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Moriah and Schultens have demonstrated that an irreducible Heegaard splitting of an orientable Seifert fibered space over an orientable base surface is either vertical or horizontal. In this paper it is determined precisely which vertical and horizontal splittings are irreducible. Let M be a Seifert fibered space which admits a horizontal splitting at the fiber f . If the genus of the horizontal splitting at f is less than the genus of the vertical splittings, its genus will be minimal and the splitting irreducible. Otherwise, this splitting will be irreducible if and only if the multiplicity of the fiber f is strictly greater than the least common multiple of the multiplicities of the other fibers. In particular, each Seifert fibered space possesses at most one irreducible horizontal splitting. The vertical splittings will be reducible if and only if M has a horizontal splitting with genus strictly less than the genus of the vertical splittings.

1. Introduction.

Throughout this paper a Seifert fibered space, M , will denote a closed, orientable Seifert fibered space over an orientable base surface B . Then M may be written $M = \{g_0, e_0 | (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_n, \beta_n)\}$, where g_0 is the genus of the base surface B , e_0 is the Euler number of M , n is the number of exceptional fibers, and (α_i, β_i) are the Seifert invariants of the exceptional fiber f_i . In particular, α_i denotes the *multiplicity* of the exceptional fiber f_i .

If M has less than three exceptional fibers and base surface a 2-sphere then M is S^3 or a lens space (or $S^2 \times S^1$) and the irreducible Heegaard splittings of M are classified by Waldhausen [W] and Bonahon and Otal [BO], respectively. (S^3 has a unique horizontal splitting and lens spaces have a unique vertical splitting.) We therefore assume that $n \geq 3$ or $g_0 \geq 1$.

The main results of [MS] and [Schu2] imply that irreducible Heegaard splittings of Seifert fibered spaces are vertical or horizontal. Every Seifert fibered space M possesses vertical splittings and when $n \geq 2$ they may be thought of as a Heegaard splittings of M which are also Heegaard splittings of the product manifold $M - N(E)$, where $N(E)$ is a neighborhood of the

exceptional fibers. When $n < 2$ it is necessary to also remove the neighborhood of one or two regular fibers. Nevertheless, each vertical splitting of M will have the same genus, denoted g_v , and will be identified up to isotopy by a non-trivial partition of the exceptional fibers (and possibly one or two regular fibers).

A horizontal splitting exists if and only if the manifold M can be described as a Dehn filling on a surface bundle $F \times S^1$, where the surface F has a single boundary component, the bundle has finite monodromy and the meridian of the attached solid torus intersects the curve ∂F exactly once. In this case the core of the attached solid torus will be a regular or exceptional fiber f and we say that M has a horizontal splitting at f . Necessary and sufficient conditions for this to occur are given in terms of the Seifert invariants in [MS]. The genus of the horizontal splitting at f will be given by $g_h(f) = 2g(F)$, where $g(F)$ is the genus of the surface F .

In this paper we will decide the irreducibility of the vertical and horizontal splittings. Typically, $g_h(f)$ will be greater than the genus of the vertical splittings g_v . When it is not the horizontal splitting will be called *small*. As the genus of a horizontal splitting is determined by a bundle structure on the complement of the fiber f , $M_0 = M - N(f)$, the presence of a small horizontal splitting places restrictions on the manifold M_0 , and hence upon the manifold M . These restrictions will be quantified by Theorem 5.3.

Theorem. *Let M admit a horizontal splitting at f . This splitting is small if and only if one the following holds:*

- (1) $M_0 = \{g_0 | \emptyset\}$, $g_0 \geq 1$, f is regular or exceptional;
- (2) $M_0 = \{0 | (2, 1), \dots, (2, 1)\}$, $n_0 \geq 3$ and odd, f is exceptional;
- (3) $M_0 = \{0 | (2, 1), \dots, (2, 1)\}$, $n_0 \geq 3$ and odd, f is regular;
- (4) $M_0 = \{0 | (2, 1), (3, \beta_2)\}$, f is exceptional;
- (5) $M_0 = \{0 | (2, 1), (4, \beta_2)\}$, f is exceptional;
- (6) $M_0 = \{0 | (3, \beta_1), (3, \beta_2)\}$, f is exceptional.

Furthermore, if (1) or (2) holds then $g_h(f) = g_v - 1$. If one of (3)-(6) holds then $g_h(f) = g_v$.

This characterization will yield Corollary 5.5.

Corollary. *A small horizontal splitting is of minimal genus and hence irreducible.*

Non-small horizontal splittings are by definition not minimal genus and it is thus considerably more difficult to determine their irreducibility. This requires a close analysis of once punctured surface bundles and an application of a slight generalization of Casson's rectangle condition ([CG1]). The result of this analysis is Theorems 6.1 and 8.1, summarized as follows:

Theorem. *Let M admit a non-small horizontal splitting at the fiber f . Then this splitting will be irreducible if and only if the multiplicity of the fiber f is strictly greater than the least common multiple of the multiplicities of the other exceptional fibers.*

From this theorem and the characterization in Theorem 5.3 we obtain Corollary 8.2.

Corollary. *The Seifert fibered space M admits an irreducible horizontal splitting at at most one fiber f .*

Using the work of [LM], [M], and [Schu3] the irreducibility of the vertical splittings is determined in Theorem 5.6.

Theorem. *The vertical splittings will be reducible if and only if M has a horizontal splitting with genus strictly less than the genus of the vertical splittings. In this case there is a horizontal splitting of genus one less than the vertical splittings.*

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2. Preliminaries.

The following notation and definitions are standard. For more information regarding Heegaard splittings see [RS], [ST2], or [Scha].

A *handlebody* is a homeomorph of a closed regular neighborhood of a connected graph in S^3 . A *Heegaard splitting* of a closed orientable 3-manifold M is a decomposition $M = H_1 \cup_G H_2$, where H_1 and H_2 are handlebodies such that $\partial H_1 = \partial H_2 = H_1 \cap H_2 = G$. The *genus* of the Heegaard splitting is the genus of the surface G and the *genus of M* , $g(M)$ is the least genus of all Heegaard splittings of M .

An *essential disk* in a manifold with boundary M is a properly embedded disk D such that ∂D is an essential loop in ∂M . A Heegaard splitting is: *stabilized* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 \cap \partial D_2 = \{point\}$, *reducible* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 = \partial D_2$, and *weakly reducible* if there are essential disks $D_1 \subset H_1$ and $D_2 \subset H_2$ such that $\partial D_1 \cap \partial D_2 = \emptyset$.

3. Vertical Heegaard Splittings.

In this section we give a brief description of vertical Heegaard splittings of Seifert fibered spaces and state a few well known facts that will be needed in later sections. A more detailed treatment of vertical splittings is given in [BZ] (see also [MS] and [Schu1]).

By identifying each fiber to a point we obtain a projection, $\rho : M \rightarrow B$, from the Seifert fibered space onto the base surface B . If f is a fiber then

$e = \rho(f)$ is a point on the base surface B . We will call e the *anchor point* corresponding to the fiber f .

A vertical Heegaard splitting of a Seifert fibered space is identified by choosing a non-trivial partition of the exceptional fibers contained in M . (If the number of exceptional fibers is less than two then we cannot form a non-trivial partition and must add 1 or 2 regular fibers to the exceptional fibers and then partition.) Choose a minimal graph Σ in the base surface B so that $B - \Sigma$ is a disk and Σ is disjoint from the anchor points of the fibers in the partition. Take arcs on B and adjoin to Σ the anchor points of each of the fibers from the first half of the partition, and for all but one of the fibers in the second half, adjoin to Σ a loop on B encircling its anchor point. We may lift this graph to M and attach to it the fibers from the first half of the partition. A closed regular neighborhood of the resulting graph, Σ' , is clearly a handlebody. Moreover, the complement of an open regular neighborhood of Σ' is also a handlebody (see [Schu1]).

Definition 3.1. Let Σ' be obtained as above. The Heegaard splitting $M = \overline{N(\Sigma')} \cup N(\Sigma')^C$ is called a **vertical splitting** of M .

In this manner we can construct vertical splittings for any Seifert fibered space. The isotopy class of the vertical splitting obtained will be well-defined up to the choice of partition, a fact established in [LM]. However, the choice of a different partition produces a vertical splitting which may or may not be isotopic to the first. (A great deal is known about the equivalence of vertical splittings, see [L], [LM], and [M].) As the genus of a vertical splitting is independent of the partition chosen and depends solely on the genus of the base surface, g_0 , and the number of exceptional fibers, n , all vertical splittings of M have the same genus and we may define:

Definition 3.2. The **vertical genus** of M , g_v , is the genus of each of the vertical splittings.

