CONGRUENCE OF TWO-DIMENSIONAL SUBSPACES IN $M_2(K)$ (CHARACTERISTIC $\neq 2$)

B. Corbas and G.D. Williams
CONGRUENCE OF TWO-DIMENSIONAL SUBSPACES
IN $M_2(K)$ (CHARACTERISTIC $\neq 2$)

B. Corbas and G.D. Williams

The structure and classification up to isomorphism of a naturally arising class of local rings is determined. Although we are primarily interested in the case of a finite residue field $K$, our results apply in fact over any field $K$ of characteristic $\neq 2$.

The problem is shown to be equivalent to that of classifying two-dimensional subspaces of $M_2(K)$ up to congruence, and it is in these terms that the question is addressed.

1. Introduction.

In investigating the structure of finite local rings one is led to consider such a ring of the form $R = K \oplus J$ in which $K = F_q$ and the Jacobson radical $J$ is such that $J^3 = 0$ and both $J/J^2$ and $J^2$ are two-dimensional over $R/J = K$.

Rings with $J^3 = 0$ form a natural object of study, the case $J^2 = 0$ having long been settled [2, 3]. If $J = Kx_1 \oplus Kx_2 \oplus J^2$ and $J^2 = Ky_1 \oplus Ky_2$, then we may write $x_ix_j = \alpha_{ij}y_1 + \beta_{ij}y_2$ ($\alpha_{ij}, \beta_{ij} \in K$) and these four products span $J^2$. The ring structure is determined by the pair of $(2 \times 2)$ matrices $A = (\alpha_{ij}), B = (\beta_{ij})$, which are linearly independent over $K$, and any pair of independent matrices defines such a ring.

We wish to determine the number of isomorphism classes of such rings and to find normal forms for the pair of matrices $A, B$ defining them. Chikunji [1] has shown that there are 10 classes for $q = 2$ and, on the basis of computer calculations for $q = 3, 5, 7$, has conjectured that when $q$ is odd the number of classes is $3q + 5$. It is also conjectured that exactly three of these rings are commutative. Our purpose here is inter alia, to prove these conjectures.

If $(x'_1, x'_2, y'_1, y'_2)$ is a new basis of $J$ with corresponding matrices $A', B'$, then $x'_1, x'_2$ are linear combinations of $x_1, x_2, y_1, y_2$. Since $J^3 = 0$, we may assume that the coefficients of $y_1, y_2$ are zero and write $x'_i = p_{i1}x_1 + p_{i2}x_2$, so that $P = (p_{ij})$ is the transition matrix from the basis $(x_1, x_2)$ of $J/J^2$ to the basis $(x'_1, x'_2)$. Equally, let $Q = (q_{ij})$ be the transition matrix from the basis $(y_1, y_2)$ of $J^2$ to $(y'_1, y'_2)$. If we now calculate $x'_ix'_j$ and compare coefficients of $y_i$ we obtain equations which, in matrix form, are

$$
\begin{align*}
P'AP &= q_{11}A' + q_{12}B' \\
P'BP &= q_{21}A' + q_{22}B'.
\end{align*}
$$
Evidently, the problem of classifying our rings up to isomorphism amounts to that of classifying pairs of linearly independent matrices \((A, B)\) under the above relation of \textit{equivalence}, \(P\) and \(Q\) being arbitrary invertible matrices, and it is to this problem of linear algebra that the paper is devoted. We shall, in fact, solve it over an arbitrary field of characteristic \(\neq 2\) and will consider all pairs, independent or otherwise. The approach we take is to first of all deal with pairs of \textit{symmetric} matrices (corresponding to commutative rings) and then to use the fact that a general equivalence class may be represented by the sum of one of the standard symmetric pairs already found with an antisymmetric pair. This is similar to an idea used in [4] for congruence of single matrices.

