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## LOCAL REFLEXIVITY OF DUAL BANACH SPACES

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We introduce a notion of finite representability of dual Banach spaces in their subspaces preserving duality (f.d.-r. in short) which arises in a natural way in situations such as the principle of local reflexivity. We give a characterization for the f.d.-r. which yields a version of the principle of local reflexivity, and can be applied to the study of the duality theory for ultrapowers of operators. For example, we show that the kernel  $\ker(T^{**}_{\mathcal{U}})$  of an ultrapower of the second conjugate of an operator  $T$  is finitely representable in  $\ker(T_{\mathcal{U}})$ , and  $\ker(T_{\mathcal{U}}^*)$  is f.d.-r. in  $\ker(T^*_{\mathcal{U}})$ . Moreover, we study the duality properties of three semigroups of operators related with the superreflexivity and the finite representability of  $c_0$  and  $\ell_1$  in a Banach space.

### 1. Introduction.

We introduce the concept of *finite representability preserving duality* (f.d.-r. in short) of a Banach space in its subspaces, and the *polar property* and the  *$\mathcal{A}$ -polar property* for subspaces of a dual Banach space, where  $\mathcal{A}$  stands for a class of operators. Given a subspace  $Z$  of a dual space  $X^*$ , we show in Theorem 4 that  $X^*$  is f.d.-r. in  $Z$  if and only if  $Z$  has the polar property. As a consequence, if a subspace  $Z$  of a dual space  $X^*$  has the polar property, then  $Z$  is norming on  $X$ , but the converse implication is not true.

We give some consequences of the  $\mathcal{A}$ -polar property in Theorem 7, from which it follows a version of the principle of local reflexivity:  $X^{**}$  is f.d.-r. in  $X$  [8], in such a way that given an operator  $T \in \mathcal{B}(X, Y)$  we obtain estimates for the norm of the restrictions of  $T^{**}$  to finite dimensional subspaces in terms of the norm of the corresponding restrictions of  $T$ .

In Section 3 we apply these results to the study of the duality theory for ultrapowers of operators. From Theorem 4 it easily follows that the dual space  $X_{\mathcal{U}}^*$  of the ultrapower  $X_{\mathcal{U}}$  of  $X$  is f.d.-r. in  $X^*_{\mathcal{U}}$ , a result first proved by Heinrich [5]. Moreover, the general character of our result allows us to obtain some consequences which do not follow from the principle of local reflexivity or the results of [5]. Namely, in Theorem 11 we show that for every  $T \in \mathcal{B}(X, Y)$  the kernel of the conjugate operator  $T_{\mathcal{U}}^*$  is f.d.-r. in the

kernel of  $T^*_{\mathfrak{U}}$ , where  $T_{\mathfrak{U}}$  is an ultrapower of  $T$ , and the kernel  $\ker(T^{**}_{\mathfrak{U}})$  is finitely representable in  $\ker(T_{\mathfrak{U}})$ .

We study the semigroups  $\mathcal{W}^{up}_+$ ,  $\mathcal{U}^{up}_+$  and  $\mathcal{R}^{up}_+$ , introduced in [13, 4]. Denoting any of them by  $\mathcal{A}_+$ , we show that an operator  $T$  belongs to  $\mathcal{A}_+$  if and only if the second conjugate  $T^{**}$  belongs to  $\mathcal{A}_+$ . Moreover,  $T_{\mathfrak{U}}^* \in \mathcal{A}_+$  if and only if  $T^*_{\mathfrak{U}} \in \mathcal{A}_+$ . Observe that  $T_{\mathfrak{U}}^*$  is an extension of  $T^*_{\mathfrak{U}}$ . Finally, we give a proof of the fact, first proved in [13], that  $T^* \in \mathcal{W}^{up}_+$  if and only if  $\ker(T_{\mathfrak{U}}^*) = \ker(T^*_{\mathfrak{U}})$ .

We use standard notations:  $X$  and  $Y$  are Banach spaces,  $B_X$  the closed unit ball of  $X$ , and  $S_X$  the unit sphere of  $X$ . The class of (bounded linear) operators from  $X$  to  $Y$  is  $\mathcal{B}(X, Y)$ , the dual of  $X$  is  $X^*$ , and given an operator  $T \in \mathcal{B}(X, Y)$ , we denote by  $T^* : Y^* \rightarrow X^*$  its conjugate operator, and by  $R(T)$ ,  $\ker(T)$  and  $\text{coker}(T) := Y/\overline{R(T)}$  the range, the kernel and the cokernel of  $T$ . Observe that  $\text{coker}(T)^*$  can be identified with  $\ker(T^*)$ .

Given a class of operators  $\mathcal{A}$  and a pair of Banach spaces  $X$  and  $Y$ , we denote by  $\mathcal{A}(X, Y)$  the component  $\mathcal{A} \cap \mathcal{B}(X, Y)$  of all operators of  $\mathcal{A}$  between  $X$  and  $Y$ . Also,  $F^\perp$  is the annihilator of a subspace  $F$ , and  $\mathbb{N}$  is the set of all positive integers. We identify  $X$  with a subspace of  $X^{**}$ . Given a subset  $A$  of  $X$ , we denote by  $\langle A \rangle$  the closure of the span of  $A$  in  $X$ .

The letters  $\mathfrak{U}, \mathfrak{V}, \dots$  will be reserved to denote ultrafilters. An ultrafilter  $\mathfrak{U}$  on a set  $I$  is said to be *countably incomplete* if there is a countable partition  $\{I_n : n \in \mathbb{N}\}$  of  $I$  such that for every positive integer  $n$ , we have that  $I_n \notin \mathfrak{U}$ . Every infinite set admits a countably incomplete ultrafilter. Throughout the paper, we assume that the ultrafilters are countably incomplete.

Given a number  $0 < \varepsilon < 1$ , an operator  $T \in \mathcal{B}(X, Y)$  is said to be an  $\varepsilon$ -isometry if  $(1 + \varepsilon)^{-1} < \|Tx\| < 1 + \varepsilon$  for all  $x \in S_X$ , and if the  $\varepsilon$ -isometry  $T$  is onto, then  $X$  is said to be  $\varepsilon$ -isometric to  $Y$ . A Banach space  $X$  is said to be *finitely representable in  $Y$*  if given  $\varepsilon > 0$  and a finite dimensional subspace  $M$  of  $X$  there exists an  $\varepsilon$ -isometry  $T : M \rightarrow Y$ . We will write  $X$  *f.r. in  $Y$*  to mean that the space  $X$  is finitely representable in  $Y$ . Given a number  $C > 1$ , two sequences  $(x_n)$  and  $(y_n)$  are said to be  $C$ -equivalent if for every sequence  $(a_k)$  of scalars and every  $n$  we have

$$C^{-1} \left\| \sum_{k=1}^n a_k x_k \right\| \leq \left\| \sum_{k=1}^n a_k y_k \right\| \leq C \left\| \sum_{k=1}^n a_k x_k \right\|.$$

**2. Finite representability preserving the duality.**

Since we need to use ultrapowers, we recall here some definitions and results, and refer to [5] for more information. Let  $I$  be an infinite set. We denote by  $\ell_\infty(I, X)$  the Banach space of bounded families  $(x_i)_{i \in I}$  in  $X$  with norm  $\|(x_i)\| := \sup\{\|x_i\| : i \in I\}$ .

Let  $\mathfrak{U}$  be an ultrafilter on  $I$  and let  $N_{\mathfrak{U}}(X)$  be the closed subspace of all families  $(x_i) \in \ell_{\infty}(I, X)$  which converge to 0 following  $\mathfrak{U}$ . The *ultrapower of  $X$  following  $\mathfrak{U}$*  is defined as follows:

$$X_{\mathfrak{U}} := \frac{\ell_{\infty}(I, X)}{N_{\mathfrak{U}}(X)}.$$

The element of  $X_{\mathfrak{U}}$  admitting the family  $(x_i) \in \ell_{\infty}(I, X)$  as a representative is denoted by  $[x_i]_{\mathfrak{U}}$ , or simply  $[x_i]$  if it does not lead to confusion. The norm of  $[x_i]$  in  $X_{\mathfrak{U}}$  is given by

$$\|[x_i]\| = \lim_{\mathfrak{U}} \|x_i\|.$$

The ultrapower  $X_{\mathfrak{U}}$  canonically contains an isometric copy of  $X$  generated by the constant families of  $\ell_{\infty}(I, X)$ . We identify this copy with  $X$ . An operator  $T \in \mathcal{B}(X, Y)$  admits an extension  $T_{\mathfrak{U}} \in \mathcal{B}(X_{\mathfrak{U}}, Y_{\mathfrak{U}})$  given by  $T_{\mathfrak{U}}[x_i] := [Tx_i]$ .