The vertical genus was first calculated by Boileau and Zieschang ([BZ]) and is given by:

$$\begin{aligned} \text{If } n < 2 \text{ then } g_v &= 2g_0 + 1. \\ \text{If } n \geq 2 \text{ then } g_v &= 2g_0 + n - 1. \end{aligned}$$

For “most” Seifert fibered spaces the vertical splittings will be of minimal genus and the vertical genus will coincide with the Heegaard genus of the manifold; the exceptions to this rule will be determined in Theorem 5.3 and Corollary 5.7.

4. Once Punctured Surface Bundles and Horizontal Heegaard Splittings.

This section contains background material regarding the horizontal construction that will be essential in proving the main theorems.

Choose a fiber, f , (regular or exceptional) and remove a solid torus neighborhood of this fiber from M . Let (α, β) be the Seifert invariants of the fiber f . Then $M_0 = M - N(f) = \{g_0 | (\alpha_1, \beta_1), \dots, (\alpha_{n_0}, \beta_{n_0})\}$ is an orientable Seifert fibered space over an orientable base surface $B_0 = B - N(pt)$ with n_0 exceptional fibers (where $n_0 = n$ or $n - 1$) and a single boundary component.

It is known ([J, VI.32]) that a Seifert fibered space with a single boundary component fibers as a periodic surface bundle over the circle, $M_0 = F \tilde{\times} S^1$, where the fiber F is a connected and orientable surface and the orbit of any point under the S^1 action is a fiber in the Seifert fibering. We can write $M_0 = F \times I / (x \times \{1\} \sim h(x) \times \{0\})$ where $h : F \rightarrow F$ is the periodic homeomorphism associated with the bundle $M_0 = F \tilde{\times} S^1$. The periodic homeomorphism h will have degree $d = lcm\{\alpha_1, \dots, \alpha_{n_0}\}$. Define the equivalence, $x \sim y$, if $y = h^i(x)$ for some i . Let $[x]$ denote the equivalence class of x . Then the map $\rho : M_0 \rightarrow B_0$ given by $\rho(x \times i) = [x]$ is the standard projection map of the Seifert fibered space M_0 . If we restrict to F , $\rho : F \rightarrow B_0$ is a branched covering. Let e_1, \dots, e_{n_0} be the anchor points on B_0 of the exceptional fibers f_1, \dots, f_{n_0} which have multiplicities $\alpha_1, \dots, \alpha_{n_0}$. Then $\rho : F \rightarrow B_0$ will have degree $d = lcm\{\alpha_1, \dots, \alpha_{n_0}\}$ and branch points e_1, \dots, e_{n_0} with corresponding branching indices $\alpha_1, \dots, \alpha_{n_0}$. In Figure 1 there are two anchor points with multiplicities $\alpha_1 = 3, \alpha_2 = 2$ ($d = 6$).

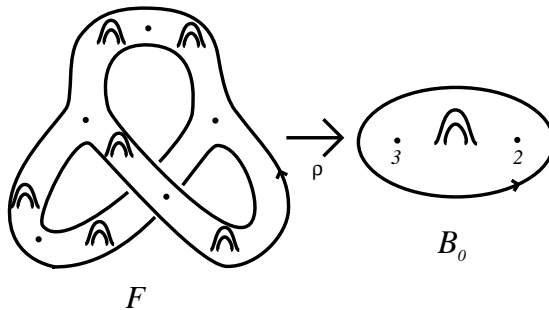


Figure 1. $\rho : F \rightarrow B_0$.

We state our first restriction: Consider only the case where the fiber F has a single boundary component. Choose two disjoint copies of the surface fiber, F_1 and F_2 , and cut along these surfaces to decompose $M_0 = F \tilde{\times} S^1$

into two pieces, $F \times I_1$ and $F \times I_2$. Label the surfaces F_1 and F_2 and orient I_1 and I_2 so that $F \times I_1^- = F_1$, $F \times I_1^+ = F_2$, $F \times I_2^- = F_2$, and $F \times I_2^+ = F_1$. We may assume that the monodromy h is the identity on $F \times I_1$ and is h on $F \times I_2$. Note that each of these pieces is a handlebody, being a punctured surface cross an interval. The boundary of M_0 is decomposed into two annuli, $\partial M_0 = A_1 \cup A_2$ where A_i is the annulus $\partial F \times I_i$.

We obtain M by gluing the solid torus neighborhood of f , $\overline{N(f)}$, to the boundary of M_0 . We now introduce the second restriction: The meridian m of the solid torus $\overline{N(f)}$ must intersect ∂F exactly once. Then $m_1 = m \cap A_1$ and $m_2 = m \cap A_2$ will each be a single arc and the manifold M may be thought of as the quotient $M_0/(A_1 = A_2)$ where the gluing of A_1 and A_2 is defined by identifying the arcs m_1 and m_2 . (See Figure 2.)

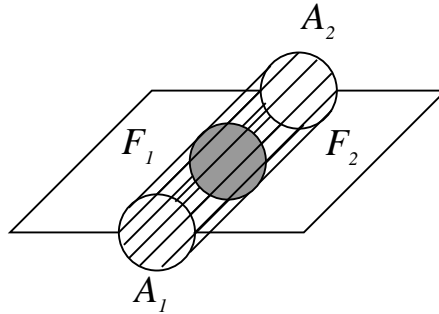


Figure 2. $M = M_0/(A_1 = A_2)$.

We have expressed M as the union of two handlebodies glued along their common boundary, this is a Heegaard splitting of M . If the surface fiber F does not have a single boundary component or the meridian disk does not meet the boundary of the surface fiber exactly one time, the manifold M will fail to have a horizontal Heegaard splitting corresponding to the fiber f . (However, it may have a horizontal splitting corresponding to removing the neighborhood of a different regular or exceptional fiber.) A test in terms of the Seifert invariants is given in [MS]. Assuming that the two restrictions are met we define:

Definition 4.1. The Heegaard splitting $M = (F \times I_1) \cup_G (F \times I_2)$, where $G = F_1 \cup A_1 \cup F_2 = F_2 \cup A_2 \cup F_1$ is called a **horizontal splitting** of M at f .

Note that the horizontal splitting at f will be determined only up to homeomorphism as the fibration $M_0 = F \tilde{\times} S^1$ is determined only up to homeomorphism of M_0 . This author suspects (but has not proven) that some Seifert fibered spaces will possess an infinite number of non-isotopic but

homeomorphic splittings obtained by twisting a given horizontal splitting in vertical tori.

We now calculate the genus of the horizontal splittings. This calculation was made in [MS] and is included here for the sake of completeness. Remove disk neighborhoods of the anchor points e_i on B_0 to obtain $B'_0 = B_0 - N(\cup_{i=1, \dots, n_0} e_i)$. Similarly remove the lifts of these neighborhoods from F to obtain $F' = F - N(\rho^{-1}(\cup_{i=1, \dots, n_0} e_i))$. The restriction, $\rho : F' \rightarrow B'_0$, is a (non-branched) covering of degree d , under which the Euler characteristic is multiplicative. Thus,

$$\begin{aligned} \chi(B'_0) &= \chi(B_0) - n_0 = 1 - 2g_0 - n_0 \\ \chi(F') &= d\chi(B_0) = d(1 - 2g_0 - n_0). \end{aligned}$$

We now replace the disks that we removed from F to get F' . In doing so we observe that the disk containing e_i will be branched covered by d/α_i disjoint disks each with branching index α_i .

$$\begin{aligned} \chi(F) &= \chi(F') + \sum_{i=1}^{n_0} d/\alpha_i \\ &= d \left(1 - 2g_0 - n_0 + \sum_{i=1}^{n_0} 1/\alpha_i \right). \end{aligned}$$

Let $g_h(f)$ denote the genus of a horizontal splitting at f . If M admits a horizontal splitting at f , $g_h(f)$ will be twice the genus of the punctured surface F .

$$\begin{aligned} g_h(f) &= 2g(F) = 1 - \chi(F) \\ &= 1 + d \left(n_0 + 2g_0 - 1 - \sum_{i=1}^{n_0} 1/\alpha_i \right). \end{aligned}$$

If M does not admit a horizontal splitting at f define

$$g_h(f) = \infty.$$

Definition 4.2. Call $g_h = \min \{g_h(f) \mid f \text{ is a regular or exceptional fiber}\}$, the **horizontal genus** of the manifold M .

If M does not admit a horizontal splitting $g_h = \infty$.

