A NOTE ON THE MOVING SPHERE METHOD

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We treat the Dirichlet problem for elliptic equations on annular regions, and show the monotonicity and symmetry properties of positive solutions with respect to the sphere. We generalize the argument of the method of moving spheres to more general partial differential equations.

1. Introduction.

Let \( A = \{ x \in \mathbb{R}^n : 1/a < |x| < a \} \) be an annulus with \( a > 1 \) and \( n \geq 2 \). In [10] Padilla proved the following theorem by employing the method of moving spheres.

**Theorem A.** Let \( n > 2 \) and let \( u \) be a solution to

\[
-\Delta u = u^{\frac{n+2}{n-2}}, \quad u > 0 \text{ in } A \quad \text{and} \quad u = 0 \text{ on } \partial A.
\]

Then \( u \) satisfies

\[
 u(x) = |x|^{2-n}u \left( \frac{x}{|x|^2} \right) \quad \text{for } x \in A \quad \text{and} \quad \left( |x|^{\frac{n-2}{2}} u \right)_r < 0 \quad \text{for } 1 < r = |x| < a.
\]

The method of moving spheres is a variant of the method of moving planes as presented in Gidas, Ni, and Nirenberg [6] or Berestycki and Nirenberg [1]. Roughly speaking, we make reflection with respect to spheres instead of planes, and then obtain the symmetry of solutions. In the works of Chou and Chu [5], Chen and Li [4], Li and Zhu [9], and Kurata and Matsuda [8], the method of moving spheres is used and is useful for solving various questions about elliptic differential equations.

In this note we generalize the argument of the method of moving spheres to more general partial differential equations. Let us consider the equation

\[
-\Delta u = f \left( x, |x|^{\frac{n-2}{2}} u, (x \cdot \nabla) \left( |x|^{\frac{n-2}{2}} u \right) \right), \quad u > 0 \quad \text{in } A,
\]

where \( f \) is a suitable function.
where \( x \cdot \nabla = \sum_{i=1}^{n} x_i \partial / \partial x_i \). We assume that \( f = f(x, s, q) \) is continuous on \( A \times [0, \infty) \times \mathbb{R}, C^1 \) in \( s \) and \( q \), and even with respect to \( q \):

\[
 f(x, s, -q) = f(x, s, q) \quad (x \in A, s \geq 0, q \in \mathbb{R}).
\]

We obtain the following theorems.

**Theorem 1.** Suppose that, for each \( 1 \leq r_0 \leq a, \omega \in S^{n-1}, s \geq 0, \) and \( q \geq 0, \)

\[
 r \frac{n+2}{2} f(r\omega, s, q) \geq r_0 \frac{n+2}{2} f(r\omega, s, q) \quad \text{for} \quad \frac{1}{r_0} \leq r \leq r_0.
\]

Then, any \( u \in C^2(A) \cap C(\overline{A}) \) satisfying (1) and \( u = 0 \) on \( |x| = a \) has the properties

\[
 |x|^{\frac{n-2}{2}} u(x) \leq \left( \frac{1}{|x|} \right)^{\frac{n-2}{2}} u \left( \frac{x}{|x|^2} \right) \quad \text{on} \quad 1 \leq |x| \leq a
\]

and

\[
 \left( |x|^{\frac{n-2}{2}} u \right)_r < 0 \quad \text{for} \quad 1 < r = |x| < a.
\]

**Theorem 2.** Suppose that, for each \( 1/a \leq r_0 \leq 1, \omega \in S^n, s \geq 0, \) and \( q \geq 0, \)

\[
 r \frac{n+2}{2} f(r\omega, s, q) \geq r_0 \frac{n+2}{2} f(r\omega, s, q) \quad \text{for} \quad r_0 \leq r \leq \frac{1}{r_0}.
\]

