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In this paper we completely characterize compact commutator of two Toeplitz operators on the Hardy space of the unit circle.

For f in L^∞ , the space of essentially bounded Lebesgue measurable functions on the unit circle, ∂D , the Toeplitz operator with symbol f is the operator T_f on the Hardy space H^2 of the unit circle defined by $T_f h = P(fh)$. Here P denotes the orthogonal projection in L^2 with range H^2 . There are many fascinating problems about Toeplitz operators ([3], [6], [7] and [20]). In this paper we shall concentrate mainly on the following problem:

Problem 0.1. *When is the commutator $[T_f, T_g] = T_f T_g - T_g T_f$ of two Toeplitz operators T_f and T_g compact?*

This problem has been of interest ever since the Fredholm theory of Toeplitz operators was studied in the 1970s [1]. If $[T_f, T_g]$ is compact, then T_f and T_g commute in the Calkin algebra and so we say T_f and T_g are essentially commuting if $[T_f, T_g]$ is compact.

The simplest Toeplitz operators, in many respects, are analytic Toeplitz operators; that is, those whose symbols are in H^∞ , the algebra of bounded analytic functions on the open unit disc D . If f is in H^∞ , then T_f is just the operator on H^2 of multiplication by f , and one easily checks that T_f commutes with T_g if

$$(0.2) \quad f \in H^\infty \quad \text{and} \quad g \in H^\infty.$$

In virtue of the equality $T_{\bar{f}} = T_f^*$, T_f commutes with T_g if

$$(0.3) \quad \bar{f} \in H^\infty \quad \text{and} \quad \bar{g} \in H^\infty.$$

Clearly, T_f commutes with T_g if there are constants a , b and c with $|a| + |b| > 0$ such that

$$(0.4) \quad af + bg = c.$$

Conversely, Halmos [15] has shown that if T_f commutes with T_g , then one of Conditions 0.2, 0.3 and 0.4 holds, i.e.,

$$(0.5) \quad f \in H^\infty \quad \text{and} \quad g \in H^\infty \quad \text{or} \quad \bar{f} \in H^\infty \quad \text{and} \quad \bar{g} \in H^\infty \quad \text{or} \quad af + bg = c$$

for some constants a , b , and c . One may guess that the answer to Problem 0.1 will be analogous to the Halmos result on commuting Toeplitz operators. In this paper we will show that Condition 0.5 holding pointwise, in some sense, on each point of the unit circle will be a necessary and sufficient condition for T_f and T_g to be essentially commuting. Without going into definitions here, we mention conditions known to imply the compactness of $[T_f, T_g]$. The reader is referred to Section 1 for the definitions of unfamiliar terms appearing below. In addition, Conditions (iii)-(v) will be stated more precisely in the next section. Conditions (iii)-(v) represent different types of localization of condition (0.5). Thus $[T_f, T_g]$ is compact if:

- (i) $f \in C$ or $g \in C$ (Coburn [5] $C = C(\partial D)$).
- (ii) f and g are piecewise continuous and have no common discontinuities (Gohberg and Krupnik [11]).
- (iii) Condition 0.5 holds at each point of ∂D (Sarason [19]).
- (iv) Condition 0.5 holds on each QC level set (Doulgas [6]). Here

$$QC = \overline{(H^\infty + C)} \cap (H^\infty + C).$$

- (v) Condition 0.5 holds on each set of antisymmetry of $H^\infty + C$ (Axler [1]).

To prove our main theorems we will also need results about Douglas algebras. A Douglas algebra is, by definition, a closed subalgebra of L^∞ which contains H^∞ . Let $H^\infty[f]$ denote the Douglas algebra generated by the function f in L^∞ , and $H^\infty[f, g]$ the Douglas algebra generated by the functions f and g in L^∞ . In terms of Douglas algebras, Condition 0.2 becomes

$$H^\infty[f, g] = H^\infty,$$

Condition 0.3 becomes

$$H^\infty[\bar{f}, \bar{g}] = H^\infty,$$

and Condition 0.4 becomes

$$H^\infty[af + b, \overline{af + bg}] = H^\infty,$$

and Condition 0.5 is equivalent to

$$H^\infty[f, g] \cap H^\infty[\bar{f}, \bar{g}] \cap \bigcap_{|a|+|b|>0} H^\infty[af + bg, \overline{af + bg}] = H^\infty.$$

The following theorem is the version of our main result in terms of Douglas algebras.

Theorem 0.6. *The commutator $[T_f, T_g]$ of two Toeplitz operators is compact if and only if*

$$(0.7) \quad H^\infty[f, g] \cap H^\infty[\bar{f}, \bar{g}] \cap \bigcap_{|a|+|b|>0} H^\infty[af + bg, \overline{af + bg}] \subseteq H^\infty + C.$$

The above theorem completely solves Problem 0.1. As a consequence, we obtain a characterization of essentially normal Toeplitz operators, thus answering a question in Douglas' paper [8]. Our work is inspired by the following beautiful theorem of Axler, Chang and Sarason [2] and Volberg [24], stated below, on the compactness of semicommutator $T_{fg} - T_f T_g$ of two Toeplitz operators.

Axler-Chang-Sarason-Volberg Theorem. *$T_{fg} - T_f T_g$ is compact if and only if*

$$H^\infty[\bar{f}] \cap H^\infty[g] \subseteq H^\infty + C.$$

It is not immediately apparent that the earlier results cited above (Conditions (i)-(v)) are consequences of Theorem 0.6. This will become clear in the next section. In Section 1 we define our terms (including the notion of support set) and we will show that (0.7) can be restated as a local version:

Theorem 0.8. *Let $f, g \in L^\infty$. Then $T_f T_g - T_g T_f$ is compact if and only if for each support set S , one of the following holds:*

1. $f|_S$ and $g|_S$ are in $H^\infty|_S$.
2. $\bar{f}|_S$ and $\bar{g}|_S$ are in $H^\infty|_S$.
3. There exist constants a, b , not both zero, such that $a f + b g|_S$ is constant.

The result in the above theorem was conjectured in [23]. An elementary condition was obtained in [25] for the compactness of $T_{fg} - T_f T_g$. An elementary equivalence of (0.7) will be given in Section 2:

Theorem 0.9. *Let f and g be in L^∞ . Then $T_f T_g - T_g T_f$ is compact if and only if*

$$(0.10) \quad \lim_{|z| \rightarrow 1} \|(H_{\bar{f}} k_z) \otimes (H_g k_z) - (H_{\bar{g}} k_z) \otimes (H_f k_z)\| = 0.$$

Here for x and y in L^2 , $x \otimes y$ denotes the following operator of rank one: for $f \in L^2$,

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

We will establish a distribution function inequality in Section 4 in order to prove Theorem 0.9. The proof of Theorem 0.9 is given in Section 3.

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1. Local Version.

