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It is shown that $H^\infty + C$ has the Shilov property. In particular any function f in $H^\infty + C$ vanishing in an open neighborhood of the zeros of another function g in $H^\infty + C$ is divisible there by g.

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let L^∞ and H^∞ denote the usual algebras on the unit circle $\partial\mathbb{D}$. The smallest closed subalgebra of L^∞ properly containing H^∞ is $H^\infty + C$, where C denotes the algebra of continuous complex valued functions on the unit circle. The algebra consisting of $H^\infty + C$ functions whose complex conjugates are also in $H^\infty + C$ is denoted by QC.

For any of the above algebras, denoted here by A, the maximal ideal space or spectrum of A is the space of nonzero multiplicative linear functionals on A and is denoted $M(A)$. When $M(A)$ is given the weak-$*$ topology, it becomes a compact Hausdorff space. Identifying each point of $\mathbb{D}$ with the multiplicative linear functional that is point evaluation, we think of $\mathbb{D}$ as a subset of $M(H^\infty)$. It is well known that $M(H^\infty + C) = M(H^\infty) \setminus \mathbb{D}$.

Factorization theorems and the study of zero sets of bounded analytic functions have been crucial to our understanding of the structure of both the algebra H^∞ and its maximal ideal space. Thus, to expand our knowledge of $H^\infty + C$ one might ask which of these properties extend to this algebra.

For H^∞ functions, zero sets in $M(H^\infty)$ play an important role in division problems. One might hope, then, that zero sets in $M(H^\infty + C)$ play an equally important role in the study of division in this algebra. However, the situation becomes more complicated here. Guillory and Sarason [**9**] have shown that there exist two inner functions, u_1, u_2 in $H^\infty + C$ with $|u_1| = |u_2|$ on $M(H^\infty + C)$, but $u_1\overline{u_2}$ is not in $H^\infty + C$.

Axler [**1**] began the study of multiplying functions into $H^\infty + C$ by showing that if f is any function in L^∞, then there exists a Blaschke product B multiplying f into $H^\infty + C$. Wolff [**19**] then showed that every unimodular function in L^∞ can be written as a quotient of Blaschke products times an invertible function in QC. Guillory and Sarason [**9**], Guillory, Izuchi and Sarason [**8**], and Axler and Gorkin [**2**] continued this work. The theorems in these papers can be restated as division theorems assuming that the divisor is a unimodular function in $H^\infty + C$. In fact, these authors show that if $f \in H^\infty + C$ and u is a unimodular function in $H^\infty + C$, then f is divisible in $H^\infty + C$ by u^n for every positive integer n if and only if $f = 0$ wherever $|u| < 1$ on $M(H^\infty + C)$.

In the present paper, as a consequence of a more general result about ideals in $H^\infty + C$, we show that if g is an arbitrary function in $H^\infty + C$ and f vanishes on an open set in $M(H^\infty + C)$ containing the zeros of g, then f is divisible in $H^\infty + C$ by g^n for every positive integer n . We remark that Izuchi and Izuchi [**13**] showed that for $f \in H^\infty + C$ and an inner function u satisfying $|f| \le |u|$ on $M(H^\infty + C)$, one obtains $f^{n+1}\overline{u^n} \in H^\infty + C$ for every positive integer n. In view of the above example, we see that one cannot expect to have $f\overline{u} \in H^\infty + C$ in general. On the other hand, if we assume a stronger hypothesis than Izuchi's, namely $f = 0$ on an open set containing the zeros of u, we are able to obtain (Theorem 1.4) the stronger conclusion $f\overline{u} \in H^\infty + C$.

These division theorems are corollaries of our main result for ideals. To state this result, we need to recall that a commutative unital Banach algebra A is said to be regular if for every closed set E in its spectrum and each point x not in E there exists a function $f \in A$ such that $f(x) = 1$ and f vanishes on E. A well known result due to Shilov ([**5**], [**14**]) states that an ideal I in a regular Banach algebra A contains any function in A that vanishes on an open subset in the spectrum of A containing the hull of I. In that case A is said to have the Shilov property. In Theorem 2.9 we show that this result can be extended to the nonregular algebra $H^\infty + C$. As a consequence we see that large classes of ideals in $H^\infty + C$, including radical ideals and intersections of primary ideals, are determined locally. This property, shared by ideals in regular algebras, is an important tool in harmonic analysis.

In a final paragraph we analyze related problems for the algebra H^∞ of bounded analytic functions. We assume that the reader is familiar with the general theory of the maximal ideal space of H^∞. As a convenient reference, we mention the book of J.B. Garnett [**4**]. We conclude this introduction with some notation.

Let $f \in H^\infty + C$. Then the zero set of f in $M(H^\infty + C)$ is denoted by $Z(f)$. The hull or zero set of an ideal I in $H^\infty + C$ is the set $Z(I) = \bigcap_{f \in I} Z(f)$. Since each nontrivial Gleason part of H^∞ is an analytic disk, we know that the functions in $H^\infty + C$ are holomorphic with respect to this analytic structure. Hence, if $f \in H^\infty + C$ vanishes at a point $x \in M(H^\infty + C)$ whose Gleason part $P(x) = \{m \in M(H^\infty) : \|m - x\| < 2\}$ is nontrivial, it is meaningful to speak of the multiplicity of x as a zero of f. In case f vanishes identically on the part of x, the multiplicity of the zero of f at x is defined to be infinite. The set of all zeros of f of infinite order, is denoted by $Z_\infty(f)$. If $q \in L^\infty$, then we denote by $H^\infty[q]$ the smallest closed subalgebra of L^∞ containing H^∞ and q. Finally, the weak-$*$ closure of a subset S of $M(H^\infty + C)$ will be denoted by $\overline{S}$; its set of interior points by S^0.

