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RESTRICTIONS OF RANK-2 SEMISTABLE VECTOR BUNDLES ON SURFACES IN POSITIVE CHARACTERISTIC

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We work over an algebraically closed field of positive characteristic. Let E be a semistable rank-2 vector bundle with respect to a very ample line bundle $\mathcal{O}(1)$ on a smooth projective surface. The purpose here is to give an effective bound d_0 such that if $d \geq d_0$ the restriction of E to a general member $C \in |\mathcal{O}(d)|$ is semistable.

1. Introduction.

Let E be a rank-r torsion free sheaf on a normal projective variety of dimension $n \geq 2$ defined over an algebraically closed field k. Assume that E is semistable with respect to a very ample line bundle $\mathcal{O}(1)$: Namely, if we set $\mu(F) = (c_1(F) \cdot \mathcal{O}(1)^{n-1})$ for a subsheaf F of E, $\mu(E) \geq \mu(F)$ holds for all subsheaf F of E.

A problem of finding a condition when the restriction $E|C$ to a member $C \in |\mathcal{O}(d)|$ is semistable on C has been considered by several authors ([**1**], [**3**], [**6**], [**7**], [**8**]): Maruyama [**6**] proved that if $r < n$ then $E|C$ is semistable for general $C \in |\mathcal{O}(d)|$ for every $d \geq 1$; Mehta and Ramanathan [**7**] proved that there exists an integer d_0 such that if $d \geq d_0$ then $E|C$ is semistable for general $C \in |\mathcal{O}(d)|$; Flenner [**3**] proved that if k is of characteristic 0 and d satisfies $\frac{\binom{d+n}{d}-d-1}{d} > (\mathcal{O}(1)^n) \cdot \max(\frac{r^2-1}{4}, 1)$ then $E|C$ is semistable for general $C \in |\mathcal{O}(d)|$. In other direction, in characteristic 0, Bogomolov [**1**] and Moriwaki [**8**] obtained an effective bound d_0 for some special restriction $E|C$ to be semistable.

The purpose here is to give an effective bound d_0 in positive characteristic when E is a rank-2 vector bundle on a surface: If $d \geq d_0$ the restriction $E|C$ of E to a general member $C \in |\mathcal{O}(d)|$ is semistable.

Our result is the following.

Theorem. *Let S be a smooth projective surface over an algebraically closed field k of characteristic* $\mathrm{char}(k) = p > 0$ *and $\mathcal{O}_S(1)$ a very ample line bundle on S. Let E be a rank-2 semistable vector bundle with respect to $\mathcal{O}_S(1)$ on S. Set* $\deg S = (\mathcal{O}_S(1)^2)$, $\Delta(E) = c_2(E) - (1/4)c_1^2(E)$, *and* $\nu = \min\{(\mathcal{M} \cdot$

$\mathcal{O}_S(1)) > 0 : \mathcal{M} \in \operatorname{Pic} S\}$. Let d be an integer with

$$d > \begin{cases} \frac{\Delta(E)}{\nu} + \frac{\sqrt{\Delta(E)}}{2\sqrt{3 \deg S}}, & \text{if } \Delta(E) > 0, \\ 0, & \text{if } \Delta(E) \leq 0. \end{cases}$$

Then the restriction $E|C$ to a general $C \in |\mathcal{O}_S(d)|$ is semistable on C.

The theorem is proved based on ideas of Ein [**2**] and Flenner [**3**].