Then, any \( u \in C^2(A) \cap C(\overline{A}) \) satisfying (1) and \( u = 0 \) on \( |x| = 1/a \) has the properties

\[
 |x|^{\frac{n-2}{2}} u(x) \geq \left( \frac{1}{|x|} \right)^{\frac{n-2}{2}} u \left( \frac{x}{|x|^2} \right) \quad \text{on} \quad 1 \leq |x| \leq a
\]

and

\[
 \left( |x|^{\frac{n-2}{2}} u \right)_r > 0 \quad \text{for} \quad \frac{1}{a} < r = |x| < 1.
\]

**Remark.** It is shown in [6, Theorem 2] by the method of moving planes that the positive solutions \( u \) of the equation \( \Delta u + f(u) = 0 \) in \( A \) with \( u = 0 \) on \( \partial A \) satisfies \( u_r < 0 \) on \( (1) \ (a + a^{-1})/2 \leq r < a \).

As a consequence of Theorems 1 and 2 we obtain the following corollary.

**Corollary 1.** Suppose that, for each \( \omega \in S^{n-1}, s \geq 0, \) and \( q \geq 0, \)

\[
 r^{(n+2)/2} f(r\omega, s, q) \quad \text{is nonincreasing in} \quad r \in (1,a) \quad \text{and}
\]

\[
 r^{\frac{n+2}{2}} f(r\omega, s, q) \equiv r^{-\frac{n+2}{2}} f(r^{-1}\omega, s, q) \quad \text{for} \quad 1 \leq r \leq a.
\]
Then, any \( u \in C^2(A) \cap C(\overline{A}) \) satisfying (1) and \( u = 0 \) on \( \partial A \) has the properties

\[
|x|^{\frac{n-2}{2}} u(x) \equiv \left( \frac{1}{|x|} \right)^{\frac{n-2}{2}} u \left( \frac{x}{|x|^2} \right) \quad \text{on} \quad 1 \leq |x| \leq a
\]

and (3).

Remark. If (1) has a solution \( u \) satisfying (4), then we must have

\[
|x|^{\frac{n+2}{2}} f(x, s(x), q(x)) \equiv |x|^{-\frac{n+2}{2}} f(x/|x|^2, s(x), q(x)) \quad \text{for} \quad 1 \leq |x| \leq a,
\]

where \( s(x) = |x|^{(n-2)/2} u(x) \) and \( q(x) = x \cdot \nabla s(x) \). In fact, \( v(x) = |x|^{(n-2)/2} u(x) \) and \( w(x) = |y|^{(n-2)/2} u(y) \), \( y = x/|x|^2 \), satisfy

\[
|x|^2 \Delta v - (n-2) x \cdot \nabla v - \frac{(n-2)^2}{4} v + |x|^{\frac{n+2}{2}} f(x, v(x), (x \cdot \nabla) v(x)) = 0
\]

and

\[
|x|^2 \Delta w - (n-2) x \cdot \nabla w - \frac{(n-2)^2}{4} w + |y|^{\frac{n+2}{2}} f(y, w(x), (x \cdot \nabla) w(x)) = 0,
\]

respectively. (See (7) and (8) below.) Then \( v(x) \equiv w(x) \) implies (5).

We consider the following typical problem

\[
\Delta u + g(x, u) = 0, \quad u > 0 \quad \text{in} \quad A, \quad u = 0 \quad \text{on} \quad \partial A,
\]

where \( g = g(x, s) \) is continuous on \( \overline{A} \times [0, \infty) \) and \( C^1 \) in \( s \). In this case we see that \( f(x, s, q) = g(x, |x|^{-(n-2)/2}s) \). Note that the existence of positive nonradial solution \( u \) of the problem (6) has been studied by many authors, see, e.g., Brezis and Nirenberg [2], Suzuki [11], Byeon [3], and the references therein. As a consequence of Corollary 1 we obtain the following corollary, which in the special case \( g(x, u) = u^{(n+2)/(n-2)} \) (and \( f(x, s, q) = |x|^{-(n+2)/2}s^{(n+2)/(n-2)} \)) yields Theorem A.