1. Division by Blaschke products.

It is well known ([**2**], [**8**]) that whenever $f \in H^\infty + C$ and b is an interpolating Blaschke product satisfying $Z(b) \subseteq Z(f)$ on $M(H^\infty + C)$, then $f\overline{b} \in H^\infty + C$. Obviously this does not hold if b is a non-interpolating Blaschke product (just take any Blaschke product B and put $f = B, b = B^2$). Guillory, Izuchi and Sarason ([**8**], Cor. 2) noticed that, even by taking multiplicities into account, no such division result holds. Assuming, however, that f vanishes in a neighborhood of the zeros of a Blaschke product B, then a positive result will be given here. To prove it, we need the following deep results of S. Axler, P. Gorkin, D. Marshall and D. Suarez.

Lemma 1.1 ([**15**]). *There exists a constant β with $0 < \beta < 1$ such that for every Blaschke product B there is an interpolating Blaschke product b so that*

$$\{z \in \mathbb{D} : b(z) = 0\} \subseteq \{z \in \mathbb{D} : |B(z)| < \beta\} \tag{1}$$

and $H^\infty[\overline{b}] = H^\infty[\overline{B}]$.

For the proof see also ([**4**], pp. 336, 379).

Lemma 1.2 ([**2**, p. 92]). *Let $h \in H^\infty + C$ and let B be a Blaschke product. Then $h\overline{B}^n \in H^\infty + C$ for every $n \in \mathbb{N}$ if and only if $h\,(1 - |B|) \equiv 0$ on $M(H^\infty + C)$.*

Remark. In fact Lemmas 1.1 and 1.2 hold in a more general setting. The interested reader is referred to [**15**], [**4**], [**7**], [**17**], [**18**] respectively [**2**] and [**20**] for further information.

Lemma 1.3 ([**18**, Th. 2.5]). *Let $E \subseteq M(H^\infty)$ be a closed set and let B be a Blaschke product with $|B| > 0$ on E. Then for every σ with $0 < \sigma < 1$ there exists a finite factorization $B = B_0 B_1 \cdots B_n$ so that $|B_j(x)| \geq \sigma$ for all $x \in E$ and $j \in \{1, 2, \cdots, n\}$ and where B_0 is a finite product of interpolating Blaschke products.*

Theorem 1.4. *Let B be a Blaschke product and suppose that $f \in H^\infty + C$ vanishes on an open subset U of $M(H^\infty + C)$ containing the zero set $Z(B)$ of B. Then $f\overline{B^\nu} \in H^\infty + C$ for every $\nu \in \mathbb{N}$.*

Proof. Obviously $B \neq 0$ on $M(H^\infty + C) \setminus U$. Let β be the constant of Lemma 1.1. Use Lemma 1.3 to factor $B = B_0 B_1 \cdots B_n$ where

$$|B_j| > \beta \quad \text{on } M(H^\infty + C) \setminus U \ (j = 1, 2, \ldots, n), \tag{2}$$

and where B_0 is a finite product of interpolating Blaschke products. Clearly f vanishes identically on every Gleason part which meets U. Hence $U \subseteq$

$Z_\infty(f)$. Since every zero of B_0 is of finite order, we deduce from $Z(B_0) \subseteq U$ that every zero of B_0 is a zero of f of infinite order. Hence by [**2**] or [**8**] we have $f\overline{B_0} \in H^\infty + C$ and $Z_\infty(f\overline{B_0}) = Z_\infty(f)$. Thus $U \subseteq Z_\infty(f\overline{B_0})$.

Next we show that $B_1 \cdots B_n$ divides $f\overline{B_0}$. To do this, we choose, according to Lemma 1.1, interpolating Blaschke products b_j such that

$$H^\infty[\overline{b_j}] = H^\infty[\overline{B_j}] \quad (j = 1, 2, \dots, n) \tag{3}$$

and

$$\{z \in \mathbb{D} : b_j(z) = 0\} \subseteq \{z \in \mathbb{D} : |B_j(z)| < \beta\} \quad (j = 1, \dots, n). \tag{4}$$

Fix $j \in \{1, \cdots, n\}$ and let $x \in Z(b_j)$. By ([**4**], p. 379), x lies in the weak-$*$ closure of $\{z \in \mathbb{D} : b_j(z) = 0\}$ and hence, by (4), $|B_j(x)| \leq \beta$. Thus, by (2), $x \in U$. In particular $Z(b_j) \subseteq Z_\infty(f\overline{B_0})$. By [**2**] or [**8**], we conclude that $(f\overline{B_0})\overline{b_j^n} \in H^\infty + C$ for ever $n \in \mathbb{N}$. Hence, by Lemma 1.2, $f\overline{B_0} = 0$ whenever $|b_j| < 1$. But by (3)

$$\{x \in M(H^\infty + C) : |b_j(x)| < 1\} = \{x \in M(H^\infty + C) : |B_j(x)| < 1\}.$$

So we see that $f\overline{B_0} = 0$ whenever $\prod_{j=1}^n |B_j| < 1$. Hence, by Lemma 1.2

$$f\overline{B_0} \prod_{j=1}^n \overline{B_j} \in H^\infty + C.$$

Thus $f\overline{B} \in H^\infty + C$. Since $Z(B) = Z(B^\nu)$, it is now clear that $f\overline{B^\nu} \in H^\infty + C$ for every $\nu \in \mathbb{N}$ (just replace B by B^ν). $\square$