We will need a more concise description of ∂M_0 before the identification of A_1 and A_2 is made. We will denote by $\langle a, b \rangle$ the oriented intersection number of the curves a and b on ∂M_0 .

Recall that the horizontal construction requires that the meridian m of the solid torus $\overline{N(f)}$ meets ∂F precisely once. That means that $(\partial F, m)$ is a basis for the homology of the boundary torus ∂M_0 . We will use this coordinate system for the remaining discussion. Let $x \in \partial B_0$. Then $f_r = \rho^{-1}(x)$ is a

regular fiber on ∂M_0 . As ρ restricts to a branched covering of degree d on F we may orient ∂F such that $\langle \partial F, f_r \rangle = d$.

Furthermore, since f is a fiber of multiplicity α we can orient m so that $\langle f_r, m \rangle = \alpha$. Recall that we have restricted the monodromy h to the piece $F \times I_2$, thus $f_r = \alpha \cdot \partial F + d \cdot m$ will appear as pictured in Figure 3.

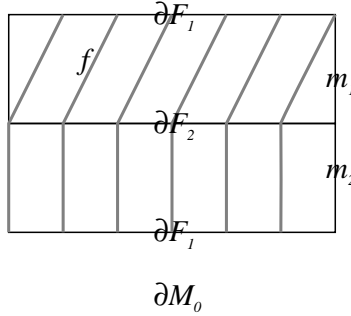


Figure 3. ∂M_0 in $(\partial F, m)$ coordinates.

As the fiber f_r consists of a single component, we have established the following lemma.

Lemma 4.3. *Let M admit a horizontal splitting at the fiber f . Then α and d are relatively prime.*

Additionally, since $d = lcm\{\alpha_1, \dots, \alpha_{n_0}\}$ we have:

Lemma 4.4. *Let M admit a horizontal splitting at the fiber f . Then α is relatively prime to the multiplicity of each of the other fibers.*

Definition 4.5. A **regular arc** is an arc of the form $x \times I_i$ where $x \in \partial F_i$.

Define a metric on ∂B_0 so that the length of ∂B_0 is one and lift it to a metric on ∂F . With this metric the length of ∂F is d .

A regular arc is a sub-arc of a regular fiber which lies in ∂M_0 . In the $(\partial F, m)$ coordinate system a regular arc on A_1 will be vertical, whereas a regular arc on A_2 will have an α shift. In the following sections it will be relevant to compare the values α and d . If $\alpha > d$ then every regular arc on A_2 will necessarily intersect the arc m_2 . Conversely, if $\alpha < d$ then there is a regular arc on A_2 that will not intersect m_2 .

5. Small Horizontal Splittings and the Irreducibility of Vertical Splittings.

In this section we classify the horizontal splittings which have genus less than the vertical splittings and demonstrate that they are irreducible. Additionally, using methods and results from [Schu1], [Schu3], and [LM] we determine under what conditions the vertical splittings are reducible.

Definition 5.1. Let M admit a horizontal splitting at the fiber f . If $g_h(f) \leq g_v$ then we say that this splitting is **small**.

A horizontal splitting is derived from a branched covering of the base surface and unless the degree of the covering, d , and the genus of the base surface, g_0 , are small, we expect the genus of the horizontal splitting to be quite large, and in particular to exceed the genus of the vertical splittings. This statement is made precise in the following lemma.

Lemma 5.2. *Let M admit a horizontal splitting at f . This splitting is small if and only if one of the following statements holds:*

- 1) $d = 1$.
- 2) $d = 2$ and $g_0 = 0$.
- 3) $g(F) = 1$ and $(g_0 = 0)$.

Proof. There are three cases according to the value of d .

Case. $d = 1$.

When $d = 1$, $n_0 = 0$ and $n \leq 1$. The horizontal genus is

$$g_h(f) = 1 + 1(0 + 2g_0 - 1) = 2g_0$$

and the vertical genus is given by

$$g_v = 2g_0 + 1.$$

Thus,

$$g_h(f) = g_v - 1$$

and the result follows in this case.

Case. $d = 2$.

When $d = 2$ we have that for all $i = 1, \dots, n_0$, $\alpha_i = 2$ and therefore $\sum_{i=1}^{n_0} 1/\alpha_i = n_0/2$. The horizontal genus is then

$$g_h(f) = 1 + 2(n_0 + 2g_0 - 1 - n_0/2) = n_0 + 4g_0 - 1.$$

If $n \geq 2$ then $g_v = 2g_0 + n - 1 \leq 2g_0 + n_0$. There is at least one multiplicity two fiber in M_0 , hence $n_0 \geq 1$. If $n < 2$ then $g_v = 2g_0 + 1 \leq 2g_0 + n_0$. Therefore,

$$g_v \leq 2g_0 + n_0.$$

Then $g_h(f) \leq g_v$ if and only if

$$n_0 + 4g_0 - 1 \leq 2g_0 + n_0$$

$$g_0 \leq 1/2$$

$$g_0 = 0,$$

as g_0 is a non-negative integer. This establishes the result for the case $d = 2$.

Case. $d \geq 3$.

When $d \geq 3$ we have that $\alpha_i \geq 2$ for every i and therefore $\sum_{i=1}^{n_0} 1/\alpha_i \leq n_0/2$. The horizontal genus is then

$$\begin{aligned} g_h(f) &\geq 1 + d \left(n_0 + 2g_0 - 1 - \sum_{i=1}^{n_0} 1/\alpha_i \right) \\ &\geq 1 + 3(n_0/2 + 2g_0 - 1) = 3/2n_0 + 6g_0 - 2. \end{aligned}$$

As in the previous case,

$$g_v \leq 2g_0 + n_0.$$

Then $g_h(f) \leq g_v$ if and only if

$$\begin{aligned} 3/2n_0 + 6g_0 - 2 &\leq 2g_0 + n_0 \\ n_0 + 4g_0 &\leq 2. \end{aligned}$$

It follows that $g_0 = 0$ and $n_0 \leq 3$. Recalling that $g_h(f) = 2g(F)$ we further calculate

$$\begin{aligned} 2g(F) = g_h(f) &\leq g_v \leq 2g_0 + n_0 \\ 2g(F) &\leq 3 \\ g(F) &\leq 3/2. \end{aligned}$$

As M is not S^3 or a lens space we always have that $g_h(f) \geq 2$ and therefore $g(F) \geq 1$. We conclude that $g(F) = 1$. By the same reasoning we always have $g_v \geq 2$. Thus, when $g(F) = 1$ we have that $g_h(f) \leq g_v$. This establishes the result for the case $d \geq 3$. □

Combining the previous lemmas with some basic facts about horizontal splittings we can give a list of the small horizontal splittings.

Theorem 5.3. *Let M have a horizontal splitting at f . This splitting is small if and only if one the following holds:*

- (1) $M_0 = \{g_0|\emptyset\}, g_0 \geq 1, f$ is regular or exceptional;
- (2) $M_0 = \{0|(2, 1), \dots, (2, 1)\}, n_0 \geq 3$ and odd, f is exceptional;
- (3) $M_0 = \{0|(2, 1), \dots, (2, 1)\}, n_0 \geq 3$ and odd, f is regular;
- (4) $M_0 = \{0|(2, 1), (3, \beta_2)\}, f$ is exceptional;
- (5) $M_0 = \{0|(2, 1), (4, \beta_2)\}, f$ is exceptional;
- (6) $M_0 = \{0|(3, \beta_1), (3, \beta_2)\}, f$ is exceptional.

Furthermore, if (1) or (2) holds then $g_h(f) = g_v - 1$. If one of (3)-(6) holds then $g_h(f) = g_v$.

Proof. There are three cases according to the value of d :

Case. $d = 1$.

Then $n_0 = 0$ and $n = 0$ or 1 . As M is not S^3 or a lens space, it follows that $g_0 \geq 1$. This is (1) where we can readily calculate that

$$\begin{aligned} g_v &= 2g_0 + 1 \\ g_h(f) &= 1 + d \left(n_0 + 2g_0 - 1 - \sum_{i=1}^{n_0} 1/\alpha_i \right) = 2g_0 \\ g_h(f) &= g_v - 1. \end{aligned}$$

Case. $d = 2$.

