RESTRICTIONS OF RANK-2 SEMISTABLE VECTOR BUNDLES ON SURFACES IN POSITIVE CHARACTERISTIC

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We work over an algebraically closed field of positive characteristic. Let $E$ be a semistable rank-2 vector bundle with respect to a very ample line bundle $O(1)$ on a smooth projective surface. The purpose here is to give an effective bound $d_0$ such that if $d \geq d_0$ the restriction of $E$ to a general member $C \in |O(d)|$ is semistable.

1. Introduction.

Let $E$ be a rank-$r$ torsion free sheaf on a normal projective variety of dimension $n \geq 2$ defined over an algebraically closed field $k$. Assume that $E$ is semistable with respect to a very ample line bundle $O(1)$: Namely, if we set $\mu(F) = (c_1(F) \cdot O(1))^{n-1}$ for a subsheaf $F$ of $E$, $\mu(E) \geq \mu(F)$ holds for all subsheaf $F$ of $E$.

A problem of finding a condition when the restriction $E|C$ to a member $C \in |O(d)|$ is semistable on $C$ has been considered by several authors ([1], [3], [6], [7], [8]): Maruyama [6] proved that if $r < n$ then $E|C$ is semistable for general $C \in |O(d)|$ for every $d \geq 1$; Mehta and Ramanathan [7] proved that there exists an integer $d_0$ such that if $d \geq d_0$ then $E|C$ is semistable for general $C \in |O(d)|$; Flenner [3] proved that if $k$ is of characteristic 0 and $d$ satisfies $(\frac{d+n}{d-1}) > (O(1)^n) \cdot \max(\frac{2}{4}, 1)$ then $E|C$ is semistable for general $C \in |O(d)|$. In other direction, in characteristic 0, Bogomolov [1] and Moriwaki [8] obtained an effective bound $d_0$ for some special restriction $E|C$ to be semistable.

The purpose here is to give an effective bound $d_0$ in positive characteristic when $E$ is a rank-2 vector bundle on a surface: If $d \geq d_0$ the restriction $E|C$ of $E$ to a general member $C \in |O(d)|$ is semistable.

Our result is the following.

Theorem. Let $S$ be a smooth projective surface over an algebraically closed field $k$ of characteristic $\text{char}(k) = p > 0$ and $O_S(1)$ a very ample line bundle on $S$. Let $E$ be a rank-2 semistable vector bundle with respect to $O_S(1)$ on $S$. Set $\deg S = (O_S(1)^2)$, $\Delta(E) = c_2(E) - (1/4)c_1^2(E)$, and $\nu = \min\{(M \cdot$
Theorem \(\mathcal{O}_S(1)) > 0 : \mathcal{M} \in \text{Pic } S \}. \) Let \(d\) be an integer with
\[
d > \begin{cases} 
\frac{\Delta(E)}{2} + \frac{\sqrt{\Delta(E)}}{2 \sqrt{\text{deg} S}}, & \text{if } \Delta(E) > 0, \\
0, & \text{if } \Delta(E) \leq 0.
\end{cases}
\]
Then the restriction \(E|_C\) to a general \(C \in |\mathcal{O}_S(d)|\) is semistable on \(C\).

The theorem is proved based on ideas of Ein [2] and Flenner [3].

2. Proof of Theorem.

Set \(L = \mathcal{O}_S(d)\). Let \(\mathbb{P}^2\) be the projective space of lines in \(\mathbb{P}^2\) and \(F\) the incidence correspondence \(\{ (x, \ell) \in \mathbb{P}^2 \times \mathbb{P}^2 : x \in \ell \}\), namely
\[
F = \mathbb{P}_{\mathbb{P}^2}(\Omega_{\mathbb{P}^2}^1(2)) \subseteq \mathbb{P}_{\mathbb{P}^2}(\wedge^2 H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes \mathcal{O}_{\mathbb{P}^2}) = \mathbb{P}^2 \times \mathbb{P}^2.
\]
Let \(\phi : S \to \mathbb{P}^2\) be a (separable) finite morphism defined by a 2-dimensional, base-point-free, linear subsystem \(\mathfrak{d}\) of \([L]\) containing a general curve \(C \in |L|\). Pulling-back the correspondence \(F\) by \(\phi\), we have the following diagram:
\[
\begin{array}{ccc}
X & \longrightarrow & F & \longrightarrow & \mathbb{P}^2\\
\pi \downarrow & & \phi_0 & & \downarrow 0\\
S & \longrightarrow & \mathbb{P}^2
\end{array}
\]
We denote the composite \(X \to F \to \mathbb{P}^2\) by \(\rho\).

