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EQUIVARIANT EMBEDDINGS OF STEIN DOMAINS  
SITTING INSIDE OF COMPLEX SEMIGROUPS

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## EQUIVARIANT EMBEDDINGS OF STEIN DOMAINS SITTING INSIDE OF COMPLEX SEMIGROUPS

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In this paper we prove an equivariant version of Hörmanders embedding theorem for Stein manifolds. More concretely, let  $G$  be a connected Lie group sitting in its complexification  $G_{\mathbb{C}}$  and  $D \subseteq G_{\mathbb{C}}$  a  $G \times G$ -invariant Stein domain. Under slight obstructions on  $D$  we construct a Hilbert space  $\mathcal{H}$  equipped with a unitary  $G \times G$ -action and a holomorphic equivariant closed embedding  $e: D \rightarrow \mathcal{H}^* \setminus \{0\}$ .

### Introduction.

An interesting problem in the field of equivariant complex analysis is: Given a connected Lie group  $G$  sitting in its universal complexification  $G_{\mathbb{C}}$ , how do the  $G \times G$ -invariant Stein domains in  $G_{\mathbb{C}}$  look like. K.-H. Neeb has shown in [Ne98] that all domains of the form

$$D = G \exp_{G_{\mathbb{C}}}(iD_h),$$

where  $D_h \subseteq \mathfrak{g}$  is a  $\text{Ad}(G)$ -invariant convex domain consisting of elliptic elements, i.e., all operators  $i \text{ad } X$ ,  $X \in D_h$ , are diagonalizable over the reals, are Stein manifolds. Moreover there is also strong evidence for that these  $D$  exhaust up to multiplication with  $N_{G_{\mathbb{C}}}(G)$  all proper bi-invariant Stein domains in  $G_{\mathbb{C}}$  (cf. [GG77], [Ne98]).

By Hörmander's Embedding Theorem one knows that every Stein manifold of dimension  $n$  can be embedded biholomorphically as a closed submanifold of  $\mathbb{C}^{2n+1}$  (cf. [Hö73]). Now the natural question is: Given a biinvariant Stein domain  $D = G \exp_{G_{\mathbb{C}}}(iD_h)$  in  $G_{\mathbb{C}}$ , does there exist a  $G \times G$ -equivariant embedding into some complex Hilbert space  $\mathcal{H}$  endowed with a unitary  $G \times G$ -action. In this paper we show that under quite natural assumptions the answer is affirmative. More concretely, if  $\text{Ad}(G)$  is closed in  $\text{Aut}(\mathfrak{g})$ , the center  $Z(G)$  is compact and the convex domain  $D_h$  is pointed, then there exists a positive definite biinvariant holomorphic kernel  $K$  on  $D$ , such that the map

$$e_K: D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

defines a  $G \times G$ -equivariant closed embedding. Here  $\mathcal{H}_K$  denotes the reproducing kernel Hilbert space and  $K_z: \mathcal{H}_K \rightarrow \mathbb{C}$ ,  $f \mapsto f(z)$  the point evaluations corresponding to  $K$ .

Our method to construct such a kernel  $K$  is to sum up kernels  $K^\lambda$  associated to unitary highest weight representations  $(\pi_\lambda, \mathcal{H}_\lambda)$  of  $G$  over a certain lattice  $\Gamma \subseteq \mathfrak{it}^*$ , where  $\mathfrak{t}$  denotes a compactly embedded Cartan subalgebra of  $\mathfrak{g}$ . More precisely, we set

$$K = \sum_{\lambda \in \Gamma} \|\lambda\|^N K^\lambda$$

with  $\|\cdot\|$  denoting a norm on  $\mathfrak{it}^*$  and  $N \in \mathbb{N}$ . These kernels  $K$  have the important property of tending to infinity at the boundary of  $D$ , i.e.,

$$\lim_{z \rightarrow \partial D} K(z, z) = \infty;$$

a result which is crucial for verifying the closedness of the map  $e_K$ .

We think that our results are a little bit surprising and we do not really understand what is actually going on. For instance, what is the reason for that one has to exclude zero in  $\mathcal{H}_K^*$  to achieve the closedness of the map  $e_K$ , or, is it possible to find an equivariant closed holomorphic embedding  $D \rightarrow E$  into a complex topological vector space endowed with a continuous  $G \times G$ -action. We hope that our results give rise to a further discussion leading to a better understanding of these phenomena.

## I. The boundary behaviour of bi-invariant kernels.

In this first section we characterize the boundary behaviour of biinvariant holomorphic positive definite kernels on a bi-invariant domain  $D = G \text{Exp}(iD_h)$  by means of the boundary behaviour on the abelian submanifold  $D_T := T \text{Exp}(i(D_h \cap \mathfrak{t}))$ . If the convex invariant set  $D_h \subseteq \mathfrak{g}$  is a pointed cone, we show that  $\lim_{z \rightarrow \partial D} K(z, z) = \infty$  if and only if  $\lim_{z \rightarrow \partial D_T} K(z, z) = \infty$ . As abelian domains are comparable easily to deal with contrary to the highly non-commutative bi-invariant domains  $D$ , this result allows us in the sequel to make quite explicit computations.

**Definition I.1.** Let  $V$  be a finite dimensional real vector space and  $V^*$  its dual.

(a) For each subset  $E \subseteq V$  we define its *dual cone* by  $E^* := \{\alpha \in V^* : (\forall x \in E) \alpha(x) \geq 0\}$ . We note that  $E^*$  is a convex closed subcone of  $V^*$ .

(b) For a convex subset  $E \subseteq V$  we set

$$H(E) := \{x \in V : x + E = E\}, \quad \text{and} \quad \lim E := \{x \in V : x + E \subseteq E\}.$$

We call  $H(E)$  the *edge* and  $\lim E$  the *limit cone* of  $E$ . Note that  $H(E)$  is a vector space,  $H(E) = H(\overline{E})$  if  $E$  is open and that  $\lim E$  is a convex cone in  $V$ .

(c) A convex set  $E$  is called *pointed* if it contains no affine lines. Note that if  $E$  is open or closed then  $E$  is pointed if and only if its edge is zero.  $\square$

**Definition I.2.** Let  $\mathfrak{g}$  be a finite dimensional Lie algebra over  $\mathbb{R}$ .

(a) An element  $X \in \mathfrak{g}$  is called *elliptic* if  $\text{ad } X$  operates semisimply with purely imaginary spectrum. A convex cone  $W \subseteq \mathfrak{g}$  is said to be *elliptic* if  $W^0 \neq \emptyset$  and all  $X \in W^0$  are elliptic.

(b) For a subalgebra  $\mathfrak{a} \subseteq \mathfrak{g}$  we write  $\text{Inn}(\mathfrak{a}) := \langle e^{\text{ad } \mathfrak{a}} \rangle \subseteq \text{Aut}(\mathfrak{g})$  for the corresponding group of inner automorphisms. A subalgebra  $\mathfrak{a} \subseteq \mathfrak{g}$  is said to be *compactly embedded* if  $\text{Inn}(\mathfrak{a})$  is relatively compact in  $\text{Aut}(\mathfrak{g})$ .

(c) Let  $\mathfrak{t} \subseteq \mathfrak{g}$  be a compactly embedded Cartan subalgebra and recall that there exists a unique maximal compactly embedded subalgebra  $\mathfrak{k}$  containing  $\mathfrak{t}$  (cf. [HHL89, A.2.40]).

(d) Associated to the Cartan subalgebra  $\mathfrak{t}_{\mathbb{C}}$  in the complexification  $\mathfrak{g}_{\mathbb{C}}$  is a root decomposition as follows. For a linear functional  $\alpha \in \mathfrak{t}_{\mathbb{C}}^*$  we set

$$\mathfrak{g}_{\mathbb{C}}^{\alpha} := \{X \in \mathfrak{g}_{\mathbb{C}} : (\forall Y \in \mathfrak{t}_{\mathbb{C}}) [Y, X] = \alpha(Y)X\}$$

and write  $\Delta := \{\alpha \in \mathfrak{t}_{\mathbb{C}}^* \setminus \{0\} : \mathfrak{g}_{\mathbb{C}}^{\alpha} \neq \{0\}\}$  for the set of roots. Then  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha}$ ,  $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$  for all  $\alpha \in \Delta$  and  $\overline{\mathfrak{g}_{\mathbb{C}}^{\alpha}} = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ , where  $X \rightarrow \overline{X}$  denotes complex conjugation on  $\mathfrak{g}_{\mathbb{C}}$  with respect to  $\mathfrak{g}$ .

(e) A root  $\alpha$  is said to be *compact* if  $\mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{k}_{\mathbb{C}}$  and *non-compact* otherwise. We write  $\Delta_k$  for the set of compact roots and  $\Delta_n$  for the non-compact ones. If  $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$  is a  $\mathfrak{k}$ -invariant Levi decomposition, then we set

$$\Delta_r := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{r}_{\mathbb{C}}\} \quad \text{and} \quad \Delta_s := \{\alpha \in \Delta : \mathfrak{g}_{\mathbb{C}}^{\alpha} \subseteq \mathfrak{s}_{\mathbb{C}}\}$$

and recall that  $\Delta = \Delta_r \dot{\cup} \Delta_s$  (cf. [Ne99, Ch. V]).