The ultrapower  $(X^*)_{\mathfrak{U}}$  is canonically contained in  $(X_{\mathfrak{U}})^*$ , but in general these spaces do not coincide. Actually, given  $(f_i) \in \ell_{\infty}(I, X^*)$ , the class  $[f_i] \in X^*_{\mathfrak{U}}$  is identified with the element  $f \in (X_{\mathfrak{U}})^*$  given by  $f([x_i]) := \lim_{\mathfrak{U}} f_i(x_i)$ . Heinrich [5] proved that  $(X^*)_{\mathfrak{U}}$  coincides with  $(X_{\mathfrak{U}})^*$  if and only if  $X$  is superreflexive.

In the following definition we introduce a concept of finite representability which is stronger than the usual one: The *finite representability preserving the duality*.

**Definition 1.** Let  $Z$  be a subspace of the dual  $X^*$  of a Banach space  $X$ . We say that  $X^*$  is *finitely representable in  $Z$  preserving the duality* (f.d.-r. in short) if for every couple of finite dimensional subspaces  $F$  of  $X^*$  and  $G$  of  $X$ , and for every  $0 < \varepsilon < 1$ , there is an  $\varepsilon$ -isometry  $L : F \rightarrow Z$  such that  $(Lf)(x) = f(x)$  for all  $x \in G$  and all  $f \in F$ .

The celebrated *principle of local reflexivity* states that the second dual  $X^{**}$  is f.d.-r. in  $X$ . We refer to [10] for an elementary proof of this principle.

A subspace  $Z$  of a dual space  $X^*$  is said to be *norming* if for every  $x \in X$  we have that  $\|x\| = \sup\{f(x) : f \in B_Z\}$ . It easily follows from the Hahn-Banach theorem that a subspace  $Z$  of  $X^*$  is norming if and only if  $\overline{B_Z}^{\sigma(X^*, X)} = B_{X^*}$  (see [1]).

Let  $X$  be a Banach space and  $k, l$  two positive integers. A linear function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$ , represented by a matrix  $(a_{ij})_{i=1}^l_{j=1}^k$ , induces an operator

$$f_X : X \times \dots \times X \rightarrow X \times \dots \times X$$

in a natural way  $f_X(x_i) := (\sum_{j=1}^k a_{ij}x_j)$ . Note that  $(f_X)^* = f^*_{X^*}$ . We denote by  $\mathcal{L}$  the class of all these operators.

$$\mathcal{L} := \{f_X : k, l \in \mathbb{N}, f : \mathbb{R}^k \rightarrow \mathbb{R}^l \text{ linear, } X \text{ Banach space}\}.$$

Henceforth we will denote by  $\ell_1^k(X)$  and  $\ell_\infty^k(X)$  the space  $X \times \cdots \times X$  endowed with the norms  $\|(x_j)_{j=1}^k\| := \sum_{j=1}^k \|x_j\|$  and  $\|(x_j)_{j=1}^k\| := \sup_{1 \leq j \leq k} \|x_j\|$ , respectively. Given two subsets  $A \subset X$  and  $B \subset X^*$ , their polar sets are defined as follows:

$$A^\circ := \{f \in X^* : |f(a)| \leq 1 \text{ for all } a \in A\};$$

$$B_\circ := \{z \in X : |f(z)| \leq 1 \text{ for all } f \in B\}.$$

Note that the sets  $A^\circ$  and  $B_\circ$  are closed in the norm topology. If, in addition,  $A$  and  $B$  are convex and symmetric then we have  $(B_\circ)^\circ = \overline{B}^{\sigma(X^*, X)}$  and  $(A^\circ)_\circ = \overline{A}$ .

**Definition 2.** A subspace  $Z$  of a dual space  $X^*$  is said to have the *polar property* if for every  $k, l$  in  $\mathbb{N}$  and every linear function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$  we have

$$\overline{fX \left( B_{\ell_1^k(X)} \right)} = \left( f^*_{X^*} \left|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)} \right. \right)_\circ.$$

Observe that the inclusion  $fX \left( B_{\ell_1^k(X)} \right) \subset \left( f^*_{X^*} \left|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)} \right. \right)_\circ$  is always true. So, taking into account that the identities  $\overline{T(B_X)} = (T^*{}^{-1}B_{X^*})_\circ$ ,  $T^*{}^{-1}B_{X^*} = (TB_X)^\circ$  hold for every operator  $T \in \mathcal{B}(X, Y)$ , we obtain the following elementary (but useful) characterization of the polar property.

**Proposition 3.** A subspace  $Z$  of a dual space  $X^*$  has the polar property if and only if for every linear function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$ , we have

$$f^*_{X^*}{}^{-1} \left( B_{\ell_\infty^l(X^*)} \right) = \overline{f^*_{X^*} \left|_{\ell_\infty^l(Z)}^{-1} \left( B_{\ell_\infty^k(Z)} \right)}^{\sigma(\ell_\infty^l(X^*), \ell_1^k(X))}.$$

The following result shows that the polar property is a useful tool to study finite representability.

**Theorem 4.** A subspace  $Z$  of a dual space  $X^*$  has the polar property if and only if  $X^*$  is finitely representable in  $Z$  preserving the duality.

*Proof.* Assume that  $X^*$  is f.d.-r. in  $Z$  and  $Z$  does not have the polar property. In virtue of Proposition 3, there are  $L \in \mathcal{L}(\ell_1^k(X), \ell_1^l(X))$ , a  $0 < \theta < 1$  and a  $l$ -tuple

$$(f_i)_{i=1}^l \in L^*{}^{-1} B_{\ell_\infty^k(X^*)} \setminus \overline{\left( L^* \left|_{\ell_\infty^l(Z)}^{-1} B_{\ell_\infty^k(Z)} \right. \right)}^{\sigma(\ell_\infty^l(X^*), \ell_1^k(X))}$$

such that  $L^*((f_i)_{i=1}^l) = (g_i)_{i=1}^k \in (1 - \theta) B_{\ell_\infty^k(X^*)}$ .

By the Hahn-Banach Theorem, there exists a number  $0 < \varepsilon < 1$  and a  $l$ -tuple  $(x_i)_{i=1}^l \in \ell_1^l(X)$  such that the  $\sigma(\ell_\infty^l(X^*), \ell_1^l(X))$ -neighborhood of  $(f_i)_{i=1}^l$ , given by

$$\mathcal{V} := \left\{ (h_i)_{i=1}^l : \left| \sum_{i=1}^l (f_i - h_i)(x_i) \right| < \varepsilon \right\}$$

verifies

$$(1) \quad \mathcal{V} \cap \overline{L^* |_{\ell_\infty^k(Z)}^{-1} B_{\ell_\infty^k(Z)}^{\sigma(\ell_\infty^k(X^*), \ell_1^k(X))}} = \emptyset.$$

Since  $X^*$  is f.d.r. in  $Z$ , given  $\delta > 0$  such that  $(1 - \theta)(1 + \delta) \leq 1$ , there are  $\tilde{f}_1, \dots, \tilde{f}_l$  in  $Z$  such that the operator

$$G : \langle \tilde{f}_i : i = 1, \dots, l \rangle \longrightarrow \langle f_i : i = 1, \dots, l \rangle$$

given by  $G(\tilde{f}_i) := f_i$  is a  $\delta$ -isometry and  $\tilde{f}_i(x_j) = f_i(x_j)$  for all  $i$  and all  $j$ .