2. The Shilov property for $H^\infty + C$.

It is a classical result (see [**10**], p. 170) that the spectrum, $M(L^\infty)$, of L^∞ is a totally disconnected compact space. Hence characteristic functions χ_E on $M(L^\infty)$ are continuous if and only if E is clopen (that is closed and open). Since we may identify L^∞ with $C(M(L^\infty))$, χ_E then is the Gelfand transform of a characteristic function χ_S for some Borel set S of $\partial\mathbb{D}$ of positive Lebesgue measure. Moreover, $M(L^\infty)$ is the Shilov boundary of H^∞ (see [**10**], p. 174).

Hoffman ([**10**], p. 184) has shown that each $m \in M(H^\infty)$ has a unique norm preserving extension to a linear functional on L^∞. Letting $\operatorname{supp} m$ in $M(L^\infty)$ denote the support set of the representing measure μ_m for m, one can show ([**4**], p. 375) that this extension is given by

$$m(f) = \int_{\operatorname{supp} m} f \, d\mu_m \quad (f \in L^\infty).$$

It follows that each function $f \in L^\infty$ can be thought of as a continuous function on $M(H^\infty)$. This point of view will be adopted throughout this paper and we write $f(m) := m(f)$. We note that this extension to $M(H^\infty)$ of $f \in L^\infty$ coincides on $\mathbb{D}$ with the Poisson integral of f.

To proceed, we need to point out several properties of the Douglas algebra $H^\infty[\chi_E]$ generated by H^∞ and χ_E. For the sake of simplicity, we simply write $\{0 < \chi_E < 1\}$ for the set

$$\{m \in M(H^\infty + C) : 0 < m(\chi_E) < 1\}.$$

By the Chang-Marshall Theorem (see [**4**], Sec. 9) we know that

$$M\left(H^\infty[\chi_E]\right) = \{m \in M(H^\infty + C) : \chi_E|_{\operatorname{supp} m} \in H^\infty|_{\operatorname{supp} m}\}.$$

Since $m(\chi_E) = \int_{\operatorname{supp} m} \chi_E \, d\mu_m$ for every $m \in M(H^\infty + C)$, we see that χ_E is real valued on $M(H^\infty + C)$ with values contained in the interval $[0, 1]$. Hence $m(\chi_E) = 0$ if and only if $\operatorname{supp} m \cap E = \emptyset$ and $m(\chi_E) = 1$ if and only if $\operatorname{supp} m \subseteq E$. Since $\operatorname{supp} m$ is a set of antisymmetry for $H^\infty + C$ (see [**3**], p. 61), we deduce that for every $m \in M\left(H^\infty[\chi_E]\right)$ the function χ_E is constant 0 or 1 on $\operatorname{supp} m$. Hence

$$M(H^\infty + C) \setminus M\left(H^\infty[\chi_E]\right) = \{0 < \chi_E < 1\}. \tag{5}$$

Moreover, by a result of Marshall [**15**] (see also [**4**], p. 398) there exists an interpolating Blaschke product b such that

$$H^\infty[\bar{b}] = H^\infty[\chi_E]. \tag{6}$$

Hence, for every clopen set E in $M(L^\infty)$ there is an interpolating Blaschke product b such that

$$\{|b| < 1\} = \{0 < \chi_E < 1\}. \tag{7}$$

The following result of K. Hoffman is used frequently throughout this paper. We list it for convenience.

Lemma 2.1 ([**10**, p. 190], [**3**, p. 33])**.** *Let $m \in M(H^\infty + C)$ and let $f \in H^\infty + C$ vanish on an open subset U in $M(L^\infty)$. Assume that $U \cap \operatorname{supp} m \neq \emptyset$. Then $f(m) = 0$.*

Lemma 2.2. *Let $f \in H^\infty + C$ and let E be a clopen subset of $M(L^\infty)$. Then*

$$f\chi_E \in H^\infty + C \Leftrightarrow f \equiv 0 \text{ on } \{0 < \chi_E < 1\}.$$

Moreover, if we let $S(E) = \{\varphi \in M(H^\infty + C) : \operatorname{supp} \varphi \subseteq E\}$ and $E^c = M(L^\infty) \setminus E$, then both statements imply that

$$Z(f\chi_{E^c}) = S(E) \cup \{0 < \chi_E < 1\} \cup \big(Z(f) \cap S(E^c)\big),$$

with an analogous formula if Z is replaced by Z_∞. In particular $Z(f) \subseteq Z(f\chi_{E^c})$ and $Z_\infty(f) \subseteq Z_\infty(f\chi_{E^c})$.

Proof. Assume that $f\chi_E \in H^\infty + C$. Then $fH^\infty[\chi_E] \subseteq H^\infty + C$. Choose an interpolating Blaschke product b satisfying (6), that is $H^\infty[\bar{b}] = H^\infty[\chi_E]$. Then $fH^\infty[\bar{b}] \subseteq H^\infty + C$. Hence we have $\overline{b^n} f \in H^\infty + C$ for every $n \in \mathbb{N}$. By Lemma 1.2, $f \equiv 0$ on $\{|b| < 1\}$. Thus, by (7), $f \equiv 0$ on $\{0 < \chi_E < 1\}$.