(f) A positive system  $\Delta^+$  of roots is a subset of  $\Delta$  for which there exists a regular element  $X_0 \in i\mathfrak{t}^*$  with  $\Delta^+ := \{\alpha \in \Delta : \alpha(X_0) > 0\}$ . A positive system is said to be  *$\mathfrak{k}$ -adapted* if the set  $\Delta_n^+ := \Delta_n \cap \Delta^+$  is invariant under the *Weyl group*  $\mathcal{W}_{\mathfrak{t}} := N_{\text{Inn}(\mathfrak{t})}(\mathfrak{t})/Z_{\text{Inn}(\mathfrak{t})}(\mathfrak{t})$  acting on  $\mathfrak{t}$ . We recall from [Ne99, Ch. V] that there exists a  $\mathfrak{k}$ -adapted positive system if and only if  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{z}(\mathfrak{k})) = \mathfrak{k}$ . In this case we call  $\mathfrak{g}$  *quasihermitian*. In this case it is easy to see that  $\mathfrak{s}$  is quasihermitian too, and so all simple ideals of  $\mathfrak{s}$  are either compact or hermitian.

(g) We associate to a positive system  $\Delta^+$  the convex cones

$$C_{\min} := \text{cone}\{i[\overline{X_{\alpha}}, X_{\alpha}] : X_{\alpha} \in \mathfrak{g}_{\mathbb{C}}^{\alpha}, \alpha \in \Delta_n^+\},$$

and  $C_{\max} := (i\Delta_n^+)^* = \{X \in \mathfrak{t} : (\forall \alpha \in \Delta_n^+) i\alpha(X) \geq 0\}$ . Note that both  $C_{\min}$  and  $C_{\max}$  are closed convex cones in  $\mathfrak{t}$ .

(h) Write  $p_{\mathfrak{t}}: \mathfrak{g} \rightarrow \mathfrak{t}$  for the orthogonal projection along  $[\mathfrak{t}, \mathfrak{g}]$  and set  $\mathcal{O}_X := \text{Inn}(\mathfrak{g}) \cdot X$  for the adjoint orbit through  $X \in \mathfrak{g}$ . We define the *minimal* and *maximal cone* associated to  $\Delta^+$  by

$$\begin{aligned} W_{\min} &:= \{X \in \mathfrak{g} : p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\min}\} & \text{and} \\ W_{\max} &:= \{X \in \mathfrak{g} : p_{\mathfrak{t}}(\mathcal{O}_X) \subseteq C_{\max}\} \end{aligned}$$

and note that both cones are convex closed and  $\text{Inn}(\mathfrak{g})$ -invariant.  $\square$

From now on we assume that  $\mathfrak{g}$  contains a compactly embedded Cartan subalgebra  $\mathfrak{t} \subseteq \mathfrak{g}$  and that there exists an elliptic cone  $W \subseteq \mathfrak{g}$ . Then there exists a  $\mathfrak{k}$ -adapted positive system  $\Delta^+$  such that

$$C_{\min} \subseteq W \cap \mathfrak{t} \subseteq C_{\max}$$

holds and  $W_{\max}$  is an elliptic cone (cf. [Ne96b, Th. II.11]). Moreover, we have  $W_{\min} \cap \mathfrak{t} = C_{\min}$  and  $W_{\max} \cap \mathfrak{t} = C_{\max}$  (cf. [Ne97, Lemma I.1]).

**Definition I.3.** (a) Let  $W \subseteq \mathfrak{g}$  be a closed elliptic cone. Let  $\tilde{G}$ , resp.  $\tilde{G}_{\mathbb{C}}$ , be the simply connected Lie groups associated to  $\mathfrak{g}$ , resp.  $\mathfrak{g}_{\mathbb{C}}$ , and set  $G_1 := \langle \exp \mathfrak{g} \rangle \subseteq \tilde{G}_{\mathbb{C}}$ . Then Lawson's Theorem (cf. [HiNe93, Th. 7.34, 35]) says that the subset  $\Gamma_{G_1}(W) := G_1 \exp(iW)$  is a closed subsemigroup of  $G_{\mathbb{C}}$  and the polar map

$$G_1 \times W \rightarrow \Gamma_{G_1}(W), \quad (g, X) \mapsto g \exp(iX)$$

is a homeomorphism.

Now the universal covering semigroup  $\Gamma_{\tilde{G}}(W) := \tilde{\Gamma}_{G_1}(W)$  has a similar structure. We can lift the exponential function  $\exp: \mathfrak{g} + iW \rightarrow \Gamma_{G_1}(W)$  to an exponential mapping  $\text{Exp}: \mathfrak{g} + iW \rightarrow \Gamma_{\tilde{G}}(W)$  with  $\text{Exp}(0) = \mathbf{1}$  and thus obtain a polar map

$$\tilde{G} \times W \rightarrow \Gamma_{\tilde{G}}(W), \quad (g, X) \mapsto g \text{Exp}(iX)$$

which is a homeomorphism.

If  $G$  is a connected Lie group associated to  $\mathfrak{g}$ , then  $\pi_1(G)$  is a discrete central subgroup of  $\Gamma_{\tilde{G}}(W)$  and we obtain a covering homomorphism  $\Gamma_{\tilde{G}}(W) \rightarrow \Gamma_G(W) := \Gamma_{\tilde{G}}(W)/\pi_1(G)$  (cf. [HiNe93, Ch. 3]). It is easy to see that there is also a polar map  $G \times W \rightarrow \Gamma_G(W)$ ,  $(g, X) \mapsto g \text{Exp}(iX)$  which is a homeomorphism. The semigroups of the type  $\Gamma_G(W)$  are called *complex Ol'shanskii semigroups*.

The subset  $\Gamma_G(W^0) \subseteq \Gamma_G(W)$  is an open semigroup carrying a complex manifold structure such that semigroup multiplication is holomorphic. Moreover there is an involution on  $\Gamma_G(W)$  given by

$$*: \Gamma_G(W) \rightarrow \Gamma_G(W), \quad s = g \text{Exp}(iX) \mapsto s^* = \text{Exp}(iX)g^{-1}$$

which is antiholomorphic on  $\Gamma_G(W^0)$  (cf. [HiNe93, Th. 9.15] for a proof of all that). Thus  $\Gamma_G(W)$  is an involutive semigroup.

(b) A *bi-invariant domain*  $D \subseteq \Gamma_G(W_{\max}^0)$  is an open connected  $G \times G$  bi-invariant subset of  $\Gamma_G(W_{\max}^0)$ . Note that

$$D = G \text{Exp}(iD_h) = G \text{Exp}(i\mathcal{D})G,$$

where  $D_h \subseteq W_{\max}^0$  and  $\mathcal{D} = D_h \cap \mathfrak{t}$ . Recall that  $D$  is a Stein manifold if and only if  $D_h$  is convex (cf. [Ne98, Th. 6.1]). In this case  $D$  is called a *bi-invariant Stein domain*. We call  $D$  *pointed* if  $D_h$  is pointed in  $\mathfrak{g}$ . The

boundary of a left  $G$ -invariant subset  $E = G \operatorname{Exp}(iE_h) \subseteq \Gamma_G(W_{\max})$  is defined as  $\partial E := G \operatorname{Exp}(i\partial E_h)$ . Note that  $\partial D = \overline{D} \setminus D$  for every bi-invariant domain  $D$ , where the closure  $\overline{D}$  is taken in  $\Gamma_G(W_{\max})$ .  $\square$

**Lemma I.4.** *Let  $W \subseteq \mathfrak{g}$  be an invariant elliptic pointed convex cone and set  $C := W \cap \mathfrak{t}$ . Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence in  $W^0$  converging to  $X \in \partial W$ . Then there exists a subsequence  $(X_{n_k})_{k \in \mathbb{N}}$  and a sequence  $(Y_{n_k})_{k \in \mathbb{N}}$  in  $C^0$  with  $Y_{n_k} \in \operatorname{Inn}(\mathfrak{g}).X_{n_k}$  and  $Y_{n_k} \rightarrow Y \in \partial C$ .*

*Proof.* W.l.o.g. we may assume that  $W$  is closed. According to [HiNe93, Th. 7.27], we can reconstruct  $W^0$  from  $C^0$ , i.e., we have  $W^0 = \operatorname{Inn}(\mathfrak{g}).C^0$ . In particular, we find a sequence  $(Y_n)_{n \in \mathbb{N}}$  in  $C^0$  and a sequence  $(g_n)_{n \in \mathbb{N}}$  in  $\operatorname{Inn}(\mathfrak{g})$  such that  $g_n.X_n = Y_n$ . We claim that  $(Y_n)_{n \in \mathbb{N}}$  is bounded.

The Convexity Theorem for Adjoint Orbits (cf. [KrNe96, Th. VIII.9]) implies that

$$(1.1) \quad p_{\mathfrak{t}}(X_n) \in \operatorname{conv}(\mathcal{W}_{\mathfrak{t}}.Y_n) + C_{\min} \subseteq C$$

for all  $n \in \mathbb{N}$ .