First we wish to reformulate Condition 0.7 in a way that will facilitate its comparison with Conditions (i)-(v). Some notation is needed. The Gelfand space (space of nonzero multiplicative linear functionals) of the Douglas

algebra B will be denoted by $M(B)$. If B is a Douglas algebra, then $M(B)$ can be identified with the set of nonzero linear functionals in $M(H^\infty)$ whose representing measures (on $M(L^\infty)$) are multiplicative on B , and we identify the function f with its Gelfand transform on $M(B)$. In particular, $M(H^\infty + C) = M(H^\infty) - D$, and a function $f \in H^\infty$ may be thought of as a continuous function on $M(H^\infty + C)$. The fiber of $M(L^\infty)$ above the point λ of ∂D is the set $\{x \in M(L^\infty) : x(z) = \lambda\}$ and will be denoted by $M_\lambda(L^\infty)$. Let $x \in M(L^\infty)$. The set E_x is called a QC level set if $E_x = \{y \in M(L^\infty) : f(y) = f(x) \text{ for all } f \in QC\}$. A subset of $M(L^\infty)$ is called a set of antisymmetry for $H^\infty + C$ if any function in $H^\infty + C$ which is real valued on the set is constant on it. Clearly, sets of antisymmetry are contained in level sets. A subset of $M(L^\infty)$ is called a support set if it is the (closed) support of the representing measure for a functional in $M(H^\infty + C)$. As we discuss later, support sets are also sets of antisymmetry. Now we are ready to state Conditions (iii)-(v) more precisely:

(iii) For each $\lambda \in D$, either $f|_{M_\lambda(L^\infty)} \in H^\infty|_{M_\lambda(L^\infty)}$ and $g|_{M_\lambda(L^\infty)} \in H^\infty|_{M_\lambda(L^\infty)}$, or $\bar{f}|_{M_\lambda(L^\infty)} \in H^\infty|_{M_\lambda(L^\infty)}$ and $\bar{g}|_{M_\lambda(L^\infty)} \in H^\infty|_{M_\lambda(L^\infty)}$, or for some constants a_λ and b_λ with $|a_\lambda| + |b_\lambda| > 0$, $(a_\lambda f + b_\lambda g)|_{M_\lambda(L^\infty)}$ is constant.

(iv) For each QC level set Q , either $f|_Q \in H^\infty|_Q$ and $g|_Q \in H^\infty|_Q$, or $\bar{f}|_Q \in H^\infty|_Q$ and $\bar{g}|_Q \in H^\infty|_Q$, or for some constants a_Q and b_Q with $|a_Q| + |b_Q| > 0$, $(a_Q f + b_Q g)|_Q$ is constant.

(v) For each set of antisymmetry S of $H^\infty + C$, either $f|_S \in H^\infty|_S$ and $g|_S \in H^\infty|_S$, or $\bar{f}|_S \in H^\infty|_S$ and $\bar{g}|_S \in H^\infty|_S$, or for some constants a_S and b_S with $|a_S| + |b_S| > 0$, $(a_S f + b_S g)|_S$ is constant.

Since each set of antisymmetry of $H^\infty + C$ is contained in a QC level set and each QC level set is contained in a single fiber of $M(L^\infty)$, it is evident that (iv) is implied by (iii) and (v) is implied by (iv). As is well known, each support set is a set of antisymmetry for $H^\infty + C$. Thus the following lemma shows that (v) implies Condition 0.7.

Lemma 1.1. *Let $f, g \in L^\infty$. Then*

$$(1.2) \quad H^\infty[f, g] \cap H^\infty[\bar{f}, \bar{g}] \cap \bigcap_{|a|+|b|>0} H^\infty[af + bg, \overline{af + bg}] \subseteq H^\infty + C$$

if and only if for each support set S one of the following holds:

- (1) $f|_S$ and $g|_S$ are in $H^\infty|_S$.
- (2) $\bar{f}|_S$ and $\bar{g}|_S$ are in $H^\infty|_S$.
- (3) There exist constants a, b not both zero such that $(af + bg)|_S$ is constant.

By the above lemma, we see that Theorem 0.6 is equivalent to Theorem 0.8.

Before giving the proof of Lemma 1.1 we make several remarks. The result of Theorem 0.8 is of interest in its own right, but it also provides more information on a long-standing open problem of Sarason. Sarason studied the algebra $H^\infty + C$. When one knows that this is a uniform algebra on the maximal ideal space of L^∞ (denoted $M(L^\infty)$) one can look at two different decompositions of $M(L^\infty)$. The first is called the Shilov decomposition. For this decomposition, we look at the real-valued functions in $H^\infty + C$ and define the level sets as we defined the QC level sets above; the largest subsets of $M(L^\infty)$ on which all such functions are constants. Studying functions restricted to these level sets provides us with a great deal of information about the algebra $H^\infty + C$. Sometimes, though, it is easier to move to the antisymmetric sets. Maximal antisymmetric sets exist, and Bishop showed (in more generality than what is stated here) that if a function $f \in L^\infty$ satisfies $f|_S \in H^\infty|_S$ for every maximal antisymmetric set S , then $f \in H^\infty + C$. Sarason [21] showed that the Shilov decomposition for $H^\infty + C$ is not equal to the Bishop decomposition; that is, he showed that there is a QC level set that is not an antisymmetric set. He raised the question [22] of whether or not every maximal antisymmetric set is, in fact, the support set of some $m \in M(H^\infty + C)$. Evidence has existed for some time to show that the answer to Sarason's question may well be in the affirmative. For example, Gorkin [12], [13] showed that there are maximal antisymmetric sets that are support sets.

Douglas [6] has shown that Condition (iii), which is Sarason's criterion for compactness of the commutator, is a consequence of a certain localization theorem to the fibers of $M(L^\infty)$. Douglas then deduced his compactness criterion ((iv)) from a refined localization theorem on the QC level sets. The Douglas localization theorem provides a faithful representation of the C^* -algebra generated by the Toeplitz operators, modulo the compact operators, into the direct sum of the local algebras corresponding to QC level sets. Axler [1] established the compactness criterion ((v)) by a yet more refined localization theorem on the maximal antisymmetric sets for $H^\infty + C$. Sarason's question about the relationship of support sets and antisymmetric sets, mentioned above, is related to the question of whether or not there is a localization theorem in which the local algebras correspond to support sets. This question was first posed in [2]. Thus far, no such localization theorem exists. So the importance of our result is that it may be added to the list of existing results that suggest that maximal antisymmetric sets for $H^\infty + C$ may well be support sets, or there is a localization theorem on the support sets.

The Chang-Marshall Theorem [4], [17] asserts that any Douglas algebra B is generated by H^∞ together with the conjugates of the interpolating Blaschke products invertible in B . This theorem also shows that the maximal ideal space $M(B)$ completely determines B . To prove Lemma 1.1 we

need to understand the maximal ideal space of the intersection of a family of Douglas algebras. The theorem below, due to Sarason, gives a description of precisely that. Since no proof of this theorem has been published, we include one below. A different proof appeared in [12]. Before doing so, recall that the pseudohyperbolic distance between two points m_1 and m_2 in $M(H^\infty)$ is given by

$$\rho(m_1, m_2) = \sup\{|f(m_2)| : f \in H^\infty, \|f\| \leq 1, f(m_1) = 0\}.$$

The Gleason part of a point $m_1 \in M(H^\infty)$, denoted by $P(m_1)$ is given by

$$P(m_1) = \{m : \rho(m, m_1) < 1\}.$$

It is well known that each Gleason part of $M(H^\infty)$ is either one point or an analytic disc. When the Gleason part of m consists of one point, m is said to be a trivial point. Otherwise m is a nontrivial point.

Lemma 1.3 (Sarason). *Let $\{A_\alpha\}$ be a family of Douglas algebras. Then*

$$M(\cap A_\alpha) = \overline{\cup M(A_\alpha)}.$$

Proof. If one of the algebras is H^∞ the result is clear. Thus we may assume that all algebras contain $H^\infty + C$ ([10], p. 376). Clearly, $M(A_\beta) \subseteq M(\cap A_\alpha)$ for every β , so it is also clear that $\overline{\cup M(A_\alpha)} \subseteq M(\cap A_\alpha)$. For the other direction, suppose that m is a point of $M(H^\infty + C)$ that is not in $\overline{\cup M(A_\alpha)}$. Since the maximal ideal space is a compact Hausdorff space there exists an open set V containing m such that the closure, \overline{V} satisfies $\overline{V} \cap \overline{\cup M(A_\alpha)} = \emptyset$.

