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The aim of this paper is to prove the existence of k-peak solutions (solutions with more than one local maximum point) for the following singularly perturbed problem without imposing any extra condition on the boundary $\partial\Omega$:

$$(1.1)\qquad \begin{cases} -\varepsilon^2\Delta u + u = u^{p-1}, & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

where ε is a small positive number, Ω is a bounded C^3-domain in R^N, n is the unit outward normal of $\partial\Omega$ at $y, 2 < p < \frac{2N}{N-2}$ if $N \geq 3$ and $2 < p < +\infty$ if $N = 2$.

1. Introduction.

Problem (1.1) appears in applied mathematics. See for example [**13**, **14**] and the references therein. For the interesting link between this problem and the modelling of activator-inhibitor systems, the authors can refer to [**11**]. In [**13**, **14**], Ni and Takagi prove that the least energy solution of (1.1) has exactly one local maximum point x_ε which lies in $\partial\Omega$, and x_ε tends to a point x_0 which attains the maximum of $H(x)$, where $H(x)$ is the mean curvature function of $\partial\Omega$. Later, Wei [**21**] proves that for a solution u_ε of (1.1) in a certain energy level, u_ε has only one local maximum point x_ε which is in $\partial\Omega$, and x_ε tends to a critical point of $H(x)$. He also gets a kind of converse, that is, for each nondegenerate critical point x_0 of $H(x)$, there exists a solution u_ε for (1.1), such that u_ε has only one local maximum point x_ε, and $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. In the recent paper [**10**], Li shows that the assumption of nondegenercy can be replaced by C^1-stable (see definition 0.1 in [**10**]). Of course, nondegenerate critical point, strictly local maximum point and strictly local minimum point are C^1-stable critical points. Thus, Li extends the results in [**8**, **21**]. Gui [**8**] and Li [**10**] also consider the existence of multipeak solutions. But locally speaking, these solutions have one local maximum point. Other results on this problem can also be found in Bates, Dancer and Shi [**5**] and Wang [**17**]

We mention here the works on the Neumann problem involving critical Sobolev exponent [**1**, **2**, **12**, **16**, **18**, **19**, **20**].

Except in [**19**, **20**], the results concerning the existence of multipeak solutions for (1.1) in the papers just mentioned were obtained by gluing some single peak solutions concentrating on different points together. So some other conditions on $\partial\Omega$ are needed and they exclude the case that Ω is a ball. In [**19**, **20**], double peak solutions have been constructed on the ball by using the special symmetric properties of the ball. But whether there exists k-peak solution for (1.1) on the ball with $k \geq 3$ is still not known. Moreover, it is impossible to use these results to get a k-peak solution u_ε such that all the local maximum points of u_ε tend to the same point.

In this paper, we just assume that $\partial\Omega$ is C^3. We will prove that for each integer $k \geq 1$, (1.1) has a k-peak solution provided ε is small enough. Before we state our main results, we introduce some notations.

Throughout this paper, we denote $H(x)$ the mean curvature function of $\partial\Omega$. Let $U(y)$ be the unique positive solution (see [**9**]) of

$$\begin{cases} -\Delta u + u = u^{p-1}, & \text{on } R^N \\ u \in \mathrm{H}^1(R^N), & \\ u(0) = \max_{y \in R^N} u(y). & \end{cases} \tag{1.2}$$

It is well known (see [**7**]) that $U(y)$ is radially symmetric about the origin, decreasing and

$$\lim_{|y| \to \infty} U(y) e^{|y|} |y|^{(N-2)/2} = c_0 > 0.$$

Define

$$\langle u, v \rangle_\varepsilon = \int_\Omega \varepsilon^2 Du \cdot Dv + uv, \qquad \forall u, v \in \mathrm{H}^1(\Omega), \tag{1.3}$$

$$\|u\|_\varepsilon = \langle u, u \rangle_\varepsilon^{\frac{1}{2}}. \tag{1.4}$$

For any $x_0 \in R^N, \varepsilon > 0$, let

$$U_{\varepsilon, x_0}(y) =: U\left(\frac{y - x_0}{\varepsilon}\right).$$

For any $x_i \in \partial\Omega, i = 1, 2, \ldots, k$, define

$$E_{\varepsilon,x,k} = \left\{ v \in \mathrm{H}^1(\Omega) : \langle U_{\varepsilon,x_i}, v \rangle_\varepsilon = \left\langle \frac{\partial U_{\varepsilon,x_i}}{\partial \tau_{ij}}, v \right\rangle_\varepsilon = 0, \right.$$
$$\left. i = 1, 2, \ldots, k, \quad j = 1, 2, \ldots, N-1 \right\},$$

where $\{\tau_{i1}, \ldots, \tau_{i(N-1)}\}$ forms an orthoganal basis of the tangent space of $\partial\Omega$ at x_i.

The main results of the paper are the following:

Theorem 1.1. *For each fixed positive integer k, there exists an $\varepsilon_0 = \varepsilon_0(k)$, such that for each $\varepsilon \in (0, \varepsilon_0]$,* (1.1) *has a solution of the form*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon i} U_{\varepsilon, x_{\varepsilon i}} + v_\varepsilon$$

where

$$\alpha_{\varepsilon i} \longrightarrow 1 \text{ as } \varepsilon \to 0,\ i = 1, 2, \dots, k, \tag{1.5}$$

$$x_{\varepsilon i} \in \partial\Omega,\ i = 1, 2, \dots, k, \tag{1.6}$$

$$\frac{|x_{\varepsilon i} - x_{\varepsilon j}|}{\varepsilon} \longrightarrow +\infty \text{ as } \varepsilon \to 0, \text{ for } i \neq j, \tag{1.7}$$

$$x_{\varepsilon i} \longrightarrow x_i \text{ and } x_i \text{ satisfies } H(x_i) = \min_{\partial\Omega} H(x), \tag{1.8}$$

$$v_\varepsilon \in E_{\varepsilon, x_\varepsilon, k}, \quad \|v\|_\varepsilon^2 = o(\varepsilon^N). \tag{1.9}$$

In particular, if $H(x)$ has exactly one global minimum point x_0, then $x_{\varepsilon i} \to x_0$ as $\varepsilon \to 0$, $i = 1, 2, \dots, k$.

Theorem 1.2. *Suppose that $x_0 \in \partial\Omega$ is a strictly local minimum point of $H(x)$. Then for any fixed integer $k > 0$, there exists an $\varepsilon_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$,* (1.1) *has a solution of the form*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon i} U_{\varepsilon, x_{\varepsilon i}} + v_\varepsilon$$

where $\alpha_{\varepsilon i}, x_{\varepsilon i}$ and v_ε satisfy (1.5)-(1.7) *and* (1.9). *Moreover, $x_{\varepsilon i} \to x_0$ as $\varepsilon \to 0, i = 1, 2, \dots, k$.*

In Section 2, we will prove Theorem 1.1. Since the proof of Theorem 1.2 is similar to that of Theorem 1.1, we just point out the necessary changes in the proof of Theorem 1.2. We put the basic estimates in Appendix A. In Appendix B, we will prove that the corresponding linear map $A_{\varepsilon,x}$ is invertible and $\|A_{\varepsilon,x}^{-1}\| \leq C$ with C independent of ε and $x \in [\partial\Omega]^k$.

