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We treat the Dirichlet problem for elliptic equations on annular regions, and show the monotonicity and symmetry properties of positive solutions with respect to the sphere. We generalize the argument of the method of moving spheres to more general partial differential equations.

1. Introduction.

Let $A = \{x \in \boldsymbol{R}^n : 1/a < |x| < a\}$ be an annulus with $a > 1$ and $n \geq 2$. In [10] Padilla proved the following theorem by emplying the method of moving spheres.

Theorem A. *Let $n > 2$ and let u be a solution to*

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \quad \textit{in} \quad A \qquad \textit{and} \qquad u = 0 \quad \textit{on} \quad \partial A.$$

Then u satisfies

$$u(x) = |x|^{2-n} u\left(\frac{x}{|x|^2}\right) \quad \textit{for} \;\; x \in A \quad \textit{and}$$
$$\left(|x|^{\frac{n-2}{2}} u\right)_r < 0 \quad \textit{for} \;\; 1 < r = |x| < a.$$

The method of moving spheres is a variant of the method of moving planes as presented in Gidas, Ni, and Nirenberg [6] or Berestycki and Nirenberg [1]. Roughly speaking, we make reflection with respect to spheres instead of planes, and then obtain the symmetry of solutions. In the works of Chou and Chu [5], Chen and Li [4], Li and Zhu [9], and Kurata and Matsuda [8], the method of moving spheres is used and is useful for solving various questions about elliptic differential equations.

In this note we generalize the argument of the method of moving spheres to more general partial differential equations. Let us consider the equation

$$(1) \qquad -\Delta u = f\left(x, |x|^{\frac{n-2}{2}} u, (x \cdot \nabla)\left(|x|^{\frac{n-2}{2}} u\right)\right), \quad u > 0 \qquad \text{in} \;\; A,$$

where $x \cdot \nabla = \sum_{i=1}^{n} x_i \partial/\partial x_i$. We assume that $f = f(x,s,q)$ is continuous on $\overline{A} \times [0,\infty) \times \boldsymbol{R}$, C^1 in s and q, and even with respect to q:

$$f(x,s,-q) = f(x,s,q) \quad (x \in A, s \geq 0, q \in \boldsymbol{R}).$$

We obtain the following theorems.

Theorem 1. *Suppose that, for each $1 \leq r_* \leq a$, $\omega \in \boldsymbol{S}^{n-1}$, $s \geq 0$, and $q \geq 0$,*

$$r^{\frac{n+2}{2}} f(r\omega, s, q) \geq r_*^{\frac{n+2}{2}} f(r_*\omega, s, q) \qquad \text{for} \quad \frac{1}{r_*} \leq r \leq r_*.$$

Then, any $u \in C^2(A) \cap C(\overline{A})$ satisfying (1) and $u = 0$ on $|x| = a$ has the properties

$$|x|^{\frac{n-2}{2}} u(x) \leq \left(\frac{1}{|x|}\right)^{\frac{n-2}{2}} u\left(\frac{x}{|x|^2}\right) \qquad \text{on } 1 \leq |x| \leq a \tag{2}$$

and

$$\left(|x|^{\frac{n-2}{2}} u\right)_r < 0 \qquad \text{for } 1 < r = |x| < a. \tag{3}$$

Theorem 2. *Suppose that, for each $1/a \leq r_* \leq 1$, $\omega \in \boldsymbol{S}^{n}$, $s \geq 0$, and $q \geq 0$,*

$$r^{\frac{n+2}{2}} f(r\omega, s, q) \geq r_*^{\frac{n+2}{2}} f(r_*\omega, s, q) \qquad \text{for } r_* \leq r \leq \frac{1}{r_*}.$$