2. The symmetric case.

We first establish some notation. Let \(X\) be the set of all pairs \((A, B)\) of \((2 \times 2)\) matrices over a field \(K\). The group \(GL_2\) acts on the right on \(X\) by congruence: \((A, B) \cdot P = (P^t A P, P^t B P)\) and on the left via \(Q \cdot (A, B) = (q_{11} A + q_{12} B, q_{21} A + q_{22} B)\), where \(Q = (q_{ij})\). These two actions are permutative and define a (left) action of \(G = GL_2 \times GL_2\) on \(X\):

\[(P, Q) \cdot (A, B) = Q \cdot (A, B) \cdot P^{-1}.
\]

By restriction, \(G\) acts on the subset \(Y\) consisting of pairs with \(A, B\) linearly independent. This amounts to studying the congruence action (via \(P\)) of \(GL_2\) on the set \(Y\) of 2-dimensional subspaces of \(M_2(K)\), \(Q\) just representing a change of basis in a given subspace. In the same way, the whole action of \(G\) on \(X\) may be reinterpreted as an action of \(GL_2\) on the set \(X\) of subspaces of dimension \(\leq 2\). Two pairs in the same \(G\)-orbit will be called \textit{equivalent}.

\(G\) also acts by restriction on the set \(S\) of pairs with \(A, B\) symmetric. \textit{Assuming henceforth that} \(\text{char } K \neq 2\), we determine these orbits first. To avoid a plague of parentheses we omit these around ordered pairs of displayed matrices.

\textbf{Theorem 1.} \textit{The following table gives a complete set of representatives for the orbits of} \(G\) \textit{on} \(S\), \textit{together with their stabilizers:}
Representative | Stabilizing elements \((P, Q)\)
---|---
1. \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) | \(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 2ac & ad \end{pmatrix}\)
2. \(\begin{pmatrix} 1 \\ \delta \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) | \(\begin{pmatrix} a & \pm \delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & \pm 2\delta ac \\ 2ac & \pm (a^2 + \delta c^2) \end{pmatrix}\)
3. \(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) | All
4. \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) | \(\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix}\)
5. \(\begin{pmatrix} 1 \\ \delta \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\) | \(\begin{pmatrix} a & \mp \delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix}\)

In 2) and 5) \(\delta\) runs through a set of coset representatives of \(K^{*2}\) in \(K^{*}\).

Write \(P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) and \(Q = \begin{pmatrix} k & l \\ m & n \end{pmatrix}\). Before giving the proof it is useful to record that if \(A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\), then:

\[
P^t A P = \begin{pmatrix} a^2 + ab + ac(\beta + \gamma) + c^2 \delta & ab + ad + b + bc\gamma + cd\delta \\ ab\alpha + ad\gamma + bc\beta + cd\delta & b^2\alpha + bd(\beta + \gamma) + d^2\delta \end{pmatrix}.
\]

In particular we have:

<table>
<thead>
<tr>
<th>(A)</th>
<th>(\begin{pmatrix} 1 \ 0 \end{pmatrix})</th>
<th>(\begin{pmatrix} 1 \ \delta \end{pmatrix})</th>
<th>(\begin{pmatrix} 1 \ 1 \end{pmatrix})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P^t A P)</td>
<td>(\begin{pmatrix} a^2 &amp; ab \ ab &amp; b^2 \end{pmatrix})</td>
<td>(\begin{pmatrix} a^2 + c^2 \delta &amp; ab + cd\delta \ ab + cd\delta &amp; b^2 + d^2 \delta \end{pmatrix})</td>
<td>(\begin{pmatrix} 2ac &amp; ad + bc \ ad + bc &amp; 2bd \end{pmatrix})</td>
</tr>
</tbody>
</table>

Note also that \((P, Q)\) fixes a pair \(\Pi = (A, B) \Leftrightarrow \Pi \cdot P = Q \cdot \Pi\).
Proof of Theorem 1. Consider first independent pairs \((A, B)\) in \(S\). We claim that any such pair is equivalent to one with \(B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\). To prove this it is enough to show that every 2-dimensional subspace \(W\) of the space \(V\) of symmetric matrices contains an isotropic matrix, in the sense that it is nonsingular and the associated quadratic form represents zero. For all isotropic matrices are congruent to this one. If \(W\) equals the space of diagonal matrices, then it contains the isotropic matrix \(\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\). If not, then, since \(\dim V = 3\), \(W\) is spanned by a diagonal matrix and a non-diagonal matrix \(\begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}\). We may clearly modify the latter so that \(\alpha\) or \(\delta\) equals 0, and then it is isotropic.

