RESTRICTED PRÉKOPA–LEINDLER INEQUALITY

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We prove a functional version of the Brunn-Minkowski inequality for restricted sums obtained by Szarek and Voiculescu.

We only consider Lebesgue-measurable subsets of $\mathbb{R}^n$, and for $A \subset \mathbb{R}^n$, we denote its volume by $|A|$. If $A, B \subset \mathbb{R}^n$, their Minkowski sum is defined by

$$A + B = \{x + y, (x, y) \in A \times B\}.$$

The classical Brunn-Minkowski inequality provides a lower bound for its volume.

**Theorem 1.** Let $A, B$ be compact, non void subsets of $\mathbb{R}^n$, one has

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

In their study of the free analogue of the entropy power inequality [SV], Szarek and Voiculescu define the notion of restricted Minkowski sum of $A$ and $B$ with respect to $\Theta \subset A \times B$:

$$A + \Theta B = \{x + y, (x, y) \in \Theta\},$$

and show that an analogue of the Brunn-Minkowski inequality holds:

**Theorem 1’.** There exists a positive constant $c$ such that for all $\rho \in ]0, 1[$, $n \in \mathbb{N}$, for all $A, B \subset \mathbb{R}^n$ and $\Theta \subset A \times B$ such that:

$$\rho \leq \left(\frac{|A|}{|B|}\right)^{\frac{1}{n}} \leq \rho^{-1} \quad \text{and} \quad \frac{|\Theta|}{|A| \cdot |B|} \geq 1 - c \min(\rho \sqrt{n}, 1),$$

one has

$$|A + \Theta B|^{\frac{2}{n}} \geq |A|^{\frac{2}{n}} + |B|^{\frac{2}{n}}.$$

It is well known that the Brunn-Minkowski inequality can be derived from the Prékopa-Leindler inequality [Pré], [Lei], which we recall here:
Theorem 2. Let \( f, g \) be non-negative functions in \( L_1(\mathbb{R}^n) \) and \( \lambda \in [0, 1] \), let \( H \) be a measurable function on \( \mathbb{R}^n \) such that
\[
H(x) \geq \sup \{ f^\lambda(u)g^{1-\lambda}(v), (u, v) \in \mathbb{R}^n \times \mathbb{R}^n \text{ and } x = \lambda u + (1 - \lambda) v \},
\]
then
\[
\int_{\mathbb{R}^n} H(x) \, dx \geq \left( \int f \right)^\lambda \left( \int g \right)^{1-\lambda}.
\]

We show that a corresponding restricted version of this statement holds.

Theorem 2'. There exist positive scalars \( c \) and \( n_0 \) such that for all \( 0 < \varepsilon \leq 1/2 \), for all \( \lambda \in [\varepsilon, 1 - \varepsilon] \) and for all \( n \geq n_0 \), if \( f, g \) are non-negative functions in \( L_1(\mathbb{R}^n) \) and if \( \Theta \) is a measurable subset of \( \mathbb{R}^{2n} \) such that
\[
\int_{\Theta} f(x)g(y) \, dx \, dy \geq \frac{1}{2} + \frac{c \log n}{\sqrt{n}},
\]
then
\[
\int_{\mathbb{R}^n} K(x) \, dx \geq \left( \int f \right)^\lambda \left( \int g \right)^{1-\lambda},
\]
as soon as the function \( K \) satisfies:
\[
K(x) \geq \sup \{ f^\lambda(u)g^{1-\lambda}(v), (u, v) \in \Theta \text{ and } x = \sqrt{\lambda} u + \sqrt{1 - \lambda} v \}.
\]

