DIVISION THEOREMS AND THE SHILOV PROPERTY FOR $H^\infty + C$

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It is shown that $H^\infty + C$ has the Shilov property. In particular any function $f$ in $H^\infty + C$ vanishing in an open neighborhood of the zeros of another function $g$ in $H^\infty + C$ is divisible there by $g$.

Let $\mathbb{D}$ be the open unit disk in the complex plane. Let $L^\infty$ and $H^\infty$ denote the usual algebras on the unit circle $\partial \mathbb{D}$. The smallest closed subalgebra of $L^\infty$ properly containing $H^\infty$ is $H^\infty + C$, where $C$ denotes the algebra of continuous complex valued functions on the unit circle. The algebra consisting of $H^\infty + C$ functions whose complex conjugates are also in $H^\infty + C$ is denoted by $QC$.

For any of the above algebras, denoted here by $A$, the maximal ideal space or spectrum of $A$ is the space of nonzero multiplicative linear functionals on $A$ and is denoted $M(A)$. When $M(A)$ is given the weak-$*$ topology, it becomes a compact Hausdorff space. Identifying each point of $\mathbb{D}$ with the multiplicative linear functional that is point evaluation, we think of $\mathbb{D}$ as a subset of $M(H^\infty)$. It is well known that $M(H^\infty + C) = M(H^\infty) \setminus \mathbb{D}$.

Factorization theorems and the study of zero sets of bounded analytic functions have been crucial to our understanding of the structure of both the algebra $H^\infty$ and its maximal ideal space. Thus, to expand our knowledge of $H^\infty + C$ one might ask which of these properties extend to this algebra.

For $H^\infty$ functions, zero sets in $M(H^\infty)$ play an important role in division problems. One might hope, then, that zero sets in $M(H^\infty + C)$ play an equally important role in the study of division in this algebra. However, the situation becomes more complicated here. Guillory and Sarason [9] have shown that there exist two inner functions, $u_1, u_2$ in $H^\infty + C$ with $|u_1| = |u_2|$ on $M(H^\infty + C)$, but $u_1 \overline{u_2}$ is not in $H^\infty + C$.

Axler [1] began the study of multiplying functions into $H^\infty + C$ by showing that if $f$ is any function in $L^\infty$, then there exists a Blaschke product $B$ multiplying $f$ into $H^\infty + C$. Wolff [19] then showed that every unimodular function in $L^\infty$ can be written as a quotient of Blaschke products times an invertible function in $QC$. Guillory and Sarason [9], Guillory, Izuchi and Sarason [8], and Axler and Gorkin [2] continued this work. The theorems in these papers can be restated as division theorems assuming that the divisor
is a unimodular function in $H^\infty + C$. In fact, these authors show that if $f \in H^\infty + C$ and $u$ is a unimodular function in $H^\infty + C$, then $f$ is divisible in $H^\infty + C$ by $u^n$ for every positive integer $n$ if and only if $f = 0$ wherever $|u| < 1$ on $M(H^\infty + C)$.

In the present paper, as a consequence of a more general result about ideals in $H^\infty + C$, we show that if $g$ is an arbitrary function in $H^\infty + C$ and $f$ vanishes on an open set in $M(H^\infty + C)$ containing the zeros of $g$, then $f$ is divisible in $H^\infty + C$ by $g^n$ for every positive integer $n$. We remark that Izuchi and Izuchi [13] showed that for $f \in H^\infty + C$ and an inner function $u$ satisfying $|f| \leq |u|$ on $M(H^\infty + C)$, one obtains $f^{n+1}u^m \in H^\infty + C$ for every positive integer $n$. In view of the above example, we see that one cannot expect to have $f \bar{u} \in H^\infty + C$ in general. On the other hand, if we assume a stronger hypothesis than Izuchi’s, namely $f = 0$ on an open set containing the zeros of $u$, we are able to obtain (Theorem 1.4) the stronger conclusion $f \bar{u} \in H^\infty + C$.

These division theorems are corollaries of our main result for ideals. To state this result, we need to recall that a commutative unital Banach algebra $A$ is said to be regular if for every closed set $E$ in its spectrum and each point $x$ not in $E$ there exists a function $f \in A$ such that $f(x) = 1$ and $f$ vanishes on $E$. A well known result due to Shilov ([5], [14]) states that an ideal $I$ in a regular Banach algebra $A$ contains any function in $A$ that vanishes on an open subset in the spectrum of $A$ containing the hull of $I$. In that case $A$ is said to have the Shilov property. In Theorem 2.9 we show that this result can be extended to the nonregular algebra $H^\infty + C$. As a consequence we see that large classes of ideals in $H^\infty + C$, including radical ideals and intersections of primary ideals, are determined locally. This property, shared by ideals in regular algebras, is an important tool in harmonic analysis.

In a final paragraph we analyze related problems for the algebra $H^\infty$ of bounded analytic functions. We assume that the reader is familiar with the general theory of the maximal ideal space of $H^\infty$. As a convenient reference, we mention the book of J.B. Garnett [4]. We conclude this introduction with some notation.

Let $f \in H^\infty + C$. Then the zero set of $f$ in $M(H^\infty + C)$ is denoted by $Z(f)$. The hull or zero set of an ideal $I$ in $H^\infty + C$ is the set $Z(I) = \bigcap_{f \in I} Z(f)$. Since each nontrivial Gleason part of $H^\infty$ is an analytic disk, we know that the functions in $H^\infty + C$ are holomorphic with respect to this analytic structure. Hence, if $f \in H^\infty + C$ vanishes at a point $x \in M(H^\infty + C)$ whose Gleason part $P(x) = \{m \in M(H^\infty) : \|m - x\| < 2\}$ is nontrivial, it is meaningful to speak of the multiplicity of $x$ as a zero of $f$. In case $f$ vanishes identically on the part of $x$, the multiplicity of the zero of $f$ at $x$ is defined to be infinite. The set of all zeros of $f$ of infinite order, is denoted by $Z_\infty(f)$. If $q \in L^\infty$, then we denote by $H^\infty[q]$ the smallest closed subalgebra
of \( L^\infty \) containing \( H^\infty \) and \( q \). Finally, the weak-* closure of a subset \( S \) of \( M(H^\infty + C) \) will be denoted by \( \overline{S} \); its set of interior points by \( S^0 \).