If m is a nontrivial point, it follows from work of Hoffman [16] that there exists an interpolating Blaschke product b such that the closure of the zeros of b lie in \overline{V} and $m(b) = 0$. Now the zeros of b do not intersect $M(A_\alpha)$ for any α , so b is invertible in A_α for every α . Therefore, $\bar{b} \in \cap A_\alpha$. Now $m(b\bar{b}) = 1$. On the other hand, since $m(b) = 0$, $m(b)m(\bar{b}) = 0$. So $m(b\bar{b})$ does not equal $m(b)m(\bar{b})$ and hence m is not in $M(\cap A_\alpha)$.

Now suppose that the Gleason part containing m is a trivial part. By Corollary 3.2 in [14] there is a nontrivial point x in V such that $\text{supp } x \subseteq \text{supp } m$. Since x is nontrivial and not in $\overline{\cup M(A_\alpha)}$ from what was done above, x is not in $M(\cap A_\alpha)$. Therefore, by the Chang-Marshall Theorem, there is an inner function u invertible in $\cap A_\alpha$ such that $|x(u)| < 1$. Thus (since u has modulus one on $\text{supp } x$) u is not constant on $\text{supp } x$ and so it is not constant on $\text{supp } m$. Hence $|m(u)| < 1$ and m is not in $M(\cap A_\alpha)$. The proof of Lemma 1.3 is now complete. \square

We will need the following known and useful observation.

Lemma 1.4. *Let u be a unimodular function and $m \in M(H^\infty + C)$, let S be the support set for m . Then $|m(u)| = 1$ if and only if $u|_S$ is constant.*

Proof. There is a probability measure dm such that

$$m(u) = \int_S u dm.$$

Note that $|u| = 1$ on S . Thus we see that $|m(u)| = 1$ if and only if $u|_S$ is constant. The proof is complete. \square

The following lemma is a consequence of the Chang-Marshall Theorem.

Lemma 1.5. *Let $m \in M(H^\infty + C)$ and let S be the support set for m . Then $m \in M(H^\infty[f])$ if and only if $f|_S \in H^\infty|_S$.*

Proof. Suppose that $m \in M(H^\infty[f])$. By Chang-Marshall Theorem ([4], [17]), there exist inner functions b_1, b_2, \dots such that

$$H^\infty[f] = H^\infty[\overline{b_j} : j = 1, 2, \dots].$$

Thus

$$1 = |m(b_j \overline{b_j})| = |m(b_j)|^2$$

for all j . Therefore, $|m(b_j)| = 1$ and so, by Lemma 1.4 we have that $b_j|_S$ is constant for all j . Since $f \in H^\infty[\overline{b_j} : j = 1, 2, \dots]$, and $H^\infty|_S$ is closed, $f|_S \in H^\infty|_S$.

Conversely, suppose that $f|_S \in H^\infty|_S$. Then the extension of m to L^∞ [16] is given by

$$m(f) = \int_S f dm$$

for $f \in L^\infty$. Let $g \in H^\infty$ be such that $g|_S = f|_S$. Thus $g^n|_S = f^n|_S$ for all n and hence $m(f^n) = m(g^n) = m(g)^n = m(f)^n$ for all n . Note that for any $h \in H^\infty$, we have

$$\begin{aligned} m(hf^n) &= \int_S hf^n dm = \int_S hg^n \\ &= m(hg^n) = m(h)m(g^n) = m(h)m(f^n). \end{aligned}$$

Hence m is a multiplicative functional on $H^\infty[f]$, and so $m \in M(H^\infty[f])$ [10]. \square

As a consequence of Lemma 1.5 we have the following corollary.

Corollary 1.6. *Let $m \in M(H^\infty + C)$, and let S be the support set for m . Then $m \in M(H^\infty[f, g])$ if and only if $f|_S \in H^\infty|_S$ and $g|_S \in H^\infty|_S$.*

Now we are ready to prove Lemma 1.1.

Proof of Lemma 1.1. Let A denote the Douglas algebra

$$H^\infty[f, g] \cap H^\infty[f, \overline{f}] \cap H^\infty[g, \overline{g}] \cap H^\infty[\overline{f}, \overline{g}] \cap \bigcap_{a \neq 0} H^\infty[f + ag, \overline{f + ag}].$$

One can easily check that A equals

$$H^\infty[f, g] \cap H^\infty[\bar{f}, \bar{g}] \cap \bigcap_{|a|+|b|>0} H^\infty[af + bg, \overline{af + bg}].$$

By Lemma 1.3, we get that $M(A)$ equals

$$M(H^\infty[f, g]) \cup M(H^\infty[\bar{f}, \bar{g}]) \cup M(H^\infty[f, \bar{f}]) \cup M(H^\infty[g, \bar{g}]) \cup \overline{\cup_{a \neq 0} M(H^\infty[f + ag, \bar{f} + ag])}.$$

Suppose that (1.2) holds. Then $A \subset H^\infty + C$, and so $M(H^\infty + C) \subset M(A)$. Let $m \in M(H^\infty + C)$. Then m is an element of

$$M(H^\infty[f, g]) \cup M(H^\infty[\bar{f}, \bar{g}]) \cup M(H^\infty[f, \bar{f}]) \cup M(H^\infty[g, \bar{g}]) \cup \overline{\cup_{a \neq 0} M(H^\infty[f + ag, \bar{f} + ag])}.$$

If m is in any of the first four sets, Corollary 1.6 gives that either (1) or (2) holds, or (3) holds with one of the constants equal to zero. Thus, we may assume that $m \in \overline{\cup_{a \neq 0} M(H^\infty[f + ag, \bar{f} + ag])}$. Hence there exist constants a_α and points $m_\alpha \in M(H^\infty[f + a_\alpha g, \bar{f} + a_\alpha g])$ such that $m_\alpha \rightarrow m$. Let S_α be the support set for m_α and S the support for m . By Corollary 1.6 $(f + a_\alpha g)|_{S_\alpha}$ and $\bar{f} + a_\alpha g|_{S_\alpha}$ are in $H^\infty|_{S_\alpha}$. Since the support of the measure is an antisymmetric set, $(f + a_\alpha g)|_{S_\alpha} = c_\alpha$ for some constant c_α .

Since $m \in \overline{\cup_{a \neq 0} M(H^\infty[f + ag, \bar{f} + ag])}$, we have

$$\begin{aligned} m &\in \overline{\cup_{|a| \leq 1} M(H^\infty[f + ag, \bar{f} + ag]) \cup \cup_{|a| \geq 1} M(H^\infty[f + ag, \bar{f} + ag])} \\ &= \overline{\cup_{|a| \leq 1} M(H^\infty[f + ag, \bar{f} + ag])} \cup \overline{\cup_{|a| \geq 1} M(H^\infty[f + ag, \bar{f} + ag])}. \end{aligned}$$

If m is in the second set, note that $H^\infty[f + ag, \bar{f} + ag] = H^\infty[(1/a)f + g, (1/a)\bar{f} + \bar{g}]$ and use the same argument that we will use below.