2. Proof of Main Results.

Let

$$I(u) = \frac{1}{2} \int_\Omega (\varepsilon^2 |Du|^2 + u^2) - \frac{1}{p} \int_\Omega |u|^p, \qquad u \in \mathrm{H}^1(\Omega). \tag{2.1}$$

For fixed integer $k > 0$, let

$$\alpha = (\alpha_1, \dots, \alpha_k) \in R^k, \tag{2.2}$$

$$x = (x_1, \dots, x_k) \in R^{kN}, \qquad x_i \in R^N, i = 1, \dots, k. \tag{2.3}$$

Define

$$D_{\varepsilon,R} = \left\{ x:\ x_i \in \partial\Omega, \quad i=1,2,\dots,k; \quad \frac{|x_i - x_j|}{\varepsilon} \geq R,\, i \neq j \right\}, \tag{2.4}$$

$$M_{\varepsilon,\delta,R} = \Big\{ (\alpha, x, v)\colon |\alpha_i - 1| \leq \delta, \\ i=1,2,\dots,k;\ x \in D_{\varepsilon,R},\ v \in E_{\varepsilon,x,k},\ \|v\|_\varepsilon \leq \delta\varepsilon^{N/2} \Big\}. \tag{2.5}$$

Let

$$J(\alpha,x,v) = I\left(\sum_{i=1}^k \alpha_i U_{\varepsilon,x_i} + v\right), \qquad (\alpha,x,v) \in M_{\varepsilon,\delta,R}. \tag{2.6}$$

First, we have the following decomposition lemma:

Lemma 2.1. *There are $\varepsilon_0 > 0, \delta > 0$ and $R > 0$, such that for each $\varepsilon \in (0,\varepsilon_0]$ and each $u \in H^1(\Omega)$, satisfying*

$$\left\| u - \sum_{i=1}^k U_{\varepsilon,z_i} \right\|_\varepsilon \leq \delta\varepsilon^{N/2}, \quad \text{for some } z \in D_{\varepsilon,R},$$

u can be uniquely decomposed into

$$u = \sum_{i=1}^k \alpha_{\varepsilon i} U_{\varepsilon,x_{\varepsilon i}} + v_\varepsilon,$$

where $(\alpha_\varepsilon, x_\varepsilon, v_\varepsilon) \in M_{\varepsilon,\delta,R}$.

The proof of Lemma 2.1 is almost identical to that of Proposition 7 in [**4**]. We thus omit it here.

As a direct consequence of Lemma 2.1, we have:

Proposition 2.2. *There exist $\varepsilon_0 > 0$ and $\delta > 0$ and $R > 0$, such that for each $\varepsilon \in (0,\varepsilon_0]$, $(\alpha,x,v) \in M_{\varepsilon,\delta,R}$ is a critical point of $J(\alpha,x,v)$ if and only if $u = \sum_{i=1}^k \alpha_i U_{\varepsilon,x_i} + v$ is a critical point of $I(u)$.*

We mention here that it is easy to prove that if $u = \sum_{i=1}^k \alpha_i U_{\varepsilon,x_i} + v$ is a solution of (1.1) with $(\alpha_\varepsilon, x_\varepsilon, v_\varepsilon) \in M_{\varepsilon,\delta,R}$, then u is positive, see for example [**15**].

In view of Proposition 2.2, the rest of this paper is devoted to find a critical point $(\alpha,x,v) \in M_{\varepsilon,\delta,R}$ for $J(\alpha,x,v)$. On the other hand, by the definition of $E_{\varepsilon,x,k}$, we know that $(\alpha,x,v) \in M_{\varepsilon,\delta,R}$ is a critical point of $J(\alpha,x,v)$ in the manifold $M_{\varepsilon,\delta,R}$ if and only if there are Lagrange multipliers $A_l, B_{li}, \quad i=1,2,\dots,k$, such that

(2.7)
$$\frac{\partial J(\alpha,x,v)}{\partial \tau_{li}} = \sum_{j=1}^{N-1} B_{lj}\left\langle \frac{\partial^2 U_{\varepsilon,x_l}}{\partial \tau_{li}\partial \tau_{lj}}, v\right\rangle_\varepsilon, \quad i=1,\dots,N-1,\ l=1,\dots,k,$$

(2.8)
$$\frac{\partial J(\alpha,x,v)}{\partial \alpha_l} = 0, \quad l=1,\dots,k,$$

(2.9)
$$\frac{\partial J(\alpha,x,v)}{\partial v} = \sum_{l=1}^{k} A_l U_{\varepsilon,x_l} + \sum_{l=1}^{k}\sum_{j=1}^{N-1} B_{lj}\frac{\partial U_{\varepsilon,x_l}}{\partial \tau_{lj}}.$$

We will proceed in a similar way as [**6**]. That is, for each fixed $x \in D_{\varepsilon,R}$, we first solve (2.8) and (2.9) simultaneously. Then we solve (2.7).

Proposition 2.3. *There are $\varepsilon_0 > 0, \delta > 0$ and $R > 0$, such that for each $\varepsilon \in (0,\varepsilon_0]$, there is a C^1-map $(\alpha_\varepsilon(x), v_\varepsilon(x)):\ D_{\varepsilon,R} \longrightarrow R^k \times E_{\varepsilon,x,k}$ satisfying*

$$\frac{\partial J(\alpha,x,v)}{\partial \alpha_l} = 0, \qquad l=1,\dots,k, \tag{2.10}$$

$$\left\langle \frac{\partial J(\alpha,x,v)}{\partial v}, \omega\right\rangle_\varepsilon = 0, \qquad \forall\, \omega \in E_{\varepsilon,x,k}. \tag{2.11}$$

Moreover,

$$|\alpha_l - 1| = O\left(\varepsilon + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right), \qquad l=1,\dots,k, \tag{2.12}$$

$$\|v_\varepsilon\|_\varepsilon = O\left(\varepsilon^{N/2}\left(\varepsilon + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right)\right), \tag{2.13}$$

where σ is some positive constant.

Before we give the proof of Proposition 2.3, we introduce some notation.
Let $\beta_l = \alpha_l - 1$, $l=1,\dots,k$, $\beta = (\beta_1,\dots,\beta_k) \in R^k$. For each fixed $\varepsilon > 0$, define

$$\langle \beta,\gamma\rangle_\varepsilon = \varepsilon^N \sum_{l=1}^{k} \beta_l\gamma_l, \qquad \forall\, \beta,\gamma \in R^k \tag{2.14}$$

$$\|\beta\|_\varepsilon = \langle \beta,\beta\rangle_\varepsilon^{\frac12}. \tag{2.15}$$

Let $w=(\beta,v)\in R^k\times E_{\varepsilon,x,k}$. Define

(2.16)
$$\left\langle w^{(1)},w^{(2)}\right\rangle_\varepsilon=\left\langle \beta^{(1)},\beta^{(2)}\right\rangle_\varepsilon+\left\langle v^{(1)},v^{(2)}\right\rangle_\varepsilon,\quad \forall w^{(1)},w^{(2)}\in R^k\times E_{\varepsilon,x,k},$$

(2.17) $$\|w\|_\varepsilon=\langle w,w\rangle_\varepsilon^{\frac12},\quad \forall w\in R^k\times E_{\varepsilon,x,k}.$$

Finally, denote

(2.18) $$H_{\varepsilon,x,k}=\sum_{l=1}^k U_{\varepsilon,x_l},$$

(2.19) $$J^*(x,w)=J(\alpha,x,v),\qquad w=(\beta,v)=(\alpha-1,v).$$

Proof of Proposition 2.3. As in Bahri [**3**], see also Rey [**15**], we expand $J^*(x,w)$ at $w=0$:

(2.20) $$J^*(x,w)=J^*(x,0)+\langle f_{\varepsilon,x},w\rangle_\varepsilon+\frac12\langle Q_{\varepsilon,x}w,w\rangle_\varepsilon+R_{\varepsilon,x}(w),$$

where $f_{\varepsilon,x}\in R^k\times E_{\varepsilon,x,k}$ satisfies

(2.21) $$\langle f_{\varepsilon,x},w\rangle_\varepsilon=-\int_\Omega H^{p-1}_{\varepsilon,x,k}v+\sum_{l=1}^k\left[\langle H_{\varepsilon,x,k},U_{\varepsilon,x_l}\rangle_\varepsilon-\int_\Omega H^{p-1}_{\varepsilon,x,k}U_{\varepsilon,x_l}\right]\beta_l,$$

$Q_{\varepsilon,x}$ is a linear map from $R^k\times E_{\varepsilon,x,k}$ to $R^k\times E_{\varepsilon,x,k}$, satisfying