It follows from Lemma 5.2 that $g_0 = 0$. Then $M_0 = \{0|(2, 1), (2, 1), \dots, (2, 1)\}$ and we note that

$$\begin{aligned} \chi(F) &= d \left(\sum_{i=1}^{n_0} 1/\alpha_i - n_0 + 1 - 2g_0 \right) \\ &= 2(n_0/2 - n_0 + 1 - 0) = 2 - n_0. \end{aligned}$$

If n_0 is even the surface F will have more than one boundary component and the horizontal construction will fail. Thus n_0 is odd. Since $g_0 = 0$ we require $n \geq 3$. Then

$$\begin{aligned} g_v &= 2g_0 + n - 1 = n - 1 \\ g_h(f) &= 1 + d \left(n_0 + 2g_0 - 1 - \sum_{i=1}^{n_0} 1/\alpha_i \right) \\ &= 1 + 2(n_0 - 1 - n_0/2) = n_0 - 1. \end{aligned}$$

Either f is an exceptional fiber, $n = n_0 + 1$ and

$$g_h(f) = n = g_v - 1,$$

or f is regular, $n = n_0$ and

$$g_h(f) = n - 1 = g_v.$$

These are the cases (2) and (3).

Case. $d \geq 3$.

It follows from Lemma 5.2 that $g(F) = 1$ and $g_0 = 0$. As M is not a lens space or S^3 , $n \geq 3$ and $n_0 \geq 2$. Also each $\alpha_i \geq 2$ implies $\sum_{i=1}^{n_0} 1/\alpha_i \leq n_0/2$. We have

$$\begin{aligned}
g_h(f) &= 2g(F) \\
1 + d \left(n_0 + 2g_0 - 1 - \sum_{i=1}^{n_0} 1/\alpha_i \right) &= 2 \\
1 + 3(n_0/2 - 1) &\leq 2 \\
n_0 &\leq 8/3.
\end{aligned}$$

Hence $n_0 = 2$, f is an exceptional fiber and $n = 3$.

It is now demonstrated that $d \leq 6$. Suppose that $d > 6$. Then one of α_1, α_2 is strictly larger than 3, say $\alpha_1 > 3$. Then

$$\begin{aligned}
2 &= 1 + d(2 - 1 - 1/\alpha_1 - 1/\alpha_2) > 1 + 6(1 - 1/3 - 1/\alpha_2) \\
1/6 &> 1 - 1/3 - 1/\alpha_2 \\
2 &> \alpha_2 \\
\alpha_2 &= 1.
\end{aligned}$$

This is a contradiction, and we conclude that $d \leq 6$.

As $d = \text{lcm}(\alpha_1, \alpha_2)$ it follows that (α_1, α_2) is one of $(2, 2), (2, 3), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (5, 5), (6, 6)$. Calculate $\chi(F)$ for:

$$\begin{aligned}
(2, 2) : \chi(F) &= d \left(\sum_{i=1}^{n_0} 1/\alpha_i - n_0 + 1 - 2g_0 \right) = 2(1/2 + 1/2 - 2 + 1) = 0; \\
(2, 6) : \chi(F) &= 6(1/2 + 1/6 - 2 + 1) = -2; \\
(3, 6) : \chi(F) &= 6(1/3 + 1/6 - 2 + 1) = -3 = 1 - 2g(F) \Rightarrow g(F) = 2; \\
(4, 4) : \chi(F) &= 4(1/4 + 1/4 - 2 + 1) = -2; \\
(5, 5) : \chi(F) &= 5(1/5 + 1/5 - 2 + 1) = -3 = 1 - 2g(F) \Rightarrow g(F) = 2; \\
(6, 6) : \chi(F) &= 6(1/6 + 1/6 - 2 + 1) = -4.
\end{aligned}$$

For the pairs $(2, 2), (2, 6), (4, 4)$ and $(6, 6)$ the Euler characteristic $\chi(F)$ is even, so F has more than one boundary component and the horizontal construction fails. For $(3, 6)$ and $(5, 5)$ $g(F) = 2$ which contradicts our assumptions. We are left with the cases where the pair $(\alpha_1, \alpha_2) = (2, 3), (2, 4)$ or $(3, 3)$. The horizontal construction can indeed work in these cases. This is $(4), (5)$ or (6) with the α 's listed.

In these cases:

$$\begin{aligned}
g_v &= 2g_0 + n - 1 = 2 \\
g_h(f) &= 2g(F) = 2 \\
g_h(f) &= g_v = 2.
\end{aligned}$$

□

We can now refer to a small horizontal splitting as being one of type (1)-(6). Using this characterization, we can easily prove

Theorem 5.4. *M admits a small horizontal splitting at at most one fiber f.*

Proof. Seeking a contradiction, suppose that M admits distinct small horizontal splittings at f and f'. We know from Lemma 4.4 that the corresponding multiplicities α and α' are relatively prime.

Suppose that one of the splittings, say the one at f, is a type (1) small splitting, i.e.

$$M_0 = \{g_0|\emptyset\}, g_0 \geq 1, f \text{ is regular or exceptional.}$$

As f' belongs to M_0 it must be a regular fiber. Furthermore, f must be an exceptional fiber or these splittings are both at regular fibers and are not distinct. Corresponding to the splitting at f' we have

$$M'_0 = \{g_0|(\alpha, \beta)\}, g_0 \geq 1, f' \text{ is regular.}$$

But then $d' = \alpha > 1$, as f is not regular, and $g_0 \geq 1$ together imply that the splitting at f' is not small. This is a contradiction.

Next, suppose that no pair of the fibers in M_0 are relatively prime. Then, no matter how we select α' it will never be relatively prime to d' . This prevents M from possessing two distinct horizontal splittings (small or not) where one is a small splitting of type (2),(3),(5) or (6).

The remaining case is when both the splitting at f and the splitting at f' are type (4),

$$M_0 = M'_0 = \{0|(2, 1), (3, \beta_2)\}, f \text{ is exceptional.}$$

Again, this is impossible, as α would have to be 2 or 3 so that M'_0 appeared as a type (4) splitting. This contradicts the fact that α and d are relatively prime.

□

Finally this establishes:

Corollary 5.5. *A small horizontal splitting is of minimal genus and hence irreducible.*

Proof. Minimal genus splittings are always irreducible. If M admits a small horizontal splitting at f, then it must be of minimal genus. If there were a splitting of smaller genus, which we may assume is irreducible, by [MS] it would either be horizontal or vertical. It cannot be vertical, given the hypothesis that the splitting at f is small. Nor can it be horizontal, for this would imply that M admitted two distinct small horizontal splittings in direct contradiction with the previous lemma.

□

The following theorem determines the irreducibility of the vertical splittings. Although this particular phrasing may be new, the theorem is essentially a corollary of the earlier work of Lustig, Moriah and Schultens ([**LM**], [**Schu1**] and [**Schu3**]).

Theorem 5.6. *The following statements hold:*

- (i) $g_h < g_v$ if and only if M admits a small horizontal splitting of type (1) or (2) if and only if the vertical splittings are reducible.
- (ii) $g_h = g_v$ if and only if M admits a small horizontal splitting of type (3)-(6). If so, then the vertical splittings are irreducible.
- (iii) $g_h > g_v$ implies that the vertical splittings are irreducible.

Proof. We start with statement (i). Theorem 5.3 establishes that $g_h < g_v$ if and only if M admits a type (1) or (2) small horizontal splitting. If the vertical splittings are reducible then there must be a splitting of smaller genus and by [**MS**] it is horizontal and we have that $g_h < g_v$.

The more difficult implication is that when $g_h < g_v$ the vertical splittings are reducible. The argument is due to Schultens ([**Schu3**]). Let M admit a small horizontal splitting of type (1) or (2). Then M has a unique (up to isotopy) vertical splitting. If this splitting is type (1) then $n \leq 1$ and there will be only one non-trivial partition of a collection of 2 fibers. If the splitting is type (2) then $\beta = 1$ for all but at most one fiber in M (the fiber not of multiplicity 2). As demonstrated in the proof of [**LM**] Theorem 2.8 M then admits only one vertical splitting. (If f is a fiber with $\beta = 1$ then there is an isotopy between any two vertical splittings whose partitions differ only by f .) In [**Schu3**] Schultens shows that a stabilization of a horizontal splitting at f is a splitting of the manifold M_0 . In [**Schu1**] it is also shown that irreducible splittings of manifolds with boundary are vertical. It follows that in this case the stabilization of a horizontal splitting at f coincides with the unique vertical splitting of M . Thus the vertical splitting is a stabilization of the unique horizontal splitting (in M , not in M_0) and is reducible.

Statement (ii) is cases (3)-(6) of Theorem 5.3. As the vertical splittings are of minimal genus they are irreducible.

In the case of statement (iii) the vertical splittings are of minimal genus and hence irreducible. □

The following corollary addresses the Heegaard genus of Seifert fibered spaces. Most of this problem was solved in [**BZ**]; the few remaining cases follow from the results in [**MS**]. We include it here for completeness.