Associated to  $L^*$ , there is a matrix of numbers  $(\lambda_{ij})_{i=1}^l_{j=1}^k$  such that

$$L^* \left( (h_i)_{i=1}^l \right) = \left( \sum_{i=1}^l \lambda_{ij} h_i \right)_{j=1}^k.$$

So, every  $g_j$  is equal to  $\sum_{i=1}^l \lambda_{ij} f_i \in (1 - \theta)B_{X^*}$ . Since  $G$  is a  $\delta$ -isometry, for every  $j = 1, \dots, k$ , we have that  $\tilde{g}_j := \sum_{i=1}^l \lambda_{ij} \tilde{f}_i \in B_Z$ , hence  $(\tilde{f}_i)_{i=1}^l \in L^* |_{\ell_\infty^k(Z)}^{-1} (B_{\ell_\infty^k(Z)})$ . On the other hand, the equalities  $\tilde{f}_i(x_j) = f_i(x_j)$  for all  $i = 1, \dots, l$  and all  $j = 1, \dots, k$  imply that  $(\tilde{f}_i)_{i=1}^l \in \mathcal{V}$ , in contradiction with (1).

For the converse implication, let  $E$  and  $F$  be finite dimensional subspaces of  $X^*$  and  $X$ , let us denote  $n := \dim E$  and  $k := \dim F$ , and let  $0 < \varepsilon < 1$ .

The Auerbach Lemma allows us to take  $(h_j, y_j)_{j=1}^n$  in  $E^* \times E$  such that  $\|h_j\| = \|y_j\| = 1$  for all  $j = 1, \dots, n$  and  $h_i(y_j) = \delta_{ij}$ . The identity  $id : E \longrightarrow X^*$  is given by  $id(e) = \sum_{j=1}^n h_j(e)y_j$ . We shall find  $z_1, \dots, z_n$  in  $Z$  for the wished  $\varepsilon$ -isometry  $L : E \longrightarrow Z$  to be defined as  $L(e) := \sum_{j=1}^n h_j(e)z_j$ .

Let us take  $0 < \alpha < \min\{2/5, (1 - \varepsilon)^{-1} - 1, \varepsilon(1 + n/2)^{-1}\}$  and choose

- a basis  $\{x_j\}_{j=1}^k$  in  $F$ ,
- an  $\alpha/4$ -net  $\{e_j\}_{j=1}^N$  in  $B_E$ , and
- vectors  $\{u_j\}_{j=1}^N$  in  $B_X$

such that  $\|e\| \leq (1 + \alpha) \sup_{1 \leq j \leq N} |e(u_j)|$  for all  $e \in E$ . The condition  $\alpha < 2/5$  guarantees the existence of the vectors  $u_j$ . We write

$$e_j = \sum_{s=1}^n \lambda_s^j y_s, \quad j = 1, \dots, N.$$

In order to simplify the notation, we denote

$$\begin{aligned} U &:= \ell_1^n(X), \\ M &:= U \oplus_1 \ell_1^N(X), \\ W &:= \ell_\infty^n(Z), \\ V &:= W \oplus_\infty \ell_\infty^N(Z). \end{aligned}$$

Let  $\Lambda^* : U^* \longrightarrow M^* = U^* \oplus_\infty \ell_\infty^N(X^*)$  and  $S^* : U^* \longrightarrow \mathbb{R}^{nk \times nN}$  be the conjugate operators given respectively by

$$\Lambda^*((g_s)_{s=1}^n) := \left( (g_s)_{s=1}^n, \left( \sum_{s=1}^n \lambda_s^j g_s \right)_{j=1}^N \right)$$

and

$$S^*((g_s)_{s=1}^n) := (g_r(x_i), g_t(u_j)).$$

Since  $Z$  has the polar property, Proposition 3 gives that

$$\Lambda^{*-1}(B_{M^*}) = \overline{\Lambda^* |_{W^{-1}}^{-1}(B_V)}^{\sigma(U^*, U)}.$$

So, the  $\sigma(U^*, U)$ -continuity of  $S^*$  yields

$$(2) \quad S^*(\Lambda^{*-1}(B_{M^*})) \subset \overline{S^*(\Lambda^* |_{W^{-1}}^{-1}(B_V))}.$$

Observe that  $R(S^*) = R(S^* |_W)$  because  $\overline{W}^{\sigma(U^*, U)} = U^*$ . Thus, by Formula 2, since  $(y_j)_{j=1}^n \in \Lambda^{*-1}(B_{M^*})$ , given any number  $\varepsilon'$  such that  $0 < \varepsilon' < \alpha \|\Lambda^*\|^{-1}$ , we can find  $(c_j)_{j=1}^n \in \Lambda^* |_{W^{-1}}^{-1}(B_V)$  and  $(b_j)_{j=1}^n \in \varepsilon' B_W$  so that

$$S^*((y_j)_{j=1}^n) = S^*((c_j)_{j=1}^n) + S^*((b_j)_{j=1}^n).$$

Taking  $z_j := c_j + b_j$  for each  $j = 1, \dots, n$  in the definition of the operator  $L$ , we obtain  $(z_j)_{j=1}^n \in (1 + \alpha)\Lambda^{*-1}(B_V)$  and  $S^*((y_j)_{j=1}^n) = S^*((z_j)_{j=1}^n)$ . Evidently, the condition  $(Le)(x) = e(x)$  holds for all  $e \in E$  and all  $x \in F$ .

In order to check that  $L$  is an  $\varepsilon$ -isometry, let  $e \in B_E$ . On the one hand, we have

$$\|Le\| \geq \sup_{1 \leq j \leq N} |(Le)(u_j)| = \sup_{1 \leq j \leq N} |e(u_j)| \geq (1 + \alpha)^{-1} \|e\| \geq 1 - \varepsilon.$$

On the other hand, first note that  $\|L\| \leq 2n$  because

$$\|Le\| = \left\| \sum_{j=1}^n h_j(e) z_j \right\| \leq \sum_{j=1}^n (1 + \alpha) \|h_j\| \leq 2n.$$

Choose now vectors  $e_k$  so that  $\|e - e_k\| \leq \alpha/4$ . Since  $\Lambda^*((z_j)_{j=1}^n) \in (1 + \alpha)B_V$  we have that  $\|Le_k\| = \|\sum_{s=1}^n \lambda_s^k z_s\| \leq 1 + \alpha$ . Therefore,

$$\|Le\| \leq \|Le_k\| + \|L(e - e_k)\| \leq 1 + \varepsilon;$$

hence  $L$  is an  $\varepsilon$ -isometry. □

**Remark.** If we take  $f = id : \mathbb{R} \rightarrow \mathbb{R}$  in Proposition 3, then we obtain that  $B_{X^*} = \overline{B_Z}^{\sigma(X^*, X)}$ . Thus, if a subspace  $Z$  of a dual space  $X$  has the polar property, then  $Z$  is norming on  $X$  [1].

The converse implication is not true. There are norming subspaces of dual Banach spaces which fail the polar property.

For example, let us consider the Rademacher-like sequence  $(x_n)$  in  $\ell_\infty$ , where  $x_1 = (1, -1, 1, -1, 1, -1, \dots)$  and for  $n \in \mathbb{N}$  the sequence  $x_{n+1}$  consists of successive repetitions of the block

$$1, (2^n), 1, -1, (2^n), -1.$$

Then  $(x_n)$  is 1-equivalent to the unit vector basis of  $\ell_1$ . Now, if we take an enumeration  $\{A_n : n \in \mathbb{N}\}$  of all the finite sequences of numbers in  $\{1, -1\}$ , with  $\text{card}(A_m) \leq \text{card}(A_n)$  for  $m < n$ , and modify each  $x_n$  in a finite number of coordinates so that the initial segment of  $x_n$  coincides with  $A_n$ , then  $(x_n)$  continues to be 1-equivalent to the unit vector basis of  $\ell_1$ , and it generates a norming subspace  $[x_n]$  of  $\ell_\infty = \ell_1^*$ .

However, since  $\ell_\infty$  is not f.r. in  $\ell_1$ , the subspace  $\langle x_n \rangle$  of  $\ell_\infty$  does not have the polar property, by Theorem 4.

The following concept is a generalization of the polar property.