2. Proof of Theorem.

Set $L = \mathcal{O}_S(d)$. Let $\mathbb{P}^{2*}$ be the projective space of lines in $\mathbb{P}^2$ and $\mathbb{F}$ the incidence correspondence $\{(x, \ell) \in \mathbb{P}^2 \times \mathbb{P}^{2*} : x \in \ell\}$, namely

$$\mathbb{F} = \mathbb{P}_{\mathbb{P}^2}(\Omega^1_{\mathbb{P}^2}(2)) \subseteq \mathbb{P}_{\mathbb{P}^2}(\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}^2 \times \mathbb{P}^{2*}.$$

Let $\phi \colon S \to \mathbb{P}^2$ be a (separable) finite morphism defined by a 2-dimensional, base-point-free, linear subsystem $\mathfrak{d}$ of $|L|$ containing a general curve $C \in |L|$. Pulling-back the correspondence $\mathbb{F}$ by ϕ, we have the following diagram:

$$\begin{array}{ccccc} X & \longrightarrow & \mathbb{F} & \xrightarrow{\rho_0} & \mathbb{P}^{2*} \\ \pi \downarrow & \square & \downarrow \pi_0 & & \\ S & \underset{\phi}{\longrightarrow} & \mathbb{P}^2 & & \end{array} .$$

We denote the composite $X \to \mathbb{F} \to \mathbb{P}^{2*}$ by ρ.

Assume that the restriction $E|C$ to a general curve $C \in \mathfrak{d} \subset |L|$ is not semistable. In other words, the restriction $\pi^* E|\rho^{-1}(\ell)$ to $\rho^{-1}(\ell)$ for a general $\ell \in \mathbb{P}^{2*}$ is not semistable, since $\rho^{-1}(\ell)$ is isomorphic to a divisor $C \in \mathfrak{d}$ and $\pi^* E|\rho^{-1}(\ell) \cong E|C$ under this isomorphism. Consider a relative Harder-Narasimhan filtration (HN-filtration) of $\pi^* E$ over ρ, which has a property that its restriction to $\rho^{-1}(\ell)$ for a general $\ell \in \mathbb{P}^{2*}$ is the HN-filtration of $\pi^* E|\rho^{-1}(\ell)$ (see [**4**, (3.2)]). By assumption, the relative HN-filtration is $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = \pi^* E$ and we may assume that $\mathcal{E}_1$ is locally free of rank 1 on X. Hence if W denotes the class of the tautological bundle of $X = \mathbb{P}_S(\phi^*(\Omega^1_{\mathbb{P}^2}(2)))$, we have $\mathcal{E}_1 \cong \mathcal{O}_X(aW) \otimes \pi^* \mathcal{M}$ for some $a \in \mathbb{Z}$ and $\mathcal{M} \in \operatorname{Pic} S$, since $\operatorname{Pic} X \cong \mathbb{Z}W \oplus \pi^* \operatorname{Pic} S$ (see [**5**, Ch. III Ex. 12.5]). Since $\mathcal{M}|\pi(\rho^{-1}(\ell)) \subset E|\pi(\rho^{-1}(\ell))$ is the HN-filtration of $E|\pi(\rho^{-1}(\ell))$ for a general $\ell \in \mathbb{P}^{2*}$, we have

$$(c_1(E) - 2\mathcal{M} \cdot L) < 0. \tag{1}$$

Consequently $H^0(\rho^{-1}(\ell), \pi^*(E \otimes \mathcal{M}^\vee)|\rho^{-1}(\ell)) \cong k$ for a general $\ell \in \mathbb{P}^{2*}$, and hence $\rho_* \pi^*(E \otimes \mathcal{M}^\vee)$ is of rank 1 and reflexive. Therefore we have $\rho_* \pi^*(E \otimes \mathcal{M}^\vee) = \mathcal{O}_{\mathbb{P}^{2*}}(-t)$ for some $t \in \mathbb{Z}$. The semistability of E implies

$$t > 0 \tag{2}$$

since $H^0(\mathbb{P}^{2*}, \rho_*\pi^*(E\otimes\mathcal{M}^\vee)) = H^0(X, \pi^*(E\otimes\mathcal{M}^\vee)) = H^0(S, E\otimes\mathcal{M}^\vee)$ and (1) holds. The natural map $\mathcal{O}_X(-tW) = \rho^*\rho_*\pi^*(E\otimes\mathcal{M}^\vee) \to \pi^*(E\otimes\mathcal{M}^\vee)$ induces an exact sequence

(3)

$$0 \to \mathcal{O}_X(-tW)\otimes\pi^*\mathcal{M} \to \pi^*E \to \mathcal{O}_X(tW)\otimes\pi^*\mathcal{O}_S(c_1(E)-\mathcal{M})\otimes\mathcal{I}_Z \to 0$$

with a closed subscheme Z of codimension 2 in X.