So suppose that m is in the first set. Let $\epsilon > 0$ be given. Note that if we cover the closed unit disc by finitely many discs D_1, \dots, D_n centered at points a_j of diameter ϵ then

$$\begin{aligned} m &\in \overline{\cup_{j=1, \dots, n} \{M(H^\infty[f + a_j g, \bar{f} + a_j g]) : a_j \in D_j\}} \\ &= \overline{\cup_{j=1, \dots, n} \{M(H^\infty[f + a_j g, \bar{f} + a_j g]) : a_j \in D_j\}}. \end{aligned}$$

Thus, there exists a disc D with center a of diameter ϵ such that

$$m \in \overline{\{M(H^\infty[f + ag, \bar{f} + ag]) : a \in D\}}.$$

Let (m_α) be a net from this set capturing m in its closure. Note that since $m_\alpha \in M(H^\infty[f + a_\alpha g, \bar{f} + a_\alpha g])$, there exists a constant c_α such that $(f + a_\alpha g)|_{S_\alpha} = c_\alpha$. Since f, g , and a_α are all bounded (independently of α),

there exists a constant M such that $|c_\alpha| \leq M$ for all α . Cover the disc of radius M by finitely many discs, B_1, \dots, B_n of diameter ϵ . Then

$$m \in \overline{\cup_{j=1, \dots, n} \{m_\alpha : (f + a_\alpha g)|_{S_\alpha} = c_\alpha \in B_j\}}.$$

Thus, we may assume that there is a net $m_\alpha \rightarrow m$ with corresponding values a_α and c_α satisfying $\text{diam}\{a_\alpha\}$ and $\text{diam}\{c_\alpha\}$ are both less than ϵ . Furthermore, we may assume that $a_\alpha \rightarrow a$, for some complex number a .

Now we claim that $S \subseteq \overline{\cup S_\alpha}$.

Let $\mathcal{S} = \overline{\cup S_\alpha}$. By ([9, p. 39]),

$$M(\overline{H^\infty|_{\mathcal{S}}}) = \{\varphi \in M(H^\infty) : \text{supp } \varphi \subseteq \mathcal{S}\}.$$

By the definition of \mathcal{S} ,

$$m_\alpha \in M(\overline{H^\infty|_{S_\alpha}})$$

for each α . Since $m_\alpha \rightarrow m, m \in M(\overline{H^\infty|_{\mathcal{S}}})$, and the claim is established.

Now if $x, y \in \cup S_\alpha$, then there exist α , and β as above with $x \in S_\alpha$ and $y \in S_\beta$. Hence

$$\begin{aligned} & |(f + ag)(x) - (f + ag)(y)| \\ & \leq |(f + ag)(x) - (f + a_\alpha g)(x)| + |(f + a_\alpha g)(x) - (f + a_\beta g)(y)| \\ & \quad + |(f + a_\beta g)(y) - (f + ag)(y)| \\ & \leq |a - a_\alpha| \|g\| + |c_\alpha - c_\beta| + |a_\beta - a| \|g\|. \end{aligned}$$

So

$$|(f + ag)(x) - (f + ag)(y)| \leq 4\epsilon \|g\|.$$

Since $S \subseteq \overline{\cup S_\alpha}$, we see that $\sup_{x, y \in S} |(f + ag)(x) - (f + ag)(y)| \leq 4\epsilon \|g\|$. Since ϵ was arbitrary, $(f + ag)|_S$ is constant. Thus $m \in H^\infty[f + ag, \overline{f + ag}]$ and so $(f + ag)|_S$ is constant, as desired.

Conversely, let S be the support set for an element $m \in M(H^\infty + C)$ and suppose that one of (1), (2) and (3) holds for m . Then by Corollary 1.6, either $m \in M(H^\infty[f, g])$ or $m \in M(H^\infty[\bar{f}, \bar{g}])$, or there exist constants a, b , not both zero such that $m \in M(H^\infty[af + bg, \overline{af + bg}])$. Thus m is in

$$M \left(H^\infty[f, g] \cap H^\infty[\bar{f}, \bar{g}] \cap \bigcap_{|a|+|b|>0} H^\infty[af + bf, \overline{af + bg}] \right).$$

Therefore, $M(H^\infty + C) \subseteq M(A)$. By the Chang-Marshall Theorem ([4], [17]) $A \subseteq H^\infty + C$. The proof of Lemma 1.1 is complete. \square

2. An Elementary Condition.

We first mention that Problem 0.1 can be reformulated as a problem about Hankel operators. The Hankel operator with symbol f in L^∞ is the operator H_f from H^2 to the orthogonal complement of H^2 defined by $H_f h = (1 - P)fh$, where P is the orthogonal projection of L^2 onto H^2 . In virtue of the

easily established identity $T_{fg} - T_f T_g = H_{\bar{f}}^* H_g$, Problem 0.1 is equivalent to the problem of when $H_{\bar{g}}^* H_f - H_{\bar{f}}^* H_g$ is compact. In [25] it is shown that $T_{fg} - T_f T_g$ is compact if and only if

$$(2.1) \quad \lim_{|z| \rightarrow 1} \|H_{\bar{f}} k_z\|_2 \|H_g k_z\|_2 = 0;$$

here k_z is the normalized reproducing kernel $\frac{(1-|z|^2)^{1/2}}{(1-\bar{z}w)}$ in H^2 . So the Axler-Chang-Sarason-Volberg condition is equivalent to Condition (2.1). In this section we will give a condition in terms of Hankel operators, which is equivalent to the condition in Lemma 1.1.

Let x and y be two vectors in L^2 . Define $x \otimes y$ to be the following operator of rank one: For $f \in L^2$,

$$(x \otimes y)(f) = \langle f, y \rangle x.$$

Note that the norm of the operator $x \otimes y$ is $\|x\|_2 \|y\|_2$. Then (2.1) is equivalent to

$$(2.2) \quad \lim_{|z| \rightarrow 1} \|(H_{\bar{f}} k_z) \otimes (H_g k_z)\| = 0.$$

The following lemma gives us one more condition to be added to the equivalence established in Lemma 1.1.

Lemma 2.3. *Let f and g be in L^∞ . Then*

$$(2.4) \quad \lim_{|z| \rightarrow 1} \|(H_{\bar{f}} k_z) \otimes (H_g k_z) - (H_{\bar{g}} k_z) \otimes (H_f k_z)\| = 0.$$

if and only if for each support set S , one of the following holds:

- (1) $f|_S$ and $g|_S$ are in $H^\infty|_S$.
- (2) $\bar{f}|_S$ and $\bar{g}|_S$ are in $H^\infty|_S$.
- (3) There exist constants a, b , not both zero, such that $af + bg|_S$ is constant.

Before giving the proof of Lemma 2.3 we need to have an interpretation of $f|_S \in H^\infty|_S$ in terms of the Hankel operator H_f . Let m be in $M(H^\infty + C)$. We use the notation $z \rightarrow m$ to mean that z converges to m in the maximal ideal space of H^∞ .

Lemma 2.5. *Let f be in L^∞ and $m \in M(H^\infty + C)$, and let S be the support set for m . Then $f|_S \in H^\infty|_S$ if and only if*

$$\varinjlim_{z \rightarrow m} \|H_f k_z\|_2 = 0.$$

Proof. Without loss of generality, assume that $\|f\| < 1$. Thus as a consequence of the Adamian-Arov-Krein Theorem [18], there exists a unimodular function $u \in f + H^\infty$ such that T_u is invertible and \bar{u} is in $H^\infty[u]$.

Suppose that $\|H_f k_z\| \rightarrow 0$ as $z \rightarrow m$. Note that since $H_f = H_u$, we get $\|H_u k_z\| \rightarrow 0$ as $z \rightarrow m$. By Lemma 3 of [25], there is a positive constant C_u such that

$$(1 - |u(z)|^2) \leq C_u \|H_u k_z\|_2^2$$

for all $z \in D$. Thus

$$\underline{\lim}_{z \rightarrow m} (1 - |u(z)|^2) \leq C_u \underline{\lim}_{z \rightarrow m} \|H_u k_z\|_2^2.$$

By a result of Hoffman [16], $u(z)$ is continuous on the maximal ideal space of H^∞ , so we see that $|m(u)| = 1$. By Lemma 1.4 we see that u must be constant on S . Thus, since $u \in f + H^\infty$, we have $f|_S \in H^\infty|_S$.

