Pacific Journal of Mathematics

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Volume 190 No. 1 September 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 190, No. 1, 1999

A SIMPLE PROOF OF AN INEQUALITY DOMINATING THE AREA INTEGRAL BY THE NONTANGENTIAL MAXIMAL FUNCTION

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In this paper, we give a simple proof for a good-λ inequality which means that nontangential maximal functions controls area integrals.

Let u be a harmonic function on $\mathbf{R}_+^{n+1}$. The nontangential maximal function and the area integral function of f are defined by

$$N_\beta(u)(x) = \sup_{(y,t)\in\Gamma_\beta(x)} |u(y,t)| \qquad (\beta \in \mathbf{R}_+^1),$$

$$A_\alpha(u)(x) = \left(\int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2\, t^{1-n} dydt\right)^{\frac{1}{2}} \qquad (\alpha \in \mathbf{R}_+^1).$$

The main aim of this paper is to give a simple proof of the inequality

$$(1) \qquad \|A_\alpha(u)\|_p \le C_{n,p,\alpha,\beta}\, \|N_\beta(u)\|_p \qquad (0 < p < \infty,\ 0 < \alpha,\ \beta < \infty).$$

As we know, this inequality is very important in H^p-theory, it is also a main difficulty in generalizing H^p-theory of one parameter to H^p -theory of several parameters, see [**2**, **6**, **7**, **8**]. The first proof of (1) is probabilistic which was given by Burkholder, Gundy and Silverstein, see [**1**]; Fefferman and Stein first got an analytic proof of (1) by dealing with a kind of Green's formula on $\mathcal{R} = \cup_{x\in E}\Gamma_\alpha(x)$, see [**4**]; a sharpened inequality was obtained in [**5**] by a different approach. In the two-parameter case, Gundy and Stein set up a similar inequality to (1) by dealing with some multi-sub-linear operators like

$$B(u,v)(x) = \left(\int_{\Gamma(x)} |\nabla_1 u|^2\, |\nabla_2 v|^2\, t_1^{1-n_1} t_2^{1-n_2}\, dx_1 dt_1\, dx_2 dt_2\right)^{\frac{1}{2}},$$

see [**9**]; Merryfield ([**8**]) and author ([**2**]) generalized Gundy-Stein's work to multi-parameter case independently and differently. In our proof ([**2**]), we introduced a kind of Carleson measure technique which does not depend on the dilation and translation structures of $\mathbf{R}^n$ such that the method works on more general case (see Chen and Wang [**3**]). Here, we shall use the idea to give a simple proof of (1).

At first, we notice that for $1 < p < \infty$, the proof of (1) is elementary, and for $0 < p \leq 1$, (1) can be followed from

$$|\{x : A_\alpha(u)(x) > \lambda\}| \leq C_{n,\alpha,\beta}\Bigg(|\{x : N_\beta(u)(x) > \lambda\}| + \lambda^{-2}\int_{N_\beta(u)(x)\leq\lambda} N_\beta(u)^2(x)dx\Bigg) \tag{2}$$

where $0 < \alpha, \beta, \lambda < \infty$ (note that, for $0 < \alpha < \beta < \infty$, (2) was set up in [**4**]). Now, we shall prove (2).

By a limitation procedure, we may assume $u(x,t) = \tilde{u}(x, t+\epsilon)$, where $N_\beta(\tilde{u}) \in L^p$. $\forall\lambda > 0$, set $E_\lambda = \{x : N_\beta(u)(x) \leq \lambda\}$, $\delta_0 = \delta(n,\beta) = \int_{|x|<\beta} p_1(x)dx \in (0,1)$, where p_t is the Poisson kernel. Take a closed subset F_λ of E_λ such that $|F_\lambda^c| \leq C_{n,\alpha,\beta}\,|E_\lambda^c|$, $p_t * \chi_{E_\lambda} \geq 1 - \frac{1}{2}\delta_0$ (on $\cup_{x\in F_\lambda}\Gamma_\alpha(x)$), which is possible by the definition of p_t and the weak type (1,1)-boundedness of nontangential maximal function operator; then, take $\varphi \in C^2(\mathbf{R}^1) \cap L^\infty(\mathbf{R}^1)$, such that $\varphi|_{(-\infty,1-\delta_o)} = 0$, $\varphi|_{(1-\frac{1}{2}\delta_o,+\infty)} = 1$, $|\varphi'| + |\varphi''| \leq c\varphi^{3/4}$ (by using $e^{-t^{-2}}$). Now, set $v = p_t * \chi_{E_\lambda}$, then

