LORENTZIAN ISOPARAMETRIC HYPERSURFACES IN $H_{1}^{n+1}$

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Lorentzian Isoparametric Hypersurfaces in $H^{n+1}_1$

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In 1981, Nomizu introduced isoparametric hypersurfaces in Lorentzian space forms and studied the Cartan identities. Later Hahn, 1984, generalized Nomizu’s work to the pseudo-Riemannian space forms and presented many examples. In general, the shape operator of a hypersurface in a pseudo-Riemannian space form may be not diagonalizable. This makes the isoparametric theory in pseudo-Riemannian space form different from that in Riemannian space forms. In 1985, Megid classified Lorentzian isoparametric hypersurfaces in $\mathbb{R}^{n+1}$. He showed that there are three types of Lorentzian isoparametric hypersurfaces in $\mathbb{R}^{n+1}$. Type I are exactly cylinders and umblic hypersurfaces while the other two types of hypersurfaces have properties close to cylinders and umblic hypersurfaces. Megid called them generalized cylinders and umblic hypersurfaces. In this paper, the local classification of Lorentzian isoparametric hypersurfaces in $H^{n+1}_1$ is obtained and the properties of them are discussed.

Introduction.

A hypersurface in $H^{n+1}_1$ is called isoparametric if the minimal polynomial of the shape operator is constant. This allows complex or non-simple principal curvatures (eigenvalues of the shape operator). In this paper, we classify Lorentzian isoparametric hypersurfaces in an anti-de Sitter sphere $H^{n+1}_1$. More precisely, we show that there are four types of such hypersurfaces. Type I hypersurfaces are determined by two orthogonal subspaces of $\mathbb{R}^{n+2}_2$ and the principal curvatures; type II and type III hypersurfaces are determined by two 1-parameter orthogonal subspaces of $\mathbb{R}^{n+2}_2$ and the principal curvatures; and the type IV hypersurfaces are homogeneous.

The classification theorem we obtain here plays an essential role in the study of isoparametric hypersurfaces in complex hyperbolic spaces $CH^n$ [9]. A connected hypersurface in $CH^n$ is called isoparametric if all parallel hypersurfaces $M_t$ for $t$ sufficiently close to zero have constant mean curvatures. In [9], we get the complete classification of isoparametric hypersurfaces in
In fact, we prove that all isoparametric hypersurfaces are homogeneous.

The paper is organized as follows. In Section 1, we recall basic definitions, notations and the structural equations of a Lorentzian hypersurface in $H^{n+1}_1$. We use a result of Megid [4] to conclude that there are four types of local isoparametric hypersurfaces in $H^{n+1}_1$. In Section 2, we study the Cartan identities and show that a Lorentzian isoparametric hypersurface has at most a pair of conjugate complex and two real principal curvatures. In Sections 3, 4 and 5, we classify hypersurfaces of type I, II, III and IV, respectively. Combining these results, we get the classification.

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1. Preliminaries.

In this section we recall the basic definitions and the structural equations of a Lorentzian hypersurface in $H^{n+1}_1$. Then we give the definition of an isoparametric hypersurface and show the forms of the shape operator.

Let $R^{n+2}_2$ be an $n + 2$-dimensional real vector space with a bilinear form of signature (2,n) given by

$$\langle x, x \rangle = -2 \sum_{i=1}^{2} x_i^2 + \sum_{i=3}^{n+2} x_i^2,$$

$H^{n+1}_1$ be the hypersurface

$$\{x \in R^{n+2}_2 \mid \langle x, x \rangle = -1\},$$

which is the anti-de Sitter sphere with constant sectional curvature $-1$. $H^{n+1}_1$ is a non-simply connected Lorentzian space form.

Let $V$ be a vector space with a Lorentzian metric $\langle , \rangle$. An orthonomal basis $\{E_1, \ldots, E_n\}$ is one satisfying $\langle E_1, E_1 \rangle = -1, \langle E_i, E_j \rangle = \delta_{ij}, \langle E_1, E_j \rangle = 0$ for $2 \leq i, j \leq n$. A pseudo-orthonormal basis is a basis $\{X, Y, E_1, \ldots, E_{n-2}\}$ such that $\langle X, X \rangle = 0 = \langle Y, Y \rangle = \langle X, E_i \rangle = \langle Y, E_i \rangle, \langle X, Y \rangle = -1$ and $\langle E_i, E_j \rangle = \delta_{ij}$ for $1 \leq i, j \leq n - 2$.

Generally, a hypersurface $M$ in $H^{n+1}_1$ is called a Lorentzian hypersurface if the induced metric has signature $(1,n - 1)$. Next, we recall the structure equations of a Lorentzian hypersurface $M$.

Let $X$ be the position vector of $M$, i.e., $X$ is the inclusion map from $M$ to $R^{n+2}_2$, $e_1, \ldots, e_{n+1}$ a local frame on $H^{n+1}_1 \subset R^{n+2}_2$ such that $e_1, \ldots, e_n$ are tangent to $M$, $e_{n+1}$ normal to $M$, and $\omega^1, \ldots, \omega^{n+1}$ the dual 1-forms.
We can write

\[ dX = \sum_{A=1}^{n+1} \omega^A e_A, \tag{1} \]

\[ de_A = \sum_{B=1}^{n+1} \omega^B e_B + \omega_A X, \tag{2} \]

where \( A = 1, \ldots, n+1 \), and \( \omega^B_A \) and \( \omega_A \) satisfy the first structural equation of \( H^{n+1}_1 \):

\[ d\omega^A + \sum_{B=1}^{n+1} \omega^B_A \wedge \omega^B = 0, \tag{3} \]

\[ dg_{AB} = \sum_{C=1}^{n+1} g_{CB} \omega^C_A + g_{AC} \omega^C_B, \]

\[ \omega_A = \sum_{B=1}^{n+1} g_{AB} \omega^B. \]

Especially for an orthonomal frame, \( \omega^1_1 = 0, \omega^1_i = \omega^1_j = \omega^2_i = \omega^2_j = 0 \), \( 2 \leq i, j \leq n+1 \), and for a pseudo-orthonomal frame, \( \omega^1_1 + \omega^2_1 = 0, \omega^1_2 = \omega^2_2 = \omega^1_i = \omega^2_i = \omega^1_j + \omega^2_j = 0 \), \( 3 \leq i, j \leq n+1 \).

The second structural equation of \( H^{n+1}_1 \) is:

\[ d\omega^B_A - \sum_{C=1}^{n+1} \omega^C_J \wedge \omega^B_C - \omega_A \wedge \omega^B = 0. \tag{4} \]

Restricting these forms to \( M \), we have

\[ \omega^{n+1}_i = 0, \quad \omega^{n+1}_{n+1} = 0. \tag{5} \]

Write

\[ \omega^i_{n+1} = \sum_{j=1}^{n} h^j_i \omega^j. \tag{6} \]

Exterior differenting (5), we get

\[ \sum_{k=1}^{n} g_{kk} h^k_j = \sum_{k=1}^{n} g_{jk} h^k_i. \tag{7} \]

The shape operator is a linear transformation for any \( x \in M \) defined by

\[ A : T_x M \to T_x M : e_i \mapsto \sum_{j=1}^{n} h^j_i e_j. \tag{8} \]
A is a symmetric linear transformation on $T_x M$ with Lorenzian product, i.e., for any $X, Y \in T_x M$, 
\[
\langle AX, Y \rangle = \langle X, AY \rangle = II(X, Y).
\]
Here $II(X, Y)$ is the second fundamental form of $M$. The eigenvalues of $A$ are called principal curvatures of $M$. Corresponding to every principal curvature $\lambda$, we have algebraic multiplicity and geometric multiplicity. Algebraic multiplicity $\nu$ is the exponent of $(x - \lambda)$ in the characteristic polynomial and geometric multiplicity $\mu$ is the dimension of the eigenspace 
\[
T_\lambda = \{ X \in T_x M \mid AX = \lambda X \}.
\]
A principal curvature $\lambda$ is called diagonalizable if $\nu = \mu$.