\textbf{Corollary 2.} Suppose that, for each \( \omega \in S^{n-1} \) and \( s \geq 0, r^{\frac{n+2}{2}} g(r\omega, r^{-\frac{n-2}{2}} s) \) is nonincreasing in \( r \in (1, a) \) and

\[
r^{\frac{n+2}{2}} g(r\omega, r^{-\frac{n-2}{2}} s) \equiv r^{-\frac{n+2}{2}} g(r^{-1} \omega, r^{\frac{n-2}{2}} s) \quad \text{for} \quad 1 \leq r \leq a.
\]

Let \( u \in C^2(A) \cap C(\overline{A}) \) be a solution of (6). Then \( u \) satisfies the properties (4) and (3).

Remark. For example,

\[
g(r\omega, s) = r^{-\frac{(n+2)+p(n-2)}{2}} h(\omega)s^p + cs^{\frac{n+2}{2}}, \quad c, \ p \in \mathbb{R}, \ p \geq 1,
\]
where \( h(\omega) \) is continuous and positive on \( S^{n-1} \), satisfies the conditions in Corollary 2. For the case \( g(r\omega, s) = r^{-2}h(\omega)s + s^{(n+2)/(n-2)} \), the existence of positive solutions for the problem (6) is investigated in [2].

In our proof we use the operator \( \Delta_g v = \frac{1}{2} \frac{n-2}{n} f(x, v, |x|) \) and \( \Delta_g v = \frac{1}{2} \frac{n-2}{n} f(x, v, |x|^2) \) for a solution \( v \) for the function \( f \) such that \( f \) is invariant under the transformation \( x \mapsto y = \lambda^2 x/|x|^2 \).

In Section 2 we prove Theorems. In fact, we only present the proof of Theorem 1 since the proof of Theorem 2 is very similar. In Appendix we show that the operator \( \Delta_g \) is invariant under the transformation by using of the property of the Kelvin transformation.

### 2. Proof of Theorems.

Due to similarity, we only give the proof of Theorem 1. Given \( \lambda \in (1, a) \), we set

\[
T_\lambda = \{ |x| = \lambda \} \quad \text{and} \quad \Sigma_\lambda = \{ \lambda < |x| < a \}.
\]

For \( x \in \Sigma_\lambda \), let \( x^\lambda = \lambda^2 x/|x|^2 \). Then we have

\[
|x| > |x^\lambda| = \frac{\lambda^2}{|x|} > \frac{1}{|x|} \quad \text{for} \quad x \in \Sigma_\lambda.
\]

Define the operator \( \Delta_g \) by \( \Delta_g v = |x|^2 \Delta v - (n-2)x \cdot \nabla v \). We note that \( \Delta_g \) is the Laplace-Beltrami operator on the Riemannian space \( (\mathbb{R}^n, dx^2/|x|^2) \). For a solution \( u \) of (1), the function \( v(x) = |x|^n u(x) \) satisfies

\[
|x|^2 \Delta v - (n-2)x \cdot \nabla v - \frac{(n-2)^2}{4} v + |x|^\frac{n+2}{2} f(x, v, (x \cdot \nabla) v) = 0
\]

in \( A \), which is written as

\[
\Delta_g v - \frac{(n-2)^2}{4} v + |x|^\frac{n+2}{2} f(x, v, (x \cdot \nabla) v) = 0 \quad \text{in} \quad A.
\]