Let us return to the example given in [SV] to show that the condition on the ratio
\[
\theta = \int_{\Theta} f(x)g(y) \, dx \, dy / \left( \int f \right) \left( \int g \right)
\]
is asymptotically optimal. Let \( B^n_2 \) be the Euclidean unit ball in \( \mathbb{R}^n \) and let \( \Theta = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n, \langle x, y \rangle \leq 0 \} \), then \( |\Theta \cap (B^n_2 \times B^n_2)| = 1/2 |B^n_2|^2 \) and the \( \Theta \)-restricted sum of a ball of radius \( r_1 \) and a ball of radius \( r_2 \) is a ball of radius \( \sqrt{r_1^2 + r_2^2} \). In particular, for all \( \lambda \in [0, 1] \),
\[
\sqrt{\lambda} B^n_2 + \Theta \sqrt{1 - \lambda} B^n_2 = B^n_2.
\]
The conclusion of Theorem 2' applied when \( f \) and \( g \) are the characteristic function of \( B^n_2 \) would be
\[
|\sqrt{\lambda} B^n_2 + \Theta \sqrt{1 - \lambda} B^n_2| \geq |B^n_2|,
\]
and actually the equality holds. It is then clear that the conclusion of Theorem 2' becomes false for ratios \( \theta < 1/2 \).
We shall first show that Theorem 2′ implies Theorem 1′, maybe with different conditions on the parameters. Let $A, B$ be two subsets of $\mathbb{R}^n$, let $\Theta \subset A \times B$ such that

$$\rho := \left( \frac{|A|}{|B|} \right)^{\frac{1}{n}} \leq 1.$$ 

Assume that the ratio $\theta = \frac{|\Theta|}{|A||B|}$ is larger than $\frac{1}{2} + c \sqrt{\frac{1 + \rho^2}{\rho^2}} \cdot \log n$. Let us define the set

$$\tilde{\Theta} = \left\{ \left( \frac{a}{|A|^\frac{1}{n}}, \frac{b}{|B|^\frac{1}{n}} \right) \in \mathbb{R}^{2n}, (a, b) \in \Theta \right\}.$$ 

Let

$$\tilde{A} = \frac{A}{|A|^\frac{1}{n}} \quad \text{and} \quad \tilde{B} = \frac{B}{|B|^\frac{1}{n}}$$

and let $f$ and $g$ be the characteristic functions of $\tilde{A}$ and $\tilde{B}$. A simple change of variables gives that

$$\int_{\tilde{\Theta}} f(x) g(y) \, dx \, dy = \frac{|\Theta|}{|A||B|} = \theta,$$

so we can apply Theorem 2′ to $f$ and $g$, with $\lambda = \frac{|A|^\frac{2}{n}}{|A|^\frac{2}{n} + |B|^\frac{2}{n}} = \frac{\rho^2}{1 + \rho^2}$ and get

$$\left| \sqrt{\lambda} \tilde{A} + \sqrt{1 - \lambda} \tilde{B} \right| \geq 1,$$

where

$$\sqrt{\lambda} \tilde{A} + \sqrt{1 - \lambda} \tilde{B} = \left\{ \sqrt{\lambda} \frac{a}{|A|^\frac{1}{n}} + \sqrt{1 - \lambda} \frac{b}{|B|^\frac{1}{n}}, (a, b) \in \Theta \right\}$$

$$= \left\{ \frac{a + b}{\sqrt{|A|^\frac{2}{n} + |B|^\frac{2}{n}}}, (a, b) \in \Theta \right\}$$

$$= \frac{A + \Theta B}{\sqrt{|A|^\frac{2}{n} + |B|^\frac{2}{n}}}.$$

Hence, we obtain

$$|A + \Theta B|^\frac{2}{n} \geq |A|^\frac{2}{n} + |B|^\frac{2}{n}.$$

Our method is based on an observation of Brascamp and Lieb [BL1]: the Prékopa-Leindler inequality is a limit case of the reverse sharp form of Young’s convolution inequality. We will first prove a restricted form of
Young’s inequality and its converse, using a modification of the method we developed in [Bar], and then take the limits in certain parameters. Our proof of Young’s inequality is based on measure-preserving mappings between measures. We use them in order to build a suitable change of variables which makes the problem simpler; then a simple arithmetico-geometric inequality gives the result. Now, we have to work with functions on $\mathbb{R}^n$, because the set $\Theta$ makes it difficult to use the classical tensorisation argument. In general, given two probability on $\mathbb{R}^n$, there are several measure-preserving mappings between them; for our purpose, the mapping built by Knothe in [Kno] fits:

**Lemma 1.** Let $f, F$ be positive continuous functions on $\mathbb{R}^n$ such that $\int f = \int F$. There exists a differentiable map $u : \mathbb{R}^n \to \mathbb{R}^n$ such that for $x \in \mathbb{R}^n$

$$\det(du(x)) \cdot f(u(x)) = F(x),$$

and for all $i \leq n$ and all $(x_i)_{i=1}^n \in \mathbb{R}^n$,

$$u((x_i)_{i=1}^n) = (u_1(x_1), u_2(x_1, x_2), \ldots, u_n(x_1, \ldots, x_n)),$$

where for all $x_1, \ldots, x_{i-1}$, the function $u_i(x_1, \ldots, x_{i-1}, \cdot)$ is increasing on $\mathbb{R}$. In particular $du(x)$ has always a lower triangular matrix with positive diagonal (in the canonical basis).

We also need a version of the arithmetico-geometric inequality for matrices of the previous form:

**Lemma 2.** Let $M, N$ be lower triangular $n \times n$-matrices with non-negative diagonal and let $t \in [0, 1]$, then

$$\det(tM + (1-t)N) \geq (\det M)^t (\det N)^{1-t}.$$

The first step of the proof is the following restricted version of Young’s inequality. For $t > 1$, we denote by $t'$ the real number such that $1/t + 1/t' = 1$.

**Lemma 3.** Let $f, F, g, G$ be positive continuous functions on $\mathbb{R}^n$, of integral 1 and dominated by some Gaussian function. Let $u$ and $v$ denote the measure preserving mappings obtained when applying Lemma 1 to $(f, F)$ and $(g, G)$ and let $T$ be the bijective map of $\mathbb{R}^n \times \mathbb{R}^n$ defined by $T(x, y) = (u(x), v(y))$. Let $p, q, r \geq 1$ such that $1/p + 1/q = 1 + 1/r$. We set

$$c = \sqrt{r'/q}, \quad s = \sqrt{r'/p'},$$

and notice that $c^2 + s^2 = 1$. Then

$$\int f(x)g(y)1_{\Theta}(x, y) \, dx \, dy = \int F(X)G(Y)1_{\Theta}(X, Y) \, dX \, dY,$$
and
\[
\left( \int \left( \int f^{\frac{1}{r}}(cx - sy)g^{\frac{1}{q}}(sx + cy)1_{T\Theta}(cx - sy, sx + cy) \right)^r \, dy \right)^{\frac{1}{r}} \leq \int \left( \int F^{\frac{1}{r}}(cX - sY)G^{\frac{1}{q}}(sX + cY)1_{\Theta}(cX - sY, sX + cY) \, dY \right)^{\frac{1}{r}} \, dX.
\]