1. Division by Blaschke products.

It is well known ([2], [8]) that whenever \( f \in H^\infty + C \) and \( b \) is an interpolating Blaschke product satisfying \( Z(b) \subseteq Z(f) \) on \( M(H^\infty + C) \), then \( fB \in H^\infty + C \). Obviously this does not hold if \( b \) is a non-interpolating Blaschke product (just take any Blaschke product \( B \) and put \( f = B, b = B^2 \)). Guillory, Izuchi and Sarason ([8], Cor. 2) noticed that, even by taking multiplicities into account, no such division result holds. Assuming, however, that \( f \) vanishes in a neighborhood of the zeros of a Blaschke product \( B \), then a positive result will be given here. To prove it, we need the following deep results of S. Axler, P. Gorkin, D. Marshall and D. Suarez.

**Lemma 1.1** ([15]). There exists a constant \( \beta \) with \( 0 < \beta < 1 \) such that for every Blaschke product \( B \) there is an interpolating Blaschke product \( b \) so that

\[
\{ z \in \mathbb{D} : b(z) = 0 \} \subseteq \{ z \in \mathbb{D} : |B(z)| < \beta \}
\]

and \( H^\infty [b] = H^\infty [B] \).

For the proof see also ([4], pp. 336, 379).

**Lemma 1.2** ([2, p. 92]). Let \( h \in H^\infty + C \) and let \( B \) be a Blaschke product. Then \( hB^n \in H^\infty + C \) for every \( n \in \mathbb{N} \) if and only if \( h(1 - |B|) \equiv 0 \) on \( M(H^\infty + C) \).

**Remark.** In fact Lemmas 1.1 and 1.2 hold in a more general setting. The interested reader is referred to [15], [4], [7], [17], [18] respectively [2] and [20] for further information.

**Lemma 1.3** ([18, Th. 2.5]). Let \( E \subseteq M(H^\infty) \) be a closed set and let \( B \) be a Blaschke product with \( |B| > 0 \) on \( E \). Then for every \( \sigma \) with \( 0 < \sigma < 1 \) there exists a finite factorization \( B = B_0B_1 \cdots B_n \) so that \( |B_j(x)| \geq \sigma \) for all \( x \in E \) and \( j \in \{1, 2, \cdots, n\} \) and where \( B_0 \) is a finite product of interpolating Blaschke products.

**Theorem 1.4.** Let \( B \) be a Blaschke product and suppose that \( f \in H^\infty + C \) vanishes on an open subset \( U \) of \( M(H^\infty + C) \) containing the zero set \( Z(B) \) of \( B \). Then \( fB^\nu \in H^\infty + C \) for every \( \nu \in \mathbb{N} \).

**Proof.** Obviously \( B \neq 0 \) on \( M(H^\infty + C) \setminus U \). Let \( \beta \) be the constant of Lemma 1.1. Use Lemma 1.3 to factor \( B = B_0B_1 \cdots B_n \) where

\[
|B_j| > \beta \quad \text{on} \quad M(H^\infty + C) \setminus U \quad (j = 1, 2, \ldots, n),
\]

and where \( B_0 \) is a finite product of interpolating Blaschke products. Clearly \( f \) vanishes identically on every Gleason part which meets \( U \). Hence \( U \subseteq \overline{S} \)
Since every zero of $B_0$ is of finite order, we deduce from $Z(B_0) \subseteq U$ that every zero of $B_0$ is a zero of $f$ of infinite order. Hence by [2] or [8] we have $fB_0 \in H^\infty + C$ and $Z_\infty(fB_0) = Z_\infty(f)$. Thus $U \subseteq Z_\infty(fB_0)$.

Next we show that $B_1 \cdots B_n$ divides $fB_0$. To do this, we choose, according to Lemma 1.1, interpolating Blaschke products $b_j$ such that

\[
H^\infty[\{b_j\}] = H^\infty[\{\overline{B_j}\}] \quad (j = 1, 2, \ldots, n)
\]

and

\[
\{z \in \mathbb{D} : b_j(z) = 0\} \subseteq \{z \in \mathbb{D} : |B_j(z)| < \beta\} \quad (j = 1, \ldots, n).
\]

Fix $j \in \{1, \ldots, n\}$ and let $x \in Z(b_j)$. By ([4], p. 379), $x$ lies in the weak-* closure of $\{z \in \mathbb{D} : b_j(z) = 0\}$ and hence, by (4), $|B_j(x)| \leq \beta$. Thus, by (2), $x \in U$. In particular $Z(b_j) \subseteq Z_\infty(fB_0)$. By [2] or [8], we conclude that $(fB_0)^n b_j \in H^\infty + C$ for ever $n \in \mathbb{N}$. Hence, by Lemma 1.2, $fB_0 = 0$ whenever $|b_j| < 1$. But by (3)

\[
\{x \in M(H^\infty + C) : |b_j(x)| < 1\} = \{x \in M(H^\infty + C) : |B_j(x)| < 1\}.
\]

So we see that $fB_0 = 0$ whenever $\prod_{j=1}^n |B_j| < 1$. Hence, by Lemma 1.2

\[
fB_0 \prod_{j=1}^n \overline{B_j} \in H^\infty + C.
\]

Thus $fB \in H^\infty + C$. Since $Z(B) = Z(B^\nu)$, it is now clear that $fB^\nu \in H^\infty + C$ for every $\nu \in \mathbb{N}$ (just replace $B$ by $B^\nu$).