As  $C$  is pointed, a sequence  $(Z_n)_{n \in \mathbb{N}}$  in  $C$  is unbounded if and only if  $\overline{\lim}_{n \rightarrow \infty} \alpha(Z_n) = \infty$  holds for one  $\alpha \in \operatorname{int} C^*$ . Thus if  $(Y_n)_{n \in \mathbb{N}}$  is unbounded, then (1.1) together with the invariance of  $C$  under  $\mathcal{W}_{\mathfrak{t}}$  implies that  $(p_{\mathfrak{t}}(X_n))_{n \in \mathbb{N}}$  is unbounded. But this contradicts the fact that  $(p_{\mathfrak{t}}(X_n))_{n \in \mathbb{N}}$  being a continuous image of a Cauchy sequence is bounded, proving the claim.

Let now  $(Y_{n_k})_{k \in \mathbb{N}}$  be a convergent subsequence of  $(Y_n)_{n \in \mathbb{N}}$  and  $Y = \lim_{k \rightarrow \infty} Y_{n_k}$  the corresponding limit in  $C$ . It remains to show that  $Y \in \partial C$ . To obtain a contradiction we assume that  $Y \in C^0$ .

We write  $\operatorname{Sl}(W)$  for the special automorphism group of the cone  $W$  and note that  $\operatorname{Inn}(\mathfrak{g}) \subseteq \operatorname{Sl}(W)$  (cf. [HiNe93, Prop. 7.3(v)]). Then [HiNe93, Prop. 1.11] implies that there exists a convergent subsequence of  $(g_{n_k})_{k \in \mathbb{N}}$  in  $\operatorname{Sl}(W)$  which we also denote by  $(g_{n_k})_{k \in \mathbb{N}}$ . Write  $g$  for the corresponding limit. Then

$$X = \lim_{k \rightarrow \infty} g_{n_k}^{-1}.Y_{n_k} = g^{-1}.Y.$$

Since  $\operatorname{Sl}(W).W^0 = W^0$  and  $Y \in W^0$ , this implies that  $X \in W^0$ ; a contradiction, concluding the proof of the lemma.  $\square$

**Definition I.5.** Let  $M$  be a complex manifold and  $\operatorname{Hol}(M)$  denote the space of holomorphic functions on  $M$ . We write  $\overline{M}$  for  $M$  equipped with the opposite complex structure.

(a) A function  $K \in \operatorname{Hol}(M \times \overline{M})$  is called a *holomorphic positive definite kernel* if for every sequence  $z_1, \dots, z_n$  in  $M$  the matrix  $(K(z_i, z_j))_{i,j}$  is positive semi-definite. We write  $\mathcal{P}(M^2)$  for the convex cone of all holomorphic positive definite kernels on  $M$ . Note that every  $K \in \mathcal{P}(M^2)$  satisfies the

inequality

$$(1.2) \quad (\forall z, w \in M) \quad |K(z, w)| \leq \sqrt{K(z, z)}\sqrt{K(w, w)}.$$

Recall that  $K \in \mathcal{P}(M^2)$  if and only if there exists a Hilbert space  $\mathcal{H} \subseteq \text{Hol}(M)$  with continuous point evaluations  $K_z: \mathcal{H} \rightarrow \mathbb{C}$ ,  $f \mapsto f(z)$  such that  $K(z, w) = \langle K_w, K_z \rangle$  holds for all  $(z, w) \in M \times M$  (cf. [Ne99, Ch. II]). In this case we also write  $\mathcal{H}_K$  instead of  $\mathcal{H}$  and refer to  $\mathcal{H}_K$  as the *reproducing kernel Hilbert space* corresponding to  $K$ .

(b) An *involutive semigroup* is a semigroup  $S$  together with an involutive antiautomorphism  $^*: S \rightarrow S$ , i.e.,  $(s^*)^* = s$  and  $(st)^* = t^*s^*$  holds for all  $s, t \in S$ .

A mapping  $\alpha: S \rightarrow \mathbb{R}^+$  is called an *absolute value* if  $\alpha(s^*) = \alpha(s)$  and  $\alpha(st) \leq \alpha(s)\alpha(t)$  hold for all  $s, t \in S$ . We denote by  $\mathcal{A}(S)$  the collection of all absolute values on  $S$ .

(c) Let  $S$  be an involutive semigroup acting on  $M$  from the left by holomorphic mappings. A positive definite kernel  $K$  is said to be  *$S$ -invariant* if  $K(s.z, w) = K(z, s^*.w)$  holds for all  $s \in S$ ,  $z, w \in M$ . We write  $\mathcal{P}_S(M^2)$  for the subcone of  $\mathcal{P}(M^2)$  of all  $S$ -invariant elements.

(d) An  $S$ -invariant positive definite kernel  $K \in \mathcal{P}_S(M^2)$  is called  *$\alpha$ -bounded* for some  $\alpha \in \mathcal{A}(S)$  if

$$K(s.z, s.z) \leq \alpha(s)K(z, z)$$

holds for all  $z \in M$ ,  $s \in S$ . The set of all  $\alpha$ -bounded positive definite kernels is denoted by  $\mathcal{P}_S(M^2, \alpha)$ . Note that each  $K \in \mathcal{P}_S(M^2, \alpha)$  gives rise to an *involutive representation* of  $S$  given by

$$\pi_K: S \rightarrow B(\mathcal{H}_K), \quad (\pi_K(s).f)(z) = f(s^*.z),$$

i.e.,  $(\pi_K, \mathcal{H}_K)$  is a representation of  $S$  satisfying  $\pi_K(s^*) = \pi_K(s)^*$  for all  $s \in S$  (cf. [Ne99, Ch. II]).  $\square$

We equip  $G \times G$  with the involution  $(g_1, g_2)^* = (g_1^{-1}, g_2^{-1})$  for  $g_1, g_2 \in G$ . Note that every  $K \in \mathcal{P}_{G \times G}$  is trivially  $\alpha$ -bounded with  $\alpha = \mathbf{1}$ , and that

$$\pi_K: G \times G \rightarrow U(\mathcal{H}_K), \quad (\pi_K(g_1, g_2).f)(z) = f(g_1^{-1}zg_2)$$

is a unitary representation of  $G \times G$  (cf. [Kr97, Lemma III.6]).

**Lemma I.6.** *Let  $V$  be a finite dimensional real vector space,  $V^\sharp := V \oplus \mathbb{R}$  and  $E \subseteq V$  a convex subset. Set  $E^\sharp := \mathbb{R}^+(E \times \{1\})$ . Then  $E^\sharp$  is a convex subcone of  $V^\sharp$  and the following assertions hold:*

- (i)  $E$  is closed in  $E^\sharp$ .
- (ii)  $E$  is pointed if and only if  $E^\sharp$  is pointed.
- (iii) If  $E$  is open or closed, then  $\partial E^\sharp = ]0, \infty[.(\partial E \times \{1\}) \cup (\lim E \times \{0\})$ .

*Proof.* (i) This is clear.

(ii) If  $E^\sharp$  is not pointed, then there exists a non-zero element  $y = (x, r) \in V^\sharp$  such that  $\mathbb{R}y \subseteq E^\sharp$ . In view of  $E^\sharp = \mathbb{R}^+(E \times \{1\})$ , we must have  $r = 0$ . Now  $\mathbb{R}y + E^\sharp = E^\sharp$  implies that  $\mathbb{R}x + E = E$ , i.e.,  $E$  is not pointed.

Conversely, if  $E$  is not pointed, then there exists a non-zero element  $x \in V$  such that  $\mathbb{R}x \subseteq H(E)$ . Then  $\mathbb{R}(x, 0) \subseteq H(E^\sharp)$ , i.e.,  $C$  is not pointed.