Then, any $u \in C^2(A) \cap C(\overline{A})$ satisfying (1) and $u = 0$ on $|x| = 1/a$ has the properties

$$|x|^{\frac{n-2}{2}} u(x) \geq \left(\frac{1}{|x|}\right)^{\frac{n-2}{2}} u\left(\frac{x}{|x|^2}\right) \qquad \text{on } 1 \leq |x| \leq a$$

and

$$\left(|x|^{\frac{n-2}{2}} u\right)_r > 0 \qquad \text{for } \frac{1}{a} < r = |x| < 1.$$

Remark. It is shown in [**6**, Theorem 2] by the method of moving planes that the positive solutions u of the equation $\Delta u + f(u) = 0$ in A with $u = 0$ on ∂A satisfies $u_r < 0$ on $(1 <)$ $(a + a^{-1})/2 \leq r < a$.

As a consequence of Theorems 1 and 2 we obtain the following corollary.

Corollary 1. *Suppose that, for each $\omega \in \boldsymbol{S}^{n-1}$, $s \geq 0$, and $q \geq 0$, $r^{(n+2)/2} f(r\omega, s, q)$ is nonincreasing in $r \in (1,a)$ and*

$$r^{\frac{n+2}{2}} f(r\omega, s, q) \equiv r^{-\frac{n+2}{2}} f(r^{-1}\omega, s, q) \qquad \text{for } 1 \leq r \leq a.$$

Then, any $u \in C^2(A) \cap C(\overline{A})$ *satisfying* (1) *and* $u = 0$ *on* ∂A *has the properties*

$$|x|^{\frac{n-2}{2}} u(x) \equiv \left(\frac{1}{|x|}\right)^{\frac{n-2}{2}} u\left(\frac{x}{|x|^2}\right) \qquad on \ \ 1 \le |x| \le a \tag{4}$$

and (3).

Remark. If (1) has a solution u satisfying (4), then we must have

$$|x|^{\frac{n+2}{2}} f(x, s(x), q(x)) \equiv |x|^{-\frac{n+2}{2}} f(x/|x|^2, s(x), q(x)) \quad \text{for} \ \ 1 \le |x| \le a, \tag{5}$$

where $s(x) = |x|^{(n-2)/2}u(x)$ and $q(x) = x \cdot \nabla s(x)$. In fact, $v(x) = |x|^{(n-2)/2}u(x)$ and $w(x) = |y|^{(n-2)/2}u(y)$, $y = x/|x|^2$, satisfy

$$|x|^2 \Delta v - (n-2) x \cdot \nabla v - \frac{(n-2)^2}{4} v + |x|^{\frac{n+2}{2}} f\left(x, v(x), (x \cdot \nabla) v(x)\right) = 0$$

and

$$|x|^2 \Delta w - (n-2) x \cdot \nabla w - \frac{(n-2)^2}{4} w + |y|^{\frac{n+2}{2}} f\left(y, w(x), (x \cdot \nabla) w(x)\right) = 0,$$

respectively. (See (7) and (8) below.) Then $v(x) \equiv w(x)$ implies (5).

We consider the following typical problem

$$\Delta u + g(x, u) = 0, \quad u > 0 \qquad \text{in} \quad A \qquad u = 0 \quad \text{on} \ \ \partial A, \tag{6}$$

where $g = g(x, s)$ is continuous on $\overline{A} \times [0, \infty)$ and C^1 in s. In this case we see that $f(x, s, q) = g(x, |x|^{-(n-2)/2}s)$. Note that the existence of positive nonradial solution u of the problem (6) has been studied by many authors, see, e.g., Brezis and Nirenberg [**2**], Suzuki [**11**], Byeon [**3**], and the references therein. As a consequence of Corollary 1 we obtain the following corollary, which in the special case $g(x, u) = u^{(n+2)/(n-2)}$ (and $f(x, s, q) = |x|^{-(n+2)/2} s^{(n+2)/(n-2)}$) yields Theorem A.