**Definition 5.** Let  $\mathcal{A}$  be a class of operators. A subspace  $Z$  of  $X^*$  is said to have the  $\mathcal{A}$ -polar property if given a Banach  $Y$ , an operator  $T \in \mathcal{A}(Y, X)$ , integers  $p, q \in \mathbb{N}$  and  $r \in \mathbb{N} \cup \{0\}$ , and a couple of matrices of real numbers  $(a_{ij})_{i=1}^q_{j=1}^p, (b_{ij})_{i=1}^r_{j=1}^p$ , we have that the operator  $L : \ell_1^q(X) \oplus_1 \ell_1^r(Y) \rightarrow \ell_1^p(X)$ , given by

$$L((x_i)_{i=1}^q, (y_j)_{j=1}^r) := \left( \sum_{i=1}^q a_{ij}x_i + \sum_{i=1}^r b_{ij}Ty_i \right)_{j=1}^p,$$

satisfies  $L \left( \overline{B_{\ell_1^q(X) \oplus_1 \ell_1^r(Y)}} \right) = \left( L^* \Big|_{\ell_\infty^p(Z)}^{-1} B_{\ell_\infty^q(Z) \oplus_\infty \ell_\infty^r(Y^*)} \right)_o$ ; equivalently,

$$L^{*-1} B_{\ell_\infty^q(X^*) \oplus_\infty \ell_\infty^r(Y^*)} = \overline{L^* \Big|_{\ell_\infty^p(Z)}^{-1} B_{\ell_\infty^q(Z) \oplus_\infty \ell_\infty^r(Y^*)}}^{\sigma(\ell_\infty^p(X^*), \ell_1^p(X))}.$$

**Remark.** In the case  $r = 0$ , we assume that the matrix  $(b_{ij})_{i=1}^r_{j=1}^p$  appearing in Definition 5 has no entries. Therefore, the  $\mathcal{A}$ -polar property implies the polar property.

The polar property is equivalent to the  $\mathcal{I}$ -polar property, where  $\mathcal{I}$  stands for the class of all identities of Banach spaces.

The difficulty to check the  $\mathcal{A}$ -polar property depends on the structure of  $\mathcal{A}$ . Sometimes it is possible to do it easily, as in the next example which will play an important role in Proposition 13.

**Example 6.** Let  $\mathcal{B}^d := \{T^* : T \in \mathcal{B}\}$  denote the class of all conjugate operators. Then every Banach space  $X$  has the  $\mathcal{B}^d$ -polar property as a subspace of  $X^{**}$ .

In fact, take an operator  $T^* : Y^* \rightarrow X^*$ , and matrices  $(a_{ij})_{i=1, j=1}^q, (b_{ij})_{i=1, j=1}^r$  of real numbers. As in Definition 5, we define  $L : \ell_1^q(X^*) \oplus_1 \ell_1^r(Y^*) \rightarrow \ell_1^p(X^*)$  by

$$L((f_i)_{i=1}^q, (g_j)_{j=1}^r) := \left( \sum_{i=1}^q a_{ij} f_i + \sum_{i=1}^r b_{ij} T^* g_i \right)_{j=1}^p.$$

Let us write  $U := \ell_\infty^q(X) \oplus_\infty \ell_\infty^r(Y)$  and  $V := \ell_\infty^p(X)$ . Clearly,  $L$  can be identified with a conjugate operator  $S^* : U^* \rightarrow V^*$ , and consequently, we have that  $S^*(B_{U^*}) = (S^{-1}B_U)_\circ$ ; therefore,

$$S^*(B_{U^*}) = \left( S^{**} \Big|_V^{-1} B_{\ell_\infty^q(X) \oplus_\infty \ell_\infty^r(Y^*)} \right)_\circ.$$

We saw in Theorem 4 that a subspace  $Z$  of  $X^*$  has the polar property if and only if  $X^*$  is f.d.-r. in  $Z$ . Now we prove a strengthening of the direct implication.

**Theorem 7.** *Let  $Z$  be a subspace of  $X^*$ , let  $E$  and  $F$  be finite dimensional subspaces of  $X^*$  and  $X$  respectively, and let  $\mathcal{A}$  be a class of operators.*

*If  $Z$  has the  $\mathcal{A}$ -polar property and  $T \in \mathcal{A}(Y, X)$ , then for every  $0 < \varepsilon < 1$  and  $\delta' > 0$  there is an  $\varepsilon$ -isometry  $L : E \rightarrow Z$  such that  $(Le)(x) = e(x)$  for all  $e \in E$  and all  $x \in F$ , and satisfies the following additional condition  $\|T^* \Big|_{L(E)}\| \leq \|T^* \Big|_E\| + \delta'$ .*

*Proof.* It is similar to the second part of the proof of Theorem 4. We shall use here the same notations.

Let  $\delta := \|T\|$ . Without loss of generality, we suppose that  $0 < \varepsilon < 2^{-3}\delta^{-1}\delta'$ . We choose  $0 < \alpha < \min\{2/5, (1 - \varepsilon)^{-1} - 1, \varepsilon(1 + n/2)^{-1}\}$  and define the operator  $\Lambda^*$  as in the proof of Theorem 4. Moreover, we write  $\widetilde{M}^* := M^* \oplus_\infty \ell_\infty^N(Y^*)$ , and define the conjugate operator  $\widetilde{\Lambda}^* : U^* \rightarrow \widetilde{M}^*$  by

$$\widetilde{\Lambda}^* ((g_s)_{s=1}^n) := \left( \Lambda^* ((g_s)_{s=1}^n), \left( \delta^{-1} T^* \left( \sum_{s=1}^n \lambda_s^j g_s \right) \right)_{j=1}^N \right).$$

Write  $\widetilde{V} := V \oplus_\infty \ell_\infty^N(Y^*)$ . Since  $Z$  has the  $\mathcal{A}$ -polar property, we have

$$\overline{\widetilde{\Lambda}^* \Big|_{\widetilde{V}}^{-1} (B_{\widetilde{V}})}^{\sigma(U^*, U)} = \widetilde{\Lambda}^{*-1} (B_{\widetilde{M}^*}).$$

Thus,  $(y_j)_{j=1}^n \in \widetilde{\Lambda}^{*-1}(B_{\widetilde{M}^*})$ , and proceeding as in the proof of Theorem 4, we find

$$(z_j)_{j=1}^n \in (1 + \alpha) \widetilde{\Lambda}^* \Big|_{\widetilde{V}}^{-1} (B_{\widetilde{V}})$$

such that  $S^*((y_j)_{j=1}^n) = S^*((z_j)_{j=1}^n)$ . Again as in the proof of Theorem 4, the condition  $\alpha < \min\{2/5, (1-\varepsilon)^{-1}-1, \varepsilon(1+n/2)^{-1}\}$  implies that the operator  $L : E \rightarrow Z$  is an  $\varepsilon$ -isometry and  $(Le)(x) = e(x)$  for  $e \in E$  and  $x \in F$ . Let us check that  $\|T^*|_{L(E)}\| \leq \delta + \delta'$ . Note that  $(z_s)_{s=1}^n \in (1 + \alpha)\tilde{\Lambda}^{*-1}(B_{\tilde{V}})$  implies that

$$\|T^*(Le_j)\| = \left\| T^* \left( \sum_{s=1}^n \lambda_s^j z_s \right) \right\| \leq \delta(1 + \alpha) \text{ for } j = 1, \dots, N.$$

Since  $L$  is an  $\varepsilon$ -isometry, if  $\beta := 4^{-1}\alpha(1 + \varepsilon) + 2\varepsilon$ , then  $\{Le_j\}_{j=1}^N$  is a  $\beta$ -net in  $(1 + \varepsilon)B_{L(E)}$ , and taking  $w_j := (1 + \varepsilon)^{-1}Le_j$  we obtain a  $(1 + \varepsilon)^{-1}\beta$ -net  $\{w_j\}_{j=1}^N$  in  $B_{L(E)}$ . Let  $w \in B_{L(E)}$  and pick  $w_j$  such that  $\|w - w_j\| \leq (1 + \varepsilon)^{-1}\beta$ . We have

$$\|T^*w\| \leq \|T^*(w - w_j)\| + \|T^*w_j\| \leq \frac{\beta}{1 + \varepsilon}\|T^*\| + \frac{\delta(1 + \alpha)}{1 + \varepsilon}.$$

Since  $\alpha < \{2/5, 2^{-2}\delta^{-1}\delta'\}$  and  $\varepsilon < 2^{-3}\delta^{-1}\delta'$ , we conclude  $\|T^*w\| \leq \delta + \delta'$ . □

As a consequence of Theorem 7, we derive a version of the principle of local reflexivity [8].