The structural equations of $M$ are
\begin{align}
    d\omega^i + \sum_{j=1}^{n} \omega^i_j \wedge \omega^j &= 0, \\
    dg_{ij} &= \sum_{k=1}^{n} (g_{ik} \omega^k_j + g_{kj} \omega^k_i), \\
    \omega^i_i &= \sum_{j=1}^{n} g_{ij} \omega^j,
\end{align}
\begin{align}
    g_{ij} \omega^i_{n+1} + \omega^i_{n+1} &= 0, \\
    d\omega^i - \sum_{k=1}^{n} \omega^k_i \wedge \omega^k_j &= \omega^i_{n+1} \wedge \omega^j_{n+1} + \omega^i \wedge \omega^j, \\
    d\omega^i_{n+1} &= \sum_{j=1}^{n} \omega^j_{n+1} \wedge \omega^i_j.
\end{align}
Among these equations, (11) and (12) are called Gauss equation and Codazzi equation of $M$, respectively.

Define
\[
\sum_{k=1}^{n} h_{j,k}^i \omega^k = dh_j^i - \sum_{k=1}^{n} h_{ik}^j \omega^k_j + \sum_{k=1}^{n} \sum_{k=1}^{n} h_{ik}^j \omega^k_i.
\]
Then Codazzi equation becomes
\[
\sum_{k=1}^{n} h_{j,k}^i = h_{k,j}^i.
\]
A hypersurface is called isoparametric if the minimal polynomial of shape operator is constant. In this paper we only consider Lorentzian isoparametric hypersurfaces. In [4], Megid showed that such a hypersurface has constant principal curvatures and the shape operator $A$ can be put into exactly one of the canonical forms I, II, III or IV.
I. \[ A = \begin{pmatrix} a_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & a_n \end{pmatrix} \]

II. \[ A = \begin{pmatrix} a_0 & 0 \\ 1 & a_0 \\ \vdots & \ddots \\ a_{n-2} & \end{pmatrix} \]

III. \[ A = \begin{pmatrix} a_0 & 0 & 0 \\ 0 & a_0 & 1 \\ -1 & 0 & a_0 \\ \vdots & \ddots & \end{pmatrix} \]

IV. \[ A = \begin{pmatrix} a_0 & b_0 \\ -b_0 & a_0 \\ \vdots & \ddots \\ a_{n-2} & \end{pmatrix} \]

Here \( b_0 \) is assumed to be non-zero. In cases I and IV \( A \) is represented with respect to an orthonormal basis while in cases II and III the basis is a pseudo-orthonormal basis. In cases I, II and III the eigenvalues are real, while \( a_0 \pm ib_0 \) are eigenvalues in case IV. Throughout this paper, a Lorentzian isoparametric hypersurface in \( H^{n+1}_1 \) is called a type I, II, III or IV hypersurface according to the form of the shape operator \( A \).

2. Cartan identities.

In this section, we use Hahn’s result on Cartan identities to study the possible number of principal curvatures of Lorentzian isoparametric hypersurfaces in \( H^{n+1}_1 \), and prove the following theorem.

**Theorem 2.1.** A type I, II or III Lorentzian isoparametric hypersurface has at most two real principal curvatures, and a type IV Lorentzian isoparametric hypersurface has a pair of conjugate complex principal curvatures and at most two real principal curvatures.

We need a couple of Lemmas to prove the theorem. The first one is proved by Hahn:
Lemma 2.2 ([2]). Let $M$ be a Lorentzian isoparametric hypersurface in $H^{n+1}_1$, and $\lambda_1, \ldots, \lambda_p$ all distinct principal curvatures of $M$ with algebraic multiplicities $\nu_1, \ldots, \nu_p$. If $\lambda_i$ is a (real) diagonalizable principal curvature, then we have Cartan identity
\[
\sum_{j=1, j \neq i}^p \nu_j \frac{-1 + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0.
\]

Lemma 2.3. Let $M$ be a type I, II or III hypersurfaces. Then $p \leq 2$. Moreover, if $p = 2$, then $\lambda_1 \lambda_2 = 1$.

Proof. If $M$ is type I, then all principal curvatures of $M$ are real and diagonalizable. By Lemma 2.2, we have
\[
\sum_{j=1, j \neq i}^p \nu_j \frac{-1 + \lambda_i \lambda_j}{\lambda_i - \lambda_j} = 0
\]
for any $i$ in $\{1, \ldots, p\}$. Without loss of generalities, we may assume $\lambda_1 < \lambda_2 < \cdots < \lambda_p$, and $\lambda_p \geq 0$. Choose the largest nonnegative $\lambda_i$ such that $\lambda_i \lambda_{i-1} \leq 1$. Then
\[
\frac{\lambda_i \lambda_j - 1}{\lambda_i - \lambda_j} \leq 0
\]
for any $j \neq 0$. Hence $\lambda_i \lambda_j = 1$ if $i \neq j$. Therefore $p \leq 2$.

If $M$ is type II or type III, then only one principal curvature of $M$ is not diagonalizable. Without loss of generalities, we may assume that $\lambda_1$ is not diagonalizable. By Lemma 2.2, we have
\[
\sum_{j=1, j \neq i}^p \nu_j = \frac{\lambda_i \lambda_j - 1}{\lambda_i - \lambda_j} = 0
\]
for any $i$ in $\{2, \ldots, p\}$.

Note that
\[
\sum_{i,j=1, i \neq j}^p \nu_i \nu_j \frac{\lambda_i \lambda_j - 1}{\lambda_i - \lambda_j}
= \sum_{i < j}^p \nu_i \nu_j (\lambda_i \lambda_j - 1) \left( \frac{1}{\lambda_i - \lambda_j} + \frac{1}{\lambda_j - \lambda_i} \right)
= 0.
\]
Combining (16) and (17), we have
\[
\sum_{j=1, j \neq i}^p \nu_j \frac{\lambda_i \lambda_j - 1}{\lambda_i - \lambda_j} = 0
\]
for any $i$ in $\{1, \ldots, p\}$, which is exactly the equation (15). Hence we know that $p \leq 2$. \qed
Lemma 2.4. Let $M$ be a type IV hypersurface, and $\lambda_1, \ldots, \lambda_p$ all distinct principal curvatures of $M$. Then $p \leq 4$.

Proof. If $M$ is a type IV, then $M$ has a pair of conjugate complex principal curvatures with algebraic multiplicities 1. We may assume $\lambda_1 = a_0 + ib_0$, $\lambda_2 = a_0 - ib_0$, $b_0 \neq 0$, $\nu_1 = \nu_2 = 1$, and $\lambda_3 < \lambda_4 < \cdots < \lambda_p$. By Lemma 2.2, $\lambda_3, \ldots, \lambda_p$ satisfy

$$
\frac{2a_0(1 + \lambda_i^2) - 2\lambda_i(a_0^2 + b_0^2 + 1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2 + b_0^2)} + \sum_{j=3,j\neq i}^{p} \nu_j = \frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = 0
$$

for any $i$ in $\{3, \ldots, p\}$.