2. The Shilov property for $H^\infty + C$.

It is a classical result (see [10], p. 170) that the spectrum, $M(L^\infty)$, of $L^\infty$ is a totally disconnected compact space. Hence characteristic functions $\chi_E$ on $M(L^\infty)$ are continuous if and only if $E$ is clopen (that is closed and open). Since we may identify $L^\infty$ with $C(M(L^\infty))$, $\chi_E$ then is the Gelfand transform of a characteristic function $\chi_S$ for some Borel set $S$ of $\partial \mathbb{D}$ of positive Lebesgue measure. Moreover, $M(L^\infty)$ is the Shilov boundary of $H^\infty$ (see [10], p. 174).

Hoffman ([10], p. 184) has shown that each $m \in M(H^\infty)$ has a unique norm preserving extension to a linear functional on $L^\infty$. Letting $\text{supp } m$ in $M(L^\infty)$ denote the support set of the representing measure $\mu_m$ for $m$, one can show ([4], p. 375) that this extension is given by

\[
m(f) = \int_{\text{supp } m} f d\mu_m \quad (f \in L^\infty).
\]

It follows that each function $f \in L^\infty$ can be thought of as a continuous function on $M(H^\infty)$. This point of view will be adopted throughout this
paper and we write \( f(m) := m(f) \). We note that this extension to \( M(H^\infty) \) of \( f \in L^\infty \) coincides on \( \mathbb{D} \) with the Poisson integral of \( f \).

To proceed, we need to point out several properties of the Douglas algebra \( H^\infty[\chi_E] \) generated by \( H^\infty \) and \( \chi_E \). For the sake of simplicity, we simply write \( \{0 < \chi_E < 1\} \) for the set

\[ \{m \in M(H^\infty + C) : 0 < m(\chi_E) < 1\}. \]

By the Chang-Marshall Theorem (see [4], Sec. 9) we know that

\[ M(H^\infty[\chi_E]) = \{m \in M(H^\infty + C) : \chi_E|_{\text{supp} m} \in H^\infty|_{\text{supp} m}\}. \]

Since \( m(\chi_E) = \int_{\text{supp} m} \chi_E \, d\mu_m \) for every \( m \in M(H^\infty + C) \), we see that \( \chi_E \) is real valued on \( M(H^\infty + C) \) with values contained in the interval \([0, 1]\). Hence \( m(\chi_E) = 0 \) if and only if \( \text{supp} m \cap E = \emptyset \) and \( m(\chi_E) = 1 \) if and only if \( \text{supp} m \subseteq E \). Since \( \text{supp} m \) is a set of antisymmetry for \( H^\infty + C \) (see [3], p. 61), we deduce that for every \( m \in M(H^\infty[\chi_E]) \) the function \( \chi_E \) is constant 0 or 1 on \( \text{supp} m \). Hence

\[ M(H^\infty + C) \setminus M(H^\infty[\chi_E]) = \{0 < \chi_E < 1\}. \]

Moreover, by a result of Marshall [15] (see also [4], p. 398) there exists an interpolating Blaschke product \( b \) such that

\[ H^\infty[b] = H^\infty[\chi_E]. \]

Hence, for every clopen set \( E \) in \( M(L^\infty) \) there is an interpolating Blaschke product \( b \) such that

\[ \{|b| < 1\} = \{0 < \chi_E < 1\}. \]

The following result of K. Hoffman is used frequently throughout this paper. We list it for convenience.

**Lemma 2.1** ([10, p. 190], [3, p. 33]). Let \( m \in M(H^\infty + C) \) and let \( f \in H^\infty + C \) vanish on an open subset \( U \) in \( M(L^\infty) \). Assume that \( U \cap \text{supp} m \neq \emptyset \). Then \( f(m) = 0 \).

**Lemma 2.2.** Let \( f \in H^\infty + C \) and let \( E \) be a clopen subset of \( M(L^\infty) \). Then

\[ f\chi_E \in H^\infty + C \iff f \equiv 0 \text{ on } \{0 < \chi_E < 1\}. \]

Moreover, if we let \( S(E) = \{\varphi \in M(H^\infty + C) : \text{supp} \varphi \subseteq E\} \) and \( E^c = M(L^\infty) \setminus E \), then both statements imply that

\[ Z(f\chi_{E^c}) = S(E) \cup \{0 < \chi_E < 1\} \cup (Z(f) \cap S(E^c)), \]

with an analogous formula if \( Z \) is replaced by \( Z_\infty \). In particular \( Z(f) \subseteq Z(f\chi_{E^c}) \) and \( Z_\infty(f) \subseteq Z_\infty(f\chi_{E^c}) \).
Proof. Assume that \( f\chi_E \in H^\infty + C \). Then \( fH^\infty[\chi_E] \subseteq H^\infty + C \). Choose an interpolating Blaschke product \( b \) satisfying (6), that is \( H^\infty[b] = H^\infty[\chi_E] \). Then \( fH^\infty[b] \subseteq H^\infty + C \). Hence we have \( \overline{b^n}f \in H^\infty + C \) for every \( n \in \mathbb{N} \).

By Lemma 1.2, \( f \equiv 0 \) on \( \{|b| < 1\} \). Thus, by (7), \( f \equiv 0 \) on \( \{0 < \chi_E < 1\} \).