**Proof.** Equality (2) is a consequence of the measure-preserving properties of \(u\) and \(v\). We give a detailed proof of the inequality. Let \(R\) be the rotation of matrix \(\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\) in the canonical basis. We are going to use the change of variable in \(\mathbb{R}^n\) given by the function \(\Phi = (R \otimes I_n)(R \otimes I_n)\), where \(I_n\) is the identity map on \(\mathbb{R}^n\). More precisely \((x, y) = \Phi(X, Y)\) means
\[
x = cu(cX - sY) + sv(sX + cY),
\]
\[
y = -su(cX - sY) + cv(sX + cY).
\]
It is clear that \(\Phi\) is a differentiable bijection of \(\mathbb{R}^n\). Its jacobian at the point \((X, Y)\) is
\[
J_{\Theta}(X, Y) = \det(du(cX - sY)) \det(dv(sX + cY)).
\]
We want an upper estimate for the integral (finite by assumption)
\[
I = \left( \int \left( \int f^{\frac{1}{r}}(cx - sy)g^{\frac{1}{q}}(sx + cy)1_{T\Theta}(cx - sy, sx + cy) \right)^r \, dy \right)^{\frac{1}{r}}.
\]
Using the \((L^r, L^q)\)-duality, there exists a positive function \(h\) on \(\mathbb{R}^n\) such that \(\|h\|_{r'} = 1\) and
\[
I = \iint f^{\frac{1}{r}}(cx - sy)g^{\frac{1}{q}}(sx + cy)1_{T\Theta}(cx - sy, sx + cy)h(y) \, dx \, dy.
\]
By the change of variable \((x, y) = \Phi(X, Y)\), we obtain that \(I\) is equal to
\[
\iint f^{\frac{1}{r}}(u(cX - sY))g^{\frac{1}{q}}(v(sX + cY))h(-su(cX - sY) + cv(sX + cY))
\]
\[
\cdot 1_{T\Theta}(u(cX - sY), v(sX + cY))
\]
\[
\cdot \det(du(cX - sY)) \det(dv(sX + cY)) \, dX \, dY.
\]
In order to shorten the formulas, denote
\[
U = u(cX - sY), \quad V = v(sX + cY),
\]
\[
U' = \det(du(cX - sY)), \quad V' = \det(dv(sX + cY)).
\]
Noticing that the definition of \(T\) implies \(1_{T\Theta}(u(cX - sY), v(sX + cY)) = 1_{\Theta}(cX - sY, sX + cY)\), and using the differential formulas
\[
\det(du(x))f(u(x)) = F(x),
\]
\[ \det(dv(x)) \cdot g(u(x)) = G(x), \]

we get

\[
I = \iint f^\frac{1}{r}(u(cX - sY))g^\frac{1}{q}(v(sX + cY))1_{\Theta}(cX - sY, sX + cY) \\
\cdot h(-sU + cV)U'V' dXdY \\
= \int \left( \int F^\frac{1}{r}(cX - sY)G^\frac{1}{q}(sX + cY)1_{\Theta}(cX - sY, sX + cY) \\
\cdot h(-sU + cV)(U')^\frac{1}{r'}(V')^\frac{1}{r'} dY \right) dX.
\]

Using Hölder's inequality for the integral in \( Y \) with parameters \( r \) and \( r' \), one has:

\[
I \leq \int \left( \int F^\frac{1}{r}(cX - sY)G^\frac{1}{q}(sX + cY)1_{\Theta}(cX - sY, sX + cY) dY \right)^\frac{1}{r'} \\
\cdot \left( \int h'^r(-sU + cV)(U')^\frac{r'}{r}(V')^\frac{r'}{r'} dY \right)^\frac{1}{r} dX.
\]

Let \( H(X) = \int h'^r(-sU + cV)(U')^\frac{r'}{r}(V')^\frac{r'}{r'} dY \), then

\[
H(X) = \int h'^r(a(X, Y))(\det du(cX - sY))^s(\det dv(sX + cY))^c dY,
\]

where

\[
a(X, Y) = -s u(cX - sY) + c v(sX + cY).
\]

It is clear that the partial differential of \( a \) with respect to \( Y \) is

\[
d_Y a(X, Y) = s^2 du(cX - sY) + c^2 dv(sX + cY).
\]

By the arithmetico-geometric inequality stated in Lemma 2,

\[
\det(dY a(X, Y)) \geq (\det du(cX - sY))^s(\det dv(sX + cY))^c,
\]

hence

\[
H(X) \leq \int h'^r(a(X, Y)) \det dY a(X, Y) dY \leq \int h'^r = 1,
\]

where we use the fact that \( a(X, Y) \) is an injective function of \( Y \) (indeed, \( u \) and \( v \) are by definition increasing for the lexicographic order on \( \mathbb{R}^n \)). This proves that

\[
I \leq \int \left( \int F^\frac{1}{r}(cX - sY)G^\frac{1}{q}(sX + cY)1_{\Theta}(cX - sY, sX + cY) dY \right)^\frac{1}{r} dX.
\]

\[ \square \]
We are going to take a limit in \( r \) to obtain an inequality similar to the Prékopa-Leindler inequality. To simplify the notations, we set \( \kappa = 1 - \lambda \).