**Corollary 8.** *Given an operator  $T \in \mathcal{B}(X, Y)$ , a pair of finite dimensional subspaces  $E \subset X^{**}$  and  $F \subset X^*$ , and numbers  $0 < \varepsilon < 1$  and  $\delta > 0$ , there exists an  $\varepsilon$ -isometry  $L : E \rightarrow X$  such that we have  $f(Le) = e(f)$  for  $f \in F$  and  $e \in E$ , and  $\|T|_{L(E)}\| < \|T^{**}|_E\| + \delta$ .*

*Proof.* We have seen in Example 6 that  $X$  has the  $\mathcal{B}^d$ -polar property as a subspace of  $X^{**}$ . Thus,  $X$  plays here the role of  $Z$  in Theorem 7 as a subspace of the dual space  $(X^*)^*$ , and the result holds. □

### 3. Applications.

Here we apply the previous results to study the duality theory for ultrapowers of operators and some semigroups of operators related with the super-reflexivity and the finite representability of  $c_0$  and  $\ell_1$  in a Banach space.

First, we show in Theorem 10 that for every operator  $T$ , the kernel  $\ker(T^{**}|_{\mathfrak{U}})$  is f.r. in  $\ker(T|_{\mathfrak{U}})$ . In order to do that, we need the following technical Lemma.

**Lemma 9.** *Given an ultrafilter  $\mathfrak{U}$  on  $I$  and a sequence  $(B_n) \subset \mathfrak{U}$  such that  $B_{n+1} \subset B_n$  for every  $n$ , there is a sequence  $(C_n) \subset \mathfrak{U}$  verifying  $C_{n+1} \subset C_n \subset B_n$ ,  $C_n \setminus C_{n+1} \neq \emptyset$  for every  $n \in \mathbb{N}$ , and  $\bigcap_{n=1}^\infty C_n = \emptyset$ .*

*Proof.* Let  $\{I_n : n \in \mathbb{N}\}$  be a partition of  $I$  such that  $\emptyset \neq I_n \notin \mathfrak{U}$  for every  $n \in \mathbb{N}$ . We build the sets  $C_n$  inductively. We take  $C_1 := B_1$ . Assume we already have the sets  $(C_n)_{n=1}^k \subset \mathfrak{U}$  verifying  $C_{n+1} \subset C_n \subset B_n$  and  $C_{n+1} \neq C_n$  for  $n = 1, \dots, k - 1$ . Let us denote  $n_k := \min \{n \geq k : I_n \cap C_k \neq \emptyset\}$  and

$$C_{k+1} := \bigcup_{n=n_k+1}^{\infty} (I_n \cap B_{k+1}).$$

It is easy to check that the sequence  $(C_n)_n$  satisfies the desired conditions. □

**Theorem 10.** *Let  $T \in \mathcal{B}(X, Y)$  be an operator and let  $\mathfrak{U}$  be an ultrafilter on  $I$ . Then the kernel  $\ker(T^{**}_{\mathfrak{U}})$  is finitely representable in  $\ker(T_{\mathfrak{U}})$ .*

*Proof.* Let  $E$  be a finite dimensional subspace of  $\ker(T^{**}_{\mathfrak{U}})$  and let  $\{\mathbf{F}_k = [F_k^i]_i : k = 1, \dots, n\}$  be a basis of  $E$ . Let us denote  $E_i := \langle F_k^i : k = 1, \dots, n \rangle$  and let  $T_i : E \rightarrow E_i$  be the operator given by  $T_i(\mathbf{F}_k) := F_k^i$ .

Taking into account that  $\lim_{\mathfrak{U}(i)} T^{**} F_k^i = 0$ , a result of Heinrich [5, Proposition 6.1] allows us to find, for each positive integer  $m$ , a set  $J_m \in \mathfrak{U}$  such that for every  $i \in J_m$ , the operators  $T_i$  are  $1/m$ -isometries and  $\|T^{**} F_k^i\| < 1/m$  for  $k = 1, \dots, n$ . By Lemma 9, we can take a decreasing sequence  $(C_n)_n$  in  $\mathfrak{U}$  such that  $\bigcap_{n=1}^{\infty} C_n = \emptyset$ ,  $C_n \neq C_{n+1}$  and  $C_n \subset J_n$  for all  $n \in \mathbb{N}$ ; add the set  $C_0 := I$  to this sequence. For every  $i \in I$ , we denote by  $m_i$  the unique element of  $\mathbb{N} \cup \{0\}$  so that  $i \in C_{m_i} \setminus C_{m_i+1}$ .

For every  $i \in C_1$ , since  $i \in C_{m_i} \setminus C_{m_i+1}$ , by Corollary 8 we can choose  $x_1^i, \dots, x_n^i$  in  $X$  such that the operator  $L_i : E_i \rightarrow F_i := \langle x_1^i, \dots, x_n^i \rangle$  given by  $L_i(F_k^i) := x_k^i$  is a  $1/m_i$ -isometry and  $\|Tx_k^i\| \leq 2/m_i$  for  $k = 1, \dots, n$ .

Define  $\mathbf{x}_k := [x_k^i]$  and write  $F := \langle \mathbf{x}_k : k = 1, \dots, n \rangle$ . Thus we have that every  $\mathbf{x}_k$  belongs to  $\ker(T_{\mathfrak{U}})$  and  $F$  is isometric to  $E$ . In fact, on the one hand  $C_m = \{i : m_i \geq m\} \in \mathfrak{U}$  for each  $m \in \mathbb{N}$ . So  $\|Tx_k^i\| \leq 2/m$  for all  $i \in C_m$ . Therefore  $\lim_{\mathfrak{U}(i)} Tx_k^i = 0$  and  $\mathbf{x}_k \in \ker(T_{\mathfrak{U}})$  for  $k = 1, \dots, n$ .

On the other hand, let  $\sum_{k=1}^n \lambda_k \mathbf{F}_k$  be a norm-one element in  $E$ . Let  $M$  be an upper bound for  $\{\sum_{k=1}^n \lambda_k F_k^i : i \in I\}$ . Take any  $m \in \mathbb{N}$ . For every  $i \in \{i : m_i \geq m\}$ , we have that  $L_i$  is a  $1/m$ -isometry; thus

$$C_m \subset \left\{ i : \left\| \sum_{k=1}^n \lambda_k F_k^i \right\| - \left\| \sum_{k=1}^n \lambda_k x_k^i \right\| \leq M/m \right\} \in \mathfrak{U}.$$

In this way we obtain that  $\lim_{\mathfrak{U}(i)} \left\| \sum_{k=1}^n \lambda_k F_k^i \right\| - \left\| \sum_{k=1}^n \lambda_k x_k^i \right\| = 0$ . Since

$$\left\| \sum_{k=1}^n \lambda_k \mathbf{x}_k \right\| = \lim_{\mathfrak{U}(i)} \left\| \sum_{k=1}^n \lambda_k x_k^i \right\| \quad \text{and} \quad \lim_{\mathfrak{U}(i)} \left\| \sum_{k=1}^n \lambda_k F_k^i \right\| = \left\| \sum_{k=1}^n \mathbf{F}_k \right\|,$$

we get  $\left\| \sum_{k=1}^n \lambda_k \mathbf{x}_k \right\| = \left\| \sum_{k=1}^n \lambda_k \mathbf{F}_k \right\|$ . Hence,  $E$  is isometric to  $F$  and therefore  $\ker(T^{**}_{\mathfrak{U}})$  is f.r. in  $\ker(T_{\mathfrak{U}})$ . □

A second application of Theorem 4: Heinrich [5] showed that  $X_{\mathfrak{U}}^*$  is f.d.-r. in  $X^*_{\mathfrak{U}}$ . This is a consequence of our Theorem 11 below, applied to the zero operator. But our Theorem 4 is stronger than the result in [5]. Indeed, it will allow us to prove in Theorem 11 that  $\ker(T_{\mathfrak{U}}^*)$  is f.d.-r. in  $\ker(T^*_{\mathfrak{U}})$  for any operator  $T \in \mathcal{B}(X, Y)$ . Note that  $\ker(T_{\mathfrak{U}}^*)$  can be identified with the dual space  $\text{coker}(T_{\mathfrak{U}})^*$ , but is not necessarily the second dual of any space. Moreover, in general  $\text{coker}(T_{\mathfrak{U}})$  is not an ultrapower [3]. Thus neither the principle of local reflexivity nor the result of Heinrich are applicable.