Conversely, suppose that \( f \equiv 0 \) on \( \{0 < \chi_E < 1\} \). Without loss of generality assume that \( \|f\| \leq 1 \). From (7) we know that \( f \equiv 0 \) on \( \{|b| < 1\} \). Hence by Lemma 1.2, \( f\overline{b^n} \in H^\infty + C \) and so \( fH^\infty[b] \subseteq H^\infty + C \). But \( fH^\infty[b] = fH^\infty[\chi_E] \). So, in particular, \( f\chi_E \in H^\infty + C \).

To prove the remaining statements, we first note that \( M(H^\infty + C) \) is the disjoint union of the three sets \( S(E), \{0 < \chi_E < 1\} \) and \( S(E^c) \). Let \( \varphi \in Z(\chi_{E^c}) \). If \( \varphi \notin S(E) \cup \{0 < \chi_E < 1\} \), then we deduce that \( \varphi \in S(E^c) \). Hence \( \chi_{E^c} \equiv 1 \) on \( \text{supp } \varphi \). Therefore

\[
0 = \varphi(f\chi_{E^c}) = \int_{\text{supp } \varphi} f\chi_{E^c} \, d\mu_{\varphi} = \int f \, d\mu_{\varphi} = \varphi(f).
\]

Therefore \( \varphi \in Z(f) \cap S(E^c) \).

To prove the converse, we distinguish three cases.

Case 1. Let \( \varphi \in S(E) \), that is \( \text{supp } \varphi \subseteq E \). Then \( \chi_{E^c} \equiv 0 \) on \( \text{supp } \varphi \). Hence

\[
\varphi(f\chi_{E^c}) = \int_{\text{supp } \varphi} f\chi_{E^c} \, d\mu_{\varphi} = 0.
\]

Case 2. Let \( 0 < \varphi(\chi_E) < 1 \). Then \( \text{supp } \varphi \cap E \neq \emptyset \). Since \( f\chi_{E^c} \) is a function in \( H^\infty + C \) vanishing on an open set \( E \) in \( M(L^\infty) \) which meets the support set of \( \varphi \), we obtain from Lemma 2.1 that \( \varphi(f\chi_{E^c}) = 0 \).

Case 3. Let \( \varphi \in Z(f) \cap S(E^c) \). Then \( \chi_{E^c} \equiv 1 \) on \( \text{supp } \varphi \). Hence

\[
\varphi(f\chi_{E^c}) = \int_{\text{supp } \varphi} f\chi_{E^c} \, d\mu_{\varphi} = \int f \, d\mu_{\varphi} = \varphi(f) = 0.
\]

The assertion for \( Z \) replaced by \( Z_\infty \) is obtained in exactly the same way. It suffices to note that all the points in a Gleason part of \( H^\infty \) have the same support set (see [3], p. 143).

The assertions that \( Z(f) \subseteq Z(f\chi_{E^c}) \) and \( Z_\infty(f) \subseteq Z_\infty(f\chi_{E^c}) \) now follow immediately. \( \square \)

**Lemma 2.3.** Let \( E \) be a clopen set in \( M(L^\infty) \). Then \( \{0 < \chi_E < 1\} \cap M(L^\infty) = \emptyset \).

Proof. Using a result of Axler [1], we may choose a Blaschke product \( B \) such that \( B\chi_E \in H^\infty + C \). By Lemma 2.2, \( B \equiv 0 \) on \( \{0 < \chi_E < 1\} \). Since a Blaschke product does not vanish on the Shilov boundary, we deduce that \( \{0 < \chi_E < 1\} \cap M(L^\infty) = \emptyset \). \( \square \)
The next lemma is well known, but for (c), we were unable to locate a convenient reference.

**Lemma 2.4** (see [4, p. 194]). (a) Given \( x \in M(H^\infty + C) \setminus M(L^\infty) \), there exists a Blaschke product \( B \) such that \( B(x) = 0 \).

(b) If \( B \) is a Blaschke product, there exists another Blaschke product \( B^* \) such that

\[
\{ x \in M(H^\infty + C) : |B(x)| < 1 \} \subseteq \{ x \in M(H^\infty + C) : B^*(x) = 0 \}.
\]

(c) If \( S \) is a closed subset of \( M(H^\infty + C) \) such that \( S \cap M(L^\infty) = \emptyset \), then there exists a Blaschke product \( B^* \) vanishing on \( S \).

**Proof.** Parts (a) and (b) are results of D.J. Newman. To prove (c), take \( x \in S \). Since \( S \cap M(L^\infty) = \emptyset \), there exists by (a) a Blaschke product \( B_x \) vanishing at \( x \). A compactness argument now yields a finite number of Blaschke products \( B_j \), \( (j = 1, \ldots, n) \), such that \( S \subseteq \bigcup_{j=1}^n \{|B_j| < 1/2\} \). Let \( B = B_1 \cdots B_n \). Then \( S \subseteq \{|B| < 1\} \). Now use (b) to get a Blaschke product \( B^* \) vanishing identically on the level set \( \{|B| < 1\} \). This yields the assertion \( S \subseteq Z(B^*) \). \( \square \)

The following result has been proven by Guillory, Izuchi and Sarason using Wolff’s factorization theorem. We include it here, because it is not explicitly stated as a theorem in [8].

**Lemma 2.5** ([8], [19]). Let \( f \in H^\infty + C \) be invertible in \( L^\infty \). Then \( f = Bq \) for some Blaschke product \( B \) and a function \( q \) invertible in \( H^\infty + C \).

Izuchi ([11, p. 55]) showed that every Blaschke product \( B \) admits a factorization of the form \( B = B_1B_2 \), where \( Z_\infty(B) = Z_\infty(B_1) = Z_\infty(B_2) \). In the case of \( H^\infty + C \) functions we have the following.

