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In this paper, we give a simple proof for a good-λ inequality which means that nontangential maximal functions controls area integrals.

Let $u$ be a harmonic function on $\mathbb{R}^{n+1}$. The nontangential maximal function and the area integral function of $f$ are defined by

$$N_\beta(u)(x) = \sup_{(y,t) \in \Gamma_\beta(x)} |u(y,t)| \quad (\beta \in \mathbb{R}^+),$$

$$A_\alpha(u)(x) = \left( \int_{\Gamma_\alpha(x)} |\nabla u(y,t)|^2 t^{1-n} dy dt \right)^{1/2} \quad (\alpha \in \mathbb{R}^+).$$

The main aim of this paper is to give a simple proof of the inequality

$$\|A_\alpha(u)\|_p \leq C_{n,p,\alpha,\beta} \|N_\beta(u)\|_p \quad (0 < p < \infty, \ 0 < \alpha, \ beta < \infty).$$

(1)

As we know, this inequality is very important in $H^p$-theory, it is also a main difficulty in generalizing $H^p$-theory of one parameter to $H^p$-theory of several parameters, see [2, 6, 7, 8]. The first proof of (1) is probabilistic which was given by Burkholder, Gundy and Silverstein, see [1]; Fefferman and Stein first got an analytic proof of (1) by dealing with a kind of Green’s formula on $\mathcal{R} = \cup_{x \in \mathcal{E}} \Gamma_\alpha(x)$, see [4]; a sharpened inequality was obtained in [5] by a different approach. In the two-parameter case, Gundy and Stein set up a similar inequality to (1) by dealing with some multi-sub-linear operators like

$$B(u,v)(x) = \left( \int_{\Gamma(x)} |\nabla_1 u|^2 |\nabla_2 v|^2 t_1^{1-n_1} t_2^{1-n_2} dx_1 dt_1 dx_2 dt_2 \right)^{1/2},$$

see [9]; Merryfield ([8]) and author ([2]) generalized Gundy-Stein’s work to multi-parameter case independently and differently. In our proof ([2]), we introduced a kind of Carleson measure technique which does not depend on the dilation and translation structures of $\mathbb{R}^n$ such that the method works
on more general case (see Chen and Wang [3]). Here, we shall use the idea to give a simple proof of (1).

At first, we notice that for \(1 < p < \infty\), the proof of (1) is elementary, and for \(0 < p \leq 1\), (1) can be followed from

\[
|x : A_\alpha(u)(x) > \lambda| \leq C_{n,\alpha,\beta} \left( |\{x : N_\beta(u)(x) > \lambda\}| + \lambda^{-2} \int_{N_\beta(u)(x) \leq \lambda} N_\beta(u)^2(x) dx \right)
\]

(2)

where \(0 < \alpha, \beta, \lambda < \infty\) (note that, for \(0 < \alpha < \beta < \infty\), (2) was set up in [4]). Now, we shall prove (2).

By a limitation procedure, we may assume \(u(x, t) = \tilde{u}(x, t + \epsilon)\), where \(N_\beta(\tilde{u}) \in L^p\). \(\forall \lambda > 0\), set \(E_\lambda = \{x : N_\beta(u)(x) \leq \lambda\}\), \(\delta_0 = \delta(n, \beta) = \int_{|x| < \beta} p_1(x) dx \in (0, 1)\), where \(p_t\) is the Poisson kernel. Take a closed subset \(F_\lambda\) of \(E_\lambda\) such that \(|F_\lambda^c| \leq C_{n,\alpha,\beta} |E_\lambda^c|\); \(p_t * \chi_{E_\lambda} \geq 1 - \frac{1}{2} \delta_0\) (on \(\cup_{x \in F_\lambda} \Gamma_\alpha(x)\)), which is possible by the definition of \(p_t\) and the weak type \((1,1)\)-boundedness of nontangential maximal function operator; then, take \(\varphi \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^1)\), such that \(\varphi|_{(-\infty,1-\delta_0)} = 0, \varphi|_{(1-\frac{1}{2} \delta_0, +\infty)} = 1, |\varphi'| + |\varphi''| \leq c\varphi^{3/4}\) (by using \(e^{-r^2}\)). Now, set \(v = p_t * \chi_{E_\lambda}\), then