The surjection in (3) induces a unique morphism $\sigma\colon X\setminus Z \to \mathbb{P}_S(E)$ with

$$\sigma^*\mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_X(tW)\otimes\pi^*\mathcal{O}_S(c_1(E)-\mathcal{M})|(X\setminus Z)$$
$$\sigma^*\Omega^1_{\mathbb{P}(E)/S} = \mathcal{O}_X(-2tW)\otimes\pi^*\mathcal{O}_S(2\mathcal{M}-c_1(E))|(X\setminus Z),$$

by the universal property of projective bundle $\tau\colon \mathbb{P}_S(E)\to S$. If the differential

$$d\sigma\colon \sigma^*\Omega^1_{\mathbb{P}(E)/S} \to \Omega^1_{X/S}|(X\setminus Z)$$

is zero, then S-morphism σ factors through the relative Frobenius $F_{(X\setminus Z)/S}\colon X\setminus Z \to (X\setminus Z)^{(1)}$ of $X\setminus Z$ over S (see [**2**, (1.4)]). Namely there exists an S-morphism $\sigma_1\colon (X\setminus Z)^{(1)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_1\circ F_{(X\setminus Z)/S}$. Here for an S-scheme Y, by $Y^{(r)}$ we denote the base change of the structure morphism $\eta\colon Y\to S$ by the rth (absolute) Frobenius $F^r_S\colon S\to S$; $F^r_{Y/S}\colon Y\to Y^{(r)}$ is the S-morphism induced by the (absolute) Frobenius $F^r_Y\colon Y\to Y$ of Y and the structure morphism η by the property of products. Furthermore, if $d\sigma_1 = 0$, then there exists a morphism $\sigma_2\colon (X\setminus Z)^{(2)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_2\circ F^2_{(X\setminus Z)/S}$. Proceeding in this way with [**2**, (1.4)], we claim that there exists a morphism $\sigma_r\colon (X\setminus Z)^{(r)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_r\circ F^r_{(X\setminus Z)/S}$ and the relative differential

$$d\sigma_r\colon \sigma_r^*\Omega^1_{\mathbb{P}(E)/S} \to \Omega^1_{X^{(r)}/S}|(X\setminus Z)^{(r)}$$

is nonzero for some $r\geq 0$. In fact, suppose that we have a morphism $\sigma_r\colon (X\setminus Z)^{(r)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_r\circ F^r_{(X\setminus Z)/S}$ for some $r\geq 0$. Here we set $\sigma_0 = \sigma$ if $r=0$. Then we have the following diagram:

$$\begin{array}{ccccc}
X\setminus Z & \overset{F^r_{(X/Z)/S}}{\longrightarrow} & (X\setminus Z)^{(r)} & & \\
\downarrow & & \downarrow & \searrow^{\sigma_r} & \\
X & \overset{F^r_{X/S}}{\longrightarrow} & X^{(r)} & & \mathbb{P}_S(E) \\
\downarrow\pi & & \downarrow\pi_r & \swarrow_{\tau} & \\
S & = & S. & &
\end{array}$$