First we claim that if $a_0 \leq 0$, then $\lambda_i \leq 0$ for any $i$ in $\{3, \ldots, p\}$.

Suppose the claim is false. Then $\lambda_p > 0$. We choose the largest positive $\lambda_i$ such that $\lambda_i\lambda_{i-1} \leq 1$. Then

$$
\frac{2a_0(1 + \lambda_i^2) - 2\lambda_i(a_0^2 + b_0^2 + 1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2 + b_0^2)} \leq 0,
$$

$$
\frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} \leq 0
$$

for any $j$ in $\{3, \ldots, p\} - \{i\}$. From (19), (20) and (21), it follows that

$$
2a_0(1 + \lambda_i^2) - 2\lambda_i(a_0^2 + b_0^2 + 1) = 0.
$$

This is a contradiction since $a_0 \leq 0$ and $\lambda_i > 0$. Therefore $\lambda_i \leq 0$ for any $i$ in $\{3, \ldots, p\}$. Similarly, we can prove that $\lambda_i \geq 0$ for any $i$ in $\{3, \ldots, p\}$ if $a_0 \geq 0$.

Note that

$$
\frac{2a_0(1 + \lambda_i^2) - 2\lambda_i(a_0^2 + b_0^2 + 1)}{\lambda_i^2 - 2a_0\lambda_i + (a_0^2 + b_0^2)} = \frac{2a_0(t)(1 + \lambda_i^2(t)) - 2\lambda_i(t)(a_0^2(t) + b_0^2(t) + 1)}{\lambda_i^2(t) - 2a_0(t)\lambda_i(t) + (a_0^2(t) + b_0^2(t))},
$$

$$
\frac{\lambda_i\lambda_j - 1}{\lambda_i - \lambda_j} = \frac{\lambda_i(t)\lambda_j(t) - 1}{\lambda_i(t) - \lambda_j(t)}.
$$

Here

$$
a_0(t) = \frac{(a_0^2 + b_0^2 + 1) \sinh t \cosh t + a_0(\cosh^2 t + \sinh^2 t)}{\cosh^2 t + 2a_0 \cosh t \sinh t + (a_0^2 + b_0^2) \sinh^2 t},
$$

$$
b_0(t) = \frac{b_0}{\cosh^2 t + 2a_0 \cosh t \sinh t + (a_0^2 + b_0^2) \sinh^2 t},
$$

$$
\lambda_i(t) = \frac{\sinh t + \lambda_i \cosh t}{\cosh t + \lambda_i \sinh t}
$$

and $t$ is any real number satisfying $\cosh t + \lambda_i \sinh t \neq 0$ for any $i$ in $\{3, \ldots, p\}$. 
Proof. We shall arrange the index as follows: Let \( \lambda_1(t) = a_0(t) + ib_0(t) \), \( \lambda_2(t) = a_0(t) - ib_0(t) \). Then \( \lambda_1(t), \lambda_2(t), \ldots, \lambda_p(t) \) are \( p \) distinct numbers satisfying the equation system (19) for any \( t \) satisfying \( \cosh t + \lambda_i \sinh t \neq 0 \). Hence if \( a_0(t) \leq 0 \) then \( \lambda_i(t) \leq 0 \) for any \( i \) in \( \{3, \ldots, p\} \) and if \( a_0(t) \geq 0 \) then \( \lambda_i(t) \geq 0 \) for any \( i \) in \( \{3, \ldots, p\} \).

If \( a_0 = 0 \), then \( \lambda_i = 0 \) for any \( i \) in \( \{3, \ldots, p\} \) which implies \( p = 3 \). Note that \( b_0 \neq 0 \). If \( a_0 \neq 0 \), then we can choose \( t_0 \in R - \{0\} \) such that

\[
a_0 + \frac{1}{a_0} + \frac{b_0^2}{a_0} = -\tanh t_0 - \frac{1}{\tanh t_0},
\]

which implies \( a_0(t_0) = 0 \). We claim that \( \lim_{t \to t_0} \lambda_i(t) = 0 \) or \( \infty \) for all \( 3 \leq i \leq p \). Suppose the claim is false. Then \( \lim_{t \to t_0} \lambda_i(t) > 0 \) or \( \lim_{t \to t_0} \lambda_i(t) < 0 \) for some \( k \) in \( \{3, \ldots, p\} \). Without loss of generalities, we may assume that \( \lim_{t \to t_0} \lambda_k(t) > 0 \). Hence we can choose a real \( t_1 \) satisfying that \( \cosh t_1 + \lambda_i \sinh t_1 \neq 0 \) for any \( i \) in \( \{3, \ldots, p\} \) such that \( a_0(t_1) < 0 \) and \( \lambda_k(t_1) > 0 \). This is a contradiction. So \( \lambda_i = -\tanh t_0 \) or \( \coth t_0 \) for any \( i \) in \( \{3, \ldots, p\} \). Hence \( p \leq 4 \) and \( \lambda_3, \lambda_4 \) satisfy the following equation

\[
2a_0(1 + \lambda_1^2) - 2\lambda_i(a_0^2 + b_0^2 + 1) = 0.
\]

As a consequence of Lemma 2.3 and 2.4, we obtain Theorem 2.1.

3. Type I hypersurfaces.

The main result of this section is the following result.

**Theorem 3.1.** Let \( M \) be a Lorentzian isoparametric hypersurfaces in \( H_1^{n+1} \). Then \( M \) is type I if and only if \( M \) is congruent to an open part of one of the following hypersurfaces:

i) \( H_1^m \left( \sqrt{1 - \lambda^2} \right) \times S^{n-m} \left( \sqrt{\frac{1 - \lambda^2}{\lambda^2}} \right) \), where \(-1 < \lambda < 1\);

ii) \( S_1^m \left( \sqrt{\lambda^2 - 1} \right) \times H^{n-m} \left( \sqrt{\frac{\lambda^2 - 1}{\lambda^2}} \right) \), where \( \lambda \) is real and \( \lambda^2 > 1 \);

iii) \( \{ x \in H_1^{n+1} \mid \langle x, C \rangle = \lambda \} \), where \( \lambda \) is real, and \( C \) is a constant vector with \( \langle C, C \rangle = 1 - \lambda^2 \).

**Proof.** We shall arrange the index as follows: \( 1 \leq i \leq m, m + 1 \leq \alpha \leq n \).

Let \( M \) be a type I hypersurface. By Lemma 2.3, we can choose a local orthonormal frame \( e_1, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1} \) such that \( e_{n+1} \) is normal to \( M \), \( \omega^i_{n+1} = \lambda \omega^i \) for \( 1 \leq i \leq m \) and \( \omega^\alpha_{n+1} = \frac{1}{\lambda} \omega^\alpha \) for \( m + 1 \leq \alpha \leq n \). Note that \( m = n \) if \( \lambda = 0 \) or \( \pm 1 \).

Consider the Codazzi equation

\[
d\omega^i_{n+1} = \sum_{A=1}^{n} \omega^A_{n+1} \wedge \omega^i_A = \lambda \sum_{j=1}^{m} \omega^j \wedge \omega^i_j + \frac{1}{\lambda} \sum_{\alpha=m+1}^{n} \omega^\alpha \wedge \omega^i_{\alpha}.
\]
On the other hand

\[ d\omega^i_{n+1} = \lambda(d\omega^i) = \lambda \sum_{j=1}^{m} \omega^j \wedge \omega^i + \lambda \sum_{\alpha=m+1}^{n} \omega^\alpha \wedge \omega^i. \]

Hence

\[ \sum_{\alpha=m+1}^{n} \omega^\alpha \wedge \omega^i = 0. \]

By Cartan’s Lemma, \( \omega^i_\alpha \) is a linear combination of \( \omega^{m+1}, \ldots, \omega^n \). Similarly we can prove that \( \omega^i_\alpha \) is a linear combination of \( \omega^1, \ldots, \omega^m \).