Corollary 5.7. *The Heegaard genus of a Seifert fibered space M is equal to the vertical genus unless M admits a type (1) or (2) small horizontal*

splitting. In the latter case, the Heegaard genus of M is one less than the vertical genus.

6. Reducibility of Horizontal Splittings.

In this section we will prove the first half of the main theorem.

Theorem 6.1. *Let M admit a horizontal splitting at f which is not small. If $\alpha < d$ then this splitting is reducible.*

The proof is postponed until later in this section. Casson and Gordon demonstrated that a weakly reducible Heegaard splitting is either reducible or the manifold in question contains an incompressible surface. Although Seifert fibered spaces usually contain incompressible surfaces, in [Schu2] Schultens demonstrates that weakly reducible horizontal splittings are indeed reducible. We therefore seek to prove that these horizontal splittings are weakly reducible. The following lemma will be our primary tool in establishing the theorem. Recall that an *essential arc* in a surface F is a properly embedded arc which does not cobound a disk on F with an arc on ∂F .

Lemma 6.2. *Let M admit a horizontal splitting at a fiber f so that $\alpha < d$. If there are essential arcs γ_1 and γ_2 in F such that*

- (i) $\gamma_1 \cap (\gamma_2 \cup h(\gamma_2)) = \emptyset$, and
- (ii) *There is an interval $I \subset \partial F$ of length less than 1 that contains, in sequence, both of the endpoints of γ_1 followed by both of the endpoints of γ_2 . (The orientation on I is given by the orientation on ∂F),*

then this splitting is weakly reducible (and hence reducible).

Proof. Note that we do not require that $\gamma_2 \cap h(\gamma_2) = \emptyset$.

Claim. The length of I can be taken to be arbitrarily small.

The endpoints of $h(\gamma_2)$ are a positive integer distance from the endpoints of γ_2 and thus cannot be contained in the interval I of length less than one. Therefore we may choose a neighborhood of I which is disjoint from $h(\gamma_2)$. Within this neighborhood we can isotope the arcs so that their endpoints are arbitrarily close together. This can be done while maintaining $\gamma_1 \cap (\gamma_2 \cup h(\gamma_2)) = \emptyset$.

Define the product disks $D_1 = \gamma_1 \times I_1$ and $D_2 = \gamma_2 \times I_2$. These disks are essential in the handlebodies $H_1 = F \times I_1$ and $H_2 = F \times I_2$. We now check that the boundaries of these disks are disjoint on the splitting surface $G = F_1 \cup A_1 \cup F_2 = F_2 \cup A_2 \cup F_1$ by analyzing the intersection $\partial D_1 \cap \partial D_2$ as restricted to each of the pieces, F_1, F_2 and $A = A_1 = A_2$.

On F_1 we have $D_1 \cap F_1 = \gamma_1$ and $D_2 \cap F_1 = h(\gamma_2)$ which are disjoint by hypothesis. Similarly on F_2 , $D_1 \cap F_2 = \gamma_1$ and $D_2 \cap F_2 = \gamma_2$ are disjoint by assumption.

Before we make the identification $A = A_1 = A_2$ we note that the boundary of D_1 on ∂M_0 is two regular arcs on A_1 which are vertical and that the boundary of D_2 on ∂M_0 is two regular arcs on A_2 which have an α shift. See Figure 4. We may take the endpoints of the arcs to be arbitrarily close together. In particular as $\alpha < d$ we may make them close enough so that after identifying A_1 to A_2 the regular arcs on A_1 will not intersect the regular arcs on A_2 . Therefore, $\partial D_1 \cap \partial D_2 = \emptyset$ and the splitting is weakly reducible.

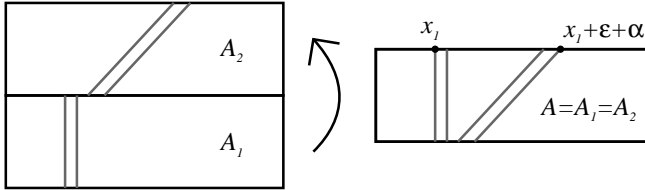


Figure 4. The regular arcs will not intersect after identifying A_1 and A_2 .

□

With Lemma 6.2, all that is needed to prove the main theorem of the section is to show that for a non-small horizontal splitting we can always find such arcs γ_1 and γ_2 . However, we know that we will never be able to produce such arcs for a small splitting (or else we could make it reducible). Unfortunately, this forces the proof of the main theorem to be quite technical; it must make exact use of the properties of non-small horizontal splittings.

Proof of Theorem 6.1. The horizontal splitting being considered is not small, according to our characterization in Lemma 5.2 this leaves us with two possibilities, either

- 1) $g_0 \geq 1$, or
- 2) $d \geq 3$ and $g(F) > 1$.

We will prove these two cases separately.

Case. $g_0 \geq 1$.

In this case we can find a non-separating essential curve on the base surface B_0 . Furthermore, we may take two disjoint copies of this curve, ϕ_1 and ϕ_2 , and adjoin to them embedded arcs, τ_1 and τ_2 (resp.), which attach them to ∂B_0 and so that regular neighborhoods of each of the resulting graphs are disjoint from each other and from the anchor points of the exceptional fibers. See Figure 5. As the total length of ∂B_0 is equal to 1, the distance between the endpoint of τ_1 and τ_2 is strictly less than 1.

Recall that $\rho : F \rightarrow B$ is a branched covering map. We may consider the preimages of our chosen graphs and neighborhoods on F . The preimage, $\rho^{-1}(\tau_i)$, of the arc τ_i will be d disjoint copies of τ_i . For each i choose

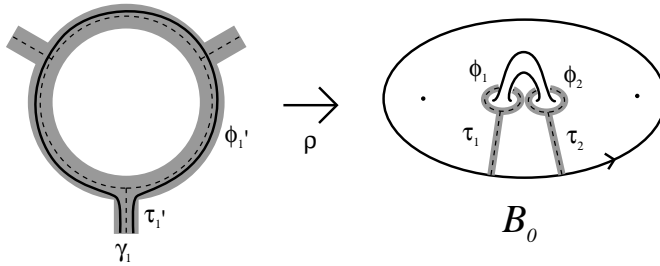


Figure 5. Lift neighborhoods of the graphs to F .

an arc $\tau'_i \subset \rho^{-1}(\tau_i)$ so that the distance from the endpoint of τ_1 to the endpoint of τ'_2 is a positive number less than 1. Each of these arcs joins a closed loop $\phi'_i \subset \rho^{-1}(\phi_i)$ to ∂F . For $i = 1, 2$ choose regular neighborhoods $N(\phi'_i \cup \tau'_i) \subset \rho^{-1}(N(\phi_i \cup \tau_i))$. One component of $\partial(N(\tau'_i \cup \phi'_i))$ will be an arc γ_i (the other component is a curve). As ϕ'_i is essential and non-boundary parallel, γ_i is an essential arc on F . By taking the regular neighborhood to be sufficiently small we have that the endpoints of γ_1 are followed by those of γ_2 and all are contained in an interval of length less than one. The arcs then satisfy hypothesis ii) of Lemma 6.2.

Moreover, γ_1 is disjoint from both γ_2 and $h(\gamma_2)$. To see this note that $\gamma_1 \subset \rho^{-1}(N(\phi_1 \cup \tau_1))$ and $\gamma_2, h(\gamma_2) \subset \rho^{-1}(N(\phi_2 \cup \tau_2))$. In turn, these sets are disjoint being preimages of disjoint neighborhoods on B_0 . Therefore, the essential arcs γ_1 and γ_2 satisfy the conditions of Lemma 6.2. In this case the horizontal splitting at f is weakly reducible.

We now to proceed to the second (and more difficult) case.

Case. $d \geq 3$ and $g(F) > 1$.

We assume that $g_0 = 0$ (otherwise the result follows from the previous case). Now, there are no essential loops or arcs on the base surface that we can lift to F and we must construct γ_1 and γ_2 by more laborious means.

Definition 6.3. An **almost essential arc** on the base surface B_0 is a properly embedded arc on B_0 which is also an essential arc on the punctured surface $B'_0 = B_0 - N(\cup_{i=1, \dots, n_0} e_i)$, where the e_i are the anchor points of exceptional fibers contained in M_0 .

Note that the preimage under ρ of an almost essential arc on B_0 will be a collection of essential arcs on F .