Conversely, suppose that $f \equiv 0$ on $\{0 < \chi_E < 1\}$. Without loss of generality assume that $\|f\| \leq 1$. From (7) we know that $f \equiv 0$ on $\{|b| < 1\}$. Hence by Lemma 1.2, $f\overline{b^n} \in H^\infty + C$ and so $fH^\infty[\bar{b}] \subseteq H^\infty + C$. But $fH^\infty[\bar{b}] = fH^\infty[\chi_E]$. So, in particular, $f\chi_E \in H^\infty + C$.

To prove the remaining statements, we first note that $M(H^\infty + C)$ is the disjoint union of the three sets $S(E), \{0 < \chi_E < 1\}$ and $S(E^c)$. Let $\varphi \in Z(f\chi_{E^c})$. If $\varphi \notin S(E) \cup \{0 < \chi_E < 1\}$, then we deduce that $\varphi \in S(E^c)$. Hence $\chi_{E^c} \equiv 1$ on supp φ. Therefore

$$0 = \varphi(f\chi_{E^c}) = \int_{\text{supp }\varphi} f\chi_{E^c}\, d\mu_\varphi = \int_{\text{supp }\varphi} f\, d\mu_\varphi = \varphi(f).$$

Therefore $\varphi \in Z(f) \cap S(E^c)$.

To prove the converse, we distinguish three cases.

Case 1. Let $\varphi \in S(E)$, that is supp $\varphi \subseteq E$. Then $\chi_{E^c} \equiv 0$ on supp φ. Hence

$$\varphi(f\chi_{E^c}) = \int_{\text{supp }\varphi} f\chi_{E^c}\, d\mu_\varphi = 0.$$

Case 2. Let $0 < \varphi(\chi_E) < 1$. Then supp $\varphi \cap E \neq \emptyset$. Since $f\chi_{E^c}$ is a function in $H^\infty + C$ vanishing on an open set E in $M(L^\infty)$ which meets the support set of φ, we obtain from Lemma 2.1 that $\varphi(f\chi_{E^c}) = 0$.

Case 3. Let $\varphi \in Z(f) \cap S(E^c)$. Then $\chi_{E^c} \equiv 1$ on supp φ. Hence

$$\varphi(f\chi_{E^c}) = \int_{\text{supp }\varphi} f\chi_{E^c}\, d\mu_\varphi = \int_{\text{supp }\varphi} f\, d\mu_\varphi = \varphi(f) = 0.$$

The assertion for Z replaced by Z_∞ is obtained in exactly the same way. It suffices to note that all the points in a Gleason part of H^∞ have the same support set (see [**3**], p. 143).

The assertions that $Z(f) \subseteq Z(f\chi_{E^c})$ and $Z_\infty(f) \subseteq Z_\infty(f\chi_{E^c})$ now follow immediately. □

Lemma 2.3. *Let E be a clopen set in $M(L^\infty)$. Then $\overline{\{0 < \chi_E < 1\}} \cap M(L^\infty) = \emptyset$.*

Proof. Using a result of Axler [**1**], we may choose a Blaschke product B such that $B\chi_E \in H^\infty + C$. By Lemma 2.2, $B \equiv 0$ on $\{0 < \chi_E < 1\}$. Since a Blaschke product does not vanish on the Shilov boundary, we deduce that $\overline{\{0 < \chi_E < 1\}} \cap M(L^\infty) = \emptyset$. □

The next lemma is well known, but for (c), we were unable to locate a convenient reference.

Lemma 2.4 (see [**4**, p. 194]). (a) *Given* $x \in M(H^\infty + C) \setminus M(L^\infty)$, *there exists a Blaschke product* B *such that* $B(x) = 0$.

(b) *If* B *is a Blaschke product, there exists another Blaschke product* B^* *such that*

$$\{x \in M(H^\infty + C) : |B(x)| < 1\} \subseteq \{x \in M(H^\infty + C) : B^*(x) = 0\}.$$

(c) *If* S *is a closed subset of* $M(H^\infty + C)$ *such that* $S \cap M(L^\infty) = \emptyset$, *then there exists a Blaschke product* B^* *vanishing on* S.

Proof. Parts (a) and (b) are results of D.J. Newman. To prove (c), take $x \in S$. Since $S \cap M(L^\infty) = \emptyset$, there exists by (a) a Blaschke product B_x vanishing at x. A compactness argument now yields a finite number of Blaschke products B_j, $(j = 1, \cdots, n)$, such that $S \subseteq \bigcup_{j=1}^n \{|B_j| < 1/2\}$. Let $B = B_1 \cdot \cdots \cdot B_n$. Then $S \subseteq \{|B| < 1\}$. Now use (b) to get a Blaschke product B^* vanishing identically on the level set $\{|B| < 1\}$. This yields the assertion $S \subseteq Z(B^*)$. □

The following result has been proven by Guillory, Izuchi and Sarason using Wolff's factorization theorem. We include it here, because it is not explicitly stated as a theorem in [**8**].

Lemma 2.5 ([**8**], [**19**]). *Let* $f \in H^\infty + C$ *be invertible in* L^∞. *Then* $f = Bq$ *for some Blaschke product* B *and a function* q *invertible in* $H^\infty + C$.

Izuchi ([**11**, p. 55]) showed that every Blaschke product B admits a factorization of the form $B = B_1 B_2$, where $Z_\infty(B) = Z_\infty(B_1) = Z_\infty(B_2)$. In the case of $H^\infty + C$ functions we have the following.