From (3), we know that \( \omega^i_\alpha = \omega^i_\alpha \) (if \( i = 1 \)) or \( -\omega^i_\alpha \) (if \( i > 1 \)) since \( e_1, \ldots, e_n \) is an orthonormal frame. Therefore

\[ \omega^i_\alpha = 0. \]

From (1), (2), (3) and (28), we get

\[ d(X - \lambda e_{n+1}) = (1 - \lambda^2)\sum_{i=1}^{m} \omega^i e_i, \]

\[ de_i = \left( \sum_{j=1, j\neq i}^{m} \omega^j e_j \right) + \omega_i (X - \lambda e_{n+1}), \]

which imply that

\[ d(e_1 \wedge e_2 \wedge \cdots e_m \wedge (X - \lambda e_{n+1})) = 0. \]

Similarly, we can prove that

\[ d(e_{m+1} \wedge e_{m+2} \wedge \cdots \wedge e_n \wedge (\lambda X - e_{n+1})) = 0. \]

Let \( W_1(x) \) be the linear span of \( \{e_1(x), e_2(x), \ldots, e_m(x), X - \lambda e_{n+1}(x)\} \), and \( W_2(x) \) the linear span of \( \{e_{m+1}(x), \ldots, e_n(x), \lambda X - e_{n+1}(x)\} \). From (30) and (31) we know that \( W_1(x) \) and \( W_2(x) \) are fixed subspaces in \( R^{2n+2}_2 \). Denote them by \( W_1 \) and \( W_2 \), respectively.

If \( \lambda \neq \pm 1 \), then

\[ R^{2n+2}_2 = W_1 + W_2 \]

is a direct sum of subspaces. Write

\[ X = X_1 + X_2, \]

where \( X \) is the position vector field of \( M \), \( X_1 \in W_1 \) and \( X_2 \in W_2 \). Since \( X - \lambda e_{n+1} \in W_1 \), \( \lambda X - e_{n+1} \in W_2 \), we know that

\[ \lambda e_{n+1} = \lambda^2 X_1 + X_2. \]

Since \( \langle X, X \rangle = -1, \langle e_{n+1}, e_{n+1} \rangle = 1 \) and \( \langle X_1, X_2 \rangle = 0 \), we have

\[ \langle X_1, X_1 \rangle = \frac{1}{\lambda^2 - 1}, \quad \langle X_2, X_2 \rangle = \frac{\lambda^2}{1 - \lambda^2}. \]
If $\lambda = \pm 1$, then from (29) we have $d(e_{n+1} - \lambda X) = 0$. Hence $C = e_{n+1} - \lambda X$ is a fixed vector in $R^{n+2}_2$ and $\langle X, C \rangle = \lambda$.

Therefore $M$ can be represented as in Theorem 3.1.

\[\square\]

4. Type II and type III hypersurfaces.

In this section, we classify the type II and type III hypersurfaces. We state the classification as a couple of theorems.

4.1. Type II hypersurfaces.

In this subsection, we arrange the index as follows: $3 \leq i, j \leq m$, $m + 1 \leq \alpha, \beta \leq n$. By a direct calculation, we have:

Theorem 4.1. Let $\gamma(s)$ be a null curve in $H^{n+1}_1 \subset R^{n+2}_2$, and $\{\gamma(s), Y(s), U_3(s), \ldots, U_m(s), V_{m+1}(s), \ldots, V_n(s), \xi(s)\}$ a pseudo-orthonormal basis of $T_{\gamma(s)}H^{n+1}_1$ such that

$$\dot{V}_\alpha(s) \in \text{span}\{Y(s), V_{m+1}(s), \ldots, V_n(s)\},$$

$$\dot{\xi}(s) = \lambda \dot{\gamma}(s) + B(s)Y(s)$$

for some nonzero $B(s)$. If $M$ is one of the following parametrized hypersurfaces in $H^{n+1}_1 \subset R^{n+2}_2$:

1. $\lambda^2 \neq 0$ or 1,

$$f(s, y, a_3, \ldots, a_m, b_{m+1}, \ldots, b_n)$$

$$= \epsilon_1(\lambda) \sqrt{\frac{1}{(\lambda^2 - 1)^2} - \sum_{i=3}^{m} \frac{a_i^2}{\lambda^2 - 1} (\gamma(s) - \lambda \xi(s))}$$

$$+ \epsilon_2(\lambda) \sqrt{\frac{\lambda^2}{(1 - \lambda^2)^2} - \sum_{\alpha=m+1}^{n} \frac{b_{\alpha}^2}{1 - \lambda^2} (\xi(s) - \lambda \gamma(s))}$$

$$+ yY(s) + \sum_{i=3}^{m} a_i U_i(s) + \sum_{\alpha=m+1}^{n} b_{\alpha} V_\alpha(s),$$

where

$$\epsilon_1(\lambda) = \begin{cases} -1, & \text{if } \lambda^2 > 1 \\ 1, & \text{if } \lambda^2 < 1, \end{cases}$$

$$\epsilon_2(\lambda) = \begin{cases} -1, & \text{if } \lambda(\lambda^2 - 1) > 0 \\ 1, & \text{if } \lambda(\lambda^2 - 1) < 0. \end{cases}$$

2. $\lambda = 0,$
\[ f(s, y, a_3, \ldots, a_n) = \sqrt{1 + \sum_{i=3}^{n} a_i^2 \gamma(s) + y Y(s) + \sum_{i=3}^{n} a_i U_i(s)}. \]

(3) \( \lambda^2 = 1, \)

\[ f(s, y, a_3, \ldots, a_n) = \left(1 + \frac{1}{2} \sum_{i=3}^{n} a_i^2 \right) \gamma(s) - \lambda \left(\frac{1}{2} \sum_{i=3}^{n} a_i^2 \right) \xi(s) + y Y(s) + \sum_{i=3}^{n} a_i U_i(s), \]

then M is type II.

Conversely, we have:

**Theorem 4.2.** Let M be a type II hypersurface in \( H^{n+1}_1 \). Then for any \( p \in M \), there is a neighborhood \( U_p \) of \( p \) in M such that \( U_p \) is exactly one of the parametrized hypersurfaces in Theorem 4.1.

Before proceeding to give the proof, we separate off the following lemma.