Conversely, assuming that $f|_S \in H^\infty|_S$, we see that $u|_S \in H^\infty|_S$. Noting that \bar{u} is also in $H^\infty[u]$, we get that $\bar{u}|_S$ is also in $H^\infty|_S$. But the support set is a set of antisymmetry for $H^\infty + C$. Thus $u|_S$ is unimodular constant, and so $|u(m)| = 1$.

By Lemma 3 of [25] again, we conclude that there exists a constant C such that $\|H_u k_z\| \leq C(1 - |u(z)|^2)$. Suppose that $z \rightarrow m$. Then $|u(z)| \rightarrow 1$. Hence, $\|H_u k_z\| \rightarrow 0$ as $z \rightarrow m$. Since $\|H_f k_z\| = \|H_u k_z\|$, we conclude

$$\underline{\lim}_{z \rightarrow m} \|H_f k_z\|_2 = 0.$$

The proof is complete. □

In fact, the proof above shows a bit more. We isolate this fact in the Lemma below for future reference.

Lemma 2.6. *Let $f \in L^\infty$ and $m \in M(H^\infty + C)$. Then*

$$\lim_{z \rightarrow m} \|H_f k_z\|_2 = 0$$

if

$$\underline{\lim}_{z \rightarrow m} \|H_f k_z\|_2 = 0.$$

Proof. As above, assume that $\|f\| < 1$. Thus as a consequence of the Adamian-Arov-Krein Theorem [18], there exists a unimodular function $u \in f + H^\infty$ such that T_u is invertible and \bar{u} is in $H^\infty[u]$.

On the other hand, by Lemma 3 of [25], there is a positive constant C_u such that

$$\frac{1}{C_u} \|H_u k_z\|_2^2 \leq (1 - |u(z)|^2) \leq C_u \|H_u k_z\|_2^2.$$

Note that u is continuous on $M(H^\infty)$. Thus we have that

$$\begin{aligned} \frac{1}{C_u} \overline{\lim}_{z \rightarrow m} \|H_u k_z\|_2^2 &\leq \lim_{z \rightarrow m} (1 - |u(z)|^2) \\ &\leq C_u \underline{\lim}_{z \rightarrow m} \|H_u k_z\|_2^2. \end{aligned}$$

Hence

$$(2.7) \quad \overline{\lim}_{z \rightarrow m} \|H_u k_z\|_2^2 \leq C_u^2 \underline{\lim}_{z \rightarrow m} \|H_u k_z\|_2^2.$$

Note that $u \in f + H^\infty$. We have that

$$\|H_u k_z\|_2 = \|H_f k_z\|_2.$$

Combining the above equation and (2.7) completes the proof. □

For a fixed z in D , let F_z denote the operator

$$(H_{\bar{f}}k_z) \otimes (H_gk_z) - (H_{\bar{g}}k_z) \otimes (H_fk_z).$$

Lemma 2.8. *If for a fixed z in D , there exists a constant λ such that H_fk_z is orthogonal to $H_{g-\lambda f}k_z$, for some constant λ , then the norm of the operator F_z is equivalent to*

$$[\|H_{g-\lambda f}k_z\|^2\|H_{\bar{f}}k_z\|^2 + \|H_fk_z\|^2\|H_{g-\lambda\bar{f}}k_z\|^2]^{1/2}.$$

Proof. One easily checks that

$$F_z = (H_{\bar{f}}k_z) \otimes (H_{g-\lambda f}k_z) - (H_{g-\lambda\bar{f}}k_z) \otimes (H_fk_z),$$

for any constant λ . For λ such that H_fk_z is orthogonal to $H_{g-\lambda\bar{f}}k_z$, we have

$$F_zF_z^* = \|H_{g-\lambda f}k_z\|^2(H_{\bar{f}}k_z) \otimes (H_{\bar{f}}k_z) + \|H_fk_z\|^2(H_{g-\lambda\bar{f}}k_z) \otimes (H_{g-\lambda\bar{f}}k_z).$$

Thus

$$\text{trace}(F_zF_z^*) = \|H_{g-\lambda f}k_z\|^2\|H_{\bar{f}}k_z\|^2 + \|H_fk_z\|^2\|H_{g-\lambda\bar{f}}k_z\|^2.$$

Noting that $F_zF_z^*$ is an operator of rank at most 2, we get

$$1/2 \text{trace}(F_zF_z^*) \leq \|F_z\|^2 \leq \text{trace}(F_zF_z^*),$$

and then the norm of F_z is equivalent to

$$[\|H_{g-\lambda f}k_z\|^2\|H_{\bar{f}}k_z\|^2 + \|H_fk_z\|^2\|H_{g-\lambda\bar{f}}k_z\|^2]^{1/2}.$$

We now return to the proof of Lemma 2.3.

Proof of Lemma 2.3. Without loss of generality we may assume that $\|f\|_\infty < 1$ and $\|g\|_\infty < 1$.

Suppose that (2.4) holds. Let m be in $M(H^\infty + C)$, and let S be the support set of m . By Carleson's Corona Theorem, there is a net z converging to m . We consider two cases.

First suppose that there is a constant c such that

$$\underline{\lim}_{z \rightarrow m} \|H_fk_z\|_2 \geq c > 0.$$

Let $\lambda_z = \langle H_gk_z, H_fk_z \rangle / \|H_fk_z\|^2$. Then $|\lambda_z| \leq \frac{1}{c}$, and so we may assume that $\lambda_z \rightarrow a$ for some constant a . Note H_fk_z is orthogonal to $H_{g-\lambda_z f}k_z$. Then by Lemma 2.8 we get

$$[\|H_{g-\lambda_z f}k_z\|^2\|H_{\bar{f}}k_z\|^2 + \|H_fk_z\|^2\|H_{g-\lambda_z\bar{f}}k_z\|^2]^{1/2}$$

tends to zero as $z \rightarrow m$. Since $\lambda_z \rightarrow a$ we get

$$(2.9) \quad \lim_{z \rightarrow m} [\|H_{g-af}k_z\|^2\|H_{\bar{f}}k_z\|^2 + \|H_fk_z\|^2\|H_{g-a\bar{f}}k_z\|^2] = 0,$$

and so

$$\begin{aligned} \lim_{z \rightarrow m} \|H_{g-af}k_z\|^2\|H_{\bar{f}}k_z\|^2 &= 0, & \text{and} \\ \lim_{z \rightarrow m} \|H_fk_z\|_2^2\|H_{g-a\bar{f}}k_z\|^2 &= 0. \end{aligned}$$

By Lemma 2.5, the second limit above gives that $(\overline{g - af})|_S$ is in $H^\infty|_S$. The first limit above gives that either

$$\begin{aligned} \underline{\lim}_{z \rightarrow m} \|H_{g-af}k_z\|^2 &= 0, \quad \text{or} \\ \underline{\lim}_{z \rightarrow m} \|H_{\bar{f}}k_z\|^2 &= 0. \end{aligned}$$

It follows from Lemma 2.5 that either $\bar{f}|_S$ or $(g - af)|_S$ is in $H^\infty|_S$. If $\bar{f}|_S$ is in $H^\infty|_S$, then using the fact that $(\overline{g - af})|_S$ is in $H^\infty|_S$ we see that $\bar{g}|_S$ is in H^∞ . This implies (2). On the other hand, if $(g - af)|_S$ is in $H^\infty|_S$, combining this with the fact that $(\overline{g - af})|_S$ is in $H^\infty|_S$, we see that (3) holds.