Corollary 2. *Suppose that, for each* $\omega \in \boldsymbol{S}^{n-1}$ *and* $s \ge 0$, $r^{\frac{n+2}{2}} g\Big(r\omega, r^{-\frac{n-2}{2}} s\Big)$ *is nonincreasing in* $r \in (1, a)$ *and*

$$r^{\frac{n+2}{2}} g(r\omega, r^{-\frac{n-2}{2}} s) \equiv r^{-\frac{n+2}{2}} g(r^{-1}\omega, r^{\frac{n-2}{2}} s) \qquad for \quad 1 \le r \le a.$$

Let $u \in C^2(A) \cap C(\overline{A})$ *be a solution of* (6). *Then* u *satisfies the properties* (4) *and* (3).

Remark. For example,

$$g(r\omega, s) = r^{-\frac{(n+2)+p(n-2)}{2}} h(\omega) s^p + c s^{\frac{n+2}{n-2}}, \qquad c, \ p \in \boldsymbol{R}, \ p \ge 1,$$

where $h(\omega)$ is continuous and positive on S^{n-1}, satisfies the conditions in Corollary 2. For the case $g(r\omega, s) = r^{-2}h(\omega)s + s^{(n+2)/(n-2)}$, the existence of positive solutions for the problem (6) is investigated in [**2**].

In our proof we use the operator $\Delta_g v = |x|^2\Delta v - (n-2)x\cdot\nabla v$. We note that Δ_g is the Laplace-Beltrami operator on the Riemannian space $(\boldsymbol{R}^n, dx^2/|x|^2)$. We find that Equation (1) is written as

$$\Delta_g v - \frac{(n-2)^2}{4}v + |x|^{\frac{n+2}{2}} f\left(x, v, (x\cdot\nabla)\, v\right) = 0 \qquad \text{in } A$$

for the function $v(x) = |x|^{(n-2)/2}u(x)$, and that the operator Δ_g is invariant under the transformation $x \mapsto y = \lambda^2 x/|x|^2$.

In Section 2 we prove Theorems. In fact, we only present the proof of Theorem 1 since the proof of Theorem 2 is very similar. In Appendix we show that the operator Δ_g is invariant under the transformation by using of the property of the Kelvin transformation.

2. Proof of Theorems.

Due to similarity, we only give the proof of Theorem 1. Given $\lambda \in (1, a)$, we set

$$T_\lambda = \{|x| = \lambda\} \qquad \text{and} \qquad \Sigma_\lambda = \{\lambda < |x| < a\}.$$

For $x \in \Sigma_\lambda$, let $x^\lambda = \lambda^2 x/|x|^2$. Then we have

$$|x| > \left|x^\lambda\right| = \frac{\lambda^2}{|x|} > \frac{1}{|x|} \qquad \text{for } x \in \Sigma_\lambda.$$

Define the operator Δ_g by $\Delta_g v = |x|^2\Delta v - (n-2)x\cdot\nabla v$. We note that Δ_g is the Laplace-Beltrami operator on the Riemannian space $(\boldsymbol{R}^n, dx^2/|x|^2)$. For a solution u of (1), the function $v(x) = |x|^{\frac{n-2}{2}}u(x)$ satisfies

$$|x|^2\Delta v - (n-2)x\cdot\nabla v - \frac{(n-2)^2}{4}v + |x|^{\frac{n+2}{2}} f\left(x, v, (x\cdot\nabla)\, v\right) = 0 \tag{7}$$

in A, which is written as

$$\Delta_g v - \frac{(n-2)^2}{4}v + |x|^{\frac{n+2}{2}} f\left(x, v, (x\cdot\nabla)\, v\right) = 0 \qquad \text{in } A.$$