(iii) Note that  $]0, \infty[(E^0 \times \{1\})$  is open and that  $]0, \infty[(\overline{E} \times \{1\})$  is closed in  $V^\sharp \setminus (V \times \{0\})$ . Hence  $\partial E^\sharp \cap V^\sharp \setminus (V \times \{0\}) = ]0, \infty[(\partial E \times \{1\})$ . By the definition of  $E^\sharp$  we have  $(x, 0) \in \partial E^\sharp$  if and only if there exists a sequence of positive real numbers  $(\lambda_n)_{n \in \mathbb{N}}$  and a sequence  $(x_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} \lambda_n = 0$  and  $\lim_{n \rightarrow \infty} \lambda_n x_n = x$ . In view of [Ne99, Prop. III.1.5(iii)], this means that  $x \in \lim E$ , concluding the proof of (iii).  $\square$

**Lemma I.7.** *Let  $V$  be a finite dimensional real vector space and  $E \subseteq V$  a convex set. Further let  $V_1 := V/H(E)$ , denote  $q: V \rightarrow V_1$  the corresponding quotient homomorphism and set  $E_1 := q(E)$ . Then we have  $q(\partial E) = \partial E_1$ .*

*Proof.* As  $E + H(E) = E$  it follows that  $E^0 + H(E) = E^0$  and  $\overline{E} + H(E) = H(E)$ . Thus  $q(\overline{E}) = \overline{E}_1$ , and  $q(E^0) = E_1^0$  since  $q$  is an open mapping. This proves the lemma.  $\square$

**Proposition I.8.** *Let  $\mathfrak{g}^\sharp = \mathfrak{g} \oplus \mathbb{R}$ ,  $G^\sharp = G \times \mathbb{R}$ , and*

$$D^\sharp := \Gamma_{G^\sharp} \left( D_h^\sharp \right) \subseteq \Gamma_{G^\sharp} \left( W_{\max}^0 \oplus \mathbb{R} \right) \cong \Gamma_G \left( W_{\max}^0 \right) \oplus \mathbb{C}.$$

(i) *The map  $j: D \rightarrow D^\sharp$ ,  $s \mapsto (s, i)$  is a  $G \times G$ -equivariant holomorphic closed embedding inducing a map*

$$\mathcal{P}_{G^\sharp \times G^\sharp} \left( D^{\sharp 2} \right) \rightarrow \mathcal{P}_{G \times G} (D^2), \quad K^\sharp \mapsto K := K^\sharp \circ j.$$

(ii) *Let  $K \in \mathcal{P}_{G \times G} (D^2)$ . Then the following statements are equivalent:*

(a)  $\lim_{z \rightarrow \partial D} K(z, z) = \infty$ .

(b)  $\lim_{X \rightarrow \partial D_h} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ . *Moreover, if  $K = K^\sharp \circ j$  with some  $K^\sharp \in \mathcal{P}_{G^\sharp \times G^\sharp} (D^{\sharp 2})$ , then (a)-(b) are implied by*

(c)  $\lim_{X \rightarrow \partial D_h^\sharp} K^\sharp(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ .

(iii) *Let  $\mathfrak{a} := H(D_h)$  be the edge of  $D_h$ ,  $\mathfrak{g}_1 := \mathfrak{g}/\mathfrak{a}$  and  $q: \mathfrak{g} \rightarrow \mathfrak{g}_1$  the corresponding quotient morphism. Further let  $A := \langle \exp(\mathfrak{a}) \rangle$ ,  $G_1 := G/A$  and  $D_1 := G_1 \text{Exp}(iD_{h,1})$ . Then the quotient morphism  $\tilde{q}: D \rightarrow D_1$  induced by  $q$  gives rise to an injection*

$$\tilde{q}^*: \mathcal{P}_{G_1 \times G_1} (D_1^2) \rightarrow \mathcal{P}_{G \times G} (D^2), \quad K_1 \mapsto K := K_1 \circ \tilde{q}$$

*and the following statements are equivalent:*

(a)  $\lim_{X \rightarrow \partial D_h} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ .

(b)  $\lim_{X \rightarrow \partial D_{h,1}} K_1(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ .

(c)  $\lim_{X \rightarrow \partial D_1} K_1(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ .

*Proof.* (i) Obviously,  $j$  is holomorphic and  $G \times G$ -equivariant. Further the closedness of  $j$  follows from Lemma I.6(i) together with the Polar Decomposition of a complex Ol'shanskiĭ semigroup. The second assertion is clear.

(ii) (a) $\Rightarrow$ (b) is obvious.

(b) $\Rightarrow$ (a): This follows from the biinvariance of  $K$ .

The remaining statement follows from Lemma I.6(iii).

(iii) (a)  $\iff$  (b): In view of  $q(\partial D_h) = \partial D_{h,1}$  (cf. Lemma I.7), this follows from the equivalence of (a) and (b) in (ii).

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (b) follows from the  $\text{Inn}(\mathfrak{g})$ -invariance of the map

$$D_{h,1} \rightarrow \mathbb{R}^+, \quad X \mapsto K_1(\text{Exp}(iX), \text{Exp}(iX))$$

together with Lemma I.4. □

## II. A kernel tending to infinity at the boundary.

In this section we construct a kernel  $K \in \mathcal{P}_{G \times G}(D^2)$  tending to infinity at the boundary. In view of Proposition I.8, this reduces to the case where  $D = \Gamma_G(W)$  is a pointed complex Ol'shanskiĭ semigroup. But Proposition I.8(iii) tells us even more: We only have to check that  $\lim_{X \rightarrow \partial C} K(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ . To start out we need some notation concerning highest weight representations and their associated characters.

**Definition II.1.** Let  $\Delta^+$  be a positive system.

(a) For a  $\mathfrak{g}_{\mathbb{C}}$ -module  $V$  and  $\beta \in (\mathfrak{t}_{\mathbb{C}})^*$  we write  $V^\beta := \{v \in V : (\forall X \in \mathfrak{t}_{\mathbb{C}}) X.v = \beta(X)v\}$  for the *weight space of weight*  $\beta$  and  $\mathcal{P}_V = \{\beta : V^\beta \neq \{0\}\}$  for the set of weights of  $V$ .

(b) Let  $V$  be a  $\mathfrak{g}_{\mathbb{C}}$ -module and  $v \in V^\lambda$  a  $\mathfrak{t}_{\mathbb{C}}$ -weight vector. We say that  $v$  is a *primitive element* of  $V$  (with respect to  $\Delta^+$ ) if  $\mathfrak{g}_{\mathbb{C}}^\alpha.v = \{0\}$  holds for all  $\alpha \in \Delta^+$ .

(c) A  $\mathfrak{g}_{\mathbb{C}}$ -module  $V$  is called a *highest weight module* with highest weight  $\lambda$  (with respect to  $\Delta^+$ ) if it is generated by a primitive element of weight  $\lambda$ .

(d) Let  $\lambda \in i\mathfrak{t}^*$  be dominant integral w.r.t.  $\Delta_k^+$  and  $F(\lambda)$  the corresponding highest weight module for  $\mathfrak{k}_{\mathbb{C}}$ . Assume that  $\Delta^+$  is  $\mathfrak{k}$ -adapted and set  $\mathfrak{p}^\pm = \bigoplus_{\alpha \in \Delta_n^\pm} \mathfrak{g}_{\mathbb{C}}^\alpha$ . We define the *generalized Verma module* by

$$N(\lambda) := \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^+)} F(\lambda).$$

Note that  $N(\lambda)$  is a highest weight module for  $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$  with highest weight  $\lambda$ . We denote by  $L(\lambda)$  the unique irreducible quotient of  $N(\lambda)$ .

(e) Let  $G$  be a connected Lie group with Lie algebra  $\mathfrak{g}$ . We write  $K$  for the analytic subgroup of  $G$  corresponding to  $\mathfrak{k}$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . A vector  $v \in \mathcal{H}$  is called *K-finite* if it is contained in

a finite dimensional  $K$ -invariant subspace. We write  $\mathcal{H}^{K,\omega}$  for the space of analytic  $K$ -finite vectors.

(f) An irreducible unitary representation  $(\pi, \mathcal{H})$  of  $G$  is called a *highest weight representation* w.r.t.  $\Delta^+$  with highest weight  $\lambda \in \mathfrak{it}^*$  if  $\mathcal{H}^{K,\omega}$  is a highest weight module for  $\mathfrak{g}_{\mathbb{C}}$  w.r.t.  $\Delta^+$  and highest weight  $\lambda$ . We say that the irreducible highest weight module  $L(\lambda)$  is *unitarizable* if there exists a unitary highest weight representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  of  $\tilde{G}$  with  $\mathcal{H}_\lambda^{K,\omega} \cong L(\lambda)$  as  $\mathfrak{g}_{\mathbb{C}}$ -modules. We write  $HW(G, \Delta^+) \subset \mathfrak{it}^*$  for the set of highest weights corresponding to unitary highest weight representations of  $G$  w.r.t.  $\Delta^+$  and write  $HW(\Delta^+) := HW(\tilde{G}, \Delta^+)$  for the set of all highest weights w.r.t.  $\Delta^+$  which correspond to a unitarizable  $L(\lambda)$ .

(g) Let  $\lambda \in HW(\Delta^+)$ . We call  $\lambda$  *singular* if the natural map  $N(\lambda) \rightarrow L(\lambda)$  has a non-trivial kernel and *non-singular* otherwise.  $\square$

For each unitary representation  $(\pi, \mathcal{H})$  of  $G$  we write  $(\pi^*, \mathcal{H}^*)$  for the corresponding dual representation. Let  $B_2(\mathcal{H})$  be the space of Hilbert Schmidt operators on  $\mathcal{H}$ . We define a representation of  $G \times G$  on  $B_2(\mathcal{H})$  by

$$\pi^c: G \times G \rightarrow U(B_2(\mathcal{H})), \quad \pi^c(g_1, g_2).A := \pi(g_2)A\pi(g_1)^*.$$

Note that there is a canonical isomorphism between  $(\pi^* \otimes \pi, \mathcal{H}^* \widehat{\otimes} \mathcal{H})$  and  $(\pi^c, B_2(\mathcal{H}))$ .

Now we fix a positive system  $\Delta^+$  associated to  $W_{\max}$ .