Assume that the restriction \(E|_C\) to a general curve \(C \in \mathfrak{d} \subset |L|\) is not semistable. In other words, the restriction \(\pi^*E|_{\rho^{-1}(\ell)}\) to \(\rho^{-1}(\ell)\) for a general \(\ell \in \mathbb{P}^2\) is not semistable, since \(\rho^{-1}(\ell)\) is isomorphic to a divisor \(C \in \mathfrak{d}\) and \(\pi^*E|_{\rho^{-1}(\ell)} \cong E|_C\) under this isomorphism. Consider a relative Harder-Narasimhan filtration (HN-filtration) of \(\pi^*E\) over \(\rho\), which has a property that its restriction to \(\rho^{-1}(\ell)\) for a general \(\ell \in \mathbb{P}^2\) is the HN-filtration of \(\pi^*E|_{\rho^{-1}(\ell)}\) (see [4, (3.2)]). By assumption, the relative HN-filtration is
\[
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = \pi^*E
\]
and we may assume that \(\mathcal{E}_1\) is locally free of rank 1 on \(X\). Hence if \(W\) denotes the class of the tautological bundle of \(X = \mathbb{P}_S(\phi^*(\Omega_{\mathbb{P}^2}(2)))\), we have \(\mathcal{E}_1 \cong \mathcal{O}_X(aW) \otimes \pi^*\mathcal{M}\) for some \(a \in \mathbb{Z}\) and \(\mathcal{M} \in \text{Pic } S\), since \(\text{Pic } X \cong \mathbb{Z}W \oplus \pi^*\text{Pic } S\) (see [5, Ch. III Ex. 12.5]). Since \(\mathcal{M}|_{\pi(\rho^{-1}(\ell))} \subset E|\pi(\rho^{-1}(\ell))\) is the HN-filtration of \(E|\pi(\rho^{-1}(\ell))\) for a general \(\ell \in \mathbb{P}^2\), we have
\[
(1) \quad (c_1(E) - 2\mathcal{M} \cdot L) < 0.
\]
Consequently \(H^0(\rho^{-1}(\ell), \pi^*(E \otimes \mathcal{M}^\vee)|_{\rho^{-1}(\ell)}) \cong k\) for a general \(\ell \in \mathbb{P}^2\), and hence \(\rho_*\pi^*(E \otimes \mathcal{M}^\vee)\) is of rank 1 and reflexive. Therefore we have \(\rho_*\pi^*(E \otimes \mathcal{M}^\vee) = \mathcal{O}_{\mathbb{P}^2}(t)\) for some \(t \in \mathbb{Z}\). The semistability of \(E\) implies
\[
(2) \quad t > 0
\]
since $H^0(\mathbb{P}^2, \rho_\ast \pi^* (E \otimes \mathcal{M}^\vee)) = H^0(X, \pi^* (E \otimes \mathcal{M}^\vee)) = H^0(S, E \otimes \mathcal{M}^\vee)$ and (1) holds. The natural map $\mathcal{O}_X(-tW) = \rho^* \rho_\ast \pi^* (E \otimes \mathcal{M}^\vee) \to \pi^* (E \otimes \mathcal{M}^\vee)$ induces an exact sequence

(3)

$0 \to \mathcal{O}_X(-tW) \otimes \pi^* \mathcal{M} \to \pi^* E \to \mathcal{O}_X(tW) \otimes \pi^* \mathcal{O}_S(c_1(E) - \mathcal{M}) \otimes \mathcal{I}_Z \to 0$

with a closed subscheme $Z$ of codimension 2 in $X$.

The surjection in (3) induces a unique morphism $\sigma: X \setminus Z \to \mathbb{P}_S(E)$ with

$\sigma^* \mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_X(tW) \otimes \pi^* \mathcal{O}_S(c_1(E) - \mathcal{M})|_{(X \setminus Z)}$

$\sigma^* \Omega^1_{\mathbb{P}(E)/S} = \mathcal{O}_X(-2tW) \otimes \pi^* \mathcal{O}_S(2\mathcal{M} - c_1(E))|_{(X \setminus Z)},$

by the universal property of projective bundle $\tau: \mathbb{P}_S(E) \to S$. If the differential

$$d\sigma: \sigma^* \Omega^1_{\mathbb{P}(E)/S} \to \Omega^1_{\mathbb{P}(E)/S}|_{(X \setminus Z)}$$