Let $v^\lambda(x) = v(x^\lambda)$ and $y = x^\lambda$. By Lemma A in Appendix, we find that $\Delta_g v^\lambda(x) = \Delta_g v(y)$. We have $x\cdot\nabla = r\partial_r$ for $r = |x|$ (see, e.g., [**7**]) and hence

$$x\cdot\nabla_x v^\lambda = r\partial_r v^\lambda = -s\partial_s v = -y\cdot\nabla_y v,$$

where $r = |x|$ and $s = |y| = \lambda^2/r$. Therefore, the property $f(x, s, -q) = f(x, s, q)$ implies the relation

$$\Delta_g v^\lambda - \frac{(n-2)^2}{4}v^\lambda + \left|x^\lambda\right|^{\frac{n+2}{2}} f\left(x^\lambda, v^\lambda, (x\cdot\nabla)\, v^\lambda\right) = 0 \qquad \text{in } A. \tag{8}$$

It follows that $|x| > \left|x^\lambda\right| > 1/|x|$ and $1 < \lambda < |x| < a$ for $x \in \Sigma_\lambda$. Then the assumption on f in Theorem 1 guarantees

$$\left|x^\lambda\right|^{\frac{n+2}{2}} f(x^\lambda, s, q) \geq |x|^{\frac{n+2}{2}} f(x, s, q)$$

for $x \in \Sigma_\lambda$, $s \geq 0$, and $q \geq 0$. Therefore, the function $w_\lambda = v^\lambda - v$ satisfies

$$\Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda + |x|^{\frac{n+2}{2}} \left(f\left(x, v^\lambda, (x\cdot\nabla)\, v^\lambda\right) - f\left(x, v, (x\cdot\nabla)\, v\right)\right) \leq 0$$

on Σ_λ. Writing

$$b_\lambda(x) = \int_0^1 f_s\left(x, tv^\lambda(x) + (1-t)v(x), (x\cdot\nabla)v^\lambda(x)\right) dt \quad \text{and}$$

$$c_\lambda(x) = \int_0^1 f_q\left(x, v(x), t(x\cdot\nabla)v^\lambda(x) + (1-t)(x\cdot\nabla)v(x)\right) dt,$$

we obtain

$$\Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda + |x|^{\frac{n+2}{2}} \left(b_\lambda(x) + c_\lambda(x) x\cdot\nabla\right) w_\lambda \leq 0 \qquad \text{on } \Sigma_\lambda.$$

Let $z_\lambda(x) = |x|^{-\frac{n-2}{2}} w_\lambda(x)$. Then we have

$$|x|^{\frac{n+2}{2}} \Delta z_\lambda = \Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda \quad \text{and} \quad |x|^{\frac{n-2}{2}} x\cdot\nabla z_\lambda = -\frac{n-2}{2} w_\lambda + x\cdot\nabla w_\lambda.$$

Define

$$\tilde{b_\lambda}(x) = |x|^{\frac{n-2}{2}} \left(b_\lambda(x) + \frac{n-2}{2} c_\lambda(x)\right) \qquad \text{and} \qquad \tilde{c_\lambda}(x) = |x|^{\frac{n-2}{2}} c_\lambda(x).$$

We have shown the following lemma.

Lemma 1. *Under the assumptions of Theorem 1, each $\lambda \in (1, a)$ admits the inequality*

(9) $$\Delta z_\lambda + \tilde{b_\lambda}(x) z_\lambda + \tilde{c_\lambda}(x) x\cdot\nabla z_\lambda \leq 0 \qquad \textit{on } \Sigma_\lambda,$$

where $z_\lambda(x) = |x|^{-\frac{n-2}{2}} \left(v^\lambda - v\right)$.

Once Lemma 1 is proven, Theorem 1 follows from the standard argument ([**1**]). Putting

$$\Lambda \equiv \{\lambda \in (1, a) : z_\lambda > 0 \quad \text{in } \Sigma_\lambda\},$$

we see that the desired consequence follows from $\Lambda = (1, a)$. We show $\Lambda = (1, a)$ by three steps.