(3) the left side of (2)

$$\begin{aligned}
&\leq |F_\lambda^c| + |F_\lambda \cap \{x : A_\alpha(u)(x) > \lambda\}| \\
&\leq C_{n,\alpha,\beta}\left\{|E_\lambda^c| + \lambda^{-2}\int_{F_\lambda}\int_{\Gamma_\alpha(x)} \varphi(v)\,|\nabla u(w,t)|^2\, t^{1-n}dwdtdx\right\} \\
&\leq C_{n,\alpha,\beta}\left\{|E_\lambda^c| + \lambda^{-2}\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\, tdwdt\right\}.
\end{aligned}$$

Note that

$$\varphi(v)\,|\nabla u|^2 = -u\varphi'(v)\nabla v\cdot\nabla u - \frac{1}{2}u^2\Delta(\varphi(v)) + \frac{1}{2}\Delta(\varphi(v)u^2);$$

and, $\|\varphi(v)u\|_\infty \leq C_\varphi\lambda$ for $v \leq 1-\delta_0$ on $(\cup_{x\in E_\lambda}\Gamma_\beta(x))^c$; in addition, it is not difficult to show that for a fixed $\psi \in C_c^\infty(\mathbf{R}^n)$ satisfying $\psi(|x| \leq 1) = 1$, $\psi(|x| \geq 2) = 0$, we have (where $\psi_r(w) := \psi(w/r)$)

$$\begin{aligned}
\int\int_{\mathbf{R}_+^{n+1}} \Delta(\varphi(v)u^2)tdwdt &= \lim_{r\to\infty}\int\int_{\mathbf{R}^n\times(0,r)} \psi_r(w)\Delta(\varphi(v)u^2)tdwdt \\
&= \int_{\mathbf{R}^n} \varphi(v(x,0))u^2(x,0)dx
\end{aligned}$$

by Green's formula, because $N_\beta(\tilde{u}) \in L^p$, and

$$\left\|t^{k+n/p}\nabla^k u\right\|_\infty + \left\|t^k\nabla^k v\right\|_\infty \leq C_{\epsilon,n,p,k}(\widetilde{u}) < \infty$$

for $k=0,1,2,\cdots$. Therefore, by Hölder's inequality, we get

$$\begin{aligned}
&\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\, tdwdt \\
&\le C_\varphi \lambda \left(\int\int_{\mathbf{R}_+^{n+1}} |\nabla v|^2\, tdwdt\right)^{\frac{1}{2}} \left(\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\, tdwdt\right)^{\frac{1}{2}} \\
&\quad + C_\varphi \lambda^2 \int\int_{\mathbf{R}_+^{n+1}} |\nabla v|^2\, tdwdt + \frac{1}{2}\int_{\mathbf{R}^n} \varphi(v(x,0))u^2(x,0)dx \\
&\le C_{\varphi,n}(\lambda^2\,|E_\lambda^c|)^{\frac{1}{2}} \left(\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\, tdwdt\right)^{\frac{1}{2}} \\
&\quad + C_{\varphi,n}\left(\lambda^2\,|E_\lambda^c| + \int_{E_\lambda} N_\beta(u)^2(x)dx\right).
\end{aligned}$$

Thus, by an elementary argument, we get

$$\int\int_{\mathbf{R}_+^{n+1}} \varphi(v)\,|\nabla u|^2\, tdwdt \le C_{\varphi,n}\left(\lambda^2\,|E_\lambda^c| + \int_{E_\lambda} N_\beta(u)^2(x)dx\right). \tag{4}$$

(3) and (4) give (2).

Acknowledgement. I would like to express my many thanks to Prof. Dr. H. König and Prof. Dr. D. Müller for their warm hospitality and fine research conditions. Thanks also to the referee for his/her helpful suggetions.

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Received November 1, 1996 and revised September 1, 1998. The author was supported by NNSFC, NSFZJ, NECC and CSC.

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