**Lemma 4.3.** Let M be a type II hypersurface, \( e_1, e_2, \ldots, e_{n+1} \) a local pseudo-orthonormal frame such that \( e_{n+1} \) is normal to M, \( \omega_{n+1}^1 = \lambda \omega^1, \omega_{n+2}^2 = \lambda \omega^2 + \omega^1, \omega_{n+1}^i = \lambda \omega^i, \omega_{n+1}^\alpha = \frac{1}{\lambda} \omega^\alpha \), and \( T_\lambda, T_{\frac{1}{\lambda}} \) the distributions defined as follows:

\[ T_\lambda = \ker(A - \lambda) = \text{span}\{e_2, e_3, \ldots, e_m\}, \]

and

\[ T_{\frac{1}{\lambda}} = \ker(A - \frac{1}{\lambda}) = \text{span}\{e_{m+1}, \ldots, e_n\}. \]

Then all distributions \( T_\lambda, T_{\frac{1}{\lambda}} \) and \( T_\lambda + T_{\frac{1}{\lambda}} \) are integrable, and \( \omega_{\alpha}^1 = \omega_{\alpha}^2 = \omega_{\alpha}^i = 0, \omega_{\alpha}^2 \wedge \omega^1 = 0 \) and \( \omega_{\alpha}^2 \wedge \omega^1 = 0. \)

**Proof.** Let \( \tilde{\omega}_{n+1}^1 = \omega_{n+1}^1 - \lambda \omega^1, \tilde{\omega}_{n+1}^2 = \omega_{n+1}^2 - \lambda \omega^2, \tilde{\omega}_{n+1}^i = \omega_{n+1}^i - \lambda \omega^i, \) and \( \tilde{\omega}_{n+1}^\alpha = \omega_{n+1}^\alpha - \lambda \omega^\alpha \). Then

\[ \tilde{\omega}_{n+1}^1 = 0, \quad \tilde{\omega}_{n+1}^2 = \omega^1, \quad \tilde{\omega}_{n+1}^i = 0, \quad \tilde{\omega}_{n+1}^\alpha = \left(\frac{1}{\lambda} - \lambda\right) \omega^\alpha. \]

By (3), we have

\[ d\tilde{\omega}_{n+1}^A = \sum_{B=1}^{n} \tilde{\omega}_{n+1}^B \wedge \omega_B^A, \]
where \( A = 1, 2, \ldots, n \). (33) and (34) are exactly the Codazzi equation, which Megid discussed in [4]. Following his results, we have Lemma 4.3. □

We are now in a position to give a:

**Proof of Theorem 4.2.** Let \( M \) be a type II hypersurface, \( x_0 \) a point of \( M \). By Lemma 2.3, there is a local pseudo-orthonormal frame \( e_1, e_2, \ldots, e_n, e_{n+1} \) defined in a neighborhood of \( x_0 \) such that \( e_{n+1} \) is normal to \( M \), and \( \omega^1_{n+1} = \lambda \omega^1, \omega^2_{n+1} = \lambda \omega^2 + \omega^1, \omega^i_{n+1} = \lambda \omega^i, \omega^n_{n+1} = \frac{1}{\lambda} \omega^0 \). Let \( \gamma(s) \) be the integral curve of \( e_1 \) through \( x_0 \), and \( N(s) \) the integral manifold of \( T_\lambda + T_\frac{1}{\lambda} \) through \( \gamma(s) \). Fixing \( s \) and restricting the forms to \( N(s) \), we have \( \omega^1 = 0 \).

From Lemma 4.3 and (3), we have

\[
de_2 = \omega^2 e_2.
\]

Denote \( Y(s) = e_2(\gamma(s)), U_i(s) = e_i(\gamma(s)) \) and \( V_\alpha(s) = e_\alpha(\gamma(s)) \). Then \( \forall x \in N(s), e_2(x) = \lambda(x) Y(s) \) for some function \( \lambda \). So the integral curve of \( e_2 \) is a straight line in \( R^{n+2}_2 \).

Define \( W_1(s) = \text{span} \{ Y(s) \} \). Then \( W_1(s) = \text{span} \{ = e_2(x) \mid \forall x \in N(s) \} \).

From (1) and (2), we have

\[
dX = \omega^2 e_2 + \sum_{i=3}^{m} \omega^i e_i + \sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha,
\]

\[
de_{n+1} = \lambda \omega^2 e_2 + \lambda \sum_{i=3}^{m} \omega^i e_i + \frac{1}{\lambda} \sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha.
\]

Computing (37)\( - \lambda (36) \), we get

\[
d(e_{n+1} - \lambda X) = \left( \frac{1}{\lambda} - \lambda \right) \sum_{\alpha=m+1}^{n} \omega^\alpha e_\alpha.
\]

From (3) and Lemma 4.3, we have

\[
de_\alpha = \sum_{\beta=m+1, \beta \neq \alpha}^{n} \omega^\beta e_\beta - \frac{1}{\lambda} \omega^\alpha e_{n+1} + \omega^0 x.
\]

It follows from (38) and (39) that

\[
d(e_{m+1} \wedge \cdots \wedge e_n \wedge (e_n - \lambda X)) = 0.
\]

Denote \( W_2(s) = \text{span} \{ e_{m+1}(\gamma(s)), \ldots, e_n(\gamma(s)), e_{n+1}(\gamma(s)) - \lambda \gamma(s) \} \). Then \( W_2(s) = \{ e_{m+1}(x), \ldots, e_n(x), e_{n+1}(x) - \lambda x \mid \forall x \in N(s) \} \). Define \( W_3(s) = \{ e_2(x), e_3(x), \ldots, e_m(x), x - \lambda e_{n+1}(x) \} \). Note that \( \langle Z(x), e_2(x) \rangle = 0 \) for any \( Z(x) \) in \( T_\lambda + T_\frac{1}{\lambda} \). Hence \( \forall x \in N(s) \), we have

\[
\langle x, Y(s) \rangle = 0.
\]
If \( m = n \), then it follows from (38) that
\[
e_{n+1}(x) - \lambda x = e_{n+1}(\gamma(s)) - \lambda \gamma(s).
\]
So \( \forall x \in N(s) \), we get
\[
\langle x, e_{n+1}(\gamma(s)) - \lambda \gamma(s) \rangle = \lambda.
\]
This includes the cases \( \lambda = 0 \) or \( \pm 1 \).

Now suppose \( \lambda \neq 0, \pm 1 \). By (38) and (39) and \( e_{n+1}(x) - \lambda x \in W_2(s) \), \( x - \lambda e_{n+1}(x) \in W_3(s) \), we can write
\[
(42) \quad x = X_1 + X_2,
\]
\[
e_{n+1} = \frac{1}{\lambda} X_1 + \lambda X_2,
\]
where \( X_1 \in W_2(s) \) and \( X_2 \in W_3(s) \). Since \( \langle X_1, X_2 \rangle = 0 \), we get
\[
(43) \quad \langle X_1, X_1 \rangle = \frac{\lambda^2}{1 - \lambda^2}, \quad \langle X_2, X_2 \rangle = \frac{1}{\lambda^2 - 1}.
\]
Hence \( M \) can be locally represented as a parametrized hypersurface in Theorem 4.1. \( \Box \)

4.2. Type III hypersurfaces.

In this subsection, we arrange the index as follows: \( 4 \leq i, j \leq m, m + 1 \leq \alpha, \beta \leq n \). By a direct calculation, we have:

**Theorem 4.4.** Let \( \gamma(s) \) be a null curve in \( H^{n+1}_1 \subset R^{n+2}_2 \), and \( \{ \dot{\gamma}(s), Y(s), U_3(s), \ldots, U_m(s), V_{m+1}(s), \ldots, V_n(s), \xi(s) \} \) a pseudo-orthonormal basis of \( T_{\gamma(s)}H^{n+1}_1 \) such that
\[
\dot{V}_\alpha(s) \in \text{span}\{Y(s), V_{m+1}(s), \ldots, V_n(s)\},
\]
\[
\dot{\xi}(s) = \lambda \dot{\gamma}(s) + B(s)U_3(s)
\]
for some nonzero \( B(s) \). If \( M \) is one of the following parametrized hypersurfaces in \( H^{n+1}_1 \subset R^{n+2}_2 \):
\[
(1) \lambda^2 \neq 0 \text{ or } 1,
\]
\[
f(s, y, a_3, \ldots, a_m, b_{m+1}, \ldots, b_n)
= \epsilon_1(\lambda) \sqrt{\frac{1}{(\lambda^2 - 1)^2} - \sum_{i=3}^{m} \frac{a_i^2}{\lambda^2 - 1}(\gamma(s) - \lambda \xi(s))}
+ \epsilon_2(\lambda) \sqrt{\frac{\lambda^2}{(1 - \lambda^2)^2} - \sum_{\alpha=m+1}^{n} \frac{b_\alpha^2}{1 - \lambda^2}(\xi(s) - \lambda \gamma(s))}
+ yY(s) + \sum_{i=3}^{m} a_i U_i(s) + \sum_{\alpha=m+1}^{n} b_\alpha V_\alpha(s),
\]
where

\[ \varepsilon_1(\lambda) = \begin{cases} 
-1, & \text{if } \lambda^2 > 1 \\
1, & \text{if } \lambda^2 < 1,
\end{cases} \]

\[ \varepsilon_2(\lambda) = \begin{cases} 
-1, & \text{if } \lambda(\lambda^2 - 1) > 0 \\
1, & \text{if } \lambda(\lambda^2 - 1) < 0.
\end{cases} \]

(2) \( \lambda = 0 \),

\[ f(s, y, a_3, \ldots, a_n) = \sqrt{1 + \sum_{i=3}^{n} a_i^2 \gamma(s) + yY(s) + \sum_{i=3}^{n} a_i U_i(s)}. \]

(3) \( \lambda^2 = 1 \),

\[ f(s, y, a_3, \ldots, a_n) = \left(1 + \frac{1}{2} \sum_{i=3}^{n} a_i^2 \right) \gamma(s) - \lambda \left(\frac{1}{2} \sum_{i=3}^{n} a_i^2 \right) \xi(s) + yY(s) + \sum_{i=3}^{n} a_i U_i(s), \]

then \( M \) is type III.

Conversely, we have:

**Theorem 4.5.** Let \( M \) be a type III hypersurface in \( H^{n+1}_1 \). Then for any \( p \in M \), there is a neighborhood \( U_p \) of \( p \) in \( M \) such that \( U_p \) is exactly one of the parametrized hypersurfaces in Theorem 4.4.

Before proving Theorem 4.5, we need the following Lemma.

**Lemma 4.6.** Let \( M \) be a type III hypersurface, and \( e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1} \) a local pseudo-orthonormal frame such that \( e_{n+1} \) is normal to \( M \), \( \omega_{n+1}^1 = \lambda \omega^1, \omega_{n+2}^2 = \lambda \omega^2 + \omega^3, \omega_{n+1}^3 = \lambda \omega^3 - \omega^1, \omega_{n+1}^i = \lambda \omega^i, \omega_{n+1}^\alpha = \frac{1}{\lambda} \omega^\alpha \), and \( T_\lambda, T_\lambda^2, T_\lambda^1 \) the distributions defined as follows:

\[ T_\lambda = \ker(A - \lambda) = \text{span}\{e_2, e_4, \ldots, e_m\}, \]
\[ T_\lambda^2 = \ker(A - \lambda)^2 = \text{span}\{e_2, e_3, \ldots, e_m\}, \]
\[ T_\lambda^1 = \ker\left(A - \frac{1}{\lambda}\right) = \text{span}\{e_{m+1}, \ldots, e_n\}. \]

Then the distribution \( T_\lambda^2 + T_\lambda^1 \) is integrable, and \( \omega_\alpha^1 = \omega_2^\alpha = \omega_3^\alpha = 0, \omega_2^i \wedge \omega^1 = 0 \) and \( \omega_\alpha^2 \wedge \omega^1 = 0 \).
Proof. Let \( \tilde{\omega}^1_{n+1} = \omega^1_{n+1} - \lambda \omega^1 \), \( \tilde{\omega}^2_{n+1} = \omega^2_{n+1} - \lambda \omega^2 \), \( \tilde{\omega}^3_{n+1} = \omega^3_{n+1} - \lambda \omega^3 \), \( \tilde{\omega}^i_{n+1} = \omega^i_{n+1} - \lambda \omega^i \), and \( \tilde{\omega}^\alpha_{n+1} = \omega^\alpha_{n+1} - \lambda \omega^\alpha \). Then
\[
\tilde{\omega}^1_{n+1} = 0, \quad \tilde{\omega}^2_{n+1} = \omega^3, \quad \tilde{\omega}^3_{n+1} = -\omega^1, \quad \tilde{\omega}^i_{n+1} = 0, \quad \tilde{\omega}^\alpha_{n+1} = \left( \frac{1}{\lambda} - \lambda \right) \omega^\alpha.
\]
By (3), we have
\[
d\tilde{\omega}^A_{n+1} = \sum_{B=1}^n \tilde{\omega}^B_{n+1} \wedge \omega^A,
\]
where \( A = 1, 2, \ldots , n \). The above equations are also the Codazzi equation, which Megid discussed in [4]. Hence the Lemma holds. \( \square \)

We can proceed to the:

Proof of Theorem 4.5. Since the proof is similar to that of Theorem 4.2, we give a sketch here.

Let \( M \) be a type III hypersurface, \( x_0 \) a point of \( M \). By Lemma 2.3, there is a local pseudo-orthonormal frame \( e_1, e_2, \ldots , e_n, e_{n+1} \) defined in a neighborhood of \( x_0 \) such that \( e_{n+1} \) is normal to \( M \), and \( \omega^1_{n+1} = \lambda \omega^1, \omega^2_{n+1} = \lambda \omega^2 + \omega^3, \omega^3_{n+1} = \lambda \omega^3 - \omega^1, \omega^i_{n+1} = \lambda \omega^i, \omega^\alpha_{n+1} = \frac{1}{\lambda} \omega^\alpha \). Let \( \gamma(s) \) be the integral curve of \( e_1 \) and \( N(s) \) be the integral manifold of \( T^2 + T^1 \) through \( \gamma(s) \). By the similar discussion for type II, we have
\[
(44) \quad d\varepsilon_2 \wedge e_2 = 0,
\]
\[
d(\varepsilon_{m+1} \wedge \cdots \wedge e_n \wedge (e_{n+1} - \lambda X)) \wedge e_2 = 0.
\]
Let \( W_1(s) \) be the linear span of \( \{ e_2(\gamma(s)) \} \), \( W_2(s) \) the span of \( \{ e_2(\gamma(s)), e_{m+1}(\gamma(s)), \ldots , e_n(\gamma(s)), e_{n+1}(\gamma(s)) - \lambda \gamma(s) \} \), and \( W_3(s) \) the span of \( \{ e_2(\gamma(s)), e_3(\gamma(s)), \ldots , e_m(\gamma(s)), \gamma(s) - \lambda e_{n+1}(\gamma(s)) \} \).