Step 1. *We have $[r_0, a) \subset \Lambda$ for r_0 close to a, that is, $\Lambda \neq \emptyset$.*

Proof. We see that the coefficents $\tilde{b_\lambda}(x)$ and $\tilde{c_\lambda}(x)$ in (9) are uniformly bounded. Then for r_0 close to a, the maximum principle holds for the

Equation (9) on any subdomain of $A \setminus \overline{B}_{r_0}$ and for any λ, where $B_{r_0} = \{x \in \boldsymbol{R}^n : |x| < r_0\}$. (See e.g. [1].) This implies $[r_0, 1) \subset \Lambda$. □

We prepare the following lemma.

Lemma 2. (i) *If* $\lambda \in \Lambda$, *then*

$$\frac{\partial z_\lambda}{\partial \nu} < 0 \qquad \text{on } T_\lambda, \tag{10}$$

where ν *denotes the outer unit normal vector on* T_λ *from* Σ_λ;

(ii) *If* $\lambda \notin \Lambda$, *then there exists some* $x_0 \in \Sigma_\lambda \cap \overline{B}_{r_0}$ *such that* $z_\lambda(x_0) \le 0$.

Proof. (i) Let $\lambda \in \Lambda$. Then we have $z_\lambda = 0$ on T_λ, and $z_\lambda > 0$ in Σ_λ. Therefore, Hopf's boundary lemma can be applied by (9) so that (10) holds.

(ii) As we have proven in Step 1, $\lambda < r_0$ and hence $\Sigma_\lambda \cap \overline{B}_{r_0} \neq \emptyset$. Suppose to the contrary that

$$z_\lambda(x) > 0 \qquad \text{on} \quad \Sigma_\lambda \cap \overline{B_{r_0}}.$$

Then we get

$$\Delta z_\lambda + \tilde{b_\lambda}(x) z_\lambda + \tilde{c_\lambda}(x) x \cdot \nabla z_\lambda \le 0 \qquad \text{in } \Sigma_\lambda \setminus \overline{B}_{r_0},$$

and

$$z_\lambda \ge 0 \qquad \text{on} \quad \partial\left(\Sigma_\lambda \setminus \overline{B}_{r_0}\right).$$

Now the maximum principle guarantees $z_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B}_{r_0}$. However, we have $z_\lambda > 0$ in $\Sigma_\lambda \cap \overline{B_{r_0}}$ and hence $z_\lambda > 0$ in Σ_λ. This means $\lambda \in \Lambda$, a contradiction. □

Step 2. Λ *is left-open.*

Proof. If Λ is not left-open, there exist $\lambda_0 \in \Lambda$ and a sequence $\{\lambda_n\}$ satisfying

$$\lambda_0 - \frac{1}{n} < \lambda_n < \lambda_0 \qquad \text{and} \qquad \lambda_n \notin \Lambda.$$

Lemma 2 (ii) guarantees the existence of $x_n \in \Sigma_{\lambda_n} \cap \overline{B}_{r_0}$ satisfying

$$z_{\lambda_n}(x_n) \le 0. \tag{11}$$

Note that $z_{\lambda_n} = 0$ on T_{λ_n}. Then we have a point y_n on the segment connecting x_n and $\lambda_n^2 x_n / |x_n|^2$ satisfying

$$\frac{\partial z_{\lambda_n}}{\partial r}(y_n) \le 0. \tag{12}$$

Taking a subsequence if necessary, we may suppose the existence of some $x_0 \in \overline{\Sigma}_{\lambda_0} \cap \overline{B}_{r_0}$ satisfying $x_n \to x_0$. By (11) we obtain $z_{\lambda_0}(x_0) \le 0$. Since