(2.22)
$$\langle Q_{\varepsilon,x}w,w\rangle_\varepsilon=A^{(1)}_{\varepsilon,x}(\beta)+A^{(2)}_{\varepsilon,x}(v)+A^{(3)}_{\varepsilon,x}(\beta,v),$$

(2.23) $$A^{(1)}_{\varepsilon,x}(\beta)=\sum_{l,h=1}^k\left[\langle U_{\varepsilon,x_h},U_{\varepsilon,x_l}\rangle_\varepsilon-(p-1)\int_\Omega H^{p-2}_{\varepsilon,x,k}U_{\varepsilon,x_h}U_{\varepsilon,x_l}\right]\beta_h\beta_l,$$

(2.24) $$A^{(2)}_{\varepsilon,x}(v)=\|v\|^2_\varepsilon-(p-1)\int_\Omega H^{p-2}_{\varepsilon,x,k}v^2,$$

(2.25)
$$A^{(3)}_{\varepsilon,x}(\beta,v)=-(p-1)\sum_{l=1}^k\int_\Omega H^{p-2}_{\varepsilon,x,k}U_{\varepsilon,x_l}v\beta_l.$$

$R_{\varepsilon,x}(w)$ is the higher order term satisfying

(2.26) $$R^{(i)}_{\varepsilon,x}(w)=O\left(\|w\|_\varepsilon^{\min(p-i,3-i)}\right),\qquad i=0,1,2.$$

Hence (2.8) and (2.9) are equivalent to

(2.27) $$f_{\varepsilon,x}+Q_{\varepsilon,x}w+R'_{\varepsilon,x}(w)=0.$$

Now we prove that $Q_{\varepsilon,x}$ is invertible and $\|Q^{-1}_{\varepsilon,x}\|\le C$ with C independent of ε and $x\in D_{\varepsilon,R}$.

From Lemmas A.1 and A.2, we get

$$
\begin{aligned}
(2.28)\qquad A^{(1)}_{\varepsilon,x}(\beta) &= \sum_{l=1}^{k}\left(\|U_{\varepsilon,x_l}\|_\varepsilon^2-(p-1)\int_\Omega U^p_{\varepsilon,x_l}\right)|\beta_l|^2+o(1)\varepsilon^N|\beta|^2\\
&= \varepsilon^N\left[(2-p)\int_{R^N}U^p+O(\varepsilon)+o(1)\right]|\beta|^2\\
&\le -c_0\varepsilon^N|\beta|^2=-c_0\|\beta\|^2_\varepsilon
\end{aligned}
$$

where $o(1)\longrightarrow 0$ as $R\longrightarrow+\infty$.

On the other hand, by Lemma A.4,

$$
\begin{aligned}
(2.29)\qquad \left|A^{(3)}_{\varepsilon,x}(\beta,v)\right| &= O\left(\sum_{l=1}^{k}\left|\int_\Omega U^{p-1}_{\varepsilon,x_l}v\right||\beta_l|+o(1)\|v\|_\varepsilon\|\beta\|_\varepsilon\right)\\
&= O\left(\varepsilon^{\frac{N}{2}+1}\|v\|_\varepsilon|\beta|+o(1)\|v\|_\varepsilon\|\beta\|_\varepsilon\right)\\
&= (O(\varepsilon)+o(1))\,\|v\|_\varepsilon\|\beta\|_\varepsilon.
\end{aligned}
$$

Define $B^{(1)}_{\varepsilon,x}: R^k\times E_{\varepsilon,x,k}\longrightarrow R^k\times E_{\varepsilon,x,k}$ as follows:

$$
\left\langle B^{(1)}_{\varepsilon,x}w,w\right\rangle_\varepsilon = A^{(1)}_{\varepsilon,x}(\beta)+A^{(2)}_{\varepsilon,x}(v).
$$

Then it follows from (2.28) and Lemma B.1 that $B^{(1)}_{\varepsilon,x}$ is invertible and

$$
\left\|\left(B^{(1)}_{\varepsilon,x}\right)^{-1}\right\|\le C.
$$

Let $B^{(2)}_{\varepsilon,x}=Q_{\varepsilon,x}-B^{(1)}_{\varepsilon,x}$. From (2.29), $\|B^{(2)}_{\varepsilon,x}\|\longrightarrow 0$ as $\varepsilon\longrightarrow 0$ and $R\longrightarrow+\infty$. Thus if we choose $R>0$ large enough, $\varepsilon>0$ small enough, $Q_{\varepsilon,x}$ is invertible and

$$
\left\|(Q_{\varepsilon,x})^{-1}\right\|\le C.
$$

Let $F(f,w)=: f+Q_{\varepsilon,x}w+R'_{\varepsilon,x}(w),\ f,w\in R^k\times E_{\varepsilon,x,k}$. Then $F(0,0)=0$ and $\frac{\partial F}{\partial w}(0,0)=Q_{\varepsilon,x}$ is invertible. So from the implicit function theorem, (2.27) has a solution $w_\varepsilon\in R^k\times E_{\varepsilon,x,k}$, and w_ε satisfies

$$
(2.30)\qquad \|w_\varepsilon\|_\varepsilon\le C\|f_{\varepsilon,x}\|.
$$

Now we estimate $\|f_{\varepsilon,x}\|$.

By Lemma A.4,

$$
\begin{aligned}
(2.31)\qquad &\int_\Omega H^{p-1}_{\varepsilon,x,k}v\\
&= \sum_{l=1}^{k}\int_\Omega U^{p-1}_{\varepsilon,x_l}v+\int_\Omega\left(H^{p-1}_{\varepsilon,x,k}-\sum_{l=1}^{k}U^{p-1}_{\varepsilon,x_l}\right)v
\end{aligned}
$$

$$= O\big(\varepsilon^{\frac{N}{2}+1}\|v\|_\varepsilon\big) + \int_\Omega \left(H^{p-1}_{\varepsilon,x,k} - \sum_{l=1}^k U^{p-1}_{\varepsilon,x_l} \right) v.$$

Suppose that $2 < p \le 3$. Then we have

$$\big||a+b|^{p-1} - |a|^{p-1} - |b|^{p-1}\big| \le \begin{cases} C|a||b|^{p-2} & \text{if } |a| \le |b|, \\ C|a|^{p-2}|b| & \text{if } |a| > |b|, \end{cases} \tag{2.32}$$
$$\le C|a|^{(p-1)/2}|b|^{(p-1)/2}.$$

We also have

$$\int_{R^N} e^{-h|y|} e^{-l|y-x|}\, dy = O(e^{-(\min(h,l)-\sigma)|x|}), \quad \text{for any } \sigma > 0. \tag{2.33}$$

So,

$$\begin{aligned} &\left| \int_\Omega \left(H^{p-1}_{\varepsilon,x_l,k} - \sum_{l=1}^k U^{p-1}_{\varepsilon,x_l} \right) v \right| \\ &\le C \sum_{i\ne j} \int_\Omega U^{\frac{p-1}{2}}_{\varepsilon,x_i} U^{\frac{p-1}{2}}_{\varepsilon,x_j} |v| \\ &\le C\varepsilon^{N/2} \|v\|_\varepsilon \sum_{i\ne j} \left[\int_{R^N} \left(U^{\frac{p-1}{2}} U^{\frac{p-1}{2}}_{1,(x_j-x_i)/\varepsilon} \right)^{p/(p-1)} \right]^{1-\frac{1}{p}} \\ &= O\left(\varepsilon^{N/2} \sum_{i\ne j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}} \right) \|v\|_\varepsilon. \end{aligned} \tag{2.34}$$

Note that in obtaining the second last inequality, we have used a change of variable, replacing x by $\frac{x}{\varepsilon}$.