**Proposition 2.6.** Let \( f \in H^\infty + C \). Assume that \( E = Z(f) \cap M(L^\infty) \) is a clopen subset of \( M(L^\infty) \). Then there exist functions \( g \) and \( h \) in \( H^\infty + C \) such that

\[
(i) \quad f = gh, \quad \quad \quad (ii) \quad Z_\infty(f) = Z_\infty(g) = Z_\infty(h).
\]

**Proof.** If \( E = \emptyset \), then \( f \) is invertible in \( L^\infty \). Hence, by Lemma 2.5, \( f \) can be written as \( f = Bq \), where \( B \) is a Blaschke product and \( q \) an invertible function in \( H^\infty + C \). The aforementioned result of Izuchi yields the desired factorization.

If \( E \neq \emptyset \), let \( \chi_E \) be the characteristic function of \( E \) in \( M(L^\infty) \). Recall that \( E^c = M(L^\infty) \setminus E \). Since \( E \) is clopen, \( \chi_E \) is continuous on \( M(L^\infty) \) and so \( \chi_E \in L^\infty \). Note also that \( f = f\chi_{E^c} \). Hence, by Lemma 2.2, \( f \) vanishes identically on \( \{0 < \chi_E < 1\} \). By a result of Axler [1] there exists a Blaschke product \( B \) such that \( B\chi_E \in H^\infty + C \). We may assume, without loss of generality, that \( \|f\| < 1 \). Since \(|f| > 0 \) on \( E^c \), we see that \( f + B\chi_E \) does not
vanish on \( M(\ell^\infty) \). Thus \( f + B\chi_E \) is invertible in \( L^\infty \). By Lemma 2.5 we can write \( f + B\chi_E = C_0q \), where \( C_0 \) is a Blaschke product and \( q \) is a function invertible in \( H^\infty + C \). Due to the result of Izuchi mentioned above ([11, p. 55]), we may factor \( C_0 \) as \( C_0 = C_1C_2 \), where the \( C_j \), \( (j = 1, 2) \), are Blaschke products such that \( Z_\infty(C_0) = Z_\infty(C_1) = Z_\infty(C_2) \). Since \( B\chi_E \in H^\infty + C \), by Lemma 2.2 we know that \( B \equiv 0 \) and \( B\chi_E \equiv 0 \) on \( \{ 0 < \chi_E < 1 \} \). Since this latter set is contained in \( Z_\infty(f) \), too, we deduce from the invertibility of \( q \) that \( C_0 \) and hence \( C_j \), \( (j = 1, 2) \), vanish identically on \( \{ 0 < \chi_E < 1 \} \). Thus, by Lemma 2.2, \( C_j\chi_E \in H^\infty + C \), \( (j = 0, 1, 2) \). So

\[
\tag{8} f = f\chi_E = (f + B\chi_E)\chi_E = (C_0\chi_E)q = (C_1\chi_E)(C_2\chi_E)\chi_Eq.
\]

We claim that for \( j = 0, 1, 2 \), \( Z_\infty(C_j\chi_E) = Z_\infty(f) \). To see this, we note that by (8) and the invertibility of \( q \), we have \( Z_\infty(f) = Z_\infty(C_0\chi_E) \). Recall that \( S(E) = \{ \varphi \in M(\ell^\infty + C) : \text{supp } \varphi \subseteq E \} \). Thus, by Lemma 2.2

\[
Z_\infty(C_j\chi_E) = S(E) \cup \{ 0 < \chi_E < 1 \} \cup (S(E^c) \cap Z_\infty(C_j)).
\]

Since \( Z_\infty(C_0) = Z_\infty(C_1) = Z_\infty(C_2) \), we obtain that \( Z_\infty(C_j\chi_E) = Z_\infty(f) \) for \( j = 0, 1, 2 \).

Hence \( f = (C_1\chi_E)(C_2\chi_E)q \) yields the desired factorization. \( \square \)

**Question Q1.** Does the factorization of Proposition 2.6 hold for every \( H^\infty + C \) function?

It is a classical result of D.J. Marshall ([15, p. 20]) that every ideal in \( H^\infty \) whose hull does not meet the Shilov boundary is generated by inner functions. In \( H^\infty + C \) we can say more:

**Proposition 2.7.** Let \( I \) be an ideal in \( H^\infty + C \). Assume that \( Z(I) \cap M(\ell^\infty) = \emptyset \). Then \( I \) is algebraically generated by Blaschke products.

**Proof.** Since \( H^\infty + C \) is a unilogmodular\(^1\) algebra on its Shilov boundary, every ideal \( I \) in \( H^\infty + C \) with \( Z(I) \cap \partial(\ell^\infty + C) = \emptyset \) contains a function \( u \) which is unimodular on the Shilov boundary [16]. By Lemma 2.5, \( u = Bq \) for some Blaschke product \( B \) and a unimodular function \( q \) invertible in \( H^\infty + C \). Thus \( B \in I \). Since for every \( f \in I \) with \( \| f \| \leq 1/2 \), the function \( B + f \) does not vanish on \( M(\ell^\infty) \), we see that \( B + f \) is invertible in \( L^\infty \). Using Lemma 2.5, we have \( B + f = B_fq_f \) for some Blaschke product \( B_f \) and an invertible function \( q_f \) in \( H^\infty + C \). Hence \( I \) is generated by \( B \) and all of the \( B_f \). \( \square \)

The last step on the way to prove our main result is the following technical lemma.

\(^1\)See ([16, p. 467]) for a definition of this term.
Lemma 2.8. Let $I$ be an ideal in $H^\infty + C$ and let $f_j \in H^\infty + C$, ($j = 1, 2$). Assume that $f_1$ and $f_2$ vanish on an open set $U$ in $M(H^\infty + C)$ which contains the hull of $I$ and that $Z(I) \cap M(L^\infty) \neq \emptyset$. Then $f_1 f_2 \in I$.