Proposition 2.6. *Let* $f \in H^\infty + C$. *Assume that* $E = Z(f) \cap M(L^\infty)$ *is a clopen subset of* $M(L^\infty)$. *Then there exist functions* g *and* h *in* $H^\infty + C$ *such that*

$$\text{(i)}\ \ f = gh, \qquad \text{(ii)}\ \ Z_\infty(f) = Z_\infty(g) = Z_\infty(h).$$

Proof. If $E = \emptyset$, then f is invertible in L^∞. Hence, by Lemma 2.5, f can be written as $f = Bq$, where B is a Blaschke product and q an invertible function in $H^\infty + C$. The aformentioned result of Izuchi yields the desired factorization.

If $E \neq \emptyset$, let χ_E be the characteristic function of E in $M(L^\infty)$. Recall that $E^c = M(L^\infty) \setminus E$. Since E is clopen, χ_E is continuous on $M(L^\infty)$ and so $\chi_E \in L^\infty$. Note also that $f = f\chi_{E^c}$. Hence, by Lemma 2.2, f vanishes identically on $\{0 < \chi_E < 1\}$. By a result of Axler [**1**] there exists a Blaschke product B such that $B\chi_E \in H^\infty + C$. We may assume, without loss of generality, that $\|f\| < 1$. Since $|f| > 0$ on E^c, we see that $f + B\chi_E$ does not vanish on $M(L^\infty)$. Thus $f+B\chi_E$ is invertible in L^∞. By Lemma 2.5 we can write $f + B\chi_E = C_0 q$, where C_0 is a Blaschke product and q is a function invertible in $H^\infty + C$. Due to the result of Izuchi mentioned above ([**11**, p. 55]), we may factor C_0 as $C_0 = C_1C_2$, where the C_j, $(j = 1, 2)$, are Blaschke products such that $Z_\infty(C_0) = Z_\infty(C_1) = Z_\infty(C_2)$. Since $B\chi_E \in H^\infty + C$, by Lemma 2.2 we know that $B \equiv 0$ and $B\chi_E \equiv 0$ on $\{0 < \chi_E < 1\}$. Since this latter set is contained in $Z_\infty(f)$, too, we deduce from the invertibility of q that C_0 and hence C_j, $(j = 1, 2)$, vanish identically on $\{0 < \chi_E < 1\}$. Thus, by Lemma 2.2, $C_j\chi_{E^c} \in H^\infty + C$, $(j = 0, 1, 2)$. So

$$f = f\chi_{E^c} = (f + B\chi_E)\chi_{E^c} = (C_0\chi_{E^c})q = (C_1\chi_{E^c})(C_2\chi_{E^c})q. \tag{8}$$

We claim that for $j = 0, 1, 2$, $Z_\infty(C_j\chi_{E^c}) = Z_\infty(f)$. To see this, we note that by (8) and the invertibility of q, we have $Z_\infty(f) = Z_\infty(C_0\chi_{E^c})$. Recall that $S(E) = \{\varphi \in M(H^\infty + C) : \operatorname{supp}\varphi \subseteq E\}$. Thus, by Lemma 2.2

$$Z_\infty(C_j\chi_{E^c}) = S(E) \cup \{0 < \chi_E < 1\} \cup \big(S(E^c) \cap Z_\infty(C_j)\big).$$

Since $Z_\infty(C_0) = Z_\infty(C_1) = Z_\infty(C_2)$, we obtain that $Z_\infty(C_j\chi_{E^c}) = Z_\infty(f)$ for $j = 0, 1, 2$.

Hence $f = (C_1\chi_{E^c})(C_2\chi_{E^c})q$ yields the desired factorization. □

Question Q1. Does the factorization of Proposition 2.6 hold for every $H^\infty + C$ function?

It is a classical result of D.J. Marshall ([**15**, p. 20]) that every ideal in H^∞ whose hull does not meet the Shilov boundary is generated by inner functions. In $H^\infty + C$ we can say more:

Proposition 2.7. *Let I be an ideal in $H^\infty + C$. Assume that $Z(I) \cap M(L^\infty) = \emptyset$. Then I is algebraically generated by Blaschke products.*

Proof. Since $H^\infty + C$ is a unilogmodular[1] algebra on its Shilov boundary, every ideal I in $H^\infty + C$ with $Z(I) \cap \partial(H^\infty + C) = \emptyset$ contains a function u which is unimodular on the Shilov boundary [**16**]. By Lemma 2.5, $u = Bq$ for some Blaschke product B and a unimodular function q invertible in $H^\infty + C$. Thus $B \in I$. Since for every $f \in I$ with $\|f\| \le 1/2$, the function $B + f$ does not vanish on $M(L^\infty)$, we see that $B + f$ is invertible in L^∞. Using Lemma 2.5, we have $B + f = B_f q_f$ for some Blaschke product B_f and an invertible function q_f in $H^\infty + C$. Hence I is generated by B and all of the B_f. □

The last step on the way to prove our main result is the following technical lemma.

[1]See ([**16**, p. 467]) for a definition of this term.