For any \( x \) in \( N(s) \),
\[
\langle x, Y(s) \rangle = 0.
\]
Here \( Y(s) = e_2(\gamma(s)) \). If \( \lambda \neq 0, \pm 1 \), by (44) and \( e_{n+1} - \lambda x \in W_2(s), x - \lambda e_{n+1} \in W_3(s) \), we can write
\[
x = X_1 + X_2,
\]
\[
e_{n+1} = \frac{1}{\lambda} X_1 + \lambda X_2,
\]
where \( X_1 \in W_2(s), X_2 \in W_3(s) \). Note that \( \langle X_1, X_2 \rangle = \langle X_1, Y(s) \rangle = \langle X_2, Y(s) \rangle = \langle Y(s), Y(s) \rangle = 0 \). So we have
\[
\langle X_1, X_1 \rangle = \frac{\lambda^2}{1 - \lambda^2}, \quad \langle X_2, X_2 \rangle = \frac{1}{\lambda^2 - 1}.
\]
If \( m = n \), then
\[
\langle x, e_{n+1}(\gamma(s)) - \lambda \gamma(s) \rangle = \lambda.
\]
This completes the proof of Theorem 4.5. \( \square \)
5. Type IV hypersurfaces.

The number $p$ of distinct principal curvatures of type IV hypersurfaces is 2, 3 or 4. In this section we classify the type IV hypersurfaces. The classification is based on the following theorems.

Let $M$ be a type IV hypersurface, $x$ a point of $M$. By Lemma 2.4, there is a local orthonormal frame $e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n$, $e_{n+1}$ defined in a neighborhood of $x$ such that $e_{n+1}$ is normal to $M$, $\omega^1_{n+1} = a_0 \omega^1 + b_0 \omega^2$, $\omega^2_{n+1} = -b_0 \omega^1 + a_0 \omega^2$ ($b_0 \neq 0$), $\omega^i_{n+1} = \lambda \omega^i$ for $3 \leq i \leq m$, and $\omega^\alpha_{n+1} = \frac{1}{\lambda} \omega^\alpha$ for $m + 1 \leq \alpha \leq n$.

Theorem 5.1. Let $M$ be a type IV hypersurface with $p = 2$. Then $M$ is congruent to an open part of a principal orbit of $G \subset O(2, 2)$ in $H_3^1$, where

$$G = \left\{ \begin{pmatrix} \cos s & \sin s & 0 & 0 \\ -\sin s & \cos s & 0 & 0 \\ 0 & 0 & \cos s & \sin s \\ 0 & 0 & -\sin s & \cos s \end{pmatrix}, \begin{pmatrix} \cosh t & 0 & \sinh t & 0 \\ 0 & \cosh t & 0 & \sinh t \\ \sinh t & 0 & \cosh t & 0 \\ 0 & \sinh t & 0 & \cosh t \end{pmatrix} \right\} \bigg| s, t \in \mathbb{R} \right\}.$$

Proof. In this case, $n = 2$ and

$$\omega^1_3 = a_0 \omega^1 + b_0 \omega^2, \quad \omega^2_3 = -b_0 \omega^1 + a_0 \omega^2.$$  

Note that $e_1, \ldots, e_n$ is an orthonormal frame. It follows from (3) that

$$\omega^2_1 = \omega^1_2.$$  

Exterior differenting (45), we get

$$\omega^1 \wedge \omega^1 = 0,$$

$$\omega^2 \wedge \omega^2 = 0.$$  

From (46) and (47), we arrive at

$$\omega^1_2 = 0.$$  

Substituting (45) and (48) to Gauss equation (11), we have

$$a_0^2 + b_0^2 = 1.$$  

From the theory of moving frame, (45) and (48) imply that $M$ is locally homogeneous. In fact $M$ is congruent to an open part of a principal orbit of $G$ defined in Theorem 5.1. \qed
Theorem 5.2. Let $M$ be a type $IV$ hypersurface with $p = 3$. Then $M$ is congruent to an open part of an orbit of $G \subset O(2, 3)$ in $H^4_1$, where the Lie algebra of $G$ is generated by

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \sqrt{3} \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\sqrt{3} & 0 & 0
\end{pmatrix} ,
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & \sqrt{3} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & \sqrt{3} & 0 & 0 & 0
\end{pmatrix} ,
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Proof. In this case, $\omega_{n+1}^1 = a_0\omega^1 + b_0\omega^2$, $\omega_{n+1}^2 = -b_0\omega^1 + a_0\omega^2$ ($b_0 \neq 0$), $\omega_{n+1}^i = \lambda\omega^i$ for $3 \leq i \leq n$. From the Codazzi equation (12), we know that

$$
\omega^2 = \omega^3 = c\omega^3
$$

for some function $c$ and

$$
\omega^3 = \frac{2cb_0}{(a_0 - \lambda)^2 + b_0^2}[(a_0 - \lambda)\omega^1 + b_0\omega^2],
$$

$$
\omega^2 = \frac{2cb_0}{(a_0 - \lambda)^2 + b_0^2}[(a_0 - \lambda)\omega^2 - b_0\omega^1],
$$

$$
\omega_i^1 = \omega_i^2 = 0
$$

for $i > 3$. Substituting (50), (51) and (52) to Gauss equation (11), we have

$$
\frac{4c^2b_0^2}{(a_0 - \lambda)^2 + b_0^2} = 1 - a_0\lambda,
$$

$$
\frac{4c^2(a_0 - \lambda)}{(a_0 - \lambda)^2 + b_0^2} = \lambda,
$$

$$
\frac{8c^2b_0^2}{(a_0 - \lambda)^2 + b_0^2} = a_0^2 + b_0^2 - 1,
$$

and

$$
-\omega_1^3 \wedge \omega_i^i = (a_0\lambda - 1)\omega^1 \wedge \omega^i + b_0\lambda\omega_2 \wedge \omega_i,
$$

$$
-\omega_3^3 \wedge \omega_i^3 = (1 - a_0\lambda)\omega^2 \wedge \omega^i + b_0\lambda\omega_1 \wedge \omega_i,
$$

where $i > 3$. But (50), (51) and (54) have no solution for any $i > 3$. This implies $n = 3$. From (49), (50), (51) and (53), we know that $M$ is locally homogeneous. In fact $M$ is congruent to an open part of a principal orbit of $G$ defined in Theorem 5.2. $\square$

Note. $G \cong SL(3, R)$. 


Theorem 5.3. Let $M$ be a type IV hypersurface with $p = 4$. Then $M$ is congruent to an open part of the hypersurface
\[
\{(x^1, \ldots, x^{2n+2}) \in H^{n+1}_1 : |-(x^1 + ix^2)^2 + (x^3 + ix^4)^2 + \cdots + (x^{2n+1} + ix^{2n+2})^2| = t\}
\]
where $t > 1$.

To prove the theorem, we need the following simple Lemma.

Lemma 5.4. Let $A_1, \ldots, A_p$ be $m \times m$ matrices in $o(1, n)$. If rank $\left( \sum_{j=1}^{p} a_j A_j \right) = 2$ and its eigenvalues are $\pm i \sqrt{\sum_{j=1}^{p} a_j^2}$, 0 for any $a_1, \ldots, a_p \in \mathbb{R}$, and $\sum_{j=1}^{p} a_j^2 \neq 0$. Then $m \geq p + 1$ and there is a invertible matrix $P$ such that $PA_jP^{-1} = e_{ij+1} - e_{j+1}i$ for all $1 \leq j \leq p$, where $e_{ij} \in \text{gl}(n+1)$ whose $ij$-th entry is 1 and all other entries are 0.