So now let \((A, B)\) be independent, with \(B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\). We may take \(A\) to be diagonal, \(A = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix}\). Under congruence by \(P = B\), if necessary, we may assume that \(\alpha \neq 0\), and then, via a suitable \(Q\), that \(\alpha = 1\).

We now determine when two pairs \(\Pi = \begin{pmatrix} 1 \\ \delta \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(\Pi' = \begin{pmatrix} 1 \\ \delta' \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) are equivalent. This happens when there exist \(P, Q\) as above such that \(\Pi \cdot P = Q \cdot \Pi'\), or in other words:

\[
\begin{pmatrix} a^2 + c^2 \delta & ab + cd \delta \\ ab + cd \delta & b^2 + d^2 \delta \end{pmatrix} \begin{pmatrix} 2ac & ad + bc \\ ad + bc & 2bd \end{pmatrix} = \begin{pmatrix} k & l \\ l & k\delta' \end{pmatrix}, \begin{pmatrix} m & n \\ n & m\delta' \end{pmatrix}.
\]

Comparing diagonal terms gives

\[
\begin{cases} b^2 + d^2 \delta = \delta'(a^2 + c^2 \delta) \\ bd = \delta'ac \end{cases}
\]

Squaring these and subtracting \(4\delta\) times the second from the first leads to \(b^2 - d^2 \delta = \pm \delta'(a^2 - c^2 \delta)\). According to the sign, there are two cases:

\[
\text{(i)} \quad \begin{cases} b^2 = \delta' a^2 \\ d^2 = \delta' c^2 \end{cases} \quad \text{or} \quad \begin{cases} b^2 = \delta' c^2 \\ d^2 = \delta' a^2 \end{cases}
\]

In either case it follows from nonsingularity of \(P\) that if \(\delta' = 0\), then \(b = 0\), \(d \neq 0\) and \(\delta = 0\). By symmetry we deduce that \(\delta = 0 \iff \delta' = 0\). The stabilizer in this case is given by the single condition \(b = 0\), and the form of \(Q\) follows from (1).

Assume now that \(\delta, \delta' \neq 0\). Case (i) cannot now arise, as is shown by the second equation of (2), the first of (i) and nonsingularity of \(P\). It follows
CONGRUENCE OF SUBSPACES (CHARACTERISTIC \( \neq 2 \)) 229

from (ii) that \( \Pi \) and \( \Pi' \) are equivalent \( \iff \delta, \delta' \) are in the same square-class.
The form of the stabilizer results at once.

We are left with the dependent pairs \((A, B)\) in \(S\). Via \(Q\) we may assume
that \(B = 0\), and then (via \(P\)) that \(A = \begin{pmatrix} \alpha \\ \delta \end{pmatrix}\).
If \(A \neq 0\), then again (via \(P\)) we may assume \(\alpha \neq 0\), and finally (via \(Q\)) that \(\alpha = 1\).
This gives the remaining types in the table. As for equivalence, these cannot be equivalent

\[ \text{to independent pairs, so we only have to examine equivalence between } \Pi = \begin{pmatrix} 1 \\ \delta \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{, and } \Pi' = \begin{pmatrix} 1 \\ \delta' \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

The condition \(\Pi \cdot P = Q \cdot \Pi'\) this time gives

\[ \begin{align*}
    b^2 + d^2\delta &= \delta'(a^2 + c^2\delta) \\
    ab &= -cd\delta
\end{align*} \tag{3} \]

(3) is of exactly the same form as (2): we merely have to interchange \(a, d\) and
replace \(\delta\) by \(-\delta\), \(\delta'\) \(\iff\) \(-\delta\).