Since $X \cong \mathbb{P}_S(\phi^*(\Omega^1_{\mathbb{P}^2}(2)))$, we have $X^{(r)} \cong \mathbb{P}_S(F_S^{r*}\phi^*(\Omega^1_{\mathbb{P}^2}(2)))$. If W' is the class of the tautological line bundle of $X^{(r)}$ over S, then $F^{r*}_{X/S}\mathcal{O}_{X^{(r)}}(W') \cong \mathcal{O}_X(p^rW)$ and $\Omega^1_{X^{(r)}/S} \cong \pi_r^*(p^rL) \otimes \mathcal{O}_{X^{(r)}}(-2W')$. On the other hand, $F^{r*}_{X/S}\pi_r^*\mathcal{A} \cong \pi^*\mathcal{A}$ for every $\mathcal{A} \in \operatorname{Pic} S$. Since $\sigma = \sigma_r \circ F^r_{(X\setminus Z)/S}$ and $\operatorname{Pic} X^{(r)} \cong \mathbb{Z}W' \oplus \pi_r^*\operatorname{Pic} S$, the morphism σ_r induces an exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_{X^{(r)}}\left(-\frac{t}{p^r}W'\right) \otimes \pi_r^*\mathcal{M} \to \pi_r^*E \\ &\to \mathcal{O}_{X^{(r)}}\left(\frac{t}{p^r}W'\right) \otimes \pi_r^*\mathcal{O}_S(c_1(E) - \mathcal{M}) \otimes \mathcal{I}_{Z'} \to 0, \end{aligned}$$

and we have $t/p^r \in \mathbb{Z}$, where Z' is a codimension 2 closed subscheme of $X^{(r)}$. Since $t > 0$, the latter implies that σ factors through the relative Frobenius over S only in finite times. Therefore for some $r \geq 0$, the morphism σ_r must have the nonzero relative differential $d\sigma_r$, as required.

We take such $r \geq 0$ and $\sigma_r\colon (X \setminus Z)^{(r)} \to \mathbb{P}_S(E)$. If $C \in \mathfrak{d} \subset |L|$, since $C \cong \rho^{-1}(\ell) \cong F^r_{X/S}(\rho^{-1}(\ell))$ for some $\ell \in \mathbb{P}^{2*}$, we can consider $C \subset X^{(r)}$. Then we have $\mathcal{O}_{X^{(r)}}(W')|C \cong \mathcal{O}_C$ and $\pi_r^*\mathcal{A}|C \cong \mathcal{A}|C$ for every $\mathcal{A} \in \operatorname{Pic} S$. The restriction $d\sigma_r|C$ to general $C \in \mathfrak{d}$ is nonzero by the choice of r. This implies that

$$(L \cdot 2\mathcal{M} - c_1(E)) \leq p^r(L^2) \leq t(L^2), \tag{4}$$

since

$$\begin{aligned} \sigma_r^*\Omega^1_{\mathbb{P}(E)/S}|C &= \mathcal{O}_C(2\mathcal{M} - c_1(E)) \\ \Omega^1_{X^{(r)}/S}|C &= \mathcal{O}_C(p^rL), \end{aligned}$$

and since $t/p^r \in \mathbb{Z}$.

Restricting the exact sequence (3) to a general member $W \in |\mathcal{O}_X(W)|$ not containing any associate points of Z, we have an exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_W(-tW) \otimes \pi^*\mathcal{M}|W \to \pi^*E|W \\ &\to \mathcal{O}_W(tW) \otimes \pi^*\mathcal{O}_S(c_1(E) - \mathcal{M})|W \otimes \mathcal{I}_{Z\cap W} \to 0. \end{aligned}$$

On the other hand, we note that $W^3 = 0$ and $W^2 \cdot \pi^*\mathcal{A} = (\mathcal{A} \cdot L)$ for any $\mathcal{A} \in \operatorname{Pic} S$, since $W^2 - \pi^*L \cdot W + (\pi^*L^2) = W^2 - c_1(\phi^*(\Omega^1_{\mathbb{P}^2}(2))) \cdot W + c_2(\phi^*(\Omega^1_{\mathbb{P}^2}(2))) = 0$. Thus from the exact sequence above, noting that