$\lambda_0 \in \Lambda$, we must have $x_0 \in T_{\lambda_0}$. In particular, $y_n \to x_0$ and $\partial z_{\lambda_0}/\partial r(x_0) \leq 0$ follows from (12). However, this is equivalent to

$$\frac{\partial z_{\lambda_0}}{\partial \nu}(x_0) \geq 0,$$

which contradicts to (10) valid for $\lambda = \lambda_0 \in \Lambda$. □

Step 3. Λ *is left-closed.*

Proof. In fact, let $\{\lambda_n\} \subset \Lambda$ be a sequence satisfying $\lambda_n \downarrow \lambda_1 > 1$. Then, we have

$$\Delta z_{\lambda_1} + \left(\tilde{b_{\lambda_1}}(x) + \tilde{c_{\lambda_1}}(x)x \cdot \nabla\right) z_{\lambda_1} \leq 0 \quad \text{and} \quad z_{\lambda_1} \geq 0 \quad \text{in } \Sigma_{\lambda_1}.$$

Since $z_{\lambda_1} > 0$ on $|x| = a$, we have $z_{\lambda_1} \not\equiv 0$ in Σ_{λ_1}. Therefore, the maximum principle implies $z_{\lambda_1} > 0$ in Σ_{λ_1}, or equivalently, $\lambda_1 \in \Lambda$. □

As a consequece of Steps 1-3, we obtain $\Lambda = (1, a)$. This implies $v^1(x) \geq v(x)$ on $1 \leq |x| \leq a$, and then (2) holds. The property (3) follows from Lemma 2 (i). This completes the proof. □

Appendix.

Let $\Delta_g v = |x|^2 \Delta v - (n-2) x \cdot \nabla v$. We show that the operator Δ_g is invariant under the transformation $x \mapsto y = \lambda^2 x/|x|^2$, that is, $\Delta_g v = \Delta_g V$ for $v(x) = V(y)$. Here we use the well-known property of the Kelvin transformation $\eta = \xi/|\xi|^2$ expressed as

$$\Delta_\eta U = |\xi|^{n+2} \Delta_\xi u \qquad \text{for} \quad U(\eta) = |\xi|^{n-2} u(\xi).$$

Lemma A. *Let $v(x) = V(y)$ and $y = \lambda^2 x/|x|^2$ with $\lambda > 0$. Then we have*

$$|x|^2 \Delta_x v - (n-2) x \cdot \nabla_x v = |y|^2 \Delta_y V - (n-2) y \cdot \nabla_y V, \tag{13}$$

where $\Delta_x = \sum_{i=1}^n \partial^2/\partial x_i^2$ and $x \cdot \nabla_x = \sum_{i=1}^n x_i \partial/\partial x_i$.

Proof. Writing $w(x) = |x|^{-\frac{n-2}{2}} v(x)$ we have

$$|x|^{\frac{n+2}{2}} \Delta_x w = |x|^2 \Delta_x v - (n-2) x \cdot \nabla_x v - \frac{(n-2)^2}{4} v. \tag{14}$$

Similarly, writing $W(y) = |y|^{-\frac{n-2}{2}} V(y)$ we have

$$|y|^{\frac{n+2}{2}} \Delta_y W = |y|^2 \Delta_y V - (n-2) y \cdot \nabla_y V - \frac{(n-2)^2}{4} V. \tag{15}$$

By $|y| = \lambda^2/|x|$ it follows that

$$|x|^{\frac{n-2}{2}} w(x) = v(x) = V(y) = |y|^{\frac{n-2}{2}} W(y) = \left(\frac{\lambda^2}{|x|}\right)^{\frac{n-2}{2}} W(y).$$

Then we obtain

$$W(y) = \left(\frac{|x|}{\lambda}\right)^{n-2} w(x).$$

By the property of the Kelvin transformation, we obtain

$$\Delta_y W = \left(\frac{|x|}{\lambda}\right)^{n+2} \Delta_x w.$$

Then we have

$$|x|^{\frac{n+2}{2}} \Delta_x w = \left(\frac{\lambda^2}{|x|}\right)^{\frac{n+2}{2}} \left(\frac{|x|}{\lambda}\right)^{n+2} \Delta_x w = |y|^{\frac{n+2}{2}} \Delta_y W.$$

Therefore, by (14) and (15), we obtain the property (13). This completes the proof. □

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