**Lemma 4.** Let \( f, g, F, G \) be as in Lemma 3. Let \( \Theta \subset \mathbb{R}^{2n} \) and denote \( \theta = \int_{\Theta} F(X)G(Y) \, dX \, dY \). Then

\[
\int \sup_{X=\sqrt{\lambda}u+\sqrt{\kappa}v} F^\lambda(u)G^\kappa(v) 1_{\Theta}(u, v) \, dX \\
\geq \inf_{A \subset \mathbb{R}^{2n}} \sup_{y \in \mathbb{R}^n} \int f^\lambda \left( \sqrt{\lambda} x - \sqrt{\kappa} y \right) g^\kappa \left( \sqrt{\kappa} x + \sqrt{\lambda} y \right) \\
\quad \cdot 1_A \left( \sqrt{\lambda} x - \sqrt{\kappa} y, \sqrt{\kappa} x + \sqrt{\lambda} y \right) \, dx,
\]

where the infimum is over the sets \( A \subset \mathbb{R}^{2n} \) such that \( \int_A f(x)g(y) \, dx \, dy \geq \theta \).

**Proof.** This lemma is a limit case of Lemma 3. For \( r > 1 \), we set

\[
p_r = \frac{r}{\lambda(r+1)}, \quad q_r = \frac{r}{\kappa(r+1)}.
\]

Then \( 1/p_r + 1/q_r = 1 + 1/r \) and when \( r \) is large enough \( p_r, q_r \) > 1. We apply Lemma 3 with \( f, g, F, G \) for this triple and take the limit when \( r \) tends to \( +\infty \). Notice that

\[
\frac{1}{p_r} \to \lambda, \quad \frac{1}{q_r} \to \kappa,
\]

and

\[
c_r = \sqrt{\frac{p_r}{q_r}} = \sqrt{\frac{1 - q_r^{-1}}{1 - r^{-1}}} \to \sqrt{\lambda}, \quad s_r \to \sqrt{\kappa}.
\]

Our strong domination hypothesis ensures that the \( r \)-norms tend to essential suprema when \( r \) tends to infinity. We get:

\[
\sup_{y \in \mathbb{R}^n} \int f^\lambda \left( \sqrt{\lambda} x - \sqrt{\kappa} y \right) g^\kappa \left( \sqrt{\kappa} x + \sqrt{\lambda} y \right) \\
\quad \cdot 1_{\mathcal{T}_r\Theta} \left( \sqrt{\lambda} x - \sqrt{\kappa} y, \sqrt{\kappa} x + \sqrt{\lambda} y \right) \, dx \\
\leq \int_{\mathbb{R}^n} \sup_{Y \in \mathbb{R}^n} F^\lambda \left( \sqrt{\lambda} X - \sqrt{\kappa} Y \right) G^\kappa \left( \sqrt{\kappa} X + \sqrt{\lambda} Y \right) \\
\quad \cdot 1_{\Theta} \left( \sqrt{\lambda} X - \sqrt{\kappa} Y, \sqrt{\kappa} X + \sqrt{\lambda} Y \right) \, dX.
\]
Noticing that \( \begin{align*}
u &= \sqrt{\lambda} X - \sqrt{\kappa} Y \\
v &= \sqrt{\kappa} X + \sqrt{\lambda} Y
\end{align*} \)
is equivalent to \( \begin{align*}X &= \sqrt{\lambda} u + \sqrt{\kappa} v \\
Y &= -\sqrt{\kappa} u + \sqrt{\lambda} v,\end{align*} \)
we can rewrite the second member of the previous inequality as
\[
\int \sup_{X=\sqrt{\lambda} u+\sqrt{\kappa} v} F^\lambda(u)G^\kappa(v)1_\Theta(u,v) \, dX.
\]

By equality (2) in Lemma 3, we have \( \int_{T\Theta} f(x)g(y) \, dx dy = \theta \), which leads to the conclusion.