Proof. Consider the ideal $J = IL^\infty$ generated by $I$ in $L^\infty$ and let $hull(J)$ be its hull in $M(L^\infty)$. We obviously have $hull(J) = Z(I) \cap M(L^\infty)$. Choose an open set $V$ in $M(L^\infty)$ such that $hull(J) \subset V \subset \overline{V} \subset U$. Since $L^\infty$ is isometrically isomorphic to $C(M(L^\infty))$, we see that there exists $q \in L^\infty$ such that $q$ is identically one on $V$ and identically zero on $M(L^\infty) \setminus U$. Thus $f_j q \equiv 0$ on $M(L^\infty)$, and hence $f_j q \in J$. But $hull(J) \cap Z(q) = \emptyset$. Thus there exist $u \in J$ and $v \in L^\infty$ so that $1 = u + vq$. Multiplying by $f_j$ yields that

$$f_j = f_j u + v(f_j q) \in J.$$  

Thus there exist functions $q^j_n \in L^\infty$ and $g_n \in I$ so that

$$f_j = \sum_{n=1}^{N} q^j_n g_n \quad (j = 1, 2).$$

By [1] there exists a Blaschke product $B$ such that $Bq^j_n \in H^\infty + C$ for $n = 1, 2, \ldots, N$ and $j = 1, 2$. It follows that $Bf_j = \sum_{n=1}^{N} (Bq^j_n)g_n \in I$.

We shall now construct a Blaschke product $D$ such that $Z(D) \subset U$ and $f_2 D \in I$. If $Z(B) \subset U$, we put $D = B$. If not, use Suarez’s result ([17, p. 244]) to choose a function $g \in I$ such that $Z(g) \subset U$. Now consider the ideal $I_1$ in $H^\infty + C$ generated by $B$ and $g$. Obviously $Z(I_1) \subset Z(g) \subset U$. But $Z(I_1) \cap M(L^\infty) = \emptyset$. Thus, by [17] again, there exists a function $h \in I_1$ such that $Z(h) \subset U$ and $Z(h) \cap M(L^\infty) = \emptyset$. In particular $h$ is invertible in $L^\infty$. By Lemma 2.5, $h = D\tilde{h}$, where $D$ is a Blaschke product and $\tilde{h}$ is invertible in $H^\infty + C$. Therefore $D = \tilde{h}^{-1}h \in I_1$. Thus there exist $x$ and $y$ in $H^\infty + C$, so that $D = xB + yg$.

Hence

$$f_2 D = f_2 (xB + yg) = x(f_2 B) + (f_2 y)g \in I + I \subset I.$$  

Moreover, $Z(D) \subset U$. Since $U \subset Z(f_1)$ we can conclude from Theorem 1.4 that $f_1 \overline{D} \in H^\infty + C$. Therefore

$$f_1 f_2 = (f_1 \overline{D})(f_2 D) \in I.$$

\[\square\]

This brings us to our main Theorem, stating that $H^\infty + C$ has the Shilov property.

Theorem 2.9. Let $I$ be an ideal in $H^\infty + C$ and let $f$ be a function in $H^\infty + C$ vanishing in an open neighborhood $U$ of the hull, $Z(I)$, of $I$. Then $f \in I$. 
Proof. Case 1. $Z(I) \cap M(L^\infty) = \emptyset$.

Let $S = M(L^\infty) \cup [M(H^\infty + C) \setminus U]$. Then $S$ is a closed subset of $M(H^\infty + C)$ which is disjoint from $Z(I)$. Hence, by ([17, p. 244]) there exists a function $g \in I$ such that $Z(g) \cap S = \emptyset$. In particular $g$ is invertible in $L^\infty$. By Lemma 2.5, $g = BG$ for a Blaschke product $B$ and a function $G$ invertible in $H^\infty + C$. Thus $B \in I$ and $Z(B) \subseteq U$. Since $U \subseteq Z(f)$, we obtain from Theorem 1.4 that $fB \in H^\infty + C$ and so $f = (fB)B \in I$.

Case 2. $Z(I) \cap M(L^\infty) \neq \emptyset$.

Let $E = \overline{Z(f)^c \cap M(L^\infty)}$. Since $M(L^\infty)$ is extremely disconnected, $E$ is a clopen set in $M(L^\infty)$ ([3, p. 18] and [4, p. 214]) contained in $Z(f)$. Let $S = \overline{Z(f)^c \cap M(L^\infty)}$. Then $S$ is a compact set containing $E$. Moreover $S \setminus E$ is compact. Since $Z(I) \cap M(L^\infty) \subseteq E$, we see that $(S \setminus E) \cap (Z(I) \cup E) = \emptyset$. Thus there is an open neighborhood $V$ of $Z(I) \cup E$ in $M(H^\infty + C)$ such that $\overline{V} \cap (S \setminus E) = \emptyset$. Let $\Omega = V \cap Z(f)^c$. Then $\Omega$ is an open subset of $M(H^\infty + C)$ satisfying

\[(9) \quad Z(I) \subseteq \Omega \subseteq Z(f)^c,\]

\[(10) \quad \overline{\Omega} \cap (S \setminus E) = \emptyset,\]

and (as will be justified below)

\[(11) \quad E = \overline{\Omega \cap M(L^\infty)} = \overline{\Omega} \cap M(L^\infty).\]

In fact, (11) is a consequence of (9), (10) and the following inclusions:

(i) $\overline{\Omega} \cap M(L^\infty) \subseteq \overline{Z(f)^c \cap M(L^\infty)} = S = (S \setminus E) \cup E$,
(ii) $Z(f)^c \cap M(L^\infty) \subseteq E \cap Z(f)^c \subseteq \overline{V} \cap Z(f)^c \cap M(L^\infty) = \Omega \cap M(L^\infty)$

and hence

\[E = \overline{Z(f)^c \cap M(L^\infty)} \subseteq \overline{\Omega \cap M(L^\infty)} \subseteq \overline{\Omega} \cap M(L^\infty).\]