Proof. Since we only need Linear algebra, we give an outline of the proof. Since rank $A_1 = 2$ and $A_1$ has eigenvalues $i, -i$ and 0, we can choose $P_1$ such that
\[
P_1A_1P_1^{-1} = \begin{pmatrix} 0 & x_{21} & \cdots & x_{m1} \\ -x_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_{m1} & 0 & \cdots & 0 \end{pmatrix}.
\]
Since rank $\left( \sum_{j=1}^{p} a_j A_j \right) = 2$ for any $a_1, \ldots, a_p$ satisfying $\sum_{j=1}^{p} a_j^2 \neq 0$, the $P_1A_1P_1^{-1}$ take the form
\[
P_1A_iP_1^{-1} = \begin{pmatrix} 0 & x_{2i} & \cdots & x_{mi} \\ -x_{2i} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -x_{mi} & 0 & \cdots & 0 \end{pmatrix}.
\]
From the fact that the eigenvalues of $\sum_{j=1}^{p} a_j A_j$ are $\pm i \sqrt{\sum_{j=1}^{p} a_j^2}$ and 0, it follows that
\[
\sum_{k=2}^{m} x_{ki}x_{kj} = \delta_{ij}.
\]
Hence $m - 1 \geq p$ and the Lemma holds. \qed

Proof of Theorem 5.3. We consider the focal manifold $N$ of $M$. Choose a local orthonormal frame $e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1}$ such that $e_1, e_2, \ldots, e_m$ are tangent to $N$ and $e_{m+1}, \ldots, e_n, e_{n+1}$ are normal to $N$. Note that $M$ is type IV. From Lemma 2.4, we know that for any (local) unit normal field $\nu$ of $N$, its principal curvatures are $ib, -ib, 0$ for some $b$, and the multiplicity of principal curvature 0 is $m - 2$. Here $i^2 = -1$. From Lemma 5.4, we have
\( m \geq n + 2 - m \), i.e., \( 2m \geq n + 2 \). Hence we can choose an orthonormal frame \( e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_n, e_{n+1} \) such that
\[
\omega^1_{m+l} = b \omega^{l+1}, \quad \omega^{l+1}_{m+l} = -b \omega^1, \quad \omega^i_{m+l} = 0
\]
for any \( l \), and \( 1 \leq l \leq n - m + 1 \). Here \( 1 \leq i \leq m \), and \( i \neq 1, l + 1 \). Note that \( n + 1 - m \geq 2 \). Using Coddazzi equation, we get
\[
\omega^j_1 = 0, \quad (j = 1, \ldots, m),
\]
and
\[
\omega^{m+k}_{m+l} = \omega^{k+1}_{l+1}, \quad (1 \leq k, l \leq n - m + 1).
\]
Substituting (56) and (57) to Gauss equation (11), we have \( b^2 = 1 \) and
\[
\omega^1 \wedge \omega^j = 0 \quad (n - m + 3 \leq j \leq m).
\]
This implies \( m \leq n - m + 2 \), i.e., \( 2m \leq n + 2 \). Therefore \( 2m = n + 2 \).

Now we prove the existence and uniqueness of \( N \).

Let \( N^m \) be the focal manifold of a type IV hypersurface \( M^n \subset H^{n+1}_1 \subset R^{n+2}_2 \). Then \( n + 2 = 2m \) and for any \( x \in N \), there is a local orthonormal frame \( e_1, e_2, \ldots, e_{n+1} \) such that \( e_1, e_2, \ldots, e_m \) are tangent to \( M \), \( e_{m+1}, \ldots, e_{n+1} \) are normal to \( M \),
\[
\omega^i_1 = 0, \quad (i = 1, \ldots, m),
\]
and
\[
\omega^{m+j}_{m+i} = \omega^{j+1}_{i+1}, \quad (i, j = 1, \ldots, m - 1),
\]
\[
\omega^j_{m+i} = \delta^j_i \omega^{i+1} - \delta^i_j \omega^1 \quad (i = 1, \ldots, m - 1, j = 1, \ldots, m).
\]
Here \( \delta^j_i = 1 \) if \( i = j \), and \( \delta^j_i = 0 \) if \( i \neq j \).

Let \( \gamma(s) \) be an integral curve of \( e_1 \). Then
\[
de_1 = \sum_{i=1}^{m} \omega^i e_i + \sum_{j=1}^{m-1} \omega^1_{m+j} - \omega^1 X,
\]
where \( X \) is the position vector field of \( N \). From (58) and (60),
\[
de_1 = \sum_{i=1}^{m-1} \omega^{i+1}_{m+i} e_{m+i} - \omega^1 X.
\]
From (62), we know that \( e_1 \) is parallel along \( \gamma \) in \( H^{n+1}_1 \). This means that \( \gamma \) is a geodesics of \( H^{n+1}_1 \). Hence
\[
\gamma(s) = \cos s \gamma(0) + \sin s e_1(\gamma(0)),
\]
and \( \omega_1 = ds \). Choose the normal vector fields \( e_{m+1}, \ldots, e_n, e_{n+1} \) such that each of them is parallel along \( \gamma(s) \), i.e., \( \omega^{m+j}_{m+i} = 0 \), \( (i, j = 1, \ldots, m - 1) \) on
\[ \gamma. \text{ So along } \gamma, \text{ we have} \]
\[ \begin{align*}
\delta e_{i+1} &= \omega^1 e_{m+i}, \quad i = 1, \ldots, m-1, \\
\delta e_{m+i} &= \omega^1 e_{i+1}, \quad i = 1, \ldots, m-1,
\end{align*} \]

which implies that
\[ \begin{align*}
\delta e_{i+1}(\gamma(s)) &= \cos se_{i+1}(\gamma(0)) + \sin se_{m+i}(\gamma(0)) \\
\end{align*} \]

for \( i \) in \( \{1, \ldots, m-1\} \). Now consider the distribution \( E = \text{span} \{ e_2, \ldots, e_m \} \). From the structural equation (3), we have
\[ \delta \omega^1 = 0, \quad \delta \omega_{m+i} = \omega^{i+1} \wedge \omega^1 \]

for \( i \) in \( \{1, \ldots, m-1\} \), which implies that \( E \) is an integrable distribution. Denote the integral manifold through \( \gamma(s) \) by \( P(s) \). Then \( e_1, e_{m+1}, \ldots, e_{n+1} \) are normal vector fields of \( P(s) \). On \( P(s) \),
\[ \begin{align*}
\delta e_1 &= \sum_{i=1}^{m-1} \omega^{i+1} e_{m+i}, \quad \delta e_{m+i} = \omega^{i+1} e_1, \quad i = 1, \ldots, m-1.
\end{align*} \]

Hence \( P(s) \) is a totally geodesic submanifold of \( H^{n+1}_1 \) for every \( s \).

Summarizing the arguments above, \( N \) is determined uniquely by \( e_1(\gamma(0)), e_2(\gamma(0)), \ldots, e_m(\gamma(0)), e_{m+1}(\gamma(0)), \ldots, e_{n+1}(\gamma(0)) \). In fact \( N \) is congruent to an open part of the submanifold:
\[ \{(x^1, \ldots, x^{2n+2}) \in H_1^{n+1} : -(x^1 + ix^2)^2 + (x^3 + ix^4)^2 + \cdots + (x^{2n+1} + ix^{2n+2})^2 \mid = 1 \}. \]

The second fundamental form of \( N \) is \( II(e_1, e_1) = 0, II(e_1, e_i) = 0, II(e_i, e_i) = e_{n+i} \) for \( 2 \leq i, j \leq n \). Since \( M \) is a tube of \( N \), we obtain Theorem 5.3. \( \square \)

Hence we have finished the classification of Lorentzian isoparametric hypersurfaces in \( H_1^{n+1} \) by combining Theorems in Sections 3, 4 and 5.

References

LORENTZIAN ISOPARAMETRIC HYPERSURFACES IN $H^n_{1}$


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