Recall from [HiNe96, Th. 3.6, Th. B] that each highest weight representation  $(\pi_\lambda, \mathcal{H}_\lambda)$  of  $G$  extends to an holomorphic representation of  $\Gamma_G(W_{\max})$  denoted by the same symbol. Moreover all operators  $\pi_\lambda(s), s \in \Gamma_G(W_{\max}^0)$ , are of trace class (cf. [Ne94, Th. III.8]), so that  $\Theta_\lambda(s) := \text{tr } \pi_\lambda(s)$  makes sense for all  $s \in \Gamma_G(W_{\max}^0)$ . We call  $\Theta_\lambda$  the *character* of  $(\pi_\lambda, \mathcal{H}_\lambda)$  and note that  $\Theta_\lambda$  is holomorphic on  $\Gamma_G(W_{\max}^0)$  (cf. [Ne94, Th. IV.11]).

Associated to a  $\mathfrak{k}$ -adapted positive system, we define the function

$$\phi: iC_{\max}^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \frac{1}{\prod_{\alpha \in \Delta_n^+} (1 - e^{-\alpha(X)})^{m_\alpha}},$$

where  $m_\alpha := \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}}^\alpha$  for all  $\alpha \in \Delta$ .

**Lemma II.2.** *Let  $\lambda \in HW(G, \Delta^+)$  be non-singular and  $(\pi_\lambda, \mathcal{H}_\lambda)$  an associated highest weight representation of  $G$ . If  $\Theta_\lambda^K$  denotes the character of  $F(\lambda)$ , then*

$$\Theta_\lambda(\text{Exp } X) = \phi(X)\Theta_\lambda^K(\text{Exp}(X))$$

for all  $X \in iC_{\max}^0$ .

*Proof.* [Kr97, Lemma IV.8(i)].  $\square$

**Proposition II.3.** *Let  $K$  be a connected Lie group with compact Lie algebra  $\mathfrak{k}$ . Let  $\mathfrak{t}$  be a Cartan subalgebra of  $\mathfrak{k}$ ,  $\Delta_k^+$  be a positive system of roots and  $\lambda \in \mathfrak{it}^*$  be a dominant analytically integral element. Further let  $\Theta_\lambda^K$  denote*

the character of the holomorphic representation  $(\pi_\lambda^K, F(\lambda))$  of  $K_{\mathbb{C}}$  and set  $d_\lambda := \dim F(\lambda)$ .

- (i)  $(\forall X \in \mathfrak{it}) e^{\lambda(X)} \leq \Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) \leq d_\lambda \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$ .
- (ii) If  $\|\cdot\|$  denotes a norm on  $\mathfrak{it}^*$ , then there exists a constant  $c > 0$  and an element  $n \in \mathbb{N}$  such that

$$d_\lambda \leq c\|\lambda\|^n + c$$

holds for all integral elements  $\lambda$ .

*Proof.* (i) Let  $\{v_j : 1 \leq j \leq d_\lambda\}$  be an orthonormal basis of weight vectors of  $F(\lambda)$ . For each  $j$  let  $\alpha_j$  be the weight corresponding to  $v_j$ . Then we have for all  $X \in \mathfrak{t}_{\mathbb{C}}$  that

$$(2.1) \quad \Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) = \sum_{j=1}^{d_\lambda} \langle \pi_\lambda^K(\exp_{K_{\mathbb{C}}}(X)).v_j, v_j \rangle = \sum_{j=1}^{d_\lambda} e^{\alpha_j(X)}.$$

Since  $\lambda = \alpha_j$  for some  $1 \leq j \leq d_\lambda$ , this proves the first inequality.

To prove the second inequality, we first observe that both  $\Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X))$  and  $d_\lambda \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$  considered as functions of  $X \in \mathfrak{it}^*$  are invariant under the Weyl group  $\mathcal{W}_k$ . Thus we may assume that  $X \in -(\Delta_k^+)^* := \{Y \in \mathfrak{it}^* : (\forall \alpha \in \Delta_k^+) \alpha(X) \leq 0\}$ . Since  $\lambda(X) = \sup_{1 \leq j \leq d_\lambda} \alpha_j(X)$  whenever  $X \in -(\Delta_k^+)^*$ , (2.1) implies that  $\Theta_\lambda^K(\exp_{K_{\mathbb{C}}}(X)) \leq d_\lambda e^{\lambda(X)}$ , concluding the proof of (i).

- (ii) This is a direct consequence of the Weyl Dimension Formula.  $\square$

**Corollary II.4.** *If  $\lambda \in HW(G, \Delta^+)$  is non-singular, then there exist a constant  $c > 0$  and  $n \in \mathbb{N}$  such that*

$$e^{\lambda(X)} \leq \Theta_\lambda(\text{Exp}(X)) \leq (c\|\lambda\|^n + c)\phi(X) \sup_{w \in \mathcal{W}_k} e^{\lambda(w.X)}$$

holds for all  $X \in iC_{\max}^0$ .

*Proof.* As  $1 = \inf_{X \in iC_{\max}^0} \phi(X)$ , the corollary follows from Lemma II.2 and Proposition II.3.  $\square$

**Lemma II.5.** *Let  $V$  be a finite dimensional real vector space,  $C \subseteq V$  a convex cone with non-empty interior,  $\Gamma \subseteq V$  a lattice and  $Q \subseteq V$  a compact subset. Then*

$$\Gamma(C, Q) := \{\gamma \in \Gamma : \gamma \in C, \gamma + Q \subseteq C\}$$

is an additive subsemigroup of  $\Gamma$ ,  $\overline{\mathbb{R}^+\Gamma(C, Q)}$  is a closed convex cone and the following equalities hold:

- (i)  $\overline{\mathbb{R}^+\Gamma(C, Q)} = \overline{C}$ .
- (ii)  $\Gamma(C, Q)^* = C^*$ .

*Proof.* First we show that  $\Gamma(C, Q)$  is an additive semigroup. Let  $\gamma_1, \gamma_2 \in \Gamma(C, Q)$ . Then  $\gamma_1 + \gamma_2 \in C$  since  $C$  is a convex cone and further we have for all  $x \in Q$

$$\gamma_1 + \gamma_2 + x = \gamma_1 + \underbrace{(\gamma_2 + x)}_{\in C} \in C + C = C,$$

proving that  $\Gamma(C, Q)$  is an additive semigroup.

It follows in particular that  $\mathbb{Q}^+\Gamma(C, Q)$  is an additive semigroup and hence the same holds for  $\overline{\mathbb{R}^+\Gamma(C, Q)} = \overline{\mathbb{Q}^+\Gamma(C, Q)}$ . This proves that  $\overline{\mathbb{R}^+\Gamma(C, Q)}$  is a closed convex cone.

(i) Since  $\Gamma(C, Q) \subseteq C$  we obtain in particular that  $\overline{\mathbb{R}^+\Gamma(C, Q)} \subseteq \overline{C}$ .

To prove the converse inclusion, we assume that  $\overline{\mathbb{R}^+\Gamma(C, Q)} \neq \overline{C}$ . Then we find an open ball  $B \subseteq V$  such that  $B \subseteq C^0 \setminus \overline{\mathbb{R}^+\Gamma(C, Q)}$ . Since  $C^0 \setminus \overline{\mathbb{R}^+\Gamma(C, Q)}$  is an open cone, this implies in particular that

$$(2.2) \quad (\forall \lambda > 0) \quad \lambda B \cap \Gamma(C, Q) = \emptyset.$$

If  $\lambda$  is sufficiently large, then we have  $\lambda B \subseteq C - x$  for all  $x \in Q$ , because  $B + \frac{1}{\lambda}x$  is contained in  $C$  for sufficiently large  $\lambda$  and  $Q$  is compact. Let  $\lambda_0 > 0$  such that  $\lambda B \subseteq C - x$  for all  $x \in Q$  and  $\lambda > \lambda_0$ .

In view of this and  $\Gamma(C, Q) = \Gamma \cap C \cap \bigcap_{x \in Q} (C - x)$ , (2.2) implies in particular that

$$(\forall \lambda > \lambda_0) \quad \Gamma \cap C \cap \lambda B = \Gamma(C, Q) \cap \lambda B = \emptyset;$$

a contradiction, concluding the proof of (i).

(ii) This follows from (i) and  $\overline{\mathbb{R}^+\Gamma(C, Q)}^* = \Gamma(C, Q)^*$ .  $\square$

Let  $\mathfrak{t}_e := \{X \in \mathfrak{t} : \exp(X) = \mathbf{1}\}$  and note that  $\mathfrak{t}_e$  is a lattice in  $\text{span}\{\mathfrak{t}_e\}$ . Hence we find a lattice  $\Gamma \in i\mathfrak{t}^*$  which is contained in the set  $\{\alpha \in i\mathfrak{t}^* : (\forall X \in \mathfrak{t}_e) \alpha(X) \in 2\pi i\mathbb{Z}\}$ . From now on we fix a lattice  $\Gamma \subseteq i\mathfrak{t}^*$  having this property.

