd). In this case $|G| = q^2(q-1)^2(q^2-1)^2$, $|X| = q^8$, $|Y| = q(q^3-1)(q^4-1)$ and $|S| = q^6$. There are two square-classes in K^*, represented by 1 and a fixed non-square ε. Over $\mathbf{F}_q$ quadratic forms of rank ≥ 2 are universal (cf. [5] for example), so that for each of $\delta = 1, \varepsilon$ the form $\alpha^2 - \delta\beta^2$ takes all values in K^*. In addition, when $\delta = 1$ it represents 0, but not when $\delta = \varepsilon$. Let χ denote the quadratic character of K.

From the previous results we can now easily determine the number of equivalence classes and their sizes:

Theorem 3. *The following table gives, for each type of representative, the sizes of the stabilizer and equivalence class and the number of classes:*

Rep.	|Stabilizer|	|Class|	Number of classes
1	$q(q-1)^2$	$q(q^2-1)^2$	1
2	$2(q-1)(q-\chi(\delta))$	$\frac{1}{2}q^2(q-1)(q^2-1)(q+\chi(\delta))$	2
3	$\|G\|$	1	1
4	$q^2(q-1)^3$	$(q+1)(q^2-1)$	1
5	$2q(q-1)^2(q-\chi(-\delta))$	$\frac{1}{2}q(q^2-1)(q+\chi(-\delta))$	2
1a	$q(q-1)^2$	$q(q^2-1)^2$	$q-1$
1b	$q-1$	$q^2(q-1)(q^2-1)^2$	1

2a	$\begin{cases} q-1 & \text{if } \alpha^2-\delta\beta^2=0 \\ 2(q-1) & \text{if not} \end{cases}$	$\begin{cases} q^2(q-1)(q^2-1)^2 \\ \frac{1}{2}q^2(q-1)(q^2-1)^2 \end{cases}$	$\begin{cases} 1 \\ 2(q-1) \end{cases}$
3a	$q^2(q-1)^2(q^2-1)$	q^2-1	1
4a	$q(q-1)^2$	$q(q^2-1)^2$	1
4b	$q^2(q-1)^2$	$(q^2-1)^2$	1
5a	$2(q-1)(q-\chi(-\delta))$	$\frac{1}{2}q^2(q-1)(q^2-1)(q+\chi(-\delta))$	2
5b	$q(q-1)^2(q-\chi(-\delta))$	$q(q^2-1)(q+\chi(-\delta))$	$q-1$

In all there are $4q+10$ *classes, of which* 7 *are symmetric. For the linearly independent pairs, the number of classes is* $3q+5$, *and* 3 *of these are symmetric.*

Proof. It is only necessary to observe, for lines 2, 5, 5a and 5b, that if $\xi \in K^*$ then the number of solutions of $\alpha^2 - \xi\beta^2 \neq 0$ is $(q-1)(q-\chi(\xi))$. Note also in line 5b that there are $\frac{1}{2}(q-1)$ classes for each of $\delta = 1, \varepsilon$. □

As a check on the arithmetic, one readily verifies that the sum of all the class sizes is $q^8 = |X|$. For the symmetric classes the sum is $q^6 = |S|$.

Corollary 2. *For the finite local rings of the Introduction, there are* $3q+5$ *isomorphism classes* (q *odd*)*. Of these,* 3 *are commutative.*

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