It follows that \(\Pi\) and \(\Pi'\) are equivalent \(\iff \delta, \delta'\) are in the same (possibly zero) square-class. Once more, the form of the
stabilizers results immediately. \(\Box\)

3. The general case.

Consider now an arbitrary pair \(\Pi = (A, B)\). This decomposes uniquely as the
sum \(\Pi = \Pi_s + \Pi_a\) of a symmetric pair \(\Pi_s = (A_s, B_s)\) and an antisymmetric
pair \(\Pi_a = (A_a, B_a)\). One checks at once that this decomposition commutes
with the action: \(((P, Q) \cdot \Pi)_s = (P, Q) \cdot \Pi_s\) and \(((P, Q) \cdot \Pi)_a = (P, Q) \cdot \Pi_a\).

In particular:

\[ (P, Q) \text{ fixes } \Pi \iff \text{it fixes each of } \Pi_s \text{ and } \Pi_a. \]

Let \(S\) be the set of symmetric representatives in Theorem 1. We now have:

**Proposition 1.** (i) Each equivalence class contains a pair \(\Sigma + T\), where
\(\Sigma \in S\) and \(T\) is antisymmetric. Moreover, the class determines \(\Sigma\) uniquely.
(ii) If \(\Pi = \Sigma + T\) and \(\Pi' = \Sigma + T'\) (similarly), then \((P, Q) \cdot \Pi = \Pi' \iff (P, Q)\)
stabilizes \(\Sigma\) and \((P, Q) \cdot T = T'\).

We also record the following evident lemma. Henceforth let \(J = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\).

**Lemma 1.** If \(T = (\alpha J, \beta J)\) and \(T' = (\alpha' J, \beta' J)\) are antisymmetric pairs and \(\Delta = \det P\), then

\[ (P, Q) \cdot T = T' \iff \begin{cases}
    k\alpha + l\beta = \Delta\alpha' \\
    m\alpha + n\beta = \Delta\beta'
\end{cases} \tag{4} \]
Prop. 1 shows that each equivalence class has an underlying type in $S$, and each type is a union of equivalence classes. We now analyze these types in turn, keeping the notation established above:

1) $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$: $(P, Q) \Pi = \Pi'$ if and only if $(P, Q)$ is as in line 1 of the table in Theorem 1 and (4) holds, which amounts to $\begin{cases} a\alpha = d\alpha' \\ 2c\alpha + d\beta = d\beta'. \end{cases}$

   If $\alpha = 0$, then $\alpha' = 0$ and $\beta' = \beta$. Thus there is one orbit for each $\beta \in K$, corresponding to $T = (0, \beta J)$. The stabilizer for each of these is all of $\text{Stab}(\Sigma)$. If $\alpha \neq 0$, we may take $a = 1$, $d = \alpha$, $c = -\beta/2$ to get $\alpha' = 1$, $\beta' = 0$, resulting in one more orbit given by $T = (J, 0)$. The stabilizer is given by the equations $a = d$, $c = 0$, hence consists of the pairs $(P, Q) = (aI, a^2I)$.

2) $\Sigma = \begin{pmatrix} 1 & \delta \\ \delta & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$: Let $O_{2, \lambda} = \left\{ \begin{pmatrix} x & \mp \lambda y \\ y & \pm x \end{pmatrix} : x^2 + \lambda y^2 = 1 \right\}$ be the orthogonal group of the quadratic form $(1, \lambda)$. The form of $(P, Q)$ shows that $Q/\Delta \in O_{2, -\delta}$ and Equations (4) say that $Q/\Delta$ sends $(\alpha, \beta)$ to $(\alpha', \beta')$. Hence these vectors have the same length with respect to the form $(1, -\delta)$, in other words $\alpha^2 - \delta\beta^2 = \alpha'^2 - \delta\beta'^2$.