Next suppose that

$$\underline{\lim}_{z \rightarrow m} \|H_f k_z\|_2 = 0.$$

It follows from Lemma 2.5 that $f|_S$ is in $H^\infty|_S$. Thus (2.9) gives

$$\underline{\lim}_{z \rightarrow m} \|H_{g-af}k_z\|^2 \|H_{\bar{f}}k_z\|^2 = 0.$$

Thus we get

$$\begin{aligned} \underline{\lim}_{z \rightarrow m} \|H_{g-af}k_z\|^2 &= 0, \quad \text{or} \\ \underline{\lim}_{z \rightarrow m} \|H_{\bar{f}}k_z\|^2 &= 0. \end{aligned}$$

It follows from Lemma 2.5 again that either $\bar{f}|_S$ or $(g - af)|_S$ is in $H^\infty|_S$. So this implies either (3) or (1) holds. Therefore we have proved that one of (1), (2), and (3) holds.

Conversely, suppose that for each support set S , one of (1), (2), and (3) holds on S . Assuming that (2.4) is false, we will derive a contradiction. Thus there are a constant $\delta > 0$ and a net z in D converging to a point $m \in M(H^\infty + C)$ such that

$$(2.10) \quad \overline{\lim}_{z \rightarrow m} \|F_z\| \geq \delta.$$

Let S denote the support set for this point m . One easily checks that

$$(2.11) \quad \|F_z\| \leq \|H_{\bar{f}}k_z\|_2 \|H_g k_z\|_2 + \|H_{\bar{g}}k_z\|_2 \|H_f k_z\|_2.$$

If (1) holds, then Lemmas 2.5 and 2.6 give

$$\begin{aligned} \lim_{z \rightarrow m} \|H_f k_z\|_2 &= 0, \quad \text{and} \\ \lim_{z \rightarrow m} \|H_g k_z\|_2 &= 0. \end{aligned}$$

Thus (2.11) shows that

$$\lim_{z \rightarrow m} \|F_z\| = 0,$$

which contradicts (2.10).

If (2) holds, using the same argument as above we can show a contradiction of (2.10).

If (3) holds, without loss of generality we may assume that there is a constant a such that $(g - af)|_S$ is a constant. By Lemmas 2.5 and 2.6 we see that

$$\begin{aligned} \lim_{z \rightarrow m} \|H_{g-af}k_z\|_2 &= 0, \quad \text{and} \\ \lim_{z \rightarrow m} \|H_{\overline{g-af}}k_z\|_2 &= 0. \end{aligned}$$

Also from the proof of Lemma 2.8, we get

$$\|F_z\| \leq \|H_{\bar{f}}k_z\|_2 \|H_{g-af}k_z\|_2 + \|H_{\overline{g-af}}k_z\|_2 \|H_fk_z\|_2.$$

So we conclude that

$$\lim_{z \rightarrow m} \|F_z\| = 0.$$

This contradicts (2.10). The proof is complete. □

3. Proof of Theorem 0.9.

Now we turn to the proof of Theorem 0.9. First we introduce some notation.

Let h be in L^2 . The Hardy-Littlewood maximal function of h will be denoted by Mh and, for $r > 1$, let

$$\Lambda_r h(w) = [M(|h|^r)(w)]^{\frac{1}{r}}.$$

For w a point of ∂D , and $0 < r < 1$, let Γ_w denote the angle with vertex w and opening $\pi/2$ which is bisected by the radius to w . The set of points z in Γ_w satisfying $|z - w| < \gamma$ will be denoted by $\Gamma_{w,\gamma}$.

Let f and g be in $L^\infty(\partial D)$ and u and v be in H^2 . Let dA denote the normalized area measure on the unit disk. Define a generalized area integral $B_\gamma(u, v)(w)$ to be

$$B_\gamma(u, v)(w) = \int_{\Gamma_{w,\gamma}} |\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)}| dA(z).$$

Here, for a function $h \in L^2$, $\text{grad } h$ refers to the usual gradient of the harmonic extension of h to D .

The following distribution function inequality encompasses the main difficulty in the proof the sufficiency part of Theorem 0.9. For a fixed $z \in D$, recall that

$$F_z = (H_{\bar{f}}k_z) \otimes (H_gk_z) - (H_{\bar{g}}k_z) \otimes (H_fk_z).$$

The Lebesgue measure of a subset E of ∂D will be denoted by $|E|$. For $z \in D$, let I_z denote the closed subarc of ∂D with center $\frac{z}{|z|}$ and measure $(1 - |z|)$ and let $\delta(z) = 1 - |z|^2$.

Distribution function inequality: *Let $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$, and let u and v be any two functions in H^2 . Let z be a point in D such that $|z| > 1/2$. Then for any $l > 2$, and for a $a > 0$ sufficiently large, there are*

positive constants K_a and r with $1 < r < 2$, depending only on l and a such that

$$\left| \left\{ w \in I_z : B_{2\delta(z)}(u, v)(w) < a^2 N_l \|F_z\|^{(l-1)/l} \inf_{w \in I_z} \Lambda_r(u)(w) \inf_{w \in I_z} \Lambda_r(v)(w) \right\} \right| \geq K_a |I_z|,$$

and $\lim_{a \rightarrow \infty} K_a = 1$. Here N_l is a positive constant depending only on l .

To prove the necessary part of Theorem 0.9 we need the following lemma from [25]. Let φ_z be the Möbius transform associated with z .

Lemma 3.1. *Let K be a compact operator on H^2 . Then*

$$\lim_{|z| \rightarrow 1} \|K - T_{\varphi_z}^* K T_{\varphi_z}\| = 0.$$

Now we are ready to present the proof of Theorem 0.9.

Proof of Theorem 0.9. Assume that $T_f T_g - T_g T_f$ is compact. Since

$$T_f T_g - T_g T_f = H_{\bar{f}}^* H_g - H_{\bar{g}}^* H_f,$$

we may use Lemma 3.1 to obtain

$$(3.2) \quad \lim_{|z| \rightarrow 1} \left\| H_{\bar{f}}^* H_g - H_{\bar{g}}^* H_f - T_{\varphi_z}^* \left(H_{\bar{f}}^* H_g - H_{\bar{g}}^* H_f \right) T_{\varphi_z} \right\| = 0.$$

We now introduce the antiunitary operator V on L^2 given by

$$Vh(w) = \overline{wh(w)}.$$

One can easily check that

$$V = V^{-1} \quad \text{and} \quad V^{-1}(1 - P)V = P.$$

From the proof of Lemma 1 of [25] we know that

$$V^* \left[H_{\bar{f}}^* H_g - T_{\varphi_z}^* H_{\bar{f}}^* H_g T_{\varphi_z} \right] V = H_{\bar{f}} k_z \otimes H_g k_z.$$

Thus we know from (3.2) above that

$$\begin{aligned} & [T_f, T_g] - T_{\varphi_z}^* [T_f, T_g] T_{\varphi_z} \\ &= V^* [(H_{\bar{f}} k_z) \otimes (H_g k_z) - (H_{\bar{g}} k_z) \otimes (H_f k_z)] V. \end{aligned}$$

By Lemma 3.1 we obtain (0.10) to finish the proof of the necessary part of Theorem 0.9.

We turn now to the converse. Assume

$$\lim_{|z| \rightarrow 1} \|F_z\| = 0.$$

We will use the distribution function inequality to show that $T_f T_g - T_g T_f$ is compact. The idea to use the distribution function inequality to study Toeplitz operators first appeared in the Axler, Chang and Sarason paper [2].