To finish the proof of Theorem 2', we have to estimate the infimum given in the previous lemma for two specific functions \( f \) and \( g \).

**Lemma 5.** Let \( F, G \) be as in Lemma 3, then
\[
\int \sup_{X=\sqrt{\lambda} u+\sqrt{\kappa} v} F^\lambda(u)G^\kappa(v)1_\Theta(u,v) \, dX \\
\geq E \left( \exp \left( \sqrt{\lambda\kappa} \sum_{i=1}^n X_i \right) 1\{\sum X_i \leq M_n,\theta\} \right),
\]
where \((X_i)_{i=1}^n\) is a sequence of i.i.d. random variables, their common law being the law of a difference of squares of two independent Gaussian variables \( N(0,1/\sqrt{2}) \) and the number \( M_n,\theta \) satisfies \( \mathbb{P}(\sum X_i \leq M_n,\theta) = \theta \).

**Proof.** We apply Lemma 4 with \( f(x) = g(x) = \pi^{-n/2} e^{-x^2} \).

We denote by \( \gamma_n \) the probability measure on \( \mathbb{R}^n \) with the previous density. We want a lower estimate of
\[
\mathcal{I} = \inf_{\gamma_n(A)=\theta} \sup_{y \in \mathbb{R}^n} \int \exp \left( -\lambda \left( \sqrt{\lambda} x - \sqrt{\kappa} y \right)^2 - \kappa \left( \sqrt{\kappa} x + \sqrt{\lambda} y \right)^2 \right) \\
\cdot 1_A \left( \sqrt{\lambda} x - \sqrt{\kappa} y, \sqrt{\kappa} x + \sqrt{\lambda} y \right) \pi^{-n/2} \, dx.
\]
Since the condition on \( A \) is rotation invariant, we can replace \( A \) by \( B \) such that \((x,y) \in B\) if and only if \((\sqrt{\lambda} x - \sqrt{\kappa} y, \sqrt{\kappa} x + \sqrt{\lambda} y) \in A\). Hence
\[
\mathcal{I} = \inf_{\gamma_n(B)=\theta} \sup_{y \in \mathbb{R}^n} \int \exp \left( -\lambda \left( \sqrt{\lambda} x - \sqrt{\kappa} y \right)^2 - \kappa \left( \sqrt{\kappa} x + \sqrt{\lambda} y \right)^2 \right) \\
\cdot 1_B(x,y) \pi^{-n/2} \, dx
\]
\[
\geq \inf_{\gamma_n(B)=\theta} \int \int \exp \left( x^2 - \lambda \left( \sqrt{\lambda} x - \sqrt{\kappa} y \right)^2 - \kappa \left( \sqrt{\kappa} x + \sqrt{\lambda} y \right)^2 \right) \\
\cdot 1_B(x,y) \, d\gamma_n(x),
\]
\cdot 1_B(x, y) \, d\gamma_n(x) d\gamma_n(y).

The matrix of the quadratic form on \(\mathbb{R}^n\), \(Q(x, y) = x^2 - \lambda(\sqrt{\lambda} x - \sqrt{\kappa} y)^2 - \kappa(\sqrt{\kappa} x + \sqrt{\lambda} y)^2\) in a suitable orthonormal basis is

\[
\sqrt{\lambda \kappa} \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix},
\]

where \(I_n\) is the identity \(n \times n\) matrix. Hence, by rotation invariance of the Gaussian measure

\[
\mathcal{I} \geq \inf_{\gamma_2n(B) = \theta} \int_B \exp \left( \sqrt{\lambda \kappa} (x^2 - y^2) \right) \, d\gamma_n(x) d\gamma_n(y).
\]

This is exactly

\[
\mathcal{J} = \int_B \exp \left( \sqrt{\lambda \kappa} (x^2 - y^2) \right) \mathbf{1}_{\{x^2 - y^2 \leq M_{n, \theta}\}} \, d\gamma_n(x) d\gamma_n(y),
\]

where \(M_{n, \theta}\) is such that \(\gamma_2n(\{x^2 - y^2 \leq M_{n, \theta}\}) = \theta\). We get the conclusion of the lemma by rewriting this with \(X_i = x_i^2 - y_i^2\), where \(x_i\) and \(y_i\) are the \(i^{th}\) coordinates of \(x\) and \(y\). \(\square\)

We are going to use the central-limit theorem in the rather precise form of the Berry-Essen theorem. [Fel].