Choose an almost essential arc γ on the base surface B_0 and choose an arc $\gamma_1 \subset \rho^{-1}(\gamma)$ on F . Apply the surface homeomorphism h iteratively to obtain the essential arcs $\gamma_2 = h(\gamma_1)$, and $\gamma_3 = h(\gamma_2)$.

Claim. γ_i and γ_{i+1} are disjoint and not parallel.

Suppose that γ_i and γ_{i+1} are parallel. Then every arc in $\Gamma = \rho^{-1}(\gamma)$ will be parallel and Γ will contain at least 3 arcs ($d \geq 3$). From Γ choose three parallel and adjacent arcs: γ_a, γ_b and γ_c . Let R_1 be the disk region bounded by γ_a and γ_b and R_2 be the disk region bounded by γ_b and γ_c . We know that γ separates the disk B_0 into two subdisks B_1 and B_2 and ρ restricted to R_1 is a branched covering of one of B_1 or B_2 , say B_1 . It follows that ρ restricted to R_2 is a branched covering of B_2 . This implies that each of the disk regions R_i is a double branched cover of the corresponding B_i . So B_1 and B_2 each contain exactly one branch point of multiplicity 2 and $d = 2$, a contradiction.

Consider $F - (\gamma_1 \cup \gamma_2 \cup \gamma_3)$. We have assumed that $g(F) > 1$ and the previous claim shows that neither γ_1 nor γ_3 is parallel to γ_2 . Thus, γ_2 is on the boundary of a non-disk region E of $F - (\gamma_1 \cup \gamma_2 \cup \gamma_3)$. We may choose an essential arc $\delta \subset E \subset F$ with endpoints at $x_2 + 2\epsilon$ and $x_2 + 3\epsilon$ ($0 < \epsilon < 1/3$) where x_2 is one of the endpoints of γ_2 . (See Figure 6.) Then $h(\delta)$ is disjoint from γ_2, γ_3 and γ_4 . So δ and $h(\delta)$ may not be disjoint, but, $(\delta \cup h(\delta)) \cap (\gamma_2 \cup \gamma_3) = \emptyset$. Let $x_3 = h(x_2)$, the endpoints of $h(\delta)$ will be $x_3 + 2\epsilon$ and $x_3 + 3\epsilon$.

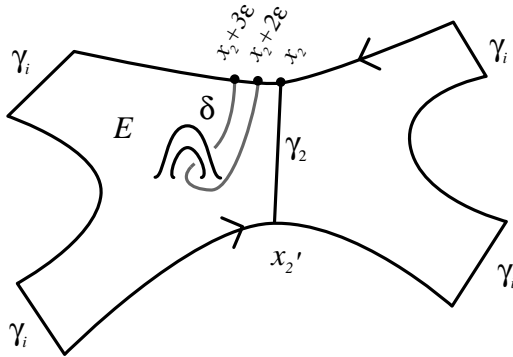


Figure 6. Choosing the arc δ .

Claim. Let R be the component of $F - (\delta \cup h(\delta))$ that contains the point x_2 . Then R is not a disk.

Suppose that R is a disk. From Figure 7 we can see that $\partial F - (\delta \cup h(\delta))$ consists of four interval components. Let J_2 be the interval containing x_2 , $J_2 = (x_3 + 3\epsilon, x_2 + 2\epsilon)$, and J_3 be the interval containing x_3 , $J_3 = (x_2 +$

$3\epsilon, x_3 + 2\epsilon$). Note that R contains the entire arc γ_2 and $J_2 \subset \partial R$ (and possibly $J_3 \subset \partial R$).

Suppose both endpoints of γ_2, x_2 and x_2' , lie on the interval J_2 . That would imply that the arc γ_2 is inessential which is a contradiction.

Therefore x_2' lies on the interval J_3 and the disk R contains the arc γ_3 as well. Then γ_3 must have also have an endpoint on each of J_2 and J_3 (or it is inessential). But then, γ_2 and γ_3 lie in the disk R and have adjacent endpoints on the intervals J_2 and J_3 and are thus parallel on the surface F . This is a contradiction, hence R is not a disk.

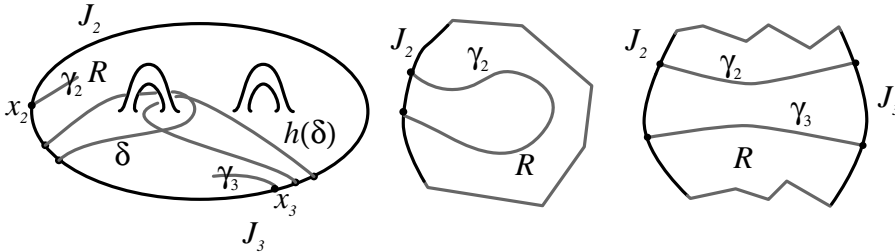


Figure 7. R is not a disk.

As R is not a disk we can find an essential arc $\zeta \subset R$ with endpoints at x_2 and $x_2 + \epsilon$. So ζ is an essential arc on F that is disjoint from $\delta \cup h(\delta)$.

The endpoints of δ are at $x_2 + 2\epsilon$ and $x_2 + 3\epsilon$ and $\epsilon < 1/3$. Then the arcs ζ and δ satisfy the hypotheses of Lemma 6.2 and a horizontal splitting at f is weakly reducible.

□

7. The Rectangle Condition.

Non-small horizontal splittings are by definition not of minimal genus. To prove their irreducibility it must be directly shown that *any* pair of essential disks, one in each handlebody, intersect at least twice. In [CG1], Casson developed the *rectangle condition* which allows one to analyze the intersection of *any* pair of disks by working with *prechosen* disk systems. Casson's rectangle condition was given by a pair of pants decomposition of a surface, here it is generalized to a planar decomposition.

Definition 7.1. Let G be a closed surface and $\mathcal{C} \subset G$ a collection of pairwise disjoint curves. Then \mathcal{C} is a **complete curve system** for G if $\mathcal{P} = G - \mathcal{C}$ is a collection of planar surfaces, none of which is a disk or an annulus. We call \mathcal{P} the **induced planar decomposition**.

Definition 7.2. Let Δ be a collection of disjoint essential disks in a handlebody H , no two of which are parallel. If $H - \Delta$ is a collection of balls then we call Δ a **complete collection of disks** for H .

Note that if Δ is a complete collection of disks for a handlebody H then $\partial\Delta$ is a complete collection of curves for $G = \partial H$ and hence induces a planar decomposition of G .

Definition 7.3. Let P be a planar surface. Suppose that $c \subset \partial P$ is a single curve and $C_1 \subset \partial P$ and $C_2 \subset \partial P$ are non-empty and disjoint collections of curves such that $\partial P = c \sqcup C_1 \sqcup C_2$. We will call the decomposition $\partial P = c \sqcup C_1 \sqcup C_2$ a **non-trivial partition** of ∂P .

Definition 7.4. Let P and P' be planar surfaces embedded on a closed surface G . If for each non-trivial partition of $\partial P = c \sqcup C_1 \sqcup C_2$ and each non-trivial partition of $\partial P' = c' \sqcup C_1' \sqcup C_2'$ there exist curves $c_1 \in C_1, c_2 \in C_2, c_1' \in C_1',$ and $c_2' \in C_2'$ such that $P \cap P'$ contains a rectangle bounded by $c_1, c_2, c_1',$ and c_2' then P and P' are said to satisfy the **rectangle condition for planar surfaces**.

Definition 7.5. Let \mathcal{C} and \mathcal{C}' be collections of curves on a surface G . Then \mathcal{C} and \mathcal{C}' **intersect essentially** if no component of $G - (\mathcal{C} \cup \mathcal{C}')$ is a *bigon*, a disk region bounded by two arcs.

Definition 7.6. Suppose that \mathcal{C} and \mathcal{C}' are complete collections of curves that intersect essentially on a surface G and that \mathcal{P} and \mathcal{P}' are their respective induced planar decompositions. Then \mathcal{C} and \mathcal{C}' are said to satisfy the **rectangle condition for curve systems** if for each $P \in \mathcal{P}$ and each $P' \in \mathcal{P}'$, P and P' satisfy the rectangle condition for planar surfaces.

Definition 7.7. Let $M = H \cup_G H'$ be a Heegaard splitting. Let Δ and Δ' be complete collections of disks for H and H' , respectively. Further assume that $\partial\Delta \subset G$ and $\partial\Delta' \subset G$ intersect essentially and that \mathcal{P} and \mathcal{P}' are their respective induced planar decompositions. If for each choice of $P \in \mathcal{P}$ and $P' \in \mathcal{P}'$, P and P' satisfy the rectangle condition for planar surfaces then we say that Δ and Δ' satisfy the **rectangle condition for disk systems**.