**Lemma II.6.** *Let  $W \subseteq \mathfrak{g}$  be a pointed closed  $\text{Inn}(\mathfrak{g})$ -invariant cone with non-empty interior,  $\Delta^+$  a positive system satisfying  $C_{\min} \subseteq C = W \cap \mathfrak{t} \subseteq C_{\max}$  and  $\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} m_\alpha \alpha$ . Then*

$$\Gamma(i \text{int } C^*, 2\rho) := \{\lambda \in \Gamma : \lambda \in i \text{int } C^*, \lambda + 2\rho \in i \text{int } C^*\}$$

*consists of non-singular  $G$ -analytically integral elements.*

*Proof.* First note that  $\Gamma$  consists of  $G$ -analytically integral elements, hence the same holds for the subset  $\Gamma(i \text{int } C^*, 2\rho)$ . We claim that  $\Gamma(i \text{int } C^*, 2\rho) \subseteq \Gamma(i \text{int } C^*, \rho)$ . In fact, Lemma II.5 implies that  $\Gamma(i \text{int } C^*, 2\rho)$  is an additive semigroup, so that  $\lambda \in \Gamma(i \text{int } C^*, 2\rho)$  implies that  $2\lambda \in \Gamma(i \text{int } C^*, 2\rho)$  which means that  $2\lambda + 2\rho \in i \text{int } C^*$  or equivalently  $\lambda + \rho \in \Gamma(i \text{int } C^*, \rho)$ . This proves the claim.

Further,  $C_{\min} \subseteq C$  implies that  $i \text{int } C^* \subseteq i \text{int } C_{\min}^*$  and we therefore get

$$\Gamma(i \text{int } C^*, 2\rho) \subseteq \{\lambda \in \Gamma : \lambda + \rho \in i \text{int } C_{\min}^*\}.$$

In view of [Ne99, Ch. IX], this implies that all elements of  $\Gamma(i \operatorname{int} C^*, 2\rho)$  are non-singular.  $\square$

**Lemma II.7.** *Let  $V$  be a finite dimensional real vector space,  $C \subseteq V$  a convex pointed cone with non-empty interior,  $\Gamma \subseteq V^*$  a lattice and  $Q \subseteq V^*$  a compact subset. We fix a norm  $\|\cdot\|$  on  $V^*$  with  $\|\gamma\| \geq 1$  for all  $\gamma \in \Gamma \setminus \{0\}$  and consider for each  $N \in \mathbb{N}_0$  the mapping*

$$F^N: V \rightarrow \mathbb{R}^+ \cup \{\infty\}, \quad x \mapsto \sum_{\gamma \in \Gamma(\operatorname{int} C^*, Q)} \|\gamma\|^N e^{-\gamma(x)}.$$

- (i) *For all  $N \in \mathbb{N}_0$  the series defining  $F^N$  converges compactly on  $C^0$ . In particular,  $F^N|_{C^0}$  is continuous.*
- (ii) *For all  $N \geq 1$  we have  $\lim_{\substack{x \rightarrow \partial C \\ x \in C^0}} F^N(x) = \infty$ .*

*Proof.* (i) Since  $C^*$  is pointed we find for every  $x \in C^0$  a constant  $C_x > 0$  such that  $\|\alpha\| \leq C_x \alpha(x)$  holds for every  $\alpha \in C^*$ . Thus we find for every compact subset  $K \subseteq C^0$  a constant  $C_K > 0$  such that

$$(\forall x \in K)(\forall \alpha \in C^*) \|\alpha\| \leq C_K \alpha(x).$$

This in turn implies that

$$\sup_{x \in K} \sum_{\gamma \in \Gamma(\operatorname{int} C^*, Q)} \|\gamma\|^N e^{-\gamma(x)} \leq \sum_{\gamma \in \Gamma(\operatorname{int} C^*, Q)} \|\gamma\|^N e^{-\frac{1}{C_K} \|\gamma\|} < \infty,$$

proving (i).

(ii) As  $\|\gamma\| \geq 1$  for all  $\gamma \in \Gamma \setminus \{0\}$ , we have  $F^1 \leq F^N$  for all  $N \in \mathbb{N}$  and hence we only have to prove the assertion for  $N = 1$ . Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $C^0$  converging to  $x \in \partial C$ . Then there exists an element  $0 \neq \alpha \in \partial C^*$  such that  $\alpha(x) = 0$ . Choose  $\alpha_n \in \mathbb{R}^+ \alpha$  with  $\alpha_n(x_n) = 1$ . We claim that  $\lim_{n \rightarrow \infty} \|\alpha_n\| = \infty$ . Indeed, otherwise we find a number  $L > 0$  and a subsequence  $(\alpha_{n_k})_{k \in \mathbb{N}}$  such that  $\alpha_{n_k} \in [0, L]\alpha$ . But then

$$1 = \alpha_{n_k}(x_{n_k}) \leq L \alpha(x_{n_k}) \rightarrow 0$$

yields a contradiction and proves the claim.

Next we claim that there exists a constant  $c > 0$  and elements  $\gamma_n \in \Gamma(\operatorname{int} C^*, Q)$  such that  $\|\gamma_n - \alpha_n\| < c$  for all  $n \in \mathbb{N}$ . Let  $r > 0$  such that  $B(\beta, r) := \{\mu \in V^* : \|\beta - \mu\| \leq r\}$  intersects  $\Gamma$  for all  $\beta \in V^*$ . Now choose  $\mu \in \operatorname{int} C^*$  such that  $B(\mu + t\alpha, r) + Q \subseteq \operatorname{int} C^*$  holds for all  $t \geq 0$ . Then we find elements  $\gamma_n \in \Gamma(\operatorname{int} C^*, Q) \cap B(\mu + \alpha_n, r)$ . These elements  $\gamma_n$  satisfy  $\|\gamma_n - \alpha_n\| \leq c$  with  $c = r + \|\mu\|$ , proving our second claim.

Now we get

$$\begin{aligned} F_1(x_n) &= \sum_{\gamma \in \Gamma(\text{int } C^*, Q)} \|\gamma\| e^{-\gamma(x_n)} \geq \|\gamma_n\| e^{-(\gamma_n(x_n) - \alpha_n(x_n))} e^{-\alpha_n(x_n)} \\ &\geq \frac{\|\gamma_n\|}{e} e^{-\|\gamma_n - \alpha_n\| \cdot \|x_n\|} \geq \frac{1}{e} e^{-c\|x_n\|} (\|\alpha_n\| - c), \end{aligned}$$

and so

$$\lim_{n \rightarrow \infty} F_1(x_n) \geq \lim_{n \rightarrow \infty} \frac{1}{e} e^{-c\|x_n\|} (\|\alpha_n\| - c) = \infty.$$

This proves (ii).  $\square$

**Theorem II.8.** *Let  $S = \Gamma_G(W)$  be a pointed complex Ol'shanskiĭ semi-group. Then for all  $N \in \mathbb{N}$  the prescription*

$$K^N: S^0 \times S^0 \rightarrow \mathbb{C}, \quad (z, w) \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N \Theta_\lambda(zw^*)$$

defines an element of  $\mathcal{P}_{G \times G}(S^{0^2})$  satisfying

$$\lim_{z \mapsto \partial S^0} K^N(z, z) = \infty.$$

*Proof.* First we show that  $K^N \in \mathcal{P}_{G \times G}(S^{0^2})$  for all  $N \in \mathbb{N}$ . Since all kernels

$$K^\lambda: S^0 \times S^0 \rightarrow \mathbb{C}, \quad (z, w) \mapsto \Theta_\lambda(zw^*)$$

belong to  $\mathcal{P}_{G \times G}(S^{0^2})$  (cf. [Ne94, Th. IV.11]), we only have to show that the series defining  $K^N$  converges uniformly on compact subsets. In view of (1.2), a series of holomorphic positive definite kernels on a complex manifold converges compactly if and only if it converges uniformly on compact subsets on the diagonal. Therefore the bi-invariance of the kernels  $K^\lambda$  together with the Polar Decomposition of  $S^0$  imply that it suffices to prove the compact convergence of the series defining the function

$$\Theta^N: iW^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N \Theta_\lambda(\text{Exp}(X)).$$

If  $C(W^0)^G$  denotes the  $\text{Ad}(G)$ -invariant continuous functions on  $W^0$  endowed with the topology of uniform convergence on compact subsets, then [Ne96b, Prop. III.6] entails that the restriction mapping  $C(W^0)^G \rightarrow C(C^0)^{\mathcal{W}_t}$  is an isomorphism of Fréchet spaces. Thus we have reduced the problem to showing the compact convergence of the series defining  $\Theta^N|_{C^0}$ . In view of Corollary II.4, we therefore have to prove that the series defined by

$$F^N|_{C^0}: iC^0 \rightarrow \mathbb{R}^+, \quad X \mapsto \sum_{\lambda \in \Gamma(i \text{int } C^*, 2\rho)} \|\lambda\|^N e^{\lambda(X)}$$

converges compactly. But this is exactly the contents of Lemma II.7(i).