is zero, then $S$-morphism $\sigma$ factors through the relative Frobenius $F_{(X \setminus Z)/S}: X \setminus Z \to (X \setminus Z)^{(1)}$ of $X \setminus Z$ over $S$ (see [2, (1.4)]). Namely there exists an $S$-morphism $\sigma_1: (X \setminus Z)^{(1)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_1 \circ F_{(X \setminus Z)/S}$. Here for an $S$-scheme $Y$, by $Y^{(r)}$ we denote the base change of the structure morphism $\eta: Y \to S$ by the $r$th (absolute) Frobenius $F_X: S \to S$; $F_Y^{(r)}: Y \to Y^{(r)}$ is the $S$-morphism induced by the (absolute) Frobenius $F_Y: Y \to Y$ of $Y$ and the structure morphism $\eta$ by the property of products. Furthermore, if $d\sigma_1 = 0$, then there exists a morphism $\sigma_2: (X \setminus Z)^{(2)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_2 \circ F^2_{(X \setminus Z)/S}$. Proceeding in this way with [2, (1.4)], we claim that there exists a morphism $\sigma_r: (X \setminus Z)^{(r)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_r \circ F^r_{(X \setminus Z)/S}$ and the relative differential

$$d\sigma_r: \sigma_r^* \Omega^1_{\mathbb{P}(E)/S} \to \Omega^1_{\mathbb{P}(E)/S}|_{(X \setminus Z)^{(r)}}$$

is nonzero for some $r \geq 0$. In fact, suppose that we have a morphism $\sigma_r: (X \setminus Z)^{(r)} \to \mathbb{P}_S(E)$ such that $\sigma = \sigma_r \circ F^r_{(X \setminus Z)/S}$ for some $r \geq 0$. Here we set $\sigma_0 = \sigma$ if $r = 0$. Then we have the following diagram:

$$
\begin{array}{ccc}
X \setminus Z & \xrightarrow{F^r_{(X \setminus Z)/S}} & (X \setminus Z)^{(r)} \\
\downarrow & & \downarrow \quad \sigma_r \\
X & \xrightarrow{F^r_X} & X^{(r)} \\
\downarrow \pi & & \downarrow \pi_r \\
S & = & S.
\end{array}
$$
Since $X \cong \mathbb{P}_S(\phi^*(\Omega^1_{\mathbb{P}_2}(2)))$, we have $X^{(r)} \cong \mathbb{P}_S(F^r_S \phi^*(\Omega^1_{\mathbb{P}_2}(2)))$. If $W'$ is the class of the tautological line bundle of $X^{(r)}$ over $S$, then $F^r_{X/S} \mathcal{O}_{X^{(r)}}(W') \cong \mathcal{O}_X(p^r W)$ and $\Omega^1_{X^{(r)}/S} \cong \pi^*_r(p^r L) \otimes \mathcal{O}_{X^{(r)}}(-2W')$. On the other hand, $F^r_{X/S} \pi^*_r \mathcal{A} \cong \pi^*_r \mathcal{A}$ for every $\mathcal{A} \in \text{Pic } S$. Since $\sigma = \sigma_r \circ F^r_{(X \setminus Z)/S}$ and $\text{Pic } X^{(r)} \cong ZW' \oplus \pi^*_r \text{Pic } S$, the morphism $\sigma_r$ induces an exact sequence

$$0 \to \mathcal{O}_{X^{(r)}} \left(-\frac{t}{p^r} W'\right) \otimes \pi^*_r \mathcal{M} \to \pi^*_r \mathcal{E}$$

and we have $t/p^r \in \mathbb{Z}$, where $Z'$ is a codimension 2 closed subscheme of $X^{(r)}$. Since $t > 0$, the latter implies that $\sigma$ factors through the relative Frobenius over $S$ only in finite times. Therefore for some $r \geq 0$, the morphism $\sigma_r$ must have the nonzero relative differential $d\sigma_r$, as required.

We take such $r \geq 0$ and $\sigma_r \colon (X \setminus Z)^{(r)} \to \mathbb{P}_S(E)$. If $C \in \mathfrak{d} \subset |L|$, since $C \cong \rho^{-1}(\ell) \cong F^r_{X/S}(\rho^{-1}(\ell))$ for some $\ell \in \mathbb{P}^{2*}$, we can consider $C \subset X^{(r)}$. Then we have $\mathcal{O}_{X^{(r)}}(W')|C \cong \mathcal{O}_C$ and $\pi^*_r \mathcal{A}|C \cong \mathcal{A}|C$ for every $\mathcal{A} \in \text{Pic } S$. The restriction $d\sigma_r|C$ to general $C \in \mathfrak{d}$ is nonzero by the choice of $r$. This implies that

$$(4) \quad (L \cdot 2\mathcal{M} - c_1(E)) \leq p^r(L^2) \leq t(L^2),$$

since

$$\sigma^*r \mathcal{O}_{E/S}|C = \mathcal{O}_C(2\mathcal{M} - c_1(E))$$

$$\Omega^1_{X^{(r)}/S}|C = \mathcal{O}_C(p^r L),$$

and since $t/p^r \in \mathbb{Z}$.