Let $S(E) = \{ \varphi \in M(H^\infty + C) : \varphi(\chi_E) = 1 \}$. We claim that

\[(12) \quad \overline{\Omega \setminus S(E)} \cap M(L^\infty) = \emptyset.\]

To see this, let $x \in \overline{\Omega \setminus S(E)}$. Then there is a net of points $(x_\alpha)$ from $\Omega \setminus S(E)$ with $(x_\alpha)$ converging to $x$. By the definition of $S(E)$ we know that $0 \leq x_\alpha(\chi_E) < 1$ for every $\alpha$. Now if $x \in \{ 0 < \chi_E < 1 \}$, then, by Lemma 2.3, $x \not\in M(L^\infty)$, so we are done. If $x \not\in \{ 0 < \chi_E < 1 \}$, then $M(H^\infty + C) \setminus \{ 0 < \chi_E < 1 \}$ is an open neighborhood of $x$. We may assume that this neighborhood contains all the $x_\alpha$. Hence $x_\alpha(\chi_E) = 0$ or $x_\alpha(\chi_E) = 1$. Since $0 \leq x_\alpha(\chi_E) < 1$, we conclude that $x_\alpha(\chi_E) = 0$ for all $\alpha$. Hence $x(\chi_E) = 0$.

So $x \not\in E$. But $E = \overline{\Omega \cap M(L^\infty)}$. Since $x \in \overline{\Omega}$, we deduce that also in this case $x \not\in M(L^\infty)$. This proves (12).
Let \( U_1 = [\Omega \cup \{0 < \chi_E < 1\}] \setminus S(E) \). We claim that \( U_1 \) is an open set such that
\[
(13) \quad U_1 \cap M(L^\infty) = \emptyset,
\]
and
\[
(14) \quad U_1 \subseteq Z(f).
\]

To see this, we note that \( U_1 = (\Omega \setminus S(E)) \cup \{0 < \chi_E < 1\} \). Hence, by (12) and Lemma 2.3
\[
\overline{U}_1 = \overline{\Omega \setminus S(E)} \cup \{0 < \chi_E < 1\}
\]
has property (13). To prove (14), we first note that if \( 0 < \varphi(\chi_E) < 1 \), then \( \text{supp} \varphi \) meets the clopen set \( E \) on which \( f \) vanishes identically. Thus by Lemma 2.1, \( \varphi(f) = 0 \). Together with (9) we obtain \( U_1 \subseteq Z(f) \).

By Lemma 2.4 and (13) we may choose a Blaschke product \( B \equiv 0 \) on \( U_1 \). By Lemma 2.2, \( \{0 < \chi_E < 1\} \subseteq Z(B) \) implies that \( B\chi_{E^c} \in H^\infty + C \) (where as usual \( E^c = M(L^\infty) \setminus E \)). Consider \( f + B\chi_{E^c} \). We may assume without loss of generality that \( \|f\| < 1 \). We claim that
\[
(15) \quad f + B\chi_{E^c} = 0 \quad \text{on} \quad E,
\]
\[
(16) \quad f + B\chi_{E^c} \neq 0 \quad \text{on} \quad M(L^\infty) \setminus E
\]
and
\[
(17) \quad f + B\chi_{E^c} = 0 \quad \text{on} \quad U_1.
\]

Since (15) and (16) are trivial, we will turn to the proof of (17). First note that on \( U_1 \) we have \( f \equiv 0 \) and \( B \equiv 0 \). Since by Lemma 2.2 \( Z(B) \subseteq Z(B\chi_{E^c}) \), we obtain (17).

Next we apply Proposition 2.6 and write \( f + B\chi_{E^c} = f_1f_2 \), where \( f_j \in H^\infty + C \) and \( f_j = 0 \) on \( U_1 \supseteq \{0 < \chi_E < 1\} \). Notice that \( U_1 \subseteq Z_{\infty}(f + B\chi_{E^c}) \).

Now \( f = f\chi_{E^c} \), so
\[
(18) \quad f + B\chi_{E^c} = (f + B\chi_{E^c})\chi_{E^c} = (f_1\chi_{E^c})(f_2\chi_{E^c}).
\]

Note that by Lemma 2.2, \( f_j\chi_{E^c} \in H^\infty + C \). Next we claim that
\[
(19) \quad f_j\chi_{E^c} \equiv 0 \quad \text{on} \quad \Omega \cup \{0 < \chi_E < 1\}.
\]

In fact, if \( 0 < \varphi(\chi_E) < 1 \), then \( \varphi(f_j\chi_{E^c}) = 0 \) by Lemma 2.2. Moreover, by the same Lemma, \( \Omega \setminus S(E) \subseteq U_1 \subseteq Z(f_j) \subseteq Z(f_j\chi_{E^c}) \) and \( \Omega \cap S(E) \subseteq S(E) \subseteq Z(f_j\chi_{E^c}) \). This yields (19).