Lemma 2.8. *Let I be an ideal in $H^\infty + C$ and let $f_j \in H^\infty + C$, $(j = 1, 2)$. Assume that f_1 and f_2 vanish on an open set U in $M(H^\infty + C)$ which contains the hull of I and that $Z(I) \cap M(L^\infty) \neq \emptyset$. Then $f_1 f_2 \in I$.*

Proof. Consider the ideal $J = IL^\infty$ generated by I in L^∞ and let $hull(J)$ be its hull in $M(L^\infty)$. We obviously have $hull(J) = Z(I) \cap M(L^\infty)$. Choose an open set V in $M(L^\infty)$ such that $hull(J) \subseteq V \subseteq \overline{V} \subseteq U$. Since L^∞ is isometrically isomorphic to $C\,(M(L^\infty))$, we see that there exists $q \in L^\infty$ such that q is identically one on V and identically zero on $M(L^\infty) \setminus U$. Thus $f_j q \equiv 0$ on $M(L^\infty)$, and hence $f_j q \in J$. But $hull(J) \cap Z(q) = \emptyset$. Thus there exist $u \in J$ and $v \in L^\infty$ so that $1 = u + vq$. Multiplying by f_j yields that $f_j = f_j u + v(f_j q) \in J$. Thus there exist functions $q_n^j \in L^\infty$ and $g_n \in I$ so that

$$f_j = \sum_{n=1}^{N} q_n^j g_n \qquad (j = 1, 2).$$

By [**1**] there exists a Blaschke product B such that $Bq_n^j \in H^\infty + C$ for $n = 1, 2, \cdots, N$ and $j = 1, 2$. It follows that $Bf_j = \sum_{n=1}^{N} (Bq_n^j) g_n \in I$.

We shall now construct a Blaschke product D such that $Z(D) \subseteq U$ and $f_2 D \in I$. If $Z(B) \subseteq U$, we put $D = B$. If not, use Suarez's result ([**17**, p. 244]) to choose a function $g \in I$ such that $Z(g) \subseteq U$. Now consider the ideal I_1 in $H^\infty + C$ generated by B and g. Obviously $Z(I_1) \subseteq Z(g) \subseteq U$. But $Z(I_1) \cap M(L^\infty) = \emptyset$. Thus, by [**17**] again, there exists a function $h \in I_1$ such that $Z(h) \subseteq U$ and $Z(h) \cap M(L^\infty) = \emptyset$. In particular h is invertible in L^∞. By Lemma 2.5, $h = D\tilde{h}$, where D is a Blaschke product and $\tilde{h}$ is invertible in $H^\infty + C$. Therefore $D = \tilde{h}^{-1} h \in I_1$. Thus there exist x and y in $H^\infty + C$, so that $D = xB + yg$.

Hence

$$f_2 D = f_2(xB + yg) = x(f_2 B) + (f_2 y)g \in I + I \subseteq I.$$

Moroever, $Z(D) \subseteq U$. Since $U \subseteq Z(f_1)$ we can conclude from Theorem 1.4 that $f_1 \overline{D} \in H^\infty + C$. Therefore

$$f_1 f_2 = (f_1 \overline{D})(f_2 D) \in I.$$

□

This brings us to our main Theorem, stating that $H^\infty + C$ has the Shilov property.

Theorem 2.9. *Let I be an ideal in $H^\infty + C$ and let f be a function in $H^\infty + C$ vanishing in an open neighborhood U of the hull, $Z(I)$, of I. Then $f \in I$.*

Proof. *Case* 1. $Z(I) \cap M(L^\infty) = \emptyset$.

Let $S = M(L^\infty) \bigcup [M(H^\infty + C) \setminus U]$. Then S is a closed subset of $M(H^\infty + C)$ which is disjoint from $Z(I)$. Hence, by ([**17**, p. 244]) there exists a function $g \in I$ such that $Z(g) \cap S = \emptyset$. In particular g is invertible in L^∞. By Lemma 2.5, $g = BG$ for a Blaschke product B and a function G invertible in $H^\infty + C$. Thus $B \in I$ and $Z(B) \subseteq U$. Since $U \subseteq Z(f)$, we obtain from Theorem 1.4 that $f\overline{B} \in H^\infty + C$ and so $f = (f\overline{B})B \in I$.

Case 2. $Z(I) \cap M(L^\infty) \neq \emptyset$.

Let $E = \overline{Z(f)^\circ \cap M(L^\infty)}$. Since $M(L^\infty)$ is extremely disconnected, E is a clopen set in $M(L^\infty)$ ([**3**, p. 18] and [**4**, p. 214]) contained in $Z(f)$. Let $S = \overline{Z(f)^\circ} \cap M(L^\infty)$. Then S is a compact set containing E. Moreover $S \setminus E$ is compact. Since $Z(I) \cap M(L^\infty) \subseteq E$, we see that $(S \setminus E) \bigcap (Z(I) \cup E) = \emptyset$. Thus there is an open neighborhood V of $Z(I) \cup E$ in $M(H^\infty + C)$ such that $\overline{V} \cap (S \setminus E) = \emptyset$. Let $\Omega = V \cap Z(f)^\circ$. Then Ω is an open subset of $M(H^\infty + C)$ satisfying

$$Z(I) \subseteq \Omega \subseteq Z(f)^\circ, \tag{9}$$

$$\overline{\Omega} \cap (S \setminus E) = \emptyset, \tag{10}$$

and (as will be justified below)

$$E = \overline{\Omega \cap M(L^\infty)} = \overline{\Omega} \cap M(L^\infty). \tag{11}$$

In fact, (11) is a consequence of (9), (10) and the following inclusions:

(i) $\overline{\Omega} \cap M(L^\infty) \subseteq \overline{Z(f)^\circ} \cap M(L^\infty) = S = (S \setminus E) \cup E$,