In order to prove the second assertion, Proposition I.8(iii) implies that we only have to check that  $\lim_{\substack{X \rightarrow \partial G \\ X \in C^0}} K^N(\text{Exp}(iX), \text{Exp}(iX)) = \infty$ . According to Corollary II.4, this follows from  $\lim_{\substack{X \rightarrow \partial G \\ X \in C^0}} F^N(X) = \infty$  which is the contents of Lemma II.7(ii). This proves the theorem.  $\square$

**Corollary II.9.** *Let  $D \subseteq \Gamma_G(W_{\max}^0)$  be a bi-invariant Stein domain. Then there exists a kernel  $K \in \mathcal{P}_{G \times G}(D^2)$  such that*

$$\lim_{z \rightarrow \partial D} K(z, z) = \infty.$$

*Proof.* In view of Proposition I.8(ii), we may assume that  $D_h = D_h^\sharp$  is a cone, and hence Proposition I.8(iii) implies that we even may assume that  $D_h = D_{h,1}^\sharp$  is a pointed cone. Now the corollary follows from Theorem II.8.  $\square$

**Remark II.10.** One can modify  $\Gamma$  a little bit without affecting the contents Theorem II.8 as follows. If one takes  $\Gamma' = \Gamma \cup F$ , where  $F \subseteq HW(G, \Delta^+)$  is a finite set of highest weights, then Theorem II.8 remains true with  $\Gamma$  replaced by  $\Gamma'$ .

### III. The equivariant embedding theorem.

In this section we apply the results of Section II to construct a kernel  $K \in \mathcal{P}_{G \times G}(D^2)$  such that the map

$$e_K: D \mapsto \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

defines a  $G \times G$ -equivariant holomorphic embedding with closed image.

**Proposition III.1.** *Let  $M$  be a complex manifold,  $S$  an involutive semi-group acting on  $M$  by holomorphic maps and  $K \in \mathcal{P}_S(M^2)$ . Then*

$$e_K: M \rightarrow \mathcal{H}_K^*, \quad z \mapsto K_z$$

*is an  $S$ -equivariant holomorphic map.*

*Proof.* From the  $S$ -invariance of  $K$  it follows that

$$e_K(s.z) = K_{s.z} = \pi_K(s).K_z = \pi_K(s).e_K(z)$$

for all  $s \in S$  and  $z \in M$ , proving the  $S$ -equivariance of the map  $e_K$ . It remains to show that  $e_K$  is holomorphic. As  $\mathcal{H}_K$  is a Hilbert space,  $e_K$  is holomorphic if and only if the following conditions are satisfied:

- (1)  $e_K$  is locally bounded.
- (2) There exists a total subset  $T \subseteq \mathcal{H}_K$  such that the mappings  $M \rightarrow \mathbb{C}$ ,  $z \mapsto K_z(f) = \langle f, K_z \rangle$  are holomorphic for all  $f \in T$ .

If  $Q \subseteq D$  is a compact subset, then we have

$$\sup_{z \in Q} \|K_z\|^2 = \sup_{z \in Q} K(z, z) < \infty,$$

proving (1). Finally  $T = \{K_w : w \in M\}$  is a total subset in  $\mathcal{H}_K$  and  $z \mapsto K_z(K_w) = \langle K_w, K_z \rangle = K(z, w)$  is holomorphic because  $K \in \text{Hol}(M \times \overline{M})$ . This proves (2) and concludes the proof of the lemma.  $\square$

**Corollary III.2.** *If  $K \in \mathcal{P}_{G \times G}(D^2)$  is non-zero, then  $K_z \neq 0$  for all  $z \in D$  and the mapping*

$$e_K : D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

*is  $G \times G$ -equivariant and holomorphic.*

*Proof.* It follows from [Ne97, Lemma III.6] that  $K_z \neq 0$  for all  $z \in D$  and thus the map  $e_K$  is well defined. Now the assertions follow from Proposition III.1.  $\square$

**Definition III.3.** A connected Lie group is called a (CA)-Lie group if  $\text{Ad}(G) \subseteq \text{Aut}(\mathfrak{g})$  is closed. Note that all connected reductive and nilpotent Lie groups are (CA)-Lie groups.  $\square$

The next lemma is our key observation. The proof depends heavily on the special choice of the lattice  $\Gamma(i \text{int } C^*, 2\rho)$  and is a little bit tricky.

**Lemma III.4.** *Let  $S = \Gamma_G(W)$  be a pointed complex Ol'shanskiĭ semi-group and suppose that  $G$  is a (CA)-Lie group. Let  $K^N$ ,  $N \in \mathbb{N}$ , be as in Theorem II.8 and  $(K_{s_n}^N)_{n \in \mathbb{N}}$  a convergent sequence in  $\mathcal{H}_{K^N}$  with limit different from zero. Further let  $Z$  denote the center of  $G$ .*

- (i) *The set  $\{s_n Z : n \in \mathbb{N}\}$  is relatively compact in  $S^0/Z$ .*
- (ii) *If, in addition,  $Z$  is compact, then  $\{s_n : n \in \mathbb{N}\}$  is relatively compact in  $S^0$ .*

*Proof.* (i) For simplicity, we write  $K$  instead of  $K^N$ . Since  $K = \sum_{\lambda} \|\lambda\|^N K^{\lambda}$  is a direct sum of positive definite kernels corresponding to inequivalent irreducible unitary representations of  $G \times G$ , it follows in particular that  $\mathcal{H}_K = \widehat{\bigoplus}_{\lambda} \mathcal{H}_{K^{\lambda}}$  (cf. [Ne99, Th. I.11, Rem. I.12(a)]). Thus  $(K_{s_n})_{n \in \mathbb{N}}$  being a convergent sequence with non-zero limit implies in particular that all sequences  $(K_{s_n}^{\lambda})_{n \in \mathbb{N}}$  are convergent and at least one limit  $f^{\lambda} = \lim_{n \rightarrow \infty} K_{s_n}^{\lambda}$  is different from zero.

Step 1: The set  $\{s_n Z : n \in \mathbb{N}\}$  is relatively compact in  $S/Z$ .

Let  $\lambda \in \Gamma(i \text{int } C^*, 2\rho)$  be such that  $(K_{s_n}^{\lambda})_{n \in \mathbb{N}}$  converges with limit different from zero. Let

$$\chi_{\lambda} : Z \rightarrow \mathbb{S}^1, \quad z = \exp(X) \mapsto e^{-\lambda(X)}$$

and note that  $\chi_{\lambda}$  is an element of  $\widehat{Z}$ .

The *Bergman space* corresponding to the character  $\chi_\lambda$  is defined as

$$\mathcal{B}^2(S/Z, \chi_\lambda) = \left\{ f \in \text{Hol}(S^0) : (\forall z \in Z, s \in S^0) f(sz) = \chi_\lambda(z)^{-1} f(s), \right. \\ \left. \|f\|_2^2 := \int_{S^0/Z} |f(s)|^2 d\mu_{S^0/Z}(sZ) < \infty \right\},$$

where  $\mu_{S^0/Z}$  denotes the canonical left  $S$ -invariant measure on  $S^0/Z$  (cf. [Kr98, Sect. II]). Recall from [Kr98, Prop. II.4, Th. IV.5] that  $\mathcal{B}^2(S/Z, \chi_\lambda)$  is a closed subspace of the Hilbert space  $L^2(S/Z, \chi_\lambda)$  and that there exists a positive constant  $c > 0$  such that the prescription

$$\mathcal{H}_{K^\lambda} \rightarrow \mathcal{B}^2(S/Z, \chi_\lambda), \quad K_z^\lambda \mapsto cK_z^\lambda$$

defines an  $S \times S$ -equivariant isometric embedding.

We obtain in particular that  $(K_{s_n}^\lambda)_{n \in \mathbb{N}}$  is a convergent sequence in  $\mathcal{B}^2(S/Z, \chi_\lambda)$  with limit  $f^\lambda \neq 0$ . To obtain a contradiction, we now assume that there exists a subsequence of  $(s_n Z)_{n \in \mathbb{N}}$  leaving every compact subset of  $S/Z$ . To avoid further notation we denote this subsequence again by  $(s_n Z)_{n \in \mathbb{N}}$ . Note that  $(s_n^* Z)_{n \in \mathbb{N}}$  also leaves every compact subset of  $S/Z$ , since the involution on  $S$  induces an involution  $*$ :  $S/Z \rightarrow S/Z$ . We write

$$\rho: S \rightarrow B(\mathcal{B}^2(S/Z, \chi_\lambda)), \quad (\rho(s).f)(z) = f(zs)$$

for the right regular representation of  $S$  on  $\mathcal{B}^2(S/Z, \chi_\lambda)$  and note that  $(\rho, \mathcal{B}^2(S/Z, \chi_\lambda))$  is a holomorphic contraction representation of  $S$  (cf. [Kr98, Prop. II.4]). Let  $s_0 \in S^0$ . Then  $\rho(s_0).K_{s_n}^\lambda \rightarrow \rho(s_0).f^\lambda$  and  $\rho(s_0).f^\lambda \neq 0$  since  $\rho(s_0)$  is an injective operator. It follows in particular that there exists a convergent subsequence of  $(\rho(s_0).K_{s_n}^\lambda)_{n \in \mathbb{N}}$  converging to  $\rho(s_0).f^\lambda$  pointwise. Note that  $|K_s^\lambda| \in C_0(S/Z) \upharpoonright_{S^0/Z}$  for all  $s \in S^0$  (cf. [Kr98, Prop. II.4]). Thus we obtain from

$$\rho(s_0).K_{s_n}^\lambda(z) = K_{s_n}^\lambda(zs_0) = K^\lambda(zs_0, s_n) = K^\lambda(z, s_n s_0^*) \\ = K^\lambda(s_n^* z, s_0^*) = K_{s_0^*}^\lambda(s_n^* z)$$

for all  $z \in S^0$  that  $\rho(s_0).K_{s_n}^\lambda \rightarrow 0$  pointwise. This is a contradiction to  $\rho(s_0).f^\lambda \neq 0$  and proves our first step.