By ([11, p. 55]), we can write \( B = C_1C_2 \), where the zero sets of infinite order of \( B, C_1 \) and \( C_2 \) coincide. In particular, since \( B \) vanishes identically on the open set \( U_1 \), so do \( C_1 \) and \( C_2 \). Thus, by (18), we obtain
\[
f = (f_1\chi_{E^c})(f_2\chi_{E^c}) - B\chi_{E^c} = (f_1\chi_{E^c})(f_2\chi_{E^c}) - (C_1\chi_{E^c})(C_2\chi_{E^c}).
\]
Because for \( j = 1, 2, \) \( \{0 < \chi_E < 1\} \subseteq U_1 \subseteq Z(C_j) \), we conclude from Lemma 2.2 that \( C_j \chi_{Ec} \in H^\infty + C \). Moreover, as above, we see that \( C_j \chi_{Ee} \equiv 0 \) on \( \Omega \). Thus we have factorized \( f \) as a sum of two factors, each of them admits a factorization of type \( gh \), where both \( g \) and \( h \) vanish on \( \Omega \). Since the hull of \( I \), \( Z(I) \), satisfies \( Z(I) \cap M(L^\infty) \neq \emptyset \) and \( Z(I) \subseteq \Omega \), Lemma 2.8 implies that 

\[
(f_1 \chi_{Ec})(f_2 \chi_{Ec}) \in I \quad \text{and} \quad (C_1 \chi_{Ec})(C_2 \chi_{Ec}) \in I.
\]

Thus \( f \in I \). \( \square \)

As a corollary, we obtain the following generalization of Theorem 1.4.

**Corollary 2.10.** Let \( f \) and \( g \) be two functions in \( H^\infty + C \). Assume that \( f \) vanishes identically on an open neighborhood of the zeros of \( g \). Then \( f \) is divisible in \( H^\infty + C \) by \( g \).

**Proof.** Take \( I \) to be the principal ideal generated by \( g \) and apply Theorem 2.9. \( \square \)

Let \( A \) be a commutative unital Banach algebra and let \( I \) be an ideal in \( A \). An element \( f \in A \) is said to belong locally to \( I \) if for every \( m \in M(A) \) there exists a neighborhood \( U \) of \( m \) in \( M(A) \) such that \( \hat{f}|_U \in \hat{I}|_U \). An important result in the theory of Banach algebras is that in regular algebras every ideal is locally determined ([5, p. 201] and [14, p. 224]); that is if \( f \in A \) belongs locally to an ideal \( I \), then actually \( f \in I \). As another corollary of Theorem 2.9 we prove that a large class of ideals in the non-regular algebra \( H^\infty + C \) is locally uniquely determined.

**Corollary 2.11.** Every intersection of primary ideals and every radical ideal in \( H^\infty + C \) is locally uniquely determined.

**Proof.** Since the case of intersections of primary ideals is an immediate consequence of Theorem 2.9, it remains to look at the case of radical ideals. So let \( f \in H^\infty + C \) belong locally to the radical ideal \( I \). Then, by a compactness argument, there exists finitely many functions \( g_j \in I \) and open sets \( U_j \), \( (j = 1, \ldots, n) \), such that \( Z(I) \subseteq \bigcup U_j \) and \( f|_{U_j} = g_j \). Hence \( \prod_{j=1}^{n} (f - g_j) \equiv 0 \) in a neighborhood of \( Z(I) \). Thus, by Theorem 2.9, we can conclude that \( f^n \in I \) and hence \( f \in I \). \( \square \)

We list below two questions we are unable to answer.

**Q2.** Is every ideal in \( H^\infty + C \) locally uniquely determined?

**Q3.** Assume that a continuous function \( q \) on \( M(H^\infty + C) \) locally belongs to \( H^\infty + C \). Is \( q \in H^\infty + C \)? In other word, is \( H^\infty + C \) a local algebra on its spectrum?
We return now to the Shilov property. Comparing that with the algebra $H^\infty + C$, the situation in $H^\infty$ is a bit different. There do exist ideals $I$ with hull contained in $M(H^\infty + C)$, such that not every function vanishing in an $M(H^\infty + C)$ neighborhood of the hull belongs to $I$. In fact, let $I$ be the ideal generated by the $n$-th roots of the atomic inner function $S(z) = \exp \left( -\frac{1+z}{1-z} \right)$. By Lemma 2.4 there exists a Blaschke product vanishing on the set \( \{ x \in M(H^\infty + C) : |S(x)| \leq 1/2 \} \). But clearly $B \notin I$.

On the other hand, we have the following result:

**Theorem 2.12.** Let $I$ be a closed ideal in $H^\infty$ whose weak-* closure in $H^\infty$ is $H^\infty$. Then $I$ contains every function vanishing in an $M(H^\infty + C)$ neighborhood of the hull of $I$.

**Proof.** The hypothesis on $I$ says that the greatest common divisor of the inner parts of the elements in $I$ is a unimodular constant and that $Z(I) \subseteq M(H^\infty + C)$. Thus by [6] there exists a unique closed ideal $J$ in $H^\infty + C$ such that $I = J \cap H^\infty$. The result now follows from Theorem 2.9. \( \square \)

Finally, let us mention that, of course, every ideal in $H^\infty$ contains every function vanishing in a $M(H^\infty)$ neighborhood of its hull, because only the zero function satisfies this hypothesis. Thus, in that case, the “real” extension of Theorem 2.9, namely that $H^\infty$ has the Shilov property, holds in $H^\infty$, too. This raises the following questions:

Let $A$ be a commutative unital Banach algebra and let $E$ be a closed subset of $M(A)$ with the property that, via the restriction map, $\hat{A}|_E$ is isometrically isomorphic to $A$; in other words, let $E$ be a closed boundary for $A$. Say that $A$ has the $E$-restricted Shilov property if any ideal, with hull, $\mathcal{H}$, contained in $E$, contains every function vanishing in a relative neighborhood of $\mathcal{H}$ in $E$.

**Q4.** For which closed boundaries $E$ in $M(A)$ does $A$ have the $E$-restricted Shilov property? What happens if one restricts to certain classes of ideals, closed ones for example?

**Q5.** Do the algebras $P(K)$ and $R(K)$ have the Shilov property? (Here $K$ is a compact subset in $\mathbb{C}$.)

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**References**


\(^2\)Note that in the above example the weak-* closure of $I$ does coincide with $H^\infty$. 

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