Restricting the exact sequence (3) to a general member $W \in |\mathcal{O}_X(W)|$ not containing any associate points of $Z$, we have an exact sequence

$$0 \to \mathcal{O}_W(-tW) \otimes \pi^* \mathcal{M}|W \to \pi^* \mathcal{E}|W$$

$$\to \mathcal{O}_W(tW) \otimes \pi^* \mathcal{O}_S(c_1(E) - \mathcal{M})|W \otimes \mathcal{I}_{Z \cap W} \to 0.$$
$W \to S$ is birational via $\pi$, we have
\[
c_2(E) = c_2(\pi^*E|W)
\]
\[
= -t^2(O_W(W)^2) + t(O_W(W) \cdot \pi^*O_S(2M - c_1(E))|W)
\]
\[
+ (\pi^*M|W \cdot \pi^*O_S(c_1(E) - M)|W) + \deg(Z \cap W)
\]
\[
= t(L \cdot 2M - c_1(E)) - (M \cdot M - c_1(E)) + \deg(Z \cap W)
\]
\[
\geq t(L \cdot 2M - c_1(E)) - (M \cdot M - c_1(E)),
\]
and hence
\[
\Delta(E) \geq 2t(L \cdot M - (1/2)c_1(E)) - ((M - (1/2)c_1(E))^2).
\]

By the Hodge index theorem for $L$ and $M - (1/2)c_1(E)$, we have
\[
\Delta(E) \geq 2t(L \cdot M - (1/2)c_1(E)) - \frac{(L \cdot M - (1/2)c_1(E))^2}{(L^2)}.
\]

From (4) and (5), it follows that
\[
\Delta(E) \geq 3\frac{(L \cdot M - (1/2)c_1(E))^2}{(L^2)}.
\]

When $\Delta(E) \leq 0$, this contradicts (1). When $\Delta(E) > 0$, we have
\[
(L \cdot M - (1/2)c_1(E)) \leq \sqrt{\frac{\Delta(E)(L^2)}{3}}.
\]

On the other hand, from (5), it follows
\[
\frac{\Delta(E)}{(L \cdot M - (1/2)c_1(E))} + \frac{(L \cdot M - (1/2)c_1(E))}{(L^2)} \geq 2t.
\]

Since $L = O_S(d)$, by using the assumption $(O_S(1) \cdot M - (1/2)c_1(E)) \geq \nu/2$ to the first term and (6) to the second term, we have
\[
\frac{1}{d} \left( \frac{\Delta(E)}{\nu} + \frac{\sqrt{\Delta(E)}}{2\sqrt{3} \deg S} \right) \geq t.
\]

By assumption of $d$, we have $t < 1$ hence $t \leq 0$, which contradicts (2). This completes the proof. \qed

**Remark.** Let $S, O_S(1), E$ and $d$ be as in Theorem.

1. Let $d$ be a 2-dimensional linear subsystem of $|O_S(d)|$ defining a separable, finite morphism from $S$ to $\mathbb{P}^2$. The proof of theorem implies that the restriction $E|C$ is semistable for a general member $C \in d$.

2. Assume that $\Delta(E) > 0$ and that the restriction $E|C$ to be a general member of $C \in |O_S(1)|$ is not semistable with HN-filtration $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 = E|C$. Then it follows from (6) that
\[
\deg \mathcal{E}_1 - (1/2) \deg(E|C) \leq \sqrt{\deg S \cdot \Delta(E)/3}
\]
holds. This inequality is exactly that of Ein in [2, (4.1)] when $S = \mathbb{P}^2$ and $\mathcal{O}_S(1) = \mathcal{O}_{\mathbb{P}^2}(1)$.

(3) I do not know the bound in Theorem is optimal or not. For example, let $E$ be the $m$th Frobenius pull-back $F^m_{\mathbb{P}^2} (\Omega_{\mathbb{P}^2}(2))$ of the twisted cotangent bundle on $\mathbb{P}^2$, which plays an important role in the proof of Theorem. We know that $E$ is semistable (see for example [2]) and $\Delta(E) = p^{2m}/4$. Thus Theorem implies that $E|C$ is semistable on a general curve $C$ of degree $d$ if $d > p^{2m}/4 + p^m/(4\sqrt{3})$. On the other hand, from a calculation of $H^0(C, E(-(p^m + 1)/2))(C)$ by using the Euler sequence, it follows that $E|C$ is semistable for general $C$ of degree $d$ if $d > (3p^m + 5)/4$ for $p \neq 2$.

References


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