If $p > 3$, then

$$\begin{aligned} &\left| \int_\Omega \left(H^{p-1}_{\varepsilon,x,k} - \sum_{l=1}^k U^{p-1}_{\varepsilon,x_l} \right) v \right| \\ &\le C \sum_{i\ne j} \int_\Omega U^{p-2}_{\varepsilon,x_i} U_{\varepsilon,x_j} |v| = O\left(\varepsilon^{N/2} \sum_{i\ne j} e^{-\frac{|x_i-x_j|}{\varepsilon}} \right) \|v\|_\varepsilon. \end{aligned} \tag{2.35}$$

Combining (2.31) and (2.34)-(2.35), we obtain

$$\int_\Omega H_{\varepsilon,x,k}^{p-1} v = O\left(\varepsilon^{N/2}\left(\varepsilon + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right)\right)\|v\|_\varepsilon. \tag{2.36}$$

On the other hand, from Lemma A.5

$$\begin{aligned}\sum_{l=1}^k &\left[\langle H_{\varepsilon,x,k}, U_{\varepsilon,x_l}\rangle_\varepsilon - \int_\Omega H_{\varepsilon,x,k}^{p-1} U_{\varepsilon,x_l}\right]\beta_l \\ &= -\sum_{l=1}^k \int_\Omega \left(H_{\varepsilon,x,k}^{p-1} - \sum_{h=1}^k U_{\varepsilon,x_h}^{p-1}\right) U_{\varepsilon,x_l}\beta_l \\ &\quad + \sum_{h,l=1}^k \varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x_h}}{\partial n} U_{\varepsilon,x_l}\beta_l \\ &= -\sum_{l=1}^k \int_\Omega \left(H_{\varepsilon,x,k}^{p-1} - \sum_{h=1}^k U_{\varepsilon,x_h}^{p-1}\right) U_{\varepsilon,x_l}\beta_l + O(\varepsilon^{N+1}|\beta|).\end{aligned} \tag{2.37}$$

As in the proof of (2.34) and (2.35), we easily get

$$\left|\int_\Omega \left(H_{\varepsilon,x,k}^{p-1} - \sum_{h=1}^k U_{\varepsilon,x_h}^{p-1}\right) U_{\varepsilon,x_l}\right| = O\left(\varepsilon^N \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right). \tag{2.38}$$

Putting (2.38) into (2.37), we get

$$\begin{aligned}\sum_{l=1}^k &\left[\langle H_{\varepsilon,x,k}, U_{\varepsilon,x_l}\rangle_\varepsilon - \int_\Omega H_{\varepsilon,x,k}^{p-1} U_{\varepsilon,x_l}\right]\beta_l \\ &= O\left[\varepsilon^{N/2}\left(\varepsilon + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right)\right]\|\beta\|_\varepsilon.\end{aligned} \tag{2.39}$$

Finally, it follows from (2.21), (2.36) and (2.39) that

$$\|f_{\varepsilon,x}\| \le O\left(\varepsilon^{N/2}\left(\varepsilon + \sum_{i\neq j} e^{-\frac{1+\sigma}{2}\frac{|x_i-x_j|}{\varepsilon}}\right)\right).$$

So we have completed the proof of Proposition 2.3. □

Let $(\alpha_\varepsilon(x), v_\varepsilon(x))$ be the function attained in Proposition 2.3. Consider

$$\sup\{J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) : \quad x \in D_{\varepsilon,R}\}. \tag{2.40}$$

Then Problem (2.40) is attained by some $x_\varepsilon \in D_{\varepsilon,R}$.

Proposition 2.4. *Let x_ε be the point which attains (2.40). Then, as $\varepsilon \longrightarrow 0$,*

$$\frac{|x_{\varepsilon i} - x_{\varepsilon j}|}{\varepsilon} \longrightarrow +\infty, \qquad i \neq j \tag{2.41}$$

$$x_{\varepsilon i} \longrightarrow x_i, \qquad i = 1, 2, \ldots, k, \tag{2.42}$$

where x_i is some point in $\partial\Omega$, satisfying $H(x_i) = \min_{x \in \partial\Omega} H(x)$.

Proof. From Proposition 2.3, for any $x \in D_{\varepsilon,R}$, we have

$$J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) = J^*(x, 0) + O\left[\varepsilon^N\left(\varepsilon^2 + \sum_{i \neq j} e^{-(1+\sigma)\frac{|x_i - x_j|}{\varepsilon}}\right)\right]. \tag{2.43}$$

Since x_ε is a maximum point of $J(\alpha_\varepsilon(x), x, v_\varepsilon(x))$, for any $z_\varepsilon \in D_{\varepsilon,R}$, the following relation holds:

$$J(\alpha_\varepsilon(x_\varepsilon), x_\varepsilon, v_\varepsilon(x_\varepsilon)) \geq J(\alpha_\varepsilon(z_\varepsilon), z_\varepsilon, v_\varepsilon(z_\varepsilon)).$$

It follows from (2.43) that

$$\begin{aligned} &J^*(x_\varepsilon, 0) + O\left[\varepsilon^N\left(\varepsilon^2 + \sum_{i \neq j} e^{-(1+\sigma)\frac{|x_{\varepsilon i} - x_{\varepsilon j}|}{\varepsilon}}\right)\right] \\ \geq &J^*(z_\varepsilon, 0) + O\left[\varepsilon^N\left(\varepsilon^2 + \sum_{i \neq j} e^{-(1+\sigma)\frac{|z_{\varepsilon i} - z_{\varepsilon j}|}{\varepsilon}}\right)\right]. \end{aligned} \tag{2.44}$$

Fix a $x_0 \in \partial\Omega$ with $H(x_0) = \min_{x \in \partial\Omega} H(x)$. Let e_i, $i = 1, 2, \ldots, k$ be a tangent vector of $\partial\Omega$ at x_0 with $e_i \neq e_j$ for $i \neq j$. Let $z_i(t)$ be a curve in $\partial\Omega$ at x_0 satisfying $z_i(0) = x_0$, $z_i'(0) = e_i$. Let

$$z_{\varepsilon i} = z_i(\varepsilon^{\frac{1}{2}}), \qquad i = 1, 2, \ldots, k. \tag{2.45}$$

Then $|z_{\varepsilon i} - z_{\varepsilon j}|/\varepsilon = (|e_i - e_j| + o(1))/\varepsilon^{\frac{1}{2}} \longrightarrow +\infty$ as $\varepsilon \longrightarrow 0$. Thus $z_\varepsilon \in D_{\varepsilon,R}$ if $\varepsilon > 0$ is small enough.

It follows from Lemma A.3 that

$$\begin{aligned} J^*(z_\varepsilon, 0) &= \frac{1}{2}\left\|\sum_{i=1}^k U_{\varepsilon, z_{\varepsilon i}}\right\|_\varepsilon^2 - \frac{1}{p}\int_\Omega \left|\sum_{i=1}^k U_{\varepsilon, z_{\varepsilon i}}\right|^p \\ &= \frac{1}{2}\sum_{i=1}^k \|U_{\varepsilon, z_{\varepsilon i}}\|_\varepsilon^2 - \frac{1}{p}\sum_{i=1}^k \int_\Omega U_{\varepsilon, z_{\varepsilon i}}^p + O\left(\varepsilon^N e^{-c_0/\varepsilon^{\frac{1}{2}}}\right) \\ &= \varepsilon^N\left\{\sum_{i=1}^k \left[\left(\frac{1}{2} - \frac{1}{p}\right)A - BH(z_{\varepsilon i})\varepsilon\right] + O(\varepsilon^2)\right\} \\ &= k\varepsilon^N\left[\left(\frac{1}{2} - \frac{1}{p}\right)A - BH(x_0)\varepsilon + O(\varepsilon^{\frac{3}{2}})\right]. \end{aligned} \tag{2.46}$$