(ii) $Z(f)^\circ \cap M(L^\infty) \subseteq E \cap Z(f)^\circ \subseteq [V \cap Z(f)^\circ] \cap M(L^\infty) = \Omega \cap M(L^\infty)$

and hence

$$E = \overline{Z(f)^\circ \cap M(L^\infty)} \subseteq \overline{\Omega \cap M(L^\infty)} \subseteq \overline{\Omega} \cap M(L^\infty).$$

Let $S(E) = \{\varphi \in M(H^\infty + C) : \varphi(\chi_E) = 1\}$. We claim that

$$\overline{\Omega \setminus S(E)} \bigcap M(L^\infty) = \emptyset. \tag{12}$$

To see this, let $x \in \overline{\Omega \setminus S(E)}$. Then there is a net of points (x_α) from $\Omega \setminus S(E)$ with (x_α) converging to x. By the definition of $S(E)$ we know that $0 \leq x_\alpha(\chi_E) < 1$ for every α. Now if $x \in \overline{\{0 < \chi_E < 1\}}$, then, by Lemma 2.3, $x \notin M(L^\infty)$, so we are done. If $x \notin \overline{\{0 < \chi_E < 1\}}$, then $M(H^\infty + C) \setminus \overline{\{0 < \chi_E < 1\}}$ is an open neighborhood of x. We may assume that this neighborhood contains all the x_α. Hence $x_\alpha(\chi_E) = 0$ or $x_\alpha(\chi_E) = 1$. Since $0 \leq x_\alpha(\chi_E) < 1$, we conclude that $x_\alpha(\chi_E) = 0$ for all α. Hence $x(\chi_E) = 0$. So $x \notin E$. But $E \overset{(11)}{=} \overline{\Omega} \cap M(L^\infty)$. Since $x \in \overline{\Omega}$, we deduce that also in this case $x \notin M(L^\infty)$. This proves (12).

Let $U_1 = [\Omega \cup \{0 < \chi_E < 1\}] \setminus S(E)$. We claim that U_1 is an open set such that

$$\overline{U_1} \cap M(L^\infty) = \emptyset, \tag{13}$$

and

$$U_1 \subseteq Z(f). \tag{14}$$

To see this, we note that $U_1 = (\Omega \setminus S(E)) \cup \{0 < \chi_E < 1\}$. Hence, by (12) and Lemma 2.3

$$\overline{U_1} = \overline{\Omega \setminus S(E)} \cup \overline{\{0 < \chi_E < 1\}}$$

has property (13). To prove (14), we first note that if $0 < \varphi(\chi_E) < 1$, then $\operatorname{supp} \varphi$ meets the clopen set E on which f vanishes identically. Thus by Lemma 2.1, $\varphi(f) = 0$. Together with (9) we obtain $U_1 \subseteq Z(f)$.

By Lemma 2.4 and (13) we may choose a Blaschke product B such that $B \equiv 0$ on U_1. By Lemma 2.2, $\{0 < \chi_E < 1\} \subseteq Z(B)$ implies that $B\chi_{_{E^c}} \in H^\infty + C$ (where as usual $E^c = M(L^\infty) \setminus E$). Consider $f + B\chi_{_{E^c}}$. We may assume without loss of generality that $\|f\| < 1$. We claim that

$$f + B\chi_{_{E^c}} = 0 \quad \text{on} \;\; E, \tag{15}$$

$$f + B\chi_{_{E^c}} \neq 0 \quad \text{on} \;\; M(L^\infty) \setminus E \tag{16}$$

and

$$f + B\chi_{_{E^c}} = 0 \quad \text{on} \;\; U_1. \tag{17}$$

Since (15) and (16) are trivial, we will turn to the proof of (17). First note that on U_1 we have $f \equiv 0$ and $B \equiv 0$. Since by Lemma 2.2 $Z(B) \subseteq Z(B\chi_{_{E^c}})$, we obtain (17).

Next we apply Proposition 2.6 and write $f + B\chi_{_{E^c}} = f_1 f_2$, where $f_j \in H^\infty + C$ and $f_j = 0$ on $U_1 \supseteq \{0 < \chi_E < 1\}$. Notice that $U_1 \subseteq Z_\infty(f + B\chi_{_{E^c}})$. Now $f = f\chi_{_{E^c}}$, so

$$f + B\chi_{_{E^c}} = (f + B\chi_{_{E^c}})\chi_{_{E^c}} = (f_1\chi_{_{E^c}})(f_2\chi_{_{E^c}}). \tag{18}$$

Note that by Lemma 2.2, $f_j\chi_{_{E^c}} \in H^\infty + C$. Next we claim that

$$f_j\chi_{_{E^c}} \equiv 0 \text{ on } \Omega \cup \{0 < \chi_{\mathrm{E}} < 1\}. \tag{19}$$

In fact, if $0 < \varphi(\chi_E) < 1$, then $\varphi(f_j\chi_{_{E^c}}) = 0$ by Lemma 2.2. Moreover, by the same Lemma, $\Omega \setminus S(E) \subseteq U_1 \subseteq Z(f_j) \subseteq Z(f_j\chi_{_{E^c}})$ and $\Omega \cap S(E) \subseteq S(E) \subseteq Z(f_j\chi_{_{E^c}})$. This yields (19).