\[
(3) \quad \text{the left side of (2)}
\]

\[
\leq |F_\lambda^c| + |F_\lambda \cap \{x : A_\alpha(u)(x) > \lambda\}|
\leq C_{n,\alpha,\beta} \left\{ \left| E_\lambda^c \right| + \lambda^{-2} \int_{F_\lambda} \int_{\Gamma_\alpha(x)} \varphi(v) |\nabla u(w, t)|^2 t^{1-n} dwdtdx \right\}
\leq C_{n,\alpha,\beta} \left\{ \left| E_\lambda^c \right| + \lambda^{-2} \int_{\mathbb{R}^{n+1}} \varphi(v) |\nabla u|^2 tdwdt \right\}
\]

Note that

\[
\varphi(v) |\nabla u|^2 = -u \varphi'(v) \nabla v \cdot \nabla u - \frac{1}{2} u^2 \Delta(\varphi(v)) + \frac{1}{2} \Delta(\varphi(v) u^2);
\]

and, \(\|\varphi(v) u\|_\infty \leq C_{\varphi}\lambda\) for \(v \leq 1 - \delta_0\) on \((\cup_{x \in E_\lambda} \Gamma_\beta(x))\); in addition, it is not difficult to show that for a fixed \(\psi \in C^\infty_c(\mathbb{R}^n)\) satisfying \(\psi(|x| \leq 1) = 1, \psi(|x| \geq 2) = 0\), we have (where \(\psi_r(w) := \psi(w/r)\))

\[
\int_{\mathbb{R}^{n+1}} \Delta(\varphi(v) u^2) tdwdt = \lim_{r \to \infty} \int_{\mathbb{R}^{n} \times (0,r)} \psi_r(w) \Delta(\varphi(v) u^2) tdwdt
= \int_{\mathbb{R}^n} \varphi(v(x,0)) u^2(x,0) dx
\]

by Green’s formula, because \(N_\beta(\tilde{u}) \in L^p\), and

\[
\left\| t^{k+n/p} \nabla^k u \right\|_\infty + \left\| t^k \nabla v \right\|_\infty \leq C_{\epsilon,n,p,k}(\tilde{u}) < \infty
\]
for \( k = 0, 1, 2, \ldots \). Therefore, by Hölder’s inequality, we get
\[
\int \int_{\mathbb{R}^{n+1}_+} \varphi(v) |\nabla u|^2 \, t \, dwdt \\
\leq C_{\varphi,n} \left( \int \int_{\mathbb{R}^{n+1}_+} |\nabla v|^2 \, t \, dwdt \right)^{\frac{1}{2}} \left( \int \int_{\mathbb{R}^{n+1}_+} \varphi(v) |\nabla u|^2 \, t \, dwdt \right)^{\frac{1}{2}} \\
+ C_{\varphi} \lambda^2 \int \int_{\mathbb{R}^{n+1}_+} |\nabla v|^2 \, t \, dwdt + \frac{1}{2} \int_{\mathbb{R}_+} \varphi(v(x,0)) u^2(x,0) \, dx \\
\leq C_{\varphi,n} \left( \lambda^2 |E_\lambda|^1 \right)^{\frac{1}{2}} \left( \int \int_{\mathbb{R}^{n+1}_+} \varphi(v) |\nabla u|^2 \, t \, dwdt \right)^{\frac{1}{2}} \\
+ C_{\varphi,n} \left( \lambda^2 |E_\lambda^e| + \int_{E_\lambda} N_\beta(u)^2(x) \, dx \right).
\]
Thus, by an elementary argument, we get
\[
\int \int_{\mathbb{R}^{n+1}_+} \varphi(v) |\nabla u|^2 \, t \, dwdt \leq C_{\varphi,n} \left( \lambda^2 |E_\lambda^e| + \int_{E_\lambda} N_\beta(u)^2(x) \, dx \right).
\]
(3) and (4) give (2).

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