On the other hand,

(2.47)

$$
\begin{aligned}
&J^*(x_\varepsilon, 0)\\
&= \sum_{i=1}^k I(U_{\varepsilon,x_{\varepsilon i}}) + \frac{1}{2}\sum_{i\neq j}\langle U_{\varepsilon,x_{\varepsilon i}}, U_{\varepsilon,x_{\varepsilon j}}\rangle_\varepsilon - \frac{1}{p}\int_\Omega\left(\left|\sum_{i=1}^k U_{\varepsilon,x_{\varepsilon i}}\right|^p - \sum_{i=1}^k U^p_{\varepsilon,x_{\varepsilon i}}\right)\\
&= \varepsilon^N\left[k\left(\frac{1}{2}-\frac{1}{p}\right)A - B\sum_{i=1}^k H(x_{\varepsilon i})\varepsilon + O(\varepsilon^2)\right]\\
&+ \sum_{i<j}\langle U_{\varepsilon,x_{\varepsilon i}}, U_{\varepsilon,x_{\varepsilon j}}\rangle_\varepsilon - \frac{1}{p}\int_\Omega\left(\left|\sum_{i=1}^k U_{\varepsilon,x_{\varepsilon i}}\right|^p - \sum_{i=1}^k U^p_{\varepsilon,x_{\varepsilon i}}\right).
\end{aligned}
$$

By Lemma A.5,

$$
\begin{aligned}
&\sum_{i<j}\langle U_{\varepsilon,x_{\varepsilon i}}, U_{\varepsilon,x_{\varepsilon j}}\rangle_\varepsilon \\
&= \sum_{i<j}\varepsilon^2\int_{\partial\Omega}\frac{\partial U_{\varepsilon,x_{\varepsilon i}}}{\partial n}U_{\varepsilon,x_{\varepsilon j}} + \sum_{i<j}\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon i}}U_{\varepsilon,x_{\varepsilon j}}\\
&= \sum_{i<j}\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon i}}U_{\varepsilon,x_{\varepsilon j}} + O\left(\varepsilon^{N+1}\sum_{i\neq j}e^{-\frac{(1-\theta)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right).
\end{aligned}
\tag{2.48}
$$

Using the following inequality,

$$
\begin{aligned}
&\big||a+b|^p - a^p - b^p - pa^{p-1}b - pab^{p-1}\big|\\
&\le \begin{cases} Ca^{p/2}b^{p/2} & \text{if } 2<p\le 3,\\ C(a^{p-2}b^2 + a^2b^{p-2}) & \text{if } p>3,\end{cases}
\end{aligned}
$$

we get,

$$
\begin{aligned}
&\int_\Omega\left(\left|\sum_{i=1}^k U_{\varepsilon,x_{\varepsilon i}}\right|^p - \sum_{i=1}^k U^p_{\varepsilon,x_{\varepsilon i}}\right)\\
&= \int_\Omega\left(\left|\sum_{i=2}^k U_{\varepsilon,x_{\varepsilon i}}\right|^p - \sum_{i=2}^k U^p_{\varepsilon,x_{\varepsilon i}}\right) + p\int_\Omega\left|\sum_{i=2}^k U_{\varepsilon,x_{\varepsilon i}}\right|^{p-1} U_{\varepsilon,x_{\varepsilon 1}}\\
&\quad + p\int_\Omega U^{p-1}_{\varepsilon,x_{\varepsilon 1}}\sum_{i=2}^k U_{\varepsilon,x_{\varepsilon i}} + O\left(\varepsilon^N\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right)\\
&= p\int_\Omega\sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_{\varepsilon i}}\right)^{p-1} U_{\varepsilon,x_{\varepsilon j}} + p\int_\Omega\sum_{i<j}U^{p-1}_{\varepsilon,x_{\varepsilon i}}U_{\varepsilon,x_{\varepsilon j}}
\end{aligned}
\tag{2.49}
$$

$$+O\left(\varepsilon^N\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right).$$

Combining (2.47)-(2.49), we obtain

(2.50)

$$J^*(x_\varepsilon,0)=\varepsilon^N\left[k\left(\frac{1}{2}-\frac{1}{p}\right)A-B\sum_{i=1}^k H(x_{\varepsilon i})\varepsilon+O(\varepsilon^2)\right]$$
$$-\int_\Omega\sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_{\varepsilon i}}\right)^{p-1}U_{\varepsilon,x_{\varepsilon j}}+O\left(\varepsilon^N\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right).$$

Putting (2.46) and (2.50) into (2.44), we are led to

$$-B\sum_{i=1}^k H(x_{\varepsilon i})\varepsilon+O(\varepsilon^2)-\varepsilon^{-N}\int_\Omega\sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_{\varepsilon i}}\right)^{p-1}U_{\varepsilon,x_{\varepsilon j}}$$
$$+O\left(\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right)\geq -kBH(x_0)\varepsilon+O(\varepsilon^{3/2}). \tag{2.51}$$

But

$$\varepsilon^{-N}\int_\Omega\sum_{j=1}^{k-1}\left(\sum_{i=j+1}^k U_{\varepsilon,x_{\varepsilon i}}\right)^{p-1}U_{\varepsilon,x_{\varepsilon j}}$$
$$\geq c_0\sum_{i\neq j}e^{-\frac{(1+\sigma/2)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\qquad\text{for some } c_0>0.$$

As a result,

(2.52)

$$B\sum_{i=1}^k H(x_{\varepsilon i})\varepsilon+O(\varepsilon^2)+c_0\sum_{i\neq j}e^{-\frac{(1+\sigma/2)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}+O\left(\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\right)$$
$$\leq kBH(x_0)\varepsilon+O(\varepsilon^{3/2}).$$

If we choose R sufficiently large, then the third term in (2.52) is much smaller than the second term. So

$$\sum_{i=1}^k H(x_{\varepsilon i})\leq kH(x_0)+O(\varepsilon^{\frac{1}{2}}), \tag{2.53}$$

$$\sum_{i\neq j}e^{-\frac{(1+\sigma)|x_{\varepsilon i}-x_{\varepsilon j}|}{\varepsilon}}\leq O(\varepsilon). \tag{2.54}$$

Clearly, (2.54) implies (2.41). Suppose that $x_{\varepsilon i} \longrightarrow x_i$ as $\varepsilon \longrightarrow 0$. Then (2.53) gives

$$\sum_{i=1}^{k} H(x_i) \le kH(x_0) = k \min_{x\in\partial\Omega} H(x),$$

which clearly implies (2.42). So we have completed the proof of Proposition 2.4. □

Proof of Theorem 1.1. We only need to prove that $(\alpha_\varepsilon(x_\varepsilon), x_\varepsilon, v_\varepsilon(x_\varepsilon))$ satisfies (2.7).

From (2.41), we know that x_ε is an interior point of $D_{\varepsilon,R}$. Conseqently,

$$\begin{aligned} 0 &= \sum_{h=1}^{k} \frac{\partial J}{\partial \alpha_h}\frac{\partial \alpha_h}{\partial \tau_{li}} + \frac{\partial J}{\partial \tau_{li}} + \left\langle \frac{\partial J}{\partial v}, \frac{\partial v}{\partial \tau_{li}} \right\rangle_\varepsilon \\ &= \frac{\partial J}{\partial \tau_{li}} + \sum_{h=1}^{k}\sum_{j=1}^{N-1} B_{hj} \left\langle \frac{\partial U_{\varepsilon,x_{\varepsilon h}}}{\partial \tau_{hj}}, \frac{\partial v}{\partial \tau_{li}} \right\rangle_\varepsilon \\ &= \frac{\partial J}{\partial \tau_{li}} - \sum_{j=1}^{N-1} B_{hj} \left\langle \frac{\partial^2 U_{\varepsilon,x_{\varepsilon l}}}{\partial \tau_{lj}\partial \tau_{li}}, v \right\rangle_\varepsilon . \end{aligned} \tag{2.55}$$

Thus, (2.7) holds. □

In order to prove Theorem 1.2, we only need to consider

$$\sup\{\, J(\alpha_\varepsilon(x), x, v_\varepsilon(x)) :\ x \in D_{\varepsilon,R},\ x_i \in \overline{B_\delta(x_0)},\ i = 1, 2, \dots, k\}.$$

Then we see that the maximum x_ε satisfies $x_{\varepsilon i} \longrightarrow x_0$, $i = 1, \dots, k$, and $\frac{|x_{\varepsilon i} - x_{\varepsilon j}|}{\varepsilon} \longrightarrow +\infty$ for $i \ne j$ as $\varepsilon \longrightarrow 0$.