Let u and v be two functions in H^2 so that $\|u\|_2 \leq 1$ and $\|v\|_2 \leq 1$. Then

$$\begin{aligned} \langle (T_f T_g - T_g T_f)u, v \rangle &= \langle (H_{\bar{g}}^* H_f - H_{\bar{f}}^* H_g)u, v \rangle \\ &= \langle H_f u, H_{\bar{g}} v \rangle - \langle H_g u, H_{\bar{f}} v \rangle. \end{aligned}$$

Since $H_f u$ is orthogonal to H^2 , we see that $H_f u(0) = 0$. Thus by the Littlewood-Paley formula ([10], p. 236), we have

$$\begin{aligned} &\langle (T_f T_g - T_g T_f)u, v \rangle \\ &= \int_D \left[\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)} \right] \log \frac{1}{|z|} dA(z). \end{aligned}$$

For $1/2 < R < 1$, we let

$$I_R = \int_{|z|>R} \left[\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)} \right] \log \frac{1}{|z|} dA(z)$$

and

$$II_R = \int_{|z|\leq R} \left[\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)} \right] \log \frac{1}{|z|} dA(z).$$

One easily checks that there is a compact operator K_R such that

$$II_R = \langle K_R u, v \rangle.$$

Thus, if we show that $I_R \rightarrow 0$ uniformly for u and v as $R \rightarrow 1$, then $\|T_f T_g - T_g T_f - K_R\| \rightarrow 0$, and we are done. The rest of the proof will be devoted to showing that $I_R \rightarrow 0$ as $R \rightarrow 1$.

Using the l and the corresponding r from the distribution function inequality, choose $z \in D$, and a fixed constant $a \geq 1$ for which the Distribution Inequality holds; that is

$$\left| \left\{ w \in I_z : B_\gamma(u, v)(w) \leq a^2 N_l \|F_z\|^{(l-1)/l} \inf_{w \in I_z} \Lambda_r u(w) \inf_{w \in I_z} \Lambda_r v(w) \right\} \right| \geq K_a |I_z|.$$

For $w \in \partial D$, let

$$\rho(w) = \max \left\{ \gamma : B_\gamma(u, v)(w) \leq a^2 N_l \sup_{|z|>R} \|F_z\|^{(l-1)/l} \Lambda_r u(w) \Lambda_r v(w) \right\}.$$

Let dw be the normalized arc length measure on the unit circle. Then

$$\begin{aligned} & \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \\ & \leq a^2 N_l \sup_{|z|>R} \|F_z\|^{(l-1)/l} \int_{\partial D} \Lambda_r u(w) \Lambda_r v(w) dw \\ & \leq a^2 N_l \sup_{|z|>R} \|F_z\|^{(l-1)/l} \|\Lambda_r u\|_2 \|\Lambda_r v\|_2. \end{aligned}$$

Now r was chosen so that $1 < r < 2$, so $\frac{2}{r} > 1$, and hence ([10], p. 24) there exists a positive constant A_r such that

$$\|\Lambda_r u\|_2 = \|M(|u|^r)^{1/r}\|_2 = \|M(|u|^r)\|_{2/r}^{1/r} \leq A_r (\| |u|^r \|_{2/r})^{1/r}.$$

So

$$\|\Lambda_r u\|_2 \leq A_r \|u\|_2.$$

Similarly, there is a constant $A'_r > 0$ such that

$$\|\Lambda_r v\|_2 \leq A'_r \|v\|_2.$$

Thus

$$(3.3) \quad \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \leq a^2 N_l A_r A'_r \sup_{|z|>R} \|F_z\|^{(l-1)/l} \|u\|_2 \|v\|_2.$$

On the other hand, let χ_w denote the characteristic function of $\Gamma_{w, \rho(w)}$. Then

$$\begin{aligned} & \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \\ & = \int_{\partial D} \int_{\Gamma_{w, \rho(w)}} |\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)}| dA(z) dw. \end{aligned}$$

So

$$\begin{aligned} & \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \\ & \geq \int_{\partial D} \int_{|z|>R} \chi_w(z) |\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)}| dA(z) dw. \end{aligned}$$

Now the distribution function inequality tells us that $\rho(w) \geq 2(1 - |z|^2)$ on a subset E_z of I_z satisfying

$$|E_z| \geq K_a |I_z|.$$

Now for $w \in E_z$, we have $w \in I_z$. Thus recalling the definition of I_z and noting that $\rho(w) \geq 2(1 - |z|^2)$ we have

$$|w - z| \leq \left| w - \frac{z}{|z|} \right| + \left| \frac{z}{|z|} - z \right| \leq 2(1 - |z|) \leq \rho(w).$$

Therefore, for $w \in E_z$, we have $z \in \Gamma_{w,\rho(w)}$. Hence $\chi_w(z) = 1$ on E_z . So,

$$\begin{aligned} & \int_{\partial D} \int_{|z|>R} \chi_w(z) \left| \text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g(u)) \overline{\text{grad}(H_{\bar{f}} v)} \right| dA(z) dw \\ &= \int_{|z|>R} \left[\int_{\partial D} \chi_w(z) dw \right] \left| \text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} \right. \\ & \quad \left. - \text{grad}(H_g(u)) \overline{\text{grad}(H_{\bar{f}} v)} \right| dA(z) dw. \end{aligned}$$

Since $\chi_w(z) = 1$ on E_z , we have

$$\begin{aligned} & \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \\ & \geq \int_{|z|>R} |E_z| \left| \text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g(u)) \overline{\text{grad}(H_{\bar{f}} v)} \right| dA(z). \end{aligned}$$

But, $|E_z| \geq K_a(1 - |z|^2)$, so

$$\begin{aligned} & \int_{\partial D} B_{\rho(w)}(u, v)(w) dw \\ & \geq K_a \int_{|z|>R} \left| \text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} \right. \\ & \quad \left. - \text{grad}(H_g(u)) \overline{\text{grad}(H_{\bar{f}} v)} \right| (1 - |z|^2) dA(z). \end{aligned}$$

Since

$$I_R = \int_{|z|>R} \left[\text{grad}(H_f u) \overline{\text{grad}(H_{\bar{g}} v)} - \text{grad}(H_g u) \overline{\text{grad}(H_{\bar{f}} v)} \right] \log \frac{1}{|z|} dA(z),$$

we have,

$$\int_{\partial D} B_{\rho(w)}(u, v)(w) dw \geq K_a |I_R|.$$

Putting this together with (3.3), we see that

$$|I_R| \leq C \sup_{|z|>R} a^2 \|F_z\|^{(l-1)/l} \|u\|_2 \|v\|_2.$$

Note that $\lim_{|z| \rightarrow 1} \|F_z\| = 0$. Thus we get that $|I_R| \rightarrow 0$ as $R \rightarrow 1$, concluding this part of the proof.