**Theorem 11.** *Let  $T : X \rightarrow Y$  be an operator and  $\mathfrak{U}$  an ultrafilter on  $I$ . Then the kernel  $\ker(T_{\mathfrak{U}}^*)$  is finitely representable in  $\ker(T^*_{\mathfrak{U}})$  preserving the duality.*

*Proof.* By Theorem 4, it is enough to prove that  $\ker(T^*_{\mathfrak{U}})$  has the polar property as a subspace of  $\ker(T_{\mathfrak{U}}^*)$ . We identify isometrically the kernel  $\ker(T_{\mathfrak{U}}^*)$  with the dual of the cokernel  $\text{coker}(T_{\mathfrak{U}})$ . Along this proof, when we say that an operator  $A : U_1 \rightarrow U_2$  is identified with  $B : V_1 \rightarrow V_2$  we mean that there are isometries onto  $J_1 : U_1 \rightarrow V_1$  and  $J_2 : U_2 \rightarrow V_2$  so that  $A = J_2^{-1}BJ_1$ .

Pick a linear function  $f : \mathbb{R}^k \rightarrow \mathbb{R}^l$  and write

$$f_{\text{coker}(T_{\mathfrak{U}})} : \ell_1^k(\text{coker}(T_{\mathfrak{U}})) \rightarrow \ell_1^l(\text{coker}(T_{\mathfrak{U}})).$$

We only have to check the inclusion

$$(3) \quad \left( (f_{\text{coker}(T_{\mathfrak{U}})})^* \Big|_{\ell_{\infty}^{-1}(\ker(T^*_{\mathfrak{U}}))} B_{\ell_{\infty}^k(\ker(T^*_{\mathfrak{U}}))} \right)_{\circ} \subset \overline{f_{\text{coker}(T_{\mathfrak{U}})} B_{\ell_1^k(\text{coker}(T_{\mathfrak{U}}))}}$$

The proof is divided into three cases. The main one is the case  $k = l$ . The cases  $k < l$  and  $k > l$  will be obtained as a consequence of the main case.

*Case a)  $k = l$ .* We consider the operator  $T^k : \ell_1^k(X) \rightarrow \ell_1^k(Y)$ , defined by  $T^k((x_j)_{j=1}^k) := (Tx_j)_{j=1}^k$ , and write  $U := T^k$ ,  $V := \ell_1^k(X)$  and  $W := \ell_1^k(Y)$ . We can identify  $U_{\mathfrak{U}} : V_{\mathfrak{U}} \rightarrow W_{\mathfrak{U}}$  with

$$(T_{\mathfrak{U}})^k : \ell_1^k(X_{\mathfrak{U}}) \rightarrow \ell_1^k(Y_{\mathfrak{U}}).$$

The operator  $\varphi : W_{\mathfrak{U}} \rightarrow \ell_1^k(\text{coker}(T_{\mathfrak{U}}))$  which sends  $[(y_i^j)_{j=1}^k]_i$  to  $[(y_i^j)]_i + \overline{R(T_{\mathfrak{U}})}_{j=1}^k$  is surjective,  $\ker(\varphi) = \overline{R(U_{\mathfrak{U}})}$  and it is easy to check that the induced operator

$$\tilde{\varphi} : \text{coker}(U_{\mathfrak{U}}) \rightarrow \ell_1^k(\text{coker}(T_{\mathfrak{U}}))$$

is an isometry. Now, since  $(f_Y)_{\mathfrak{U}} : W_{\mathfrak{U}} \rightarrow W_{\mathfrak{U}}$  satisfies  $(f_Y)_{\mathfrak{U}}(R(U_{\mathfrak{U}})) \subset R(U_{\mathfrak{U}})$ , it induces an operator  $L : \text{coker}(U_{\mathfrak{U}}) \rightarrow \text{coker}(U_{\mathfrak{U}})$  given by

$$L \left( [y_i] + \overline{R(U_{\mathfrak{U}})} \right) := (f_Y)_{\mathfrak{U}}([y_i]) + \overline{R(U_{\mathfrak{U}})}.$$

The operator  $L$  is identified with

$$f_{\text{coker}(T_{\mathfrak{U}})} : \ell_1^k(\text{coker}(T_{\mathfrak{U}})) \rightarrow \ell_1^k(\text{coker}(T_{\mathfrak{U}}))$$

because  $L = \tilde{\varphi}^{-1} f_{\text{coker}(T_{\mathfrak{U}})} \tilde{\varphi}$ . Since  $\ell_1^k(\text{coker}(T_{\mathfrak{U}}))^* = \ell_\infty^k(\ker(T_{\mathfrak{U}}^*))$ , we can identify

$$(f_{\text{coker}(T_{\mathfrak{U}})})^* : \ell_\infty^k(\ker(T_{\mathfrak{U}}^*)) \rightarrow \ell_\infty^k(\ker(T_{\mathfrak{U}}^*))$$

with  $L^* : \ker(U_{\mathfrak{U}}^*) \rightarrow \ker(U_{\mathfrak{U}}^*)$ . Under the previous identification we have that  $\ker(U_{\mathfrak{U}}^*) = \ell_\infty^k(\ker(T_{\mathfrak{U}}^*))$ . So the result that we want to prove (Formula 3) is equivalent to the inclusion

$$\left( L^* \Big|_{\ker(U_{\mathfrak{U}}^*)}^{-1} B_{\ker(U_{\mathfrak{U}}^*)} \right)_\circ \subset \overline{LB_{\text{coker}(U_{\mathfrak{U}})}}.$$

Suppose that this inclusion is false. Then there exists  $\mathbf{h} = [h_i] \in W_{\mathfrak{U}}$  such that

$$\mathbf{h} + \overline{R(U_{\mathfrak{U}})} \in (L^{*-1} B_{\ker(U_{\mathfrak{U}}^*)})_\circ \setminus \overline{LB_{\text{coker}(U_{\mathfrak{U}})}}.$$

We can assume that  $\|h_i\| = K > 0$  for all  $i$ . Write  $S := f_Y$ ,  $A := S(B_W)$  and  $\mathbf{A} := S_{\mathfrak{U}}(B_{W_{\mathfrak{U}}}) = \{[a_i] : a_i \in A, i \in I\}$ . Take  $\varepsilon > 0$  such that

$$\text{dist}(\mathbf{h} - \mathbf{A}, R(U_{\mathfrak{U}})) > \varepsilon > 0.$$

Thus, for every  $m \in \mathbb{N}$ , we have  $\text{dist}(\mathbf{h} - \mathbf{A}, mU_{\mathfrak{U}}B_{V_{\mathfrak{U}}}) > \varepsilon$ , hence

$$D_m := \{i \in I : \text{dist}(h_i - A, mUB_V) > \varepsilon\} \in \mathfrak{U}.$$

By Lemma 9 we can take a sequence  $(C_n)_n$  in  $\mathfrak{U}$  such that  $C_{n+1} \subset C_n \subset D_n$  and  $C_n \neq C_{n+1}$  for every  $n \in \mathbb{N}$ , and  $\bigcap_{n=1}^\infty C_n = \emptyset$ .