Appendix A.

Lemma A.1. *Let $x \in \partial\Omega$. Then*

$$\int_\Omega U^p_{\varepsilon,x} = \varepsilon^N \left(A - \frac{1}{2}\varepsilon H(x) \int_{R^{N-1}} U^p(y', 0)|y'|^2\, dy' + O(\varepsilon^2) \right)$$

where $A = \int_{R^N} U^p$.

Proof. Choose a coordinate system such that $x = 0$ and

$$\Omega \cap B_\tau(0) = \{y_N > f(y')\}, \tag{A.1}$$

$$\partial\Omega \cap B_\tau(0) = \{y_N = f(y')\}, \tag{A.2}$$

where $\tau > 0$ is a small constant and $f(y')$ satisfies

$$f(y') = \frac{1}{2}\sum_{i=1}^{N-1} \rho_i y_i^2 + O(|y'|^3), \qquad y' \in B^{N-1}_\tau(0) = \{|y'| \le \tau\}.$$

Then $H(0) = \frac{1}{N-1}\sum_{i=1}^{N-1}\rho_i$. Let

$$\Omega_{\varepsilon,x} = \{y:\ \varepsilon y + x \in \Omega\},\ \Omega_\varepsilon = \Omega_{\varepsilon,0}, \tag{A.3}$$

$$U_\varepsilon(y) = U_{\varepsilon,0}(y), \tag{A.4}$$

$$B_\tau^+(0) = B_\tau(0)\cap\{y_N > 0\}. \tag{A.5}$$

Since $U(y)$ is exponentially small at infinity, we have

(A.6)

$$\begin{aligned}
\int_{B_\tau^+(0)\setminus\Omega} U_\varepsilon^p &= \varepsilon^N \int_{B_{\frac{\tau}{\varepsilon}}^+(0)\setminus\Omega_\varepsilon} U^p = \varepsilon^N \int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)} \int_0^{f(\varepsilon y')/\varepsilon} U^p(y', y_N)\, dy_N dy' \\
&= \varepsilon^N \int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)} \int_0^{f(\varepsilon y')/\varepsilon} \left[U^p(y',0) + O(|y_N| U^p(y',0))\right]\, dy_N dy' \\
&= \varepsilon^N \left[\frac{1}{2}\int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)} U^p(y',0)\sum_{i=1}^{N-1}\rho_i y_i^2 \varepsilon + O(\varepsilon^2)\right] \\
&= \varepsilon^N \left[\frac{1}{2}\int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)} U^p(y',0)|y'|^2 \frac{1}{N-1}\sum_{i=1}^{N-1}\rho_i \varepsilon + O(\varepsilon^2)\right] \\
&= \varepsilon^N \left[\frac{1}{2}H(0)\varepsilon \int_{R^{N-1}} U^p(y',0)|y'|^2 + O(\varepsilon^2)\right].
\end{aligned}$$

As a result,

$$\begin{aligned}
\int_\Omega U_\varepsilon^p &= \int_{B_\tau(0)\cap\Omega} U_\varepsilon^p + O(\varepsilon^N e^{-\tau/\varepsilon}) \\
&= \int_{B_\tau^+(0)} U_\varepsilon^p - \int_{B_\tau^+(0)\setminus\Omega} U_\varepsilon^p + O(\varepsilon^N e^{-\tau/\varepsilon}) \\
&= \varepsilon^N \left(A - \frac{1}{2}\varepsilon H(0)\int_{R^{N-1}} U^p(y',0)|y'|^2\, dy' + O(\varepsilon^2)\right).
\end{aligned} \tag{A.7}$$

□

Lemma A.2.

$$\begin{aligned}
&\int_\Omega \varepsilon^2 |DU_{\varepsilon,x}|^2 + U_{\varepsilon,x}^2 \\
&= \varepsilon^N \Bigg(A - \frac{1}{2}\varepsilon H(x)\int_{R^{N-1}} U^p(y',0)|y'|^2\, dy' \\
&\quad + \frac{1}{2}\varepsilon H(x)\int_{R^{N-1}} U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|\, dy' + O(\varepsilon^2)\Bigg).
\end{aligned} \tag{A.8}$$

Proof. Choose the coordinate systems as in Lemma A.1. Then

$$\text{(A.9)}\qquad \int_\Omega \varepsilon^2|DU_{\varepsilon,x}|^2+U_{\varepsilon,x}^2=\varepsilon^2\int_{\partial\Omega}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon+\int_\Omega(-\varepsilon^2\Delta U_\varepsilon+U_\varepsilon)U_\varepsilon$$
$$=\varepsilon^2\int_{\partial\Omega}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon+\int_\Omega U_\varepsilon^p.$$

$$\text{(A.10)}\qquad \varepsilon^2\int_{\partial\Omega}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon=\varepsilon^2\int_{\partial\Omega\cap B_\tau(0)}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon+O(e^{-\tau/\varepsilon}).$$

For $y\in\partial\Omega\cap B_\tau(0)$,

$$n=\left(\frac{\partial f}{\partial y_1},\dots,\frac{\partial f}{\partial y_{N-1}},-1\right)\Big/\left(1+\sum_{i=1}^{N-1}\left|\frac{\partial f}{\partial y_i}\right|^2\right)^{\frac12}.$$

Hence,

$$\frac{\partial U_\varepsilon}{\partial n}=\frac{1}{\varepsilon}\left[\sum_{i=1}^{N-1}\frac{\partial f}{\partial y_i}\frac{\partial U}{\partial z_i}\left(\frac{y}{\varepsilon}\right)-\frac{\partial U}{\partial z_N}\left(\frac{y}{\varepsilon}\right)\right]\Big/\left(1+\sum_{i=1}^{N-1}\left|\frac{\partial f}{\partial y_i}\right|^2\right)^{\frac12}.$$

Thus,

(A.11)

$$\varepsilon^2\int_{\partial\Omega}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon$$
$$=\varepsilon\int_{B_\tau^{N-1}(0)}\left[\sum_{i=1}^{N-1}\frac{\partial f}{\partial y_i}\frac{\partial U}{\partial z_i}\left(\frac{y}{\varepsilon}\right)-\frac{\partial U}{\partial z_N}\left(\frac{y}{\varepsilon}\right)\right]U_\varepsilon\,dy'+O\left(\varepsilon^Ne^{-\frac{\tau}{\varepsilon}}\right)$$
$$=\varepsilon^N\int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)}U(y',f(\varepsilon y')/\varepsilon)\left[\sum_{i=1}^{N-1}\left(\rho_iy_i\varepsilon+O(\varepsilon^2|y'|^2)\right)\frac{\partial U(y',f(\varepsilon y')/\varepsilon)}{\partial y_i}\right.$$
$$\left.-\frac{\partial U(y',f(\varepsilon y')/\varepsilon)}{\partial y_N}\right]dy'+O\left(\varepsilon^Ne^{-\frac{\tau}{\varepsilon}}\right)$$
$$=\varepsilon^N\left[\int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)}U(y',0)\frac{\partial U(y',0)}{\partial r}\sum_{i=1}^{N-1}\rho_iy_i^2|y'|^{-1}\varepsilon\right.$$
$$\left.-\int_{B_{\frac{\tau}{\varepsilon}}^{N-1}(0)}U(y',0)\frac{\partial^2U(y',0)}{\partial y_N^2}\sum_{i=1}^{N-1}\frac12\rho_iy_i^2\varepsilon+O(\varepsilon^2)\right].$$

But

$$\frac{\partial^2U(y',0)}{\partial y_N^2}=\frac{\partial U(y',0)}{\partial r}|y'|^{-1}.$$