   Conversely, let $(\alpha, \beta)$ and $(\alpha', \beta')$ be non-zero vectors satisfying this condition. Then by Witt’s Extension Theorem (cf. [4, Prop. 3]) there exists $R = \begin{pmatrix} x & \pm \delta y \\ y & \pm x \end{pmatrix}$ in $O_{2, -\delta}$ (so that $x^2 - \delta y^2 = 1$) sending $(\alpha, \beta)$ to $(\alpha', \beta')$. We can now choose $a, c$ such that $R = Q/\Delta$. Namely, if $x \neq \mp 1$, let $a = \delta^{-1}(1 \pm x)$, $c = \pm \delta^{-1}y$ and if $x = \mp 1$, let $a = 0$, $c = 1$. Now (4) holds, so $\Pi$ and $\Pi'$ are equivalent.

   Thus, apart from the symmetric class (given by $T = (0, 0)$), there is one orbit for each element of $K$ represented (non-trivially) by the form $\alpha^2 - \delta\beta^2$, corresponding to $T = (\alpha J, \beta J)$.

   The stabilizers are easily found from (4), with $\alpha' = \alpha$, $\beta' = \beta$.

   If $P = \begin{pmatrix} a & \delta c \\ c & a \end{pmatrix}$, this condition becomes $\begin{cases} c(a\alpha + a\beta) = 0 \\ c(a\alpha + \delta c\beta) = 0 \end{cases}$, which reduces to $c = 0$, $P$ being nonsingular. Thus $(P, Q) = (aI, a^2I)$.

   If $P = \begin{pmatrix} a & -\delta c \\ c & -a \end{pmatrix}$, it amounts to $a\alpha = \delta c\beta$, so that $(a, c) = \mu(\delta \beta, \alpha)$ ($\mu \neq 0$). The only other condition which must be met is that $\Delta \neq 0$, or equivalently $\alpha^2 - \delta\beta^2 \neq 0$. Provided this is so, the stabilizer contains elements of this second type, namely $(P, Q) = \mu(\begin{pmatrix} \delta \beta & -\delta \alpha \\ \alpha & -\delta \beta \end{pmatrix}, \delta \mu^2(\begin{pmatrix} \alpha^2 + \delta \beta^2 & -2\delta\alpha\beta \\ 2\alpha\beta & -(\alpha^2 + \delta \beta^2) \end{pmatrix})$. Otherwise such elements do not arise.
3) \( \Sigma = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \): Taking \( P = I \) and \( Q \) arbitrary shows that in addition to the symmetric class there is just one orbit with \( T \neq 0 \). We may, for example, take \( T = (J, 0) \). The stabilizer then consists of all pairs \( (P, Q) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \Delta & l \\ 0 & n \end{pmatrix} \).

4) \( \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \): Here \( (4) \) becomes

\[
\begin{align*}
a^2\alpha + l\beta &= ad\alpha' \\
n\beta &= ad\beta'
\end{align*}
\]

which implies that \( \beta = 0 \Leftrightarrow \beta' = 0 \). As well as the symmetric class we have the cases:

(i) \( \beta' \neq 0 \): This is equivalent to the case \( (\alpha, \beta) = (0, 1) \) as follows by taking \( a = d = 1, l = \alpha' - \alpha, n = \beta' \). So we get one orbit corresponding to \( T = (0, J) \). The stabilizer consists of the pairs \( (P, Q) = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} a^2 & 0 \\ 0 & ad \end{pmatrix} \).

(ii) \( \alpha' \neq 0, \beta' = 0 \): This is equivalent to \( (\alpha, \beta) = (1, 0) \) (take \( a = \alpha', d = 1 \)), and there is again one orbit, given by \( T = (J, 0) \). The stabilizer consists of the pairs \( (P, Q) = \begin{pmatrix} a & 0 \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 & l \\ 0 & n \end{pmatrix} \).