**Theorem 3.** Let \((X_i)_{i \in \mathbb{N}}\) be a sequence of i.i.d. random variables, let

\[
m = \mathbb{E}X_i, \quad \sigma = \left(\mathbb{E}X_i^2\right)^{1/2} \quad \text{and} \quad \beta = \mathbb{E}|X_i|^3.
\]

For all \(t \in \mathbb{R}\), let

\[
F_n(t) = \mathbb{P}\left(\frac{\sum_{i=1}^n X_i - nm}{\sigma \sqrt{n}} < t\right)
\]

and

\[
G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-s^2/2} \, ds.
\]

There exists a universal constant \(c > 0\) such that for all \(t\) and for all \(n\),

\[
|F_n(t) - G(t)| \leq \frac{c\beta}{\sigma^3 \sqrt{n}}.
\]

**Proof of Theorem 2'.** By homogeneity, we may assume \(\int F = \int G = 1\). Comparing the assertions of Lemma 5 and Theorem 2%', we see that to prove the latter, it is enough to show that the expectation from the former is \(\geq 1\) provided the parameter \(\theta = \int_{\mathbb{R}^2} F(x) G(y) \, dx \, dy\) exceeds

\[
\frac{1}{2} + \frac{c}{\sqrt{2}} \frac{\log n}{\sqrt{n}}.
\]
To this end, we apply Theorem 3 to the variables $X_i$ defined in Lemma 5, and notice that $m = 0$ and $\beta, \sigma$ and $c$ are universal constants. We set

$$\xi_n = \frac{\log n}{\sigma \sqrt{\lambda \kappa n}}$$

and

$$\alpha = \frac{c \beta}{\sigma^3}.$$ 

We fix $\lambda$ and prove that, for $n$ large enough and for

$$\theta = G(\xi_n) + \frac{\alpha}{\sqrt{n}},$$

the quantity

$$J = E \left( \exp \left( \sqrt{\lambda \kappa} \sum_{i=1}^{n} X_i \right) 1_{\{\sum X_i \leq M_{n,\theta}\}} \right)$$

is larger than 1.

As $E X_i = 0$, we get from the Berry-Essen theorem

$$P \left( \sum_{i=1}^{n} X_i < \xi_n \sigma \sqrt{n} \right) \leq G(\xi_n) + \frac{c \beta}{\sigma^3 \sqrt{n}} = \theta,$$

so $M_{n,\theta} \geq \xi_n \sigma \sqrt{n}$. We set $Z_n = \frac{\sum_{i=1}^{n} X_i}{\sigma \sqrt{n}}$, it is clear that

$$J \geq E \left( \exp \left( \sigma \sqrt{\lambda \kappa} Z_n \right) 1_{\{Z_n \leq \xi_n\}} \right).$$

Let $n_1(\lambda)$ be the smallest integer $n$ such that $\xi_n \leq 1$, notice that it is a non-increasing function of $\lambda \in [0, 1/2]$. We work with $n \geq n_1(\lambda)$. When $n$ is large, $Z_n$ behaves like a normal Gaussian $g$. So we can almost estimate this expectation by replacing $Z_n$ by $g$.