Consequently,

(A.12)

$$\varepsilon^2\int_{\partial\Omega}\frac{\partial U_\varepsilon}{\partial n}U_\varepsilon=\varepsilon^N\left[\frac{1}{2}\varepsilon H(0)\int_{R^{N-1}}U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|\,dy'+O(\varepsilon^2)\right].$$

Clearly, (A.9), (A.10), (A.12) and Lemma A.1 give the desired result. □

Lemma A.3.

$$I(U_{\varepsilon,x})=\varepsilon^N\left[\left(\frac{1}{2}-\frac{1}{p}\right)A-BH(x)\varepsilon+O(\varepsilon^2)\right],$$

where

$$\begin{aligned}B=&\frac{1}{2}\left(\frac{1}{2}-\frac{1}{p}\right)\int_{R^{N-1}}U^p(y',0)|y'|^2\,dy\\&-\frac{1}{2}\int_{R^{N-1}}U(y',0)\frac{\partial U(y',0)}{\partial r}|y'|\,dy'>0.\end{aligned}$$

Proof. This is just a direct consequence of Lemmas A.1 and A.2. □

Lemma A.4.

$$\left|\int_\Omega U_{\varepsilon,x}^{p-1}v\right|=O(\varepsilon^{\frac{N}{2}+1})\|v\|_\varepsilon,\qquad\forall\, v\ \textit{with}\ \langle U_{\varepsilon,x},v\rangle_\varepsilon=0.$$

Proof. Suppose that $x=0$. Then

$$\begin{aligned}\int_\Omega U_{\varepsilon,x}^{p-1}v&=\varepsilon^N\int_{\Omega_\varepsilon}U^{p-1}v(\varepsilon y)=-\varepsilon^N\int_{\partial\Omega_\varepsilon}\frac{\partial U}{\partial n}v(\varepsilon y)\\&=O(\varepsilon^N)\left(\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U}{\partial n}\right|^2\right)^{\frac{1}{2}}\|v(\varepsilon y)\|_{H^1(\Omega_\varepsilon)}\\&=O(\varepsilon^{N/2})\left(\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U}{\partial n}\right|^2\right)^{\frac{1}{2}}\|v\|_\varepsilon.\end{aligned}\tag{A.13}$$

But

$$\begin{aligned}\int_{\partial\Omega_\varepsilon}\left|\frac{\partial U}{\partial n}\right|^2&=\int_{\partial\Omega_\varepsilon\cap B_{\frac{\tau}{\varepsilon}}(0)}\left|\frac{\partial U}{\partial n}\right|^2+O(e^{-\frac{\tau}{\varepsilon}})\\&=\int_{B^{N-1}_{\frac{\tau}{\varepsilon}}(0)}\Bigg|\sum_{i=1}^{N-1}\left[\rho_iy_i\varepsilon+O(\varepsilon^2|y'|^2)\right]\frac{\partial U}{\partial y_i}\\&\qquad-\frac{\partial U(y',f(\varepsilon y')/\varepsilon)}{\partial y_N}\Bigg|^2dy'+O(\varepsilon^3)\\&=O(\varepsilon^2).\end{aligned}\tag{A.14}$$

So, from (A.13) and (A.14), we get the desired result. □

Lemma A.5. *For any $\theta > 0$, we have*

$$\varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x_i}}{\partial n} U_{\varepsilon,x_j} = \begin{cases} O(\varepsilon^{N+1}) & \text{if } i = j, \\ O\left(\varepsilon^{N+1} e^{-(1-\theta)\frac{|x_i - x_j|}{\varepsilon}}\right) & \text{if } i \neq j. \end{cases}$$

Proof.

(A.15)

$$\begin{aligned}
&\varepsilon^2 \int_{\partial\Omega} \frac{\partial U_{\varepsilon,x_i}}{\partial n} U_{\varepsilon,x_j} = \varepsilon^N \int_{\partial\Omega_\varepsilon} \frac{\partial U}{\partial n} U\left(y - \frac{x_j - x_i}{\varepsilon}\right) \\
&= \varepsilon^N \left[\int_{\partial\Omega_\varepsilon \cap B_{\frac{\tau}{\varepsilon}}(0)} \frac{\partial U}{\partial n} U\left(y - \frac{x_j - x_i}{\varepsilon}\right) + O(e^{-\frac{\tau}{\varepsilon}}) \right] \\
&= \varepsilon^N \Bigg[\int_{\partial\Omega_\varepsilon \cap B_{\frac{\tau}{\varepsilon}}(0)} \left(\sum_{i=1}^{N-1} \frac{\partial U}{\partial r} \frac{y_i}{|y|} \frac{\partial (f(\varepsilon y')/\varepsilon)}{\partial y_i} - \frac{\partial U}{\partial r} \frac{y_N}{|y|} \right) U\left(y - \frac{x_j - x_i}{\varepsilon}\right) \\
&\qquad + O(e^{-\frac{\tau}{\varepsilon}}) \Bigg] \\
&\le C\varepsilon^{N+1} \int_{\partial\Omega_\varepsilon \cap B_{\frac{\tau}{\varepsilon}}(0)} \left(\left|\frac{\partial U}{\partial r}\right| |y| + \frac{f(\varepsilon y')}{\varepsilon^2} \frac{1}{|y|} \left|\frac{\partial U}{\partial r}\right| \right) U\left(y - \frac{x_j - x_i}{\varepsilon}\right) \\
&\qquad + O(\varepsilon^N e^{-\frac{\tau}{\varepsilon}}) \\
&\le C\varepsilon^{N+1} \int_{\partial\Omega_\varepsilon \cap B_{\frac{\tau}{\varepsilon}}(0)} e^{-|y|} e^{-|y - \frac{x_j - x_i}{\varepsilon}|} + O(\varepsilon^N e^{-\frac{\tau}{\varepsilon}}) \\
&\le C\varepsilon^{N+1} e^{-(1-\theta)\frac{|x_i - x_j|}{\varepsilon}} \int_{\partial\Omega_\varepsilon \cap B_{\frac{\tau}{\varepsilon}}(0)} e^{-\theta|y|} + O(\varepsilon^N e^{-\frac{\tau}{\varepsilon}}) \\
&= O\left(\varepsilon^{N+1} e^{-(1-\theta)\frac{|x_i - x_j|}{\varepsilon}}\right).
\end{aligned}$$

□

Appendix B.

For $x \in D_{\varepsilon,R}$, let $A_{\varepsilon,x} : E_{\varepsilon,x,k} \longrightarrow E_{\varepsilon,x,k}$ be defined as follows:

$$\langle A_{\varepsilon,x} v, w \rangle_\varepsilon = \langle v, w \rangle_\varepsilon - (p-1) \int_\Omega \left| \sum_{i=1}^k U_{\varepsilon,x_i} \right|^{p-2} vw.$$

Lemma B.1. *There exist $\varepsilon_0 > 0, R > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, $x \in D_{\varepsilon,R}$, $A_{\varepsilon,x}$ is invertible and*

$$\|A_{\varepsilon,x}^{-1}\| \le C,$$

where C is independent of ε and x.