Let \( v^\lambda(x) = v(x^\lambda) \) and \( y = x^\lambda \). By Lemma A in Appendix, we find that \( \Delta_g v^\lambda(x) = \Delta_g v(y) \). We have \( x \cdot \nabla = r \partial_r \) for \( r = |x| \) (see, e.g., [7]) and hence

\[
x \cdot \nabla x^\lambda = r \partial_r x^\lambda = -s \partial_s x^\lambda = -y \cdot \nabla y,
\]

where \( r = |x| \) and \( s = |y| = \lambda^2/r \). Therefore, the property \( f(x, s, -q) = f(x, s, q) \) implies the relation

\[
\Delta_g v^\lambda - \frac{(n-2)^2}{4} v^\lambda + |x|^\frac{n+2}{2} f(x^\lambda, v^\lambda, (x \cdot \nabla) v^\lambda) = 0 \quad \text{in} \quad A.
\]
It follows that $|x| > |x^\lambda| > 1/|x|$ and $1 < \lambda < |x| < a$ for $x \in \Sigma_\lambda$. Then the assumption on $f$ in Theorem 1 guarantees
\[ |x|^{\frac{n+2}{2}} f (x^\lambda, s, q) \geq |x|^{\frac{n+2}{2}} f (x, s, q) \]
for $x \in \Sigma_\lambda$, $s \geq 0$, and $q \geq 0$. Therefore, the function $w_\lambda = v^\lambda - v$ satisfies
\[ \Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda + |x|^{\frac{n+2}{2}} \left( f \left( x, v^\lambda, (x \cdot \nabla) v^\lambda \right) - f \left( x, v, (x \cdot \nabla) v \right) \right) \leq 0 \]
on $\Sigma_\lambda$. Writing
\[ b_\lambda (x) = \int_0^1 f_s \left( x, tv^\lambda(x) + (1-t)v(x), (x \cdot \nabla)v^\lambda(x) \right) dt \quad \text{and} \]
\[ c_\lambda (x) = \int_0^1 f_q \left( x, v(x), t(x \cdot \nabla)v^\lambda(x) + (1-t)(x \cdot \nabla)v(x) \right) dt, \]
we obtain
\[ \Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda + |x|^{\frac{n+2}{2}} (b_\lambda (x) + c_\lambda (x) x \cdot \nabla) w_\lambda \leq 0 \quad \text{on } \Sigma_\lambda. \]
Let $z_\lambda (x) = |x|^{\frac{n-2}{2}} w_\lambda (x)$. Then we have
\[ |x|^{\frac{n+2}{2}} \Delta z_\lambda = \Delta_g w_\lambda - \frac{(n-2)^2}{4} w_\lambda \quad \text{and} \quad |x|^{\frac{n-2}{2}} x \cdot \nabla z_\lambda = - \frac{n-2}{2} w_\lambda + x \cdot \nabla w_\lambda. \]
Define
\[ \tilde{b}_\lambda (x) = |x|^{\frac{n-2}{2}} \left( b_\lambda (x) + \frac{n-2}{2} c_\lambda (x) \right) \quad \text{and} \quad \tilde{c}_\lambda (x) = |x|^{\frac{n-2}{2}} c_\lambda (x). \]
We have shown the following lemma.

**Lemma 1.** *Under the assumptions of Theorem 1, each $\lambda \in (1, a)$ admits the inequality*
\[ \Delta z_\lambda + \tilde{b}_\lambda (x) z_\lambda + \tilde{c}_\lambda (x) x \cdot \nabla z_\lambda \leq 0 \quad \text{on } \Sigma_\lambda, \]
*where $z_\lambda (x) = |x|^{\frac{n-2}{2}} (v^\lambda - v)$.*

Once Lemma 1 is proven, Theorem 1 follows from the standard argument ([1]). Putting
\[ \Lambda = \{ \lambda \in (1, a) : z_\lambda > 0 \quad \text{in } \Sigma_\lambda \}, \]
we see that the desired consequence follows from $\Lambda = (1, a)$. We show
\[ \Lambda = (1, a) \] by three steps.

**Step 1.** *We have $[r_0, a) \subset \Lambda$ for $r_0$ close to $a$, that is, $\Lambda \neq \emptyset$.***

*Proof.* We see that the coefficients $\tilde{b}_\lambda (x)$ and $\tilde{c}_\lambda (x)$ in (9) are uniformly bounded. Then for $r_0$ close to $a$, the maximum principle holds for the
Equation (9) on any subdomain of \( A \setminus \overline{B}_{r_0} \) and for any \( \lambda \), where \( B_{r_0} = \{ x \in \mathbb{R}^n : |x| < r_0 \} \). (See e.g. [1].) This implies \([r_0, 1) \subset \Lambda. \)

We prepare the following lemma.