Step 2: Every cluster point of  $(s_n Z)_{n \in \mathbb{N}}$  lies in  $S^0/Z$ .

As the Polar Decomposition of  $S$  is inherited by  $S/Z$ , i.e., the mapping

$$G/Z \times W \rightarrow S/Z, \quad (gZ, X) \mapsto g \text{Exp}(iX)Z$$

is a homeomorphism, we can write  $s_n = g_n \text{Exp}(X_n)$ , where  $g_n \in G$  and  $X_n \in iW^0$ . According to Step 1, we now may assume that both  $(g_n Z)_{n \in \mathbb{N}}$  and  $(X_n)_{n \in \mathbb{N}}$  converge. Let  $X \in iW$  be the limit of  $(X_n)_{n \in \mathbb{N}}$ . Note that it suffices to show that  $X \in iW^0$ . As  $(K_{s_n}^\lambda)_{n \in \mathbb{N}}$  is convergent with non-zero

limit,  $(\|K_{s_n}\|^2)_{n \in \mathbb{N}}$  is a convergent sequence in  $\mathbb{R}^+$  with positive limit. The bi-invariance of  $K$  further implies that

$$\begin{aligned} \|K_{s_n}\|^2 &= K(s_n, s_n) = K(g_n \text{Exp}(X_n), g_n \text{Exp}(X_n)) \\ &= K(\text{Exp}(X_n), \text{Exp}(X_n)), \end{aligned}$$

so that Step 2 follows from Theorem II.8.

(ii) This is a direct consequence of (i) and the compactness of  $Z$ .  $\square$

**Lemma III.5.** *If  $S = \Gamma_G(W)$  is a pointed complex Ol'shanskii semigroup, then there exists a finite set  $F \subseteq HW(G, \Delta^+)$  such that all representations  $\pi_{KN} : S \rightarrow B(\mathcal{H}_{KN})$ ,  $N \in \mathbb{N}$ , associated to  $\Gamma' = \Gamma \cup F$  (cf. Remark II.10) are injective.*

*Proof.* Since  $S$  is pointed, it follows from [Kr98, Prop. V.7] that  $\pi_{KN}$  is injective if and only if  $\pi_{KN}|_T$  is injective, where  $T = \exp \mathfrak{t}$ . Evaluation of the operators  $\pi_{KN}(t)$ ,  $t \in T$ , on highest weight vectors  $v_\lambda$ ,  $\lambda \in \Gamma(i \text{int } C^*, 2\rho)$ , now easily shows how one can choose  $F$  to obtain injective representations. For more details we refer to [Ne96a, Sect. V].  $\square$

**Theorem III.6.** *Let  $D \subseteq \Gamma_G(W_{\max}^0)$  be a pointed bi-invariant Stein domain and suppose that  $G$  is a (CA)-Lie group and  $Z$  is compact. Then there exists a kernel  $K \in \mathcal{P}_{G \times G}(D^2)$  such that the mapping*

$$e_K : D \rightarrow \mathcal{H}_K^* \setminus \{0\}, \quad z \mapsto K_z$$

*defines a  $G \times G$ -equivariant holomorphic embedding with closed range.*

*Proof.* In view of Proposition I.8(i) and Lemma I.6(ii), we may assume that  $D = S^0 = \Gamma_G(W^0)$  is a pointed open complex Ol'shanskii semigroup. Now let  $\Gamma'$  as in Lemma III.5 and  $K = K^N$  for some  $N \in \mathbb{N}$ . As  $K \neq 0$ , the map  $e_K$  is a well defined  $G \times G$ -equivariant holomorphic map (cf. Corollary III.2). Lemma III.5 implies that the representation  $\pi_K : S \rightarrow B(\mathcal{H}_K)$  is injective so that  $\mathcal{H}_K$  separates points by [Kr98, Prop. V.10], which in turn means that  $e_K$  is injective.

Next we show that  $\text{ime}_K$  is closed. In fact, if  $K_{s_n} \rightarrow f \neq 0$ , then Lemma III.4 implies that  $(s_n)_{n \in \mathbb{N}}$  is a bounded sequence in  $S^0$  with all accumulation points in  $S^0$ . Thus we find a convergent subsequence  $(s_{n_k})_{k \in \mathbb{N}}$  with limit  $s \in S^0$ . Now we get

$$f = \lim_{n \rightarrow \infty} K_{s_n} = \lim_{k \rightarrow \infty} K_{s_{n_k}} = \lim_{k \rightarrow \infty} e_K(s_{n_k}) = e_K(s) = K_s,$$

proving the closedness of  $\text{ime}_K$ .

Finally another easy application of Lemma III.4 shows that  $e_K$  is homeomorphic onto its image, concluding the proof of the theorem.  $\square$

*Example III.7* (The Bergman kernel associated to  $\mathrm{Sl}(2, \mathbb{R})$ ). Let  $G = \mathrm{Sl}(2, \mathbb{R})$  and  $\mathfrak{g} := \mathfrak{sl}(2, \mathbb{R})$ . We choose

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as a basis for  $\mathfrak{g}$ .

Then  $\mathfrak{t} := \mathbb{R}U$  is a compactly embedded Cartan subalgebra. Let  $\alpha \in i\mathfrak{t}^*$  be defined by  $\alpha(U) = -2i$ . The root system of  $\mathfrak{g}$  is given by  $\Delta = \{\pm\alpha\}$  with root spaces  $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathbb{C}(T + iH)$  and  $\mathfrak{g}_{\mathbb{C}}^{-\alpha} = \mathbb{C}(T - iH)$ . We define a positive system by  $\Delta^+ := \{\alpha\}$  and write  $\kappa$  for the Cartan-Killing form on  $\mathfrak{g}$ . Then the upper light cone

$$\begin{aligned} W &:= \{X = uU + tT + hH : u \geq 0, \kappa(X, X) \leq 0\} \\ &= \{X = uU + tT + hH : u \geq 0, h^2 + t^2 - u^2 \leq 0\} \end{aligned}$$

is an invariant pointed cone in  $\mathfrak{g}$ . Moreover,  $W$  is up to sign the unique invariant elliptic cone in  $\mathfrak{g}$  (cf. [HiNe93, Th. 7.25]). Thus up to isomorphism  $S := \Gamma_G(W)$  is the unique complex Ol'shanskiĭ semigroup corresponding to  $G$ .

In the following we identify  $\mathfrak{t}_{\mathbb{C}}$  with  $\mathbb{C}$  via the isomorphism  $\mathfrak{t}_{\mathbb{C}} \rightarrow \mathbb{C}$ ,  $\lambda \mapsto \lambda(iU)$ . Then  $HW(G, \Delta^+) = HW(G, W) = \{\lambda \in \mathbb{Z} : \lambda \leq 0\}$ ,  $\lambda + 2\rho \in i \operatorname{int} C^*$  if and only if  $\lambda \leq -3$ , and thus  $\Gamma(i \operatorname{int} C^*, 2\rho) = \{\lambda \in \mathbb{Z} : \lambda \leq -3\}$ . The main point is that  $\Gamma(i \operatorname{int} C^*, 2\rho)$  coincides with the weights in the decomposition of the Bergman kernel  $B$  of the *Bergman space*

$$\mathcal{B}^2(S) := \left\{ f \in \operatorname{Hol}(S^0) : \|f\|_2^2 := \int_{S^0} |f(s)|^2 d\mu_{G_{\mathbb{C}}}(s) < \infty \right\}$$

(cf. [Kr98, Th. IV.7]). Further one knows that

$$B = \sum_{\lambda \leq -3} \lambda(1 + \lambda)^2(4 - \lambda^2)K^{\lambda}$$

(cf. [Kr98, Th. IV.7, Ex. IV.8]) so that Theorem II.8 and Theorem III.6 imply that

$$\lim_{z \rightarrow \partial S} B(z, z) = \infty$$

and that the map

$$e_B : S^0 \rightarrow \mathcal{B}^2(S)^* \setminus \{0\}, \quad s \mapsto B_s$$

is a  $G \times G$ -equivariant holomorphic embedding with closed range.  $\square$

**Problems III.8.** (a) What is the reason for that one has to exclude zero in  $\mathcal{H}_K^*$  to obtain the closedness of the map  $e_K$ ?

(b) Given a pointed biinvariant Stein domain  $D$ , does there exist an equivariant closed embedding of  $D$  into a complex topological vector space  $E$  endowed with a continuous  $G \times G$ -action?

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