Lemma 7.8. *Let $H \cup_F H'$ be a Heegaard splitting. Let $D \subset H$ and $D' \subset H'$ be essential disks. If there are complete collections of disks, $\Delta \subset H$ and $\Delta' \subset H'$, which satisfy the rectangle condition for disk systems then $|\partial D \cap \partial D'| \geq 4$. In particular, the splitting $H \cup_F H'$ is strongly irreducible.*

Proof. First isotope the disks D and D' so that the quantity $|\partial D \cap \partial D' \cap \partial\Delta \cap \partial\Delta'|$ is minimal.

Claim. No two curves in the collection $\partial D \cup \partial D' \cup \partial\Delta \cup \partial\Delta'$ intersect inessentially.

If so, then there is an innermost bigon (other arcs may be contained in the bigon, but they join distinct boundary arcs of the bigon). As $\partial\Delta \cap \partial\Delta'$ is essential by the definition of the rectangle condition, one of these disks must be D or D' , say D . The bigon can be used to construct an isotopy of D that reduces $|\partial D \cap \partial D' \cap \partial\Delta \cap \partial\Delta'|$. But, we have assumed that this quantity is minimal. The claim holds.

As there are no innermost bigons, there are no bigons at all and the curves ∂D and $\partial D'$ must intersect minimally. Furthermore, as handlebodies are irreducible we may isotope the interiors of the disk so that there are no closed loops of intersection in $D \cap \Delta$ and $D' \cap \Delta'$.

Consider an outermost subdisk D_0 of $D - \Delta$. Let δ designate the portion of ∂D that is coincident with ∂D_0 . Either $D_0 \subsetneq D$ and δ is an arc or $D_0 = D$ and $\delta = \partial D$, a loop. See Figure 8. δ is properly embedded in a single component $P \in \mathcal{P}$.

Claim. δ is an essential loop or arc.

If δ is an arc and inessential then it forms a bigon with some $c \in \partial P$ and there is an inessential intersection which contradicts the previous claim. If δ is a loop and inessential then the disk D is inessential, again a contradiction.

Similarly, we can find an essential arc (or loop) δ' that lies entirely and essentially in a planar piece P' .

Case. δ and δ' are both arcs.

As δ belongs to an outermost disk both of its endpoints belong to a single loop $c \in \partial P$. As δ is essential in ∂P it separates the remaining curves, $\partial P - c$, into two non-empty sets C_1 and C_2 . We obtain the non-trivial partition $\partial P = c \sqcup C_1 \sqcup C_2$. Similarly δ' yields the non-trivial partition $\partial P' = c' \sqcup C_1' \sqcup C_2'$ where δ' separates C_1' from C_2' .

By assumption \mathcal{C} and \mathcal{C}' satisfy the rectangle condition for curve systems which in turn implies that the pieces P and P' satisfy the rectangle condition for planar surfaces. Thus there exist curves $c_1 \in C_1, c_2 \in C_2, c_1' \in C_1'$, and $c_2' \in C_2'$ which bound a rectangle on $P \cap P'$. As δ separates c_1 and c_2 on P (it separates C_1 from C_2) and δ' separates c_1' and c_2' on P' , the arcs δ and δ' must meet in the rectangle. See Figure 8.

Any such outermost arc δ must meet any outermost arc δ' at least once. There will be at least two outermost subdisks on D and thus we obtain at least two distinct choices for δ . Similarly there are at least two distinct choices for δ' . For each pair we obtain a distinct point of intersection. Thus $\partial D \cap \partial D' \geq 4$.

Case. δ is a loop and δ' is an arc.

As δ is essential and entirely contained in P it separates ∂P into two collections C_1 and C_2 . One collection, say C_1 , contains more than one curve (P is not an annulus). Select $c \in C_1$ and form the non-trivial partition

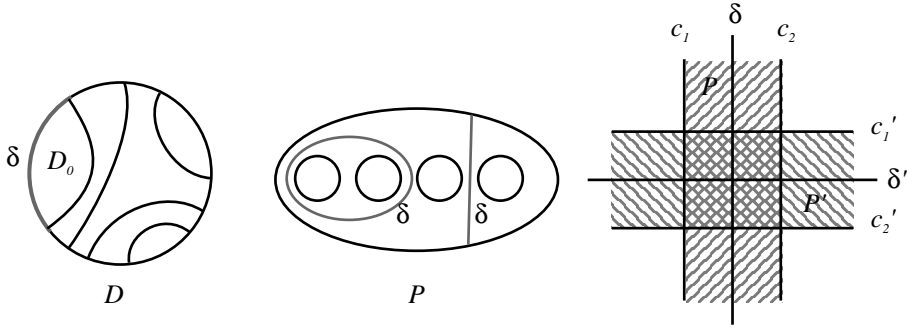


Figure 8. δ and δ' must meet in the rectangle.

$\partial P = c \sqcup (C_1 - c) \sqcup C_2$. δ separates the curves in $C_1 - c$ from those in C_2 . As before the arc δ' will induce a non-trivial partition $\partial P' = c' \sqcup C_1' \sqcup C_2'$ where it separates separates C_1' from C_2' .

As our curve systems satisfy the rectangle condition, we can find curves $c_1 \in C_1, c_2 \in C_2, c_1' \in C_1',$ and $c_2' \in C_2'$ which bound a rectangle on $P \cap P'$. Again δ will separate c_1 and c_2 on P and δ' will separate c_1' and c_2' on P' . Then δ and δ' will have a point of intersection x_1 .

Let $\delta_1' = \delta' \cap P$ be the subarc of δ' that contains the point x_1 . Then δ_1' is a properly embedded arc on P that joins the curves c_1 and c_2 . See Figure 8.

Note that δ also separates the curves in $C_1 - c_1$ (non-empty) from the curves in C_2 and we obtain the non-trivial partition $\partial P = c_1 \sqcup (C_1 - c_1) \sqcup C_2$. Then there are curves $d_1 \in C_1 - c_1, d_2 \in C_2$ and $d_1' \in C_1',$ and $d_2' \in C_2'$ which bound a rectangle on $P \cap P'$. Because δ and δ' separate the curves in the chosen non-trivial partitions they must meet at a point x_2 in this rectangle.

Let $\delta_2' = \delta' \cap P$ be the subarc of δ' that contains the point x_2 . Then δ_2' is a properly embedded arc on P that joins the curves d_1 and d_2 . Note that $c_1 \neq d_1$ ($d_1 \in C_1 - c_1$) so the subarcs δ_1' and δ_2' are distinct. Therefore the points of intersection x_1 and x_2 are distinct.

As before there is a distinct choice for δ' that will provide two additional points of intersection. Again, $|\partial D \cap \partial D'| \geq 4$.

Case. δ and δ' are both loops.

We proceed as in the previous case. For each loop there are at least two choices of a non-trivial partition. There are four combinations each yielding a point of intersection. Again, we can check that the points of intersection are distinct by checking that they are contained in different subarcs of $\partial \Delta \cap P'$ and $\partial \Delta' \cap P$.

□

8. Irreducibility of Horizontal Splittings.

In this section we complete the proof of the main theorem using the rectangle condition.

Theorem 8.1. *Let M admit a horizontal splitting at f . If $\alpha > d$ then this splitting is irreducible.*

Before proving this theorem we will note and prove the following corollary.

Corollary 8.2. *A Seifert fibered space admits an irreducible horizontal splitting at at most one fiber f .*

Proof. Suppose that M admits two irreducible splittings, at distinct fibers f and f' with multiplicities α and α' . We have already demonstrated (Theorem 5.4) that both of these splittings cannot be small.

Nor can it be that neither splitting is small. In that case, our condition for irreducibility implies that $\alpha > lcm\{\alpha_i\}_{i=1,\dots,n_0}$. In particular $\alpha > \alpha'$. Symmetrically $\alpha' > \alpha$. This is a contradiction.

We then suppose that one of the splittings, say the one at f , is a small splitting and the other, at f' , is not small. We may assume that $\alpha \leq \alpha'$ (otherwise the above argument works).

If the splitting at f is a type (1) splitting then

$$M_0 = \{g_0|\emptyset\}, g_0 \geq 1, f \text{ is regular or exceptional,}$$

and in particular $\alpha' = 1$. Now $\alpha \leq \alpha'$ implies that $\alpha = 1$. Then both f and f' are regular fibers and these two splittings are the same.