**Lemma 2.** (i) If \( \lambda \in \Lambda \), then

\[
\frac{\partial z_\lambda}{\partial \nu} < 0 \quad \text{on} \quad T_\lambda,
\]

where \( \nu \) denotes the outer unit normal vector on \( T_\lambda \) from \( \Sigma_\lambda \);

(ii) If \( \lambda \notin \Lambda \), then there exists some \( x_0 \in \Sigma_\lambda \cap \overline{B}_{r_0} \) such that \( z_{\lambda}(x_0) \leq 0 \).

**Proof.** (i) Let \( \lambda \in \Lambda \). Then we have \( z_{\lambda} = 0 \) on \( T_\lambda \), and \( z_\lambda > 0 \) in \( \Sigma_\lambda \). Therefore, Hopf’s boundary lemma can be applied by (9) so that (10) holds.

(ii) As we have proven in Step 1, \( \lambda < r_0 \) and hence \( \Sigma_\lambda \cap \overline{B}_{r_0} \neq \emptyset \). Suppose to the contrary that

\[
z_{\lambda}(x) > 0 \quad \text{on} \quad \Sigma_\lambda \cap \overline{B}_{r_0}.
\]

Then we get

\[
\Delta z_\lambda + \tilde{b}_\lambda(x) z_\lambda + \tilde{c}_\lambda(x) x \cdot \nabla z_\lambda \leq 0 \quad \text{in} \quad \Sigma_\lambda \setminus \overline{B}_{r_0},
\]

and

\[
z_\lambda \geq 0 \quad \text{on} \quad \partial (\Sigma_\lambda \setminus \overline{B}_{r_0}) .
\]

Now the maximum principle guarantees \( z_{\lambda} > 0 \) in \( \Sigma_\lambda \setminus \overline{B}_{r_0} \). However, we have \( z_{\lambda} > 0 \) in \( \Sigma_\lambda \cap \overline{B}_{r_0} \) and hence \( z_{\lambda} > 0 \) in \( \Sigma_\lambda \). This means \( \lambda \notin \Lambda \), a contradiction.

**Step 2.** \( \Lambda \) is left-open.

**Proof.** If \( \Lambda \) is not left-open, there exist \( \lambda_0 \in \Lambda \) and a sequence \( \{ \lambda_n \} \) satisfying

\[
\lambda_0 - \frac{1}{n} < \lambda_n < \lambda_0 \quad \text{and} \quad \lambda_n \notin \Lambda.
\]

Lemma 2 (ii) guarantees the existence of \( x_n \in \Sigma_{\lambda_n} \cap \overline{B}_{r_0} \) satisfying

\[
z_{\lambda_n}(x_n) \leq 0.
\]

Note that \( z_{\lambda_n} = 0 \) on \( T_{\lambda_n} \). Then we have a point \( y_n \) on the segment connecting \( x_n \) and \( \lambda_n^2 x_n/|x_n|^2 \) satisfying

\[
\frac{\partial z_{\lambda_n}}{\partial r}(y_n) \leq 0.
\]

Taking a subsequence if necessary, we may suppose the existence of some \( x_0 \in \Sigma_{\lambda_0} \cap \overline{B}_{r_0} \) satisfying \( x_n \rightarrow x_0 \). By (11) we obtain \( z_{\lambda_0}(x_0) \leq 0 \). Since
$\lambda_0 \in \Lambda$, we must have $x_0 \in T_{\lambda_0}$. In particular, $y_n \to x_0$ and $\partial z_{\lambda_0}/\partial r(x_0) \leq 0$ follows from (12). However, this is equivalent to

$$ \frac{\partial z_{\lambda_0}}{\partial \nu}(x_0) \geq 0, $$

which contradicts to (10) valid for $\lambda = \lambda_0 \in \Lambda$.