For every  $i \in I$ , let  $m_i$  be the unique positive integer for which  $i \in C_{m_i} \setminus C_{m_i+1}$ , and denote  $K_i := h_i - A - m_iUB_V$ . Since  $K_i$  is convex and  $K_i \cap \varepsilon B_W = \emptyset$ , by the Hahn-Banach Theorem there is a vector  $f_i \in S_{W^*}$  such that  $\inf f_i(K_i) \geq \varepsilon$ . Hence

$$|f_i(Ux)| \leq \frac{K + \|S\| + \varepsilon}{m_i} \quad \text{for all } x \in B_V;$$

equivalently,  $\|U^* f_i\| \leq m_i^{-1}(K + \|S\| + \varepsilon)$ . Since the chain  $(C_n)_n$  has empty intersection we have  $\lim_{\mathfrak{U}} m_i^{-1} = 0$ , and therefore  $\mathbf{f} := [f_i] \in \ker(U_{\mathfrak{U}}^*)$ .

On the other hand,  $\inf f_i(K_i) \geq \varepsilon$  implies  $\inf f_i(h_i - A) \geq \varepsilon$ . Thus we get  $0 < \varepsilon \leq \varepsilon_i := f_i(h_i) \leq K$  for all  $i$  and  $|f_i(a)| \leq \varepsilon_i - \varepsilon$  for all  $a \in A$ . Hence, letting  $g_i := \varepsilon_i^{-1} f_i$ , we obtain that

$$\alpha := \sup [g_i](LB_{\text{coker}(U_{\mathfrak{U}})}) \leq 1 - \varepsilon K^{-1} < 1 = [g_i]([h_i]).$$

Take  $y_i := \alpha^{-1} g_i$ . We have that  $\mathbf{y} := [y_i] = \alpha^{-1}(\lim_{\mathfrak{U}} \varepsilon_i^{-1})[f_i] \in \ker(U_{\mathfrak{U}}^*)$  and

$$\sup \mathbf{y}(LB_{\text{coker}(U_{\mathfrak{U}})}) \leq 1 < \mathbf{y}(\mathbf{h});$$

thus  $\mathbf{y} \in L^{*-1} B_{\ker(U_{\mathfrak{U}}^*)}$  and  $\mathbf{h} + \overline{R(U_{\mathfrak{U}})} \notin (L^{*-1} B_{\ker(U_{\mathfrak{U}}^*)})_\circ \setminus \overline{LB_{\text{coker}(U_{\mathfrak{U}})}}$ , a contradiction.

For the cases b) and c), we adopt the notations  $Z := \text{coker}(T_{\mathfrak{U}})$  and  $H := \ker(T_{\mathfrak{U}}^*)$ , so that  $Z^* = \ker(T_{\mathfrak{U}}^*)$ .

Case b)  $k < l$ . We consider the operator  $\tilde{L} : \ell_1^k(Z) \oplus_1 \ell_1^{l-k}(Z) \longrightarrow \ell_1^l(Z)$ , defined by  $\tilde{L}(a, b) := La$ . By case a) we have

$$\tilde{L}B_{\ell_1^k(Z) \oplus_1 \ell_1^{l-k}(Z)} = \left( \tilde{L}^* \Big|_{\ell_\infty^l(H)}^{-1} B_{\ell_\infty^k(H) \oplus_\infty \ell_\infty^{l-k}(H)} \right)_\circ.$$

Since  $LB_{\ell_1^k(Z)} = \tilde{L}B_{\ell_1^k(Z) \oplus_1 \ell_1^{l-k}(Z)}$  and

$$L^* \Big|_{\ell_\infty^l(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right) = \tilde{L}^* \Big|_{\ell_\infty^l(H)}^{-1} \left( B_{\ell_\infty^k(H) \oplus_\infty \ell_\infty^{l-k}(H)} \right),$$

we have that  $\overline{LB_{\ell_1^k(Z)}} = \left( L^* \Big|_{\ell_\infty^l(H)}^{-1} B_{\ell_\infty^k(H)} \right)_\circ$ .

Case c)  $k > l$ . We consider the operator  $\tilde{L} : \ell_1^k(Z) \longrightarrow \ell_1^l(Z) \oplus_1 \ell_1^{k-l}(Z)$ , defined by  $\tilde{L}(a) := (La, 0)$ . By case a) we have

$$\tilde{L}B_{\ell_1^k(Z)} = \left( \tilde{L}^* \Big|_{\ell_\infty^l(H) \oplus_\infty \ell_\infty^{k-l}(H)}^{-1} B_{\ell_\infty^k(H)} \right)_\circ.$$

The definition of  $\tilde{L}$  yields  $L \left( B_{\ell_1^k(Z)} \right) \times 0_{\ell_1^{k-l}(Z)} = \tilde{L} \left( B_{\ell_1^k(Z)} \right)$  and

$$L^* \Big|_{\ell_\infty^l(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right) \times \ell_\infty^{k-l}(Z^*) = \tilde{L}^* \Big|_{\ell_\infty^l(H) \oplus_\infty \ell_\infty^{k-l}(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right).$$

So we obtain

$$\left( L^* \Big|_{\ell_\infty^l(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right) \right)_\circ \times 0_{\ell_1^{k-l}(Z)} = \left( \tilde{L}^* \Big|_{\ell_\infty^l(H) \oplus_\infty \ell_\infty^{k-l}(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right) \right)_\circ.$$

Consequently, we get  $\overline{LB_{\ell_1^k(Z)}} = \left( L^* \Big|_{\ell_\infty^l(H)}^{-1} \left( B_{\ell_\infty^k(H)} \right) \right)_\circ$ . □

**Corollary 12** ([9]). *Let  $T : X \longrightarrow Y$  be an operator and let  $\mathfrak{U}$  be an ultrafilter on  $I$ . Then  $\ker(T^*_{\mathfrak{U}})$  is a norming subspace of  $\ker(T_{\mathfrak{U}}^*) = (\text{coker}(T_{\mathfrak{U}}))^*$ .*

A class of operators  $\mathcal{A}$  is said to be a *semigroup* if the composition of two operators of  $\mathcal{A}$  belongs to  $\mathcal{A}$  and if for every Banach space  $X$ , the identity operator  $I_X$  on  $X$  belongs to  $\mathcal{A}$ . As well-known examples of semigroups, we mention the classical semi-Fredholm operators, and the tauberian operators, introduced by Kalton and Wilansky [7]. An operator  $T \in \mathcal{B}(X, Y)$  is *tauberian* if  $T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y$ .

Now we apply the previous results to study the dual properties of the semigroups  $\mathcal{W}^{up}_+$ , introduced in [12], and  $\mathcal{U}^{up}_+$  and  $\mathcal{R}^{up}_+$ , introduced in [4]. Recall that  $S_X = \{x \in X : \|x\| = 1\}$ .

An operator  $T \in \mathcal{B}(X, Y)$  belongs to  $\mathcal{W}^{up}_+$  if for every  $0 < r < 1$  there exists  $n \in \mathbb{N}$  for which there are no finite sets  $\{x_1, \dots, x_n\} \subset S_X$  and  $\{f_1, \dots, f_n\} \subset S_{X^*}$  for which  $f_i(x_j) > r$  for  $i \leq j$ ,  $f_i(x_j) = 0$  for  $j < i$  and  $\|Tx_k\| < 1/k$ , for  $k = 1, \dots, n$ .

An operator  $T \in \mathcal{B}(X, Y)$  belongs to  $\mathcal{U}^{up}_+$  if for every  $C \geq 1$  there are  $\delta > 0$  and  $n \in \mathbb{N}$  for which there is no finite set  $\{x_1, \dots, x_n\} \subset S_X$  which is  $C$ -equivalent to the unit basis of  $\ell^\infty_n$  and satisfies  $\|Tx_k\| < \delta$  for  $k = 1, \dots, n$ .

An operator  $T \in \mathcal{B}(X, Y)$  belongs to  $\mathcal{R}^{up}_+$  if for every  $C \geq 1$  there are  $\delta > 0$  and  $n \in \mathbb{N}$  for which there is no finite set  $\{x_1, \dots, x_n\} \subset S_X$  which is  $C$ -equivalent to the unit basis of  $\ell^1_n$  and satisfies  $\|Tx_k\| < \delta$  for  $k = 1, \dots, n$ .