In the proof of Theorem 5.4 it was demonstrated that if M admits a small splitting of type (2),(3),(5), or (6) then this is the only horizontal splitting that M admits. The remaining case is when there is a type (4) small horizontal splitting at f . So

$$M_0 = \{0|(2, 1), (3, \beta_2)\}, f \text{ is exceptional.}$$

We know that $\alpha \leq \alpha' = 2$ or 3 . Moreover, α is relatively prime to $d = 6$ and therefore $\alpha = 1$. This contradicts the fact that f is an exceptional fiber (M is not a lens space). □

We now proceed with the proof of the second half of the main theorem.

Proof of Theorem 8.1. Let $E \subset B_0$ be the collection of anchor points of exceptional fibers. Let E_2 be the sub collection of multiplicity two anchor points. Choose the following collections of disjoint embedded arcs in B_0 .

$$\Gamma_1 = \{2g_0 \text{ arcs chosen such that } B_0 - \Gamma_1 \text{ is a single disk}\}$$

$\Gamma_2 = \{|E_2| \text{ arcs, each joining a multiplicity two anchor point to a point on } \partial B_0\}$. These arcs will be called half arcs.

When $E_2 \subsetneq E$ let $\Gamma_3 = \{ |E| - |E_2| - 1 \text{ arcs so that } B_0 - \Gamma_1 - \Gamma_2 - \Gamma_3 \text{ is a collection of disks each of which contains exactly one point from the collection } E - E_2 \}$.

When $E_2 = E$ let $\Gamma_3 = \{\emptyset\}$.

These collections can be chosen so that the union $\Gamma_0 = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is an embedded disjoint collection of arcs in B_0 . With the exception of the arcs in Γ_2 these arcs are properly embedded in B_0 .

Let $\Gamma = \rho^{-1}(\Gamma_0)$. Each arc in Γ is essential on F . If the anchor points are all of multiplicity two then $B_0 - \Gamma_0$ is a single disk which contains no branch points. Otherwise it is a collection of disks each containing exactly one branch point of multiplicity greater than or equal to 3. It follows that $F - \Gamma_0$ is a collection of disks. Furthermore,

Claim. No two arcs in Γ are parallel.

Let γ_a and γ_b be adjacent parallel arcs and R be the component of $F - \Gamma$ bounded by them. Along ∂R we see two arcs and two intervals of ∂F . The branched covering $\rho : F \rightarrow B_0$ restricts to a branched covering, $\rho : R \rightarrow R_0$, for some component R_0 in $B_0 - \Gamma_0$. If $E \neq E_2$ then R_0 contains an anchor point of multiplicity at least three and R has at least three arcs on its boundary, a contradiction. Therefore, all branch points are multiplicity two and R_0 is the sole component of $B_0 - \Gamma_0$. Also, there are no branch points in the interior of R_0 , so R is a copy of R_0 . But along ∂R_0 we see $2|\Gamma_1| + |\Gamma_2| = 4g_0 + |E_2|$ arcs (the half arcs are doubled to look like full arcs). It follows that $g_0 = 0$ and $E = E_2 = 2$. So F is a branched cover over the disk with two branch points each of multiplicity two. Then F is an annulus, and does not have a single boundary component, contradicting the horizontal construction.

Let $\Delta_1 = \Gamma \times I_1$ and $\Delta_2 = \Gamma \times I_2$ be disk systems for $H_1 = F \times I_1$ and $H_2 = F \times I_2$ respectively. As no arc $\gamma \in \Gamma$ is inessential, each disk $D \in \Delta_i$ is essential. Moreover, as no two arcs in Γ are parallel, no two disks in Δ_i are parallel. Each piece of $H_i - \Delta_i$ is a $\{\text{disk}\} \times I$, in other words a ball. Thus Δ_i is a complete collection of disks for H_i .

Note that the boundaries of the disk collections are not transverse on the surfaces F_1 and F_2 but are coincident there. In fact $\partial\Delta_1 \cap \partial\Delta_2 \cap F_1$ (and similarly for F_2) is the collection of arcs Γ . By perturbing each disk in Δ_2 slightly we may arrange that each coincident arc becomes a single point of intersection between Δ_1 and Δ_2 (See Figure 9). We treat this coincident arc as a single point of intersection and refer to it as a *long crossing*.

By examining how the disk systems Δ_1 and Δ_2 intersect on the splitting surface $G = F_1 \cup A_1 \cup F_2 = F_2 \cup A_2 \cup F_1$ it can be seen that they satisfy the rectangle condition.

Claim. The disk collections $\partial\Delta_1$ and $\partial\Delta_2$ intersect essentially.

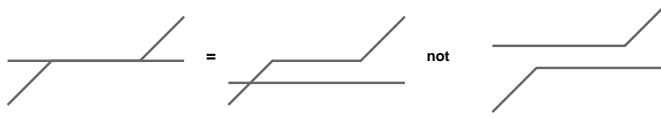


Figure 9. The disks intersect on F_i in a long crossing.

If the disk collections $\partial\Delta_1$ and $\partial\Delta_2$ do not intersect essentially then some component of $G - \partial\Delta_1 \cup \partial\Delta_2$ is a bigon. There are no components of $G - \partial\Delta_1 \cup \partial\Delta_2$ lying entirely on the surfaces F_i so we focus on $A = A_1 = A_2$. See Figure 10. Each disk in Δ_i is of the form $\{arc\} \times I_i$. Thus $\partial\Delta_i \cap A_i$ is a collection of regular arcs. On A the components of $G - \partial\Delta_1 \cup \partial\Delta_2$ are quadrilaterals contained on A or triangular regions that extend onto one the surfaces F_i . The components of the first type are not bigons. The components of the second type already have three crossings, a regular crossing and two long crossings. Such a component cannot be a bigon unless its long crossings are coincident. This is impossible, the bigon would then imply that one of the arcs in Γ (the long crossing) is inessential on F_i . Hence, there are no innermost bigons, and the intersection between $\partial\Delta_1$ and $\partial\Delta_2$ is minimal.

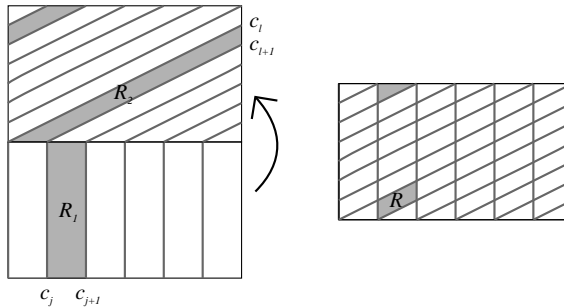


Figure 10. ∂M_0 when $\alpha > d$.

Let P_1 be any component of $\partial H_1 - \Delta_1$ and $\partial P_1 = c \sqcup C_1 \sqcup C_2$ any non-trivial partition of its boundary components. Then $P_1 = R \times I_1$ for some component R of $F - \Gamma$ and appears as in Figure 11. By following ∂R around we may cyclically label the arcs in $\partial R \cap \Gamma$ as they are encountered. Each component c of ∂P_1 corresponds to one of these arcs and we can thus cyclically label the components of ∂P_1 , $\partial P_1 = \{c_1, c_2, \dots, c_m\}$. As the components are cyclically ordered there must be consecutive boundary components $c_j \in C_1$ and $c_{j+1} \in C_2$ (or vice-versa). Select the rectangle R_1 which is the subset

of the annulus A_1 that is bounded by these adjacent curves. In the same fashion given any planar piece P_2 and a non-trivial partition of its boundary curves we choose a rectangle R_2 bounded by adjacent curves, c_l and c_{l+1} , that belong to different components of the partition.

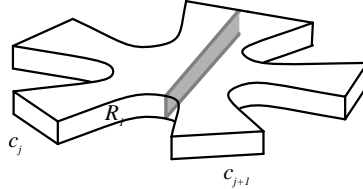


Figure 11. A piece of $F \times I_i - \Delta_i$.

Consider $\partial M_0 = A_1 \cup A_2$ as pictured in Figure 10. The rectangle R_1 is bounded by adjacent regular arcs on A_1 and is thus a vertical strip. The rectangle R_2 is also bounded by adjacent regular arcs and is therefore a strip with an α shift. As $\alpha > d$, after the identification $A_1 = A_2$ is made, $R_1 \cup R_2$ will contain a rectangle R .

The rectangle R is contained in $R_1 \cap R_2 \subset P_1 \cap P_2$ and is bounded by the curves c_j, c_{j+1}, c_l and c_{l+1} that were selected from different components of each of the given partitions. As the planar pieces and partitions were chosen arbitrarily, we have demonstrated that the disk collections Δ_1 and Δ_2 satisfy the rectangle condition, hence this horizontal splitting is irreducible. \square

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