By ([**11**, p. 55]), we can write $B = C_1C_2$, where the zero sets of infinite order of B, C_1 and C_2 coincide. In particular, since B vanishes identically on the open set U_1, so do C_1 and C_2. Thus, by (18), we obtain

$$f = (f_1\chi_{_{E^c}})(f_2\chi_{_{E^c}}) - B\chi_{_{E^c}} = (f_1\chi_{_{E^c}})(f_2\chi_{_{E^c}}) - (C_1\chi_{_{E^c}})(C_2\chi_{_{E^c}}).$$

Because for $j = 1, 2$, $\{0 < \chi_E < 1\} \subseteq U_1 \subseteq Z(C_j)$, we conclude from Lemma 2.2 that $C_j\chi_{E^c} \in H^\infty + C$. Moreover, as above, we see that $C_j\chi_{E^c} \equiv 0$ on Ω. Thus we have factorized f as a sum of two factors, each of them admits a factorization of type gh, where both g and h vanish on Ω. Since the hull of I, $Z(I)$, satisfies $Z(I) \cap M(L^\infty) \neq \emptyset$ and $Z(I) \overset{(9)}{\subseteq} \Omega$, Lemma 2.8 implies that

$$(f_1\chi_{E^c})(f_2\chi_{E^c}) \in I \quad \text{and} \quad (C_1\chi_{E^c})(C_2\chi_{E^c}) \in I.$$

Thus $f \in I$. □

As a corollary, we obtain the following generalization of Theorem 1.4.

Corollary 2.10. *Let f and g be two functions in $H^\infty + C$. Assume that f vanishes identically on an open neighborhood of the zeros of g. Then f is divisible in $H^\infty + C$ by g.*

Proof. Take I to be the principal ideal generated by g and apply Theorem 2.9. □

Let A be a commutative unital Banach algebra and let I be an ideal in A. An element $f \in A$ is said to belong locally to I if for every $m \in M(A)$ there exists a neighborhood U of m in $M(A)$ such that $\hat{f}|_U \in \hat{I}|_U$. An important result in the theory of Banach algebras is that in regular algebras every ideal is locally determined ([**5**, p. 201] and [**14**, p. 224]); that is if $f \in A$ belongs locally to an ideal I, then actually $f \in I$. As another corollary of Theorem 2.9 we prove that a large class of ideals in the non-regular algebra $H^\infty + C$ is locally uniquely determined.

Corollary 2.11. *Every intersection of primary ideals and every radical ideal in $H^\infty + C$ is locally uniquely determined.*

Proof. Since the case of intersections of primary ideals is an immediate consequence of Theorem 2.9, it remains to look at the case of radical ideals. So let $f \in H^\infty + C$ belong locally to the radical ideal I. Then, by a compactness argument, there exists finitely many functions $g_j \in I$ and open sets U_j, $(j = 1, \dots, n)$, such that $Z(I) \subseteq \bigcup U_j$ and $f|_{U_j} = g_j$. Hence $\prod_{j=1}^n (f - g_j) \equiv 0$ in a neighborhood of $Z(I)$. Thus, by Theorem 2.9, we can conclude that $f^n \in I$ and hence $f \in I$. □

We list below two questions we are unable to answer.

Q2. Is every ideal in $H^\infty + C$ locally uniquely determined?

Q3. Assume that a continuous function q on $M(H^\infty + C)$ locally belongs to $H^\infty + C$. Is $q \in H^\infty + C$? In other word, is $H^\infty + C$ a local algebra on its spectrum?

We return now to the Shilov property. Comparing that with the algebra $H^\infty + C$, the situation in H^∞ is a bit different. There do exist ideals I with hull contained in $M(H^\infty + C)$, such that not every function vanishing in an $M(H^\infty + C)$ neighborhood of the hull belongs to I. In fact, let I be the ideal generated by the n-th roots of the atomic inner function $S(z) = \exp\left(-\frac{1+z}{1-z}\right)$. By Lemma 2.4 there exists a Blaschke product vanishing on the set $\{x \in M(H^\infty + C) : |S(x)| \leq 1/2\}$. But clearly $B \notin I$.

On the other hand, we have the following result:

Theorem 2.12. *Let I be a* closed *ideal in H^∞ whose weak-$*$ closure in H^∞ is H^∞.*[2] *Then I contains every function vanishing in an $M(H^\infty + C)$ neighborhood of the hull of I.*

Proof. The hypothesis on I says that the greatest common divisor of the inner parts of the elements in I is a unimodular constant and that $Z(I) \subseteq M(H^\infty + C)$. Thus by [**6**] there exists a unique closed ideal J in $H^\infty + C$ such that $I = J \cap H^\infty$. The result now follows from Theorem 2.9. □

Finally, let us mention that, of course, every ideal in H^∞ contains every function vanishing in a $M(H^\infty)$ neighborhood of its hull, because only the zero function satisfies this hypothesis. Thus, in that case, the "real" extension of Theorem 2.9, namely that H^∞ has the Shilov property, holds in H^∞, too. This raises the following questions:

Let A be a commutative unital Banach algebra and let E be a closed subset of $M(A)$ with the property that, via the restriction map, $\hat{A}|_E$ is isometrically isomorphic to A; in other words, let E be a closed boundary for A. Say that A has the *E-restricted Shilov property* if any ideal, with hull, $\mathcal{H}$, contained in E, contains every function vanishing in a relative neighborhood of $\mathcal{H}$ in E.

Q4. For which closed boundaries E in $M(A)$ does A have the E-restricted Shilov property? What happens if one restricts to certain classes of ideals, closed ones for example?

Q5. Do the algebras $P(K)$ and $R(K)$ have the Shilov property? (Here K is a compact subset in $\mathbb{C}$.)

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