5) \( \Sigma = \begin{pmatrix} 1 & \delta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \): Now \( (4) \) is

\[
\begin{align*}
(a^2 + \delta c^2)\alpha + l\beta &= \pm(a^2 + \delta c^2)\alpha' \\
n\beta &= \pm(a^2 + \delta c^2)\beta'
\end{align*}
\]

leading again to \( \beta = 0 \Leftrightarrow \beta' = 0 \). Apart from the symmetric class we must consider:

(i) \( \beta' \neq 0 \): As before, this reduces to one orbit, given by \( T = (0, J) \). The stabilizer is the set of all \( (P, Q) = \begin{pmatrix} a & \pm\delta c \\ c & \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & 0 \\ 0 & \pm(a^2 + \delta c^2) \end{pmatrix} \).

(ii) \( \alpha' \neq 0, \beta' = 0 \): It follows that \( \alpha = \pm\alpha' \), and thus that the distinct orbits are given by \( T = (\alpha J, 0) \), \( \alpha \) running over \( K^*/\{\pm1\} \). To calculate the stabilizers we put \( \alpha = \alpha', \beta = \beta' = 0 \) in the equations above. This forces the sign to be +, and hence the stabilizer in the set of \( (P, Q) = \begin{pmatrix} a & -\delta c \\ c & a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 & l \\ 0 & n \end{pmatrix} \).

We collect our results in the next theorem. Since we have dealt already with the symmetric classes in Theorem 1, we confine ourselves to the rest:

**Theorem 2.** The following table gives a complete set of representatives for the orbits of \( G \) on \( X - S \) (the non-symmetric classes), together with their stabilizers:
<table>
<thead>
<tr>
<th>Representative</th>
<th>Stabilizing elements ((P,Q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a. \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 - \beta &amp; 1 + \beta \end{pmatrix} ) ((\beta \in K^*))</td>
<td>(\begin{pmatrix} a &amp; 0 \ c &amp; d \end{pmatrix}, \begin{pmatrix} a^2 &amp; 0 \ 2ac &amp; ad \end{pmatrix})</td>
</tr>
<tr>
<td>(1b. \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>(\begin{pmatrix} a &amp; \alpha \end{pmatrix}, \begin{pmatrix} a^2 &amp; a^2 \end{pmatrix})</td>
</tr>
<tr>
<td>(2a. \begin{pmatrix} 1 &amp; \alpha \ -\alpha &amp; \delta \end{pmatrix}, \begin{pmatrix} 1 - \beta &amp; 1 + \beta \end{pmatrix} ) (\text{in 1-1 correspondence with the values in } K) (\text{represented by } \alpha^2 - \delta \beta^2), for each (\delta \in K^<em>/K^</em>) (\text{otherwise, the above pairs plus:})</td>
<td>(\begin{pmatrix} a &amp; \alpha \end{pmatrix}, \begin{pmatrix} a^2 &amp; a^2 \end{pmatrix}) (\text{if } \alpha^2 - \delta \beta^2 = 0) (\text{Otherwise})</td>
</tr>
<tr>
<td>(3a. \begin{pmatrix} 1 &amp; 0 \ -1 &amp; 1 \end{pmatrix} )</td>
<td>(\begin{pmatrix} a &amp; b \ c &amp; d \end{pmatrix}, \begin{pmatrix} \Delta &amp; l \ 0 &amp; n \end{pmatrix})</td>
</tr>
<tr>
<td>(4a. \begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix} )</td>
<td>(\begin{pmatrix} a &amp; 0 \ c &amp; d \end{pmatrix}, \begin{pmatrix} a^2 &amp; 0 \ 0 &amp; ad \end{pmatrix})</td>
</tr>
<tr>
<td>(4b. \begin{pmatrix} 1 &amp; 0 \ -1 &amp; 1 \end{pmatrix} )</td>
<td>(\begin{pmatrix} a &amp; 0 \ c &amp; a \end{pmatrix}, \begin{pmatrix} a^2 &amp; l \ 0 &amp; n \end{pmatrix})</td>
</tr>
<tr>
<td>(5a. \begin{pmatrix} 1 &amp; 0 \ \delta &amp; -1 \end{pmatrix} )</td>
<td>(\begin{pmatrix} a &amp; \pm \delta c \ c &amp; \pm a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 \ 0 \pm (a^2 + \delta c^2) \end{pmatrix})</td>
</tr>
<tr>
<td>(5b. \begin{pmatrix} 1 &amp; \alpha \ -\alpha &amp; \delta \end{pmatrix}, \begin{pmatrix} 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix} ) ((\alpha \in K^*/{\pm 1}))</td>
<td>(\begin{pmatrix} a &amp; -\delta c \ c &amp; a \end{pmatrix}, \begin{pmatrix} a^2 + \delta c^2 &amp; l \ 0 &amp; n \end{pmatrix})</td>
</tr>
</tbody>
</table>
By inspection from Theorems 1 and 2 we also have:

**Corollary 1.** The orbits of $G$ on $Y$ (the linearly independent classes) are given by lines 1, 2, 1a, 1b, 2a, 4a and 5a.

4. Finite Fields.

We now specialize the foregoing to the finite field $K = \mathbb{F}_q$ ($q$ odd). In this case $|G| = q^2(q-1)^2(q^2-1)^2$, $|X| = q^8$, $|Y| = q(q^3-1)(q^4-1)$ and $|S| = q^6$. There are two square-classes in $K^*$, represented by 1 and a fixed non-square $\varepsilon$. Over $\mathbb{F}_q$ quadratic forms of rank $\geq 2$ are universal (cf. [5] for example), so that for each of $\delta = 1, \varepsilon$ the form $\alpha^2 - \delta \beta^2$ takes all values in $K^*$. In addition, when $\delta = 1$ it represents 0, but not when $\delta = \varepsilon$. Let $\chi$ denote the quadratic character of $K$.

From the previous results we can now easily determine the number of equivalence classes and their sizes:

**Theorem 3.** The following table gives, for each type of representative, the sizes of the stabilizer and equivalence class and the number of classes:

| Rep. | |Stabilizer| |Class| Number of classes |
|------|----------------|----------------|----------------|-------------------|
| 1    | $q(q-1)^2$     | $q(q^2-1)^2$   | 1              |
| 2    | $2(q-1)(q-\chi(\delta))$ | $\frac{1}{2}q^2(q-1)(q^2-1)(q+\chi(\delta))$ | 2              |
| 3    | $|G|$           | 1              | 1              |
| 4    | $q^2(q-1)^3$   | $(q+1)(q^2-1)$ | 1              |
| 5    | $2q(q-1)^2(q-\chi(-\delta))$ | $\frac{1}{2}q(q^2-1)(q+\chi(-\delta))$ | 2              |
| 1a   | $q(q-1)^2$     | $q(q^2-1)^2$   | $q-1$          |
| 1b   | $q-1$          | $q^2(q-1)(q^2-1)^2$ | 1              |
In all there are $4q + 10$ classes, of which 7 are symmetric. For the linearly independent pairs, the number of classes is $3q + 5$, and 3 of these are symmetric.

**Proof.** It is only necessary to observe, for lines 2, 5, 5a and 5b, that if $\xi \in K^*$ then the number of solutions of $\alpha^2 - \delta \beta^2 \neq 0$ is $\left(\frac{q}{2}(q - 1)\right)$. Note also in line 5b that there are $\frac{1}{2}(q - 1)$ classes for each of $\delta = 1, \varepsilon$. □

As a check on the arithmetic, one readily verifies that the sum of all the class sizes is $q^8 = |X|$. For the symmetric classes the sum is $q^6 = |S|$.

**Corollary 2.** For the finite local rings of the Introduction, there are $3q + 5$ isomorphism classes ($q$ odd). Of these, 3 are commutative.

### References


Received June 6, 1997.

University of Reading
Reading RG6 6AX
England

University of Reading
Reading RG6 6AX
England

E-mail address: G.D.Williams@reading.ac.uk