Proof. First we prove that there are $\varepsilon_0 > 0, R > 0$ and $c_0 > 0$, such that for each $\varepsilon \in (0, \varepsilon_0]$, $x \in D_{\varepsilon,R}$, we have

$$\|A_{\varepsilon,x} v\|_\varepsilon \geq c_0 \|v\|_\varepsilon, \qquad \forall\, v \in E_{\varepsilon,x,k}. \tag{B.1}$$

We argue by contradiction. Suppose that there are $\varepsilon_m \longrightarrow 0$, $R_m \longrightarrow +\infty$, $x^{(m)} \in D_{\varepsilon_m,R_m}$, $v_m \in E_{\varepsilon_m,x_m,k}$, such that

$$\|A_{\varepsilon_m,x^{(m)}} v_m\|_{\varepsilon_m} = o(1)\|v_m\|_{\varepsilon_m}. \tag{B.2}$$

We may assume

$$\|v_m\|_{\varepsilon_m} = \varepsilon_m^{N/2}. \tag{B.3}$$

So,

$$\left|\langle A_{\varepsilon_m,x^{(m)}} v_m, \omega\rangle_{\varepsilon_m}\right| = o(\varepsilon_m^{N/2})\|\omega\|_{\varepsilon_m}. \tag{B.4}$$

That is,

$$\begin{aligned} &\int_\Omega \varepsilon_m^2 Dv_m D\omega + v_m\omega - (p-1)\int_\Omega \left|\sum_{i=1}^k U_{\varepsilon_m, x_i^{(m)}}\right|^{p-2} v_m\omega \\ &= o(\varepsilon_m^{N/2})\|\omega\|_{\varepsilon_m}, \qquad \forall\, \omega \in E_{\varepsilon_m,x^{(m)},k}. \end{aligned} \tag{B.5}$$

For each fixed i, let

$$\bar{v}_m(y) = v_m(\varepsilon_m y + x_i^{(m)}), \tag{B.6}$$

$$\Omega_m = \{y:\ \varepsilon_m y + x_i^{(m)} \in \Omega\}. \tag{B.7}$$

Then, from (B.5), we have

(B.8)

$$\begin{aligned} &\int_{\Omega_m} D\bar{v}_m D\omega + \bar{v}_m\omega - (p-1)\int_{\Omega_m} \left|\sum_{j=1}^k U\left(y - \frac{x_j^{(m)} - x_i^{(m)}}{\varepsilon_m}\right)\right|^{p-2} \bar{v}_m\omega \\ &= o(1)\|\omega\|, \qquad \omega \in F_{\varepsilon_m,x^{(m)},k}, \end{aligned}$$

where

(B.9)

$$\begin{aligned} F_{\varepsilon_m,x^{(m)},k} = \Bigg\{\omega :\ \left\langle \omega, U\left(\cdot - \frac{x_j - x_i}{\varepsilon}\right)\right\rangle = \left\langle \omega, \frac{\partial U\left(\cdot - \frac{x_j - x_i}{\varepsilon}\right)}{\partial \tau_{jl}}\right\rangle = 0, \\ j = 1,\dots,k,\ \ l = 1,\dots,N-1\Bigg\}, \end{aligned}$$

and $\{\tau_{j1},\dots,\tau_{j(N-1)}\}$ forms an orthogonal basis for the tangent space of $\partial\Omega_{\varepsilon,x}$ at $\frac{x_i - x_j}{\varepsilon}$.

Since $\|\bar{v}_m\| = 1$, we may assume that

$$\bar{v}_m \rightharpoonup v, \qquad \text{weakly in } \mathrm{H}^1(R_+^N), \tag{B.10}$$

$$\bar{v}_m \longrightarrow v, \qquad \text{strongly in } \mathrm{L}^p_{\mathrm{loc}}(R_+^N). \tag{B.11}$$

Then it is easy to see that v satisfies

$$\langle v, U \rangle_{R_+^N} = 0, \tag{B.12}$$

$$\left\langle v, \left.\frac{\partial U_x}{\partial x_j}\right|_{x=0} \right\rangle_{R_+^N} = 0, \tag{B.13}$$

where

$$\langle v, w \rangle_{R_+^N} = \int_{R_+^N} DvDw + vw.$$

Now we claim that $v = 0$. Assume this for the moment. Since for each fixed $L > 0$, we have

$$\sum_{i=1}^k U^{p-2}_{\varepsilon_m, x_i^{(m)}} = O\left(e^{-(p-2)L}\right), \quad \text{in } \Omega \setminus \cup_{i=1}^k B_{L\varepsilon_m}\left(x_i^{(m)}\right).$$

As a result,

(B.14)

$$\begin{aligned}\int_\Omega \left|\sum_{i=1}^k U_{\varepsilon_m, x_i^{(m)}}\right|^{p-2} v_m^2 &= \sum_{i=1}^k \int_{B_{L\varepsilon_m}(x_i^{(m)})} U^{p-2}_{\varepsilon_m, x_i^{(m)}} v_m^2 + O\left(e^{-(p-2)L}\right)\varepsilon_m^N \\ &= \varepsilon_m^N\left(o(1) + O\left(e^{-(p-2)L}\right)\right),\end{aligned}$$

where $o(1) \to 0$ as $m \to +\infty$.

Letting $\omega = v_m$ in (B.5), from (B.14), we get

$$\|v_m\|^2_{\varepsilon_m} = \varepsilon_m^N\left(o(1) + O\left(e^{-(p-2)L}\right)\right).$$

This is a contradiction to (B.3) if L is chosen large enough.

So it remains to prove that $v = 0$. First we claim that v satisfies

$$\int_{R_+^N} DvD\omega + v\omega - (p-1)\int_{R_+^N} U^{p-2} v\omega = 0, \qquad \forall\, \omega \in F, \tag{B.15}$$

where

(B.16)

$$F = \left\{\omega : \ \langle \omega, U \rangle_{R_+^N} = \left\langle \omega, \left.\frac{\partial U_x}{\partial x_i}\right|_{x=0} \right\rangle_{R_+^N} = 0, \quad i = 1, \ldots, N-1\right\}.$$

In fact, for each $\omega \in F$, we can choose $\alpha_j^{(m)}, \gamma_{lj}^{(m)}$, such that

$$\eta_m = \omega - \sum_{j=1}^{k} \alpha_j^{(m)} U\left(\cdot - \frac{x_j^{(m)} - x_i^{(m)}}{\varepsilon}\right) - \sum_{j=1}^{k}\sum_{l=1}^{N-1} \gamma_{lj}^{(m)} \frac{\partial U\left(\cdot - \frac{x_j^{(m)} - x_i^{(m)}}{\varepsilon}\right)}{\partial \tau_{jl}} \in F_{\varepsilon_m, x^{(m)}, k}.$$

And it is easy to see that $\alpha_j^{(m)} \longrightarrow 0$, $\gamma_{lj}^{(m)} \longrightarrow 0$ as $m \longrightarrow +\infty$. Letting $\omega = \eta_m$ in (B.8), we easily deduce (B.15).

Define $v(y', -y_N) = v(y', y_N)$ for $y_N > 0$. Then

$$\int_{R^N} DvD\omega + v\omega - (p-1)\int_{R^N} U^{p-2} v\omega = 0, \qquad \forall\, \omega \in F_1, \tag{B.17}$$

where

$$F_1 = \left\{\omega \in \mathrm{H}^1(R^N) :\ \langle \omega, U\rangle_{R^N} = \left\langle \omega, \left.\frac{\partial U_x}{\partial x_i}\right|_{x=0}\right\rangle_{R^N} = 0, \quad i = 1, \dots, N\right\}. \tag{B.18}$$

Since $v \in F_1$ (see (B.12) and (B.13)), we know that (B.17) holds for all $\omega \in \mathrm{H}^1(R^N)$. By [**14**], there are $\alpha_i \in R^1$, $i = 1, \dots, N$, such that

$$v = \sum_{i=1}^{N} \alpha_i \left.\frac{\partial U}{\partial x_i}\right|_{x=0}.$$

So $v = 0$.

From (B.1), it is standard to prove that $A_{\varepsilon,x}$ is invertible. In fact, (B.1) implies that $A_{\varepsilon,x}$ is one to one and $A_{\varepsilon,x}$ is closed. If $A_{\varepsilon,x}E_{\varepsilon,x,k} \neq E_{\varepsilon,x,k}$, then there is $w \in (A_{\varepsilon,x}E_{\varepsilon,x,k})^{\perp}$ and $w \neq 0$. Thus,

$$\langle A_{\varepsilon,x} v, w\rangle_{\varepsilon} = 0, \qquad \forall\, v \in E_{\varepsilon,x,k}.$$

But $\langle A_{\varepsilon,x} v, w\rangle_{\varepsilon} = \langle A_{\varepsilon,x} w, v\rangle_{\varepsilon}$. Hence, $A_{\varepsilon,x} w = 0$, and thus $w = 0$. This is a contradiction. □

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