Applying our Theorem 0.8 to an essentially normal Toeplitz operator, we answer a question of R. Douglas [8]. Recall that an operator T is essentially normal if $TT^* - T^*T$ is compact. □

Corollary 3.4. *Let f be in L^∞ . Then the Toeplitz operator T_f is normal modulo the compacts if and only if for each support set S , either $f|_S$ is constant or there is a unimodular constant a_S such that $(f + a_S \bar{f})|_S$ is constant.*

Proof. By the definition of essentially normal operator, we have that T_f is essentially normal if and only if $T_f T_f^* - T_f^* T_f$ is compact. Then by Theorem 0.8, T_f is essentially normal if and only if for each support set S , either $f|_S$ is constant or there is a constant a_S such that $(f + a_S \bar{f})|_S$ is constant. To finish the proof we need to prove that if $f|_S$ is not constant, then a_S is unimodular. In this case, we have that both $(f + a_S \bar{f})|_S$ and $(\bar{f} + \overline{a_S} f)|_S$ are constant. Since $f|_S$ is not constant, we conclude that

$$\det \begin{bmatrix} 1 & a_S \\ \overline{a_S} & 1 \end{bmatrix} = 0.$$

So a_S is unimodular. This completes the proof. □

4. Distribution function inequality.

In this section we will prove the distribution function inequality. It involves the Lusin area integral. For h in $L^1(\partial D)$, we define the truncated Lusin area integral of h to be

$$A_\epsilon(h)(w) = \left[\int_{\Gamma_{w,\epsilon}} |\text{grad} h(z)|^2 dA(z) \right]^{1/2}.$$

Let f and g be in $L^\infty(\partial D)$ and u and v be in $H^2(\partial D)$. Recall that the generalized area integral was defined to be

$$\begin{aligned} & B_\gamma(u, v)(w) \\ &= \int_{\Gamma_{w,\gamma}} \left| \text{grad} (H_f u) \overline{\text{grad} (H_{\bar{g}} v)} - \text{grad} (H_g u) \overline{\text{grad} (H_{\bar{f}} v)} \right| dA(z). \end{aligned}$$

Proof of the distribution function inequality.

Assume that f, g are in L^∞ so that $\|f\|_\infty \leq 1$ and $\|g\|_\infty \leq 1$. Let u, v be in H^2 .

For a fixed $z \in D$, either $\|H_f k_z\|_2 \leq \|H_g k_z\|_2$ or $\|H_g k_z\|_2 \leq \|H_f k_z\|_2$. Without loss of generality, we may assume that $\|H_g k_z\|_2 \leq \|H_f k_z\|_2$. Let b be $-\frac{\langle H_g k_z, H_f k_z \rangle}{\|H_f k_z\|_2^2}$ if $H_f k_z$ is not zero, and 1 if $H_f k_z$ is zero. Then $|b| \leq 1$, and $H_f k_z$ is orthogonal to $H_{g+bf} k_z$. By Lemma 2.8, we see that

$$2\|F_z\| \geq \left[\|H_f k_z\|_2^2 \|H_{g+bf} k_z\|_2^2 + \|H_{\bar{f}} k_z\|_2^2 \|H_{g+bf} k_z\|_2^2 \right]^{1/2}.$$

For $h \in L^2$, let h_+ and h_- denote Ph and $(1 - P)h$, respectively. Let F and G be in L^2 . Fix $l > 2$. Then by Theorem 6 [25] we see that there are numbers $p, r \in (1, 2)$, a , and a constant $C_a > 0$ with $1/l + 1/r = 1/p$, such

that for $|z| > 1/2$ and $a > 0$ sufficiently large,

$$(4.1) \quad \left\{ w \in I_z : A_{2\delta(z)}(H_F u)(w) A_{2\delta(z)}(H_G v)(w) < a^2 [|F_- - F_-(z)|^l(z)]^{1/l} \times \right. \\ \left. \times [|G_- - G_-(z)|^l(z)]^{1/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\} \geq C_a |I_z|.$$

Moreover, the constant C_a can be chosen to satisfy $\lim_{a \rightarrow \infty} C_a = 1$. Applying the above distribution function inequality to the functions f and $\overline{g + bf}$, we get

$$\left\{ w \in I_z : A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}} v \right) (w) < a^2 [|f_- - f_-(z)|^l(z)]^{1/l} \times \right. \\ \left. \times [(g + bf)_+ - (g + bf)_+(z)]^l(z)]^{1/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\} \geq C_a |I_z|.$$

Let $Q = 1 - P$. Then Q is bounded on L^p for $1 < p < \infty$. So for any $h \in L^\infty$ and l with $2 < l < 3$, by Hölder's inequality, we have

$$\|Q(h)\|_l \leq (N_l/2)^{1/2} \|Q(h)\|_2^{(l-1)/l} \|h\|_\infty^{1/l}$$

for some constant N_l depending only on l .

Noting that $\|f\|_\infty \leq 1$, $\|g\|_\infty \leq 1$ and $|b| \leq 1$, we get

$$\begin{aligned} & [| (g + bf)_+ - (g + bf)_+(z) |^l(z)]^{1/l} [|f_- - f_-(z)|^l(z)]^{1/l} \\ & \leq (N_l/2) [| (g + bf)_+ - (g + bf)_+(z) |^2(z)]^{(l-1)/(2l)} \\ & \quad \cdot [|f_- - f_-(z)|^2(z)]^{(l-1)/(2l)} \\ & = (N_l/2) \left[\| H_{\overline{g+bf}} k_z \|_2 \right]^{(l-1)/l} [\| H_f k_z \|_2]^{(l-1)/l} \\ & \leq (N_l/2) \|F_z\|^{(l-1)/l}. \end{aligned}$$

Therefore

$$\left\{ w \in I_z : A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}} v \right) (w) < a^2 [|f_- - f_-(z)|^l(z)]^{1/l} \times \right. \\ \left. \times [(g + bf)_+ - (g + bf)_+(z)]^l(z)]^{1/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\}$$

is a subset of

$$\left\{ w \in I_z : A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}} v \right) (w) \right. \\ \left. < a^2 (N_l/2) \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\}.$$

The distribution function inequality (4.1) gives

$$(4.2) \quad \left\{ w \in I_z : A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}v} \right) (w) \right. \\ \left. < a^2(N_l/2) \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\} \geq C_a |I_z|.$$

Similarly, we can establish the following distribution function inequality

$$(4.3) \quad \left\{ w \in I_z : A_{2\delta(z)} \left(H_{\overline{f}u} \right) (w) A_{2\delta(z)}(H_{g+bf}v)(w) \right. \\ \left. < a^2(N_l/2) \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\} \geq C_a |I_z|.$$

Note that

$$B_\gamma(u, v)(w) \\ = \int_{\Gamma_{w,\gamma}} \left| \text{grad} \left(H_{g+bf}u \right) \overline{\text{grad} \left(H_{\overline{f}v} \right)} - \text{grad} \left(H_f u \right) \overline{\text{grad} \left(H_{\overline{g+bf}v} \right)} \right| dA(z),$$

for any constant b . Thus by the Hölder inequality we get

$$B_{2\delta(z)}(u, v)(w) \\ \leq A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}v} \right) (w) \\ + A_{2\delta(z)} \left(H_{\overline{f}u} \right) (w) A_{2\delta(z)}(H_{g+bf}v)(w),$$

and so the intersection of

$$\left\{ w \in I_z : A_{2\delta(z)}(H_f u)(w) A_{2\delta(z)} \left(H_{\overline{g+bf}v} \right) (w) \right. \\ \left. < a^2(N_l/2) \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\}$$

and

$$\left\{ \lambda \in I_z : A_{2\delta(z)} \left(H_{\overline{f}u} \right) (w) A_{2\delta(z)}(H_{g+bf}v)(w) \right. \\ \left. < a^2(N_l/2) \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\}$$

is contained in

$$\left\{ w \in I_z : B_\epsilon(u, v)(w) < a^2 N_l \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\}.$$

Combining (4.2) and (4.3) gives

$$\left| \left\{ w \in I_z : B_\epsilon(u, v)(w) < a^2 N_l \|F_z\|^{(l-1)/l} \inf_{\lambda \in I_z} \Lambda_r(u)(\lambda) \inf_{\lambda \in I_z} \Lambda_r(v)(\lambda) \right\} \right| \geq (2C_a - 1) |I_z|.$$

Note that $\lim_{a \rightarrow 1} C_a = 1$. We let $K_a = 2C_a - 1$, to finish the proof of the distribution function inequality.

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