$W \to S$ is birational via π, we have

$$\begin{aligned} c_2(E) &= c_2(\pi^*E|W) \\ &= -t^2(\mathcal{O}_W(W)^2) + t(\mathcal{O}_W(W) \cdot \pi^*\mathcal{O}_S(2\mathcal{M} - c_1(E))|W) \\ &\quad + (\pi^*\mathcal{M}|W \cdot \pi^*\mathcal{O}_S(c_1(E) - \mathcal{M})|W) + \deg(Z \cap W) \\ &= t(L \cdot 2\mathcal{M} - c_1(E)) - (\mathcal{M} \cdot \mathcal{M} - c_1(E)) + \deg(Z \cap W) \\ &\geq t(L \cdot 2\mathcal{M} - c_1(E)) - (\mathcal{M} \cdot \mathcal{M} - c_1(E)), \end{aligned}$$

and hence

$$\Delta(E) \geq 2t(L \cdot \mathcal{M} - (1/2)c_1(E)) - ((\mathcal{M} - (1/2)c_1(E))^2).$$

By the Hodge index theorem for L and $\mathcal{M} - (1/2)c_1(E)$, we have

$$\Delta(E) \geq 2t(L \cdot \mathcal{M} - (1/2)c_1(E)) - \frac{(L \cdot \mathcal{M} - (1/2)c_1(E))^2}{(L^2)}. \tag{5}$$

From (4) and (5), it follows that

$$\Delta(E) \geq 3\frac{(L \cdot \mathcal{M} - (1/2)c_1(E))^2}{(L^2)}.$$

When $\Delta(E) \leq 0$, this contradicts (1). When $\Delta(E) > 0$, we have

$$(L \cdot \mathcal{M} - (1/2)c_1(E)) \leq \sqrt{\frac{\Delta(E)(L^2)}{3}}. \tag{6}$$

On the other hand, from (5), it follows

$$\frac{\Delta(E)}{(L \cdot \mathcal{M} - (1/2)c_1(E))} + \frac{(L \cdot \mathcal{M} - (1/2)c_1(E))}{(L^2)} \geq 2t.$$

Since $L = \mathcal{O}_S(d)$, by using the assumption $(\mathcal{O}_S(1) \cdot \mathcal{M} - (1/2)c_1(E)) \geq \nu/2$ to the first term and (6) to the second term, we have

$$\frac{1}{d}\left(\frac{\Delta(E)}{\nu} + \frac{\sqrt{\Delta(E)}}{2\sqrt{3}\deg S}\right) \geq t.$$

By assumption of d, we have $t < 1$ hence $t \leq 0$, which contradicts (2). This completes the proof. □

Remark. Let S, $\mathcal{O}_S(1)$, E and d be as in Theorem.

(1) Let $\mathfrak{d}$ be a 2-dimensional linear subsystem of $|\mathcal{O}_S(d)|$ defining a separable, finite morphism from S to $\mathbb{P}^2$. The proof of theorem implies that the restriction $E|C$ is semistable for a general member $C \in \mathfrak{d}$.

(2) Assume that $\Delta(E) > 0$ and that the restriction $E|C$ to be a general member of $C \in |\mathcal{O}_S(1)|$ is not semistable with HN-filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = E|C$. Then it follows from (6) that

$$\deg \mathcal{E}_1 - (1/2)\deg(E|C) \leq \sqrt{\deg S \cdot \Delta(E)/3}$$

holds. This inequality is exactly that of Ein in [**2**, (4.1)] when $S = \mathbb{P}^2$ and $\mathcal{O}_S(1) = \mathcal{O}_{\mathbb{P}^2}(1)$.

(3) I do not know the bound in Theorem is optimal or not. For example, let E be the mth Frobenius pull-back $F^{m*}(\Omega_{\mathbb{P}^2}(2))$ of the twisted cotangent bundle on $\mathbb{P}^2$, which plays an important role in the proof of Theorem. We know that E is semistable (see for example [**2**]) and $\Delta(E) = p^{2m}/4$. Thus Theorem implies that $E|C$ is semistable on a general curve C of degree d if $d > p^{2m}/4 + p^m/(4\sqrt{3})$. On the other hand, from a calculation of $H^0(C, E(-(p^m+1)/2)|C)$ by using the Euler sequence, it follows that $E|C$ is semistable for general C of degree d if $d > (3p^m + 5)/4$ for $p \neq 2$.

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