More precisely, let $d = 2 \alpha \sqrt{2 \pi e}$ and let $n_2(\lambda)$ be the smallest integer such that $\xi_n/3 \geq d/\sqrt{n}$, it is a non-decreasing function of $\lambda \in [0, 1/2]$. Then for $n > \max(n_1(\lambda), n_2(\lambda))$, one has

$$P_{Z_n}([t, \xi_n]) \geq P_g \left( \left[ t, \xi_n - \frac{d}{\sqrt{n}} \right] \right).$$

This comes from the Berry-Essen theorem and from the fact that $\xi_n$ stays in $[0, 1]$ where the density of the law of $g$ is bounded from below:

$$P_{Z_n}([t, \xi_n]) \geq P_g ([t, \xi_n]) - 2 \frac{\alpha}{\sqrt{n}}$$

$$= P_g \left( \left[ t, \xi_n - \frac{d}{\sqrt{n}} \right] \right) + \frac{1}{\sqrt{2 \pi}} \int_{\xi_n - \frac{d}{\sqrt{n}}}^{\xi_n} e^{-t^2/2} dt - 2 \frac{\alpha}{\sqrt{n}}$$

$$\geq P_g \left( \left[ t, \xi_n - \frac{d}{\sqrt{n}} \right] \right) + \frac{1}{\sqrt{n}} \left( \frac{d}{\sqrt{2 \pi}} e^{-1/2} - 2 \alpha \right).$$
We are now able to compute a lower estimate of $\mathcal{J}$.

\[
\mathcal{J} \geq \int_{-\infty}^{\xi_n} \exp\left(\sigma \sqrt{\lambda \kappa n} t\right) \, dP_{Z_n}(t)
\]

\[
= \int_{-\infty}^{\xi_n} \sigma \sqrt{\lambda \kappa n} \exp\left(\sigma \sqrt{\lambda \kappa n} t\right) P_{Z_n}([t, \xi_n]) \, dt
\]

\[
\geq \int_{-\infty}^{\xi_n - \frac{d}{\sqrt{n}}} \sigma \sqrt{\lambda \kappa n} \exp\left(\sigma \sqrt{\lambda \kappa n} t\right) P_{g} \left(\left[ t, \xi_n - \frac{d}{\sqrt{n}} \right] \right) \, dt
\]

\[
= \int_{-\infty}^{\xi_n - \frac{d}{\sqrt{n}}} \exp\left(\sigma \sqrt{\lambda \kappa n} t\right) e^{-t^2/2} \, dt \frac{1}{\sqrt{2\pi}}.
\]

Because of our assumptions on $n$, we can write:

\[
\mathcal{J} \geq \int_{\xi_n/2}^{2\xi_n/3} \exp\left(\sigma \sqrt{\lambda \kappa n} t\right) e^{-t^2/2} \, dt \frac{1}{\sqrt{2\pi}}
\]

\[
\geq \frac{\xi_n}{6} \exp\left(\sigma \sqrt{\lambda \kappa n} \xi_n/2\right) e^{-1/2} \frac{1}{\sqrt{2\pi}}
\]

\[
= \frac{\log n}{6\sigma \sqrt{2\pi e \lambda \kappa n}} \exp \left( \frac{\log n}{2} \right)
\]

\[
= \frac{\log n}{6\sigma \sqrt{2\pi e \lambda \kappa}}.
\]

We denote by $n_3(\lambda)$ the smallest integer $n$ such that the previous quantity is larger than 1. It is a non-decreasing function of $\lambda \in [0, 1/2]$.

Eventually, if $\lambda \in [\varepsilon, 1/2]$, then for $n \geq \max(n_1(\varepsilon), n_2(1/2), n_3(1/2))$ the conclusion of Theorem 2' holds for

\[
\theta = G(\xi_n) + \frac{\alpha}{\sqrt{n}}.
\]

As by concavity, $G(t) \leq \frac{1}{2} + \frac{t}{\sqrt{2\pi}}$ for all $t$ positive, one easily deduces the theorem from the previous result. Theorem 2' gives some information only if the quantity $1/2 + c \log n/\sqrt{\varepsilon n}$ is smaller than one. So the condition $n \geq n_1(\varepsilon)$ is implicitly contained in Theorem 2'.

\[
\square
\]

References


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