**Step 3.** $\Lambda$ is left-closed.

Proof. In fact, let $\{\lambda_n\} \subset \Lambda$ be a sequence satisfying $\lambda_n \downarrow \lambda_1 > 1$. Then, we have

$$ \Delta z_{\lambda_1} + \left( b_{\lambda_1}(x) + c_{\lambda_1}(x) \right) z_{\lambda_1} \leq 0 \quad \text{and} \quad z_{\lambda_1} \geq 0 \quad \text{in} \quad \Sigma_{\lambda_1}. $$

Since $z_{\lambda_1} > 0$ on $|x| = a$, we have $z_{\lambda_1} \not\equiv 0$ in $\Sigma_{\lambda_1}$. Therefore, the maximum principle implies $z_{\lambda_1} > 0$ in $\Sigma_{\lambda_1}$, or equivalently, $\lambda_1 \in \Lambda$.

As a consequence of Steps 1-3, we obtain $\Lambda = (1, a)$. This implies $v^1(x) \geq v(x)$ on $1 \leq |x| \leq a$, and then (2) holds. The property (3) follows from Lemma 2 (i). This completes the proof.

**Appendix.**

Let $\Delta_g v = |x|^2 \Delta v - (n - 2)x \cdot \nabla v$. We show that the operator $\Delta_g$ is invariant under the transformation $x \mapsto y = \lambda^2 x/|x|^2$, that is, $\Delta_g v = \Delta_g V$ for $v(x) = V(y)$. Here we use the well-known property of the Kelvin transformation $\eta = \xi/|\xi|^2$ expressed as

$$ \Delta_\eta U = |\xi|^{n+2} \Delta_\xi u \quad \text{for} \quad U(\eta) = |\xi|^{n-2} u(\xi). $$

**Lemma A.** Let $v(x) = V(y)$ and $y = \lambda^2 x/|x|^2$ with $\lambda > 0$. Then we have

$$ |x|^2 \Delta_x v - (n - 2)x \cdot \nabla_x v = |y|^2 \Delta_y V - (n - 2)y \cdot \nabla_y V, $$

where $\Delta_x = \sum_{i=1}^n \partial^2/\partial x_i^2$ and $x \cdot \nabla_x = \sum_{i=1}^n x_i \partial/\partial x_i$.

Proof. Writing $w(x) = |x|^{-\frac{n-2}{2}} v(x)$ we have

$$ |x|^\frac{n+2}{2} \Delta_x w = |x|^2 \Delta_x v - (n - 2)x \cdot \nabla_x v - \frac{(n - 2)^2}{4} v. $$

Similarly, writing $W(y) = |y|^{-\frac{n-2}{2}} V(y)$ we have

$$ |y|^\frac{n+2}{2} \Delta_y W = |y|^2 \Delta_y V - (n - 2)y \cdot \nabla_y V - \frac{(n - 2)^2}{4} V. $$
By $|y| = \lambda^2/|x|$ it follows that

$$|x|^{n-2} w(x) = v(x) = V(y) = |y|^{n-2} W(y) = \left(\frac{\lambda^2}{|x|}\right)^{n-2} W(y).$$

Then we obtain

$$W(y) = \left(\frac{|x|}{\lambda}\right)^{n-2} w(x).$$

By the property of the Kelvin transformation, we obtain

$$\Delta_y W = \left(\frac{|x|}{\lambda}\right)^{n+2} \Delta_x w.$$

Then we have

$$|x|^{n+2} \Delta_x w = \left(\frac{\lambda^2}{|x|}\right)^{n+2} \left(\frac{|x|}{\lambda}\right)^{n+2} \Delta_x w = |y|^{n+2} \Delta_y W.$$ 

Therefore, by (14) and (15), we obtain the property (13). This completes the proof. 

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\textbf{References}


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