Note that  $T \in \mathcal{W}^{up}_+$  if and only if all the ultrapowers  $T_{\mathcal{U}}$  are tauberian operators [2]. Tacon [13] proved that  $T \in \mathcal{W}^{up}_+$  implies  $T^{**} \in \mathcal{W}^{up}_+$ . Nevertheless, his proof does not seem to be applicable to the semigroups  $\mathcal{U}^{up}_+$  and  $\mathcal{R}^{up}_+$ . Next we show that Theorem 10 allows us to give a unified proof of the result for the three semigroups  $\mathcal{W}^{up}_+$ ,  $\mathcal{R}^{up}_+$  and  $\mathcal{U}^{up}_+$ .

**Proposition 13.** *Let  $\mathcal{A}_+$  be any of the semigroups  $\mathcal{W}^{up}_+$ ,  $\mathcal{R}^{up}_+$  or  $\mathcal{U}^{up}_+$  and let  $T \in \mathcal{B}(X, Y)$ . Then  $T \in \mathcal{A}_+$  if and only if  $T^{**} \in \mathcal{A}_+$ .*

*Proof.* Let  $\mathcal{U}$  be an ultrafilter. It was proved in [2] (for  $\mathcal{W}^{up}_+$ ) and [4] (for  $\mathcal{R}^{up}_+$  and  $\mathcal{U}^{up}_+$ ) that  $T$  belongs to  $\mathcal{A}_+$  if and only if the zero operator  $0_{\ker(T_{\mathcal{U}})}$  on  $\ker(T_{\mathcal{U}})$  belongs to  $\mathcal{A}_+$ . In any of the three cases, the condition  $0_X \in \mathcal{A}_+$  defines a superproperty; i.e.,  $Y$  f.r. in  $X$  and  $0_X \in \mathcal{A}_+$  implies  $0_Y \in \mathcal{A}_+$ .

By Theorem 10,  $\ker(T^{**}_{\mathcal{U}})$  is f.r. in  $\ker(T_{\mathcal{U}})$ . Since  $\ker(T^{**}_{\mathcal{U}})$  contains  $\ker(T_{\mathcal{U}})$ , we conclude that  $T \in \mathcal{A}_+$  if and only if  $T^{**} \in \mathcal{A}_+$ . □

**Proposition 14.** *Let  $\mathcal{A}_+$  be any of the semigroups  $\mathcal{W}^{up}_+$ ,  $\mathcal{R}^{up}_+$  or  $\mathcal{U}^{up}_+$  and let  $T \in \mathcal{B}(X, Y)$ . Then  $T_{\mathcal{U}}^* \in \mathcal{A}_+$  if and only if  $T^*_{\mathcal{U}} \in \mathcal{A}_+$ .*

*Proof.* The direct implication is clear since  $T_{\mathcal{U}}^*$  is an extension of  $T^*_{\mathcal{U}}$ . For the converse implication, observe that Theorem 11 says that  $\ker(T_{\mathcal{U}}^*)$  is finitely representable in  $\ker(T^*_{\mathcal{U}})$ . Now, it is enough to observe that  $T^* \in \mathcal{W}^{up}_+$  if and only if  $\ker(T^*_{\mathcal{U}})$  is superreflexive [2], and  $T^*$  belongs to  $\mathcal{R}^{up}_+$  (resp.  $\mathcal{U}^{up}_+$ ) if and only if  $\ell_1$  (resp.  $c_0$ ) is not finitely representable in  $\ker(T^*_{\mathcal{U}})$  [4]. □

The following result was proved by Tacon using nonstandard analysis. Here we give a more transparent proof.

**Proposition 15** ([13, Theorem 3]). *Given an operator  $T \in \mathcal{B}(X, Y)$ , we have  $T^* \in \mathcal{W}^{up}_+$  if and only if  $\ker(T^*_{\mathcal{U}}) = \ker(T_{\mathcal{U}}^*)$ .*

*Proof.* By Corollary 12 we have that  $\ker(T^*_{\mathcal{U}})$  is  $w^*$ -dense in  $\ker(T_{\mathcal{U}}^*)$ . Moreover, it is shown in [2, Theorem 9] that  $T^*$  belongs to  $\mathcal{W}^{up}_+$  if and only if  $\ker(T^*_{\mathcal{U}})$  is reflexive. Hence the direct implication is clear.

Conversely, assume that  $T^* \notin \mathcal{W}^{up}_+$ ; equivalently,  $\ker(T^*_{\mathcal{U}})$  is not reflexive. If  $\ker(T_{\mathcal{U}}^*) = \ker(T^*_{\mathcal{U}})$ , since  $\ker(T_{\mathcal{U}}^*) = \text{coker}(T_{\mathcal{U}})^*$ , the triangular

arrays in non-reflexive spaces, discovered in [6] and [11], give normalized sequences  $(\mathbf{y}_n + \overline{R(T_{\mathfrak{U}})})_n$  in  $\text{coker}(T_{\mathfrak{U}})$  and  $(\mathbf{f}_n)$  in  $\ker(T^*_{\mathfrak{U}})$ , and  $0 < \varepsilon < 1$  so that

$$(4) \quad \mathbf{f}_k(\mathbf{y}_l) = \begin{cases} > \varepsilon, & \text{if } 1 \leq k \leq l < \infty \\ = 0, & \text{if } 1 \leq l < k < \infty. \end{cases}$$

Write  $\mathbf{y}_n = [y_n^i]_i$  and  $\mathbf{f}_n = [f_n^i]_i$ . Let  $\mathbf{g} = [g_i]$  be a  $w^*$ -cluster point of  $\{\mathbf{f}_n : n \in \mathbb{N}\}$  in  $\ker(T^*_{\mathfrak{U}})$ . Formula 4 yields  $\mathbf{g}(\mathbf{y}_n) = 0$  for all  $n \in \mathbb{N}$ , and allows us to build the sequence  $(A_n)_n$  of elements of  $\mathfrak{U}$  given inductively by

$$A_1 := \{i \in I : g_i(y_1^i) < \varepsilon/2, f_1^i(y_1^i) > \varepsilon\} \in \mathfrak{U}$$

$$A_n := A_{n-1} \cap \{i \in I : g_i(y_n^i) < \varepsilon/2, f_k^i(y_n^i) > \varepsilon, 1 \leq k \leq n\} \in \mathfrak{U}.$$

Since  $(A_n)_n$  is decreasing, by Lemma 9 we find another decreasing sequence  $(C_n)_n \subset \mathfrak{U}$  such that  $C_n \subset A_n$  and  $C_n \setminus C_{n+1} \neq \emptyset$  for all  $n \in \mathbb{N}$ , and  $\bigcap_{n=1}^{\infty} C_n = \emptyset$ . Thus, for every  $k \in \mathbb{N}$ , we take  $s_i := y_k^i$  for every  $i \in C_k \setminus C_{k+1}$ , and define  $\mathbf{s} := [s_i]$ .

On the one hand, we have that  $\mathbf{g}(\mathbf{s}) \geq \varepsilon$ . Indeed, let  $k$  be a positive integer. For every  $l > k$  and every  $i \in C_l \setminus C_{l+1}$  we have  $f_k^i(s_i) = f_k^i(y_l^i) \geq \varepsilon$ . Since  $\bigcup_{l=k}^{\infty} (C_l \setminus C_{l+1}) \in \mathfrak{U}$ , we have  $[f_k^i]([s_i]) \geq \varepsilon$ ; therefore,  $\mathbf{f}_k(\mathbf{s}) \geq \varepsilon$  for each  $k \in \mathbb{N}$ . Besides that, as  $\mathbf{g}$  is a  $w^*$ -cluster point of  $\{\mathbf{f}_n : n \in \mathbb{N}\}$ , we have  $\mathbf{g}(\mathbf{s}) \geq \varepsilon$ .

On the other hand, for every  $n \in \mathbb{N}$  and  $i \in C_n \setminus C_{n+1} \subset A_n$  we have that  $g_i(s_i) = g_i(y_n^i) < \varepsilon/2$ . Thus  $\bigcup_{n=1}^{\infty} (C_n \setminus C_{n+1}) \in \mathfrak{U}$  leads to  $\mathbf{g}(\mathbf{s}) = [g_i]([s_i]) \leq \varepsilon/2$ , and we get a contradiction.  $\square$

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