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We characterize those domains in the plane whose boundary is a chord arc curve in terms of some L^2 integrals, which are mainly a version of Green's theorem. As a consequence of this we obtain a "converse" to a theorem due to Laurentiev that states that for such domains harmonic measure and arc length are A_∞ equivalent.

Let Γ be a locally rectifiable Jordan curve in the plane that passes through ∞, and let Ω_+, Ω_- be the two domains bounded by Γ.

Given a function f defined on Γ, its Cauchy integral

$$Cf(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \notin \Gamma$$

defines an analytic function off Γ.

If C_+f, C_-f denote the restrictions of Cf to Ω_+ and Ω_-, and if f_+, f_- denote their boundary values, then

$$f_\pm(z) = \pm\frac{1}{2}f(z) + \frac{1}{2\pi i}\ \text{P.V.} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \quad z \in \Gamma.$$

G. David has shown in [**D**] that the Cauchy integral is bounded in $L^2(\Gamma)$ if and only if Γ is regular, that is, there exists a constant C such that for all $z_0 \in \mathbb{C}$ and all $R > 0$, the arclength of $B(z_0, R) \cap \Gamma$ is at most CR, where $B(z_0, R)$ denotes the ball centered at z_0 and radius R.

Several proofs have been given of the boundedness of the Cauchy integral under stronger hypothesis on Γ. We shall concentrate on the first proof presented in [**C-J-S**] which is based on complex variables methods. They show the result for Lipschitz graphs, i.e.,

$$\Gamma = \{x + iA(x) : x \in \mathbb{R}\} \text{ with } A' \in L^\infty.$$

By following their argument very closely one can notice that the theorem is a consequence of the fact that for any F holomorphic in $\Omega_\pm$ that decays to zero at ∞, the following two integrals are equivalent:

$$\iint_{\Omega_\pm} |F'(z)|^2 \delta(z)\, dx\, dy \cong \int_\Gamma |F|^2\, ds$$

where $\delta(z) = \operatorname{dist}(z, \Gamma)$.

It is a well known result, [**J-K**], that such an equivalence holds if Γ is a chord-arc curve (the length of the arc is comparable to the chord). The main purpose of this paper is to show that the chord-arc condition is also necessary for the equivalence to hold.

To avoid problems at ∞, we will assume that the curves Γ and the functions F are analytic at ∞. In particular $F(z) = O\left(\frac{1}{z}\right)$ at ∞. Note that if Γ is the real line and Ω is the upper half plane, the equivalence of the integrals is just Green's theorem applied to the functions $u(z) = |F(z)|^2$ and $v(z) = y$ in the domain $\Omega_R = \{z \in \mathbb{R}_2^+;\, |z| \leq R\}$ [**G**, p. 236]. Since $F(z) = O\left(\frac{1}{z}\right)$ the terms involving the line integral on $\{z = \mathrm{Re}^{i\theta};\, 0 < \theta < \pi\}$ will tend to 0 as R tends to ∞.

Before stating the results we need to recall a few definitions:

A function $\varphi \in L^1_{\mathrm{loc}}(\mathbb{R})$ lies in $\mathrm{BMO}(\mathbb{R})$ if

$$\sup_I \frac{1}{|I|} \int_I |\varphi - \varphi_I| \, dt = \|\varphi\|_* < \infty$$

where $I \subset \mathbb{R}$ is any bounded interval and $\varphi_I = \frac{1}{|I|} \int_I \varphi \, dt$. The space $\mathrm{BMOA}(\mathbb{R})$ denotes the space of holomorphic functions in the upper half plane that are Poisson integrals of functions in $\mathrm{BMO}(\mathbb{R})$.

A positive measure μ defined on the upper half plane is called a *Carleson measure* if there is a constant $N(\mu)$ such that

$$\mu(Q) \leq N(\mu) l(Q)$$

for all cubes

$$Q = \{x_0 < x < x_0 + l(Q),\, 0 < y < l(Q)\}.$$

There is a close connection between BMO functions and Carleson measures: A function $\varphi \in \mathrm{BMO}(\mathbb{R})$ if and only if $|\nabla\varphi(z)|^2 y \, dx \, dy$ is a Carleson measure where $\varphi(z)$ denotes the harmonic extension of φ. See [**G**, p. 240].

We are ready now to state the results:

Theorem 1. *Let Γ be a locally rectifiable Jordan curve analytic at ∞ and let Ω be a domain bounded by Γ.*

Denote by Φ the conformal mapping from $\mathbb{R}_2^+$ onto Ω with $\Phi(\infty) = \infty$. Then $\log \Phi' \in \mathrm{BMOA}(\mathbb{R})$ *if and only if there is a constant c, depending only on the BMO constant, such that*

$$\iint_\Omega |F'|^2 \delta(z) \, dx \, dy \leq c \int_\Gamma |F|^2 \, ds \tag{1}$$

for any F holomorphic in Ω with $F(z) = O\left(\frac{1}{z}\right)$ at ∞.

Note that the boundary values of Φ' are defined a.e. on $\mathbb{R}$ because of our assumptions on Γ.

Theorem 2. *Let Γ be a locally rectifiable Jordan curve bounding the domains Ω_+, Ω_-. Suppose there exists a constant c such that*

$$\int_\Gamma |F|^2\, ds \le c \iint_{\Omega_+} |F'|^2 \delta(z)\, dx\, dy \tag{2}$$

and

$$\int_\Gamma |G|^2\, ds \le c \iint_{\Omega_-} |G'|^2 \delta(z)\, dx\, dy$$

for any holomorphic function $F(G)$ on $\Omega_+(\Omega_-)$ vanishing at ∞. Then Γ is a chord-arc curve.

As we mentioned before its converse is also true. Also note that if (2) holds then (1) holds, that is because if Ω is bounded by a chord-arc curve, $\log \Phi' \in \mathrm{BMOA}(\mathbb{R})$.

It will become clear from the proof of the theorem that (2) can be replaced by

$$\int_\Gamma |\varphi|^2\, ds \cong \iint_{\mathbb{C}} |(C\varphi)'|^2 \delta(z)\, dx\, dy$$

where $\varphi = \chi_I$ for any arc $I \subset \Gamma$.

It is also interesting to see what happens if we consider functions of the form $F(z) = \frac{1}{|z-w|}$, $w \notin \Gamma$. Then the result is the following:

Theorem 3. *Let Γ be a locally rectifiable curve, then Γ is regular if and only if there exist constants c_1, c_2 such that*

$$c_1 \frac{1}{\delta(w)} \le \int_\Gamma \frac{|dz|}{|z-w|^2} \le c_2 \frac{1}{\delta(w)}, \quad \textit{for all } w \notin \Gamma. \tag{3}$$

The proofs of these theorems are contained in Section 1. Further remarks and corollaries will be given in Section 2. Finally we would like to thank, M. Melnikov for suggesting some questions and for many helpful conversations, and the referee for his comments which improved the presentation of this paper.

1. Proofs of the Theorems.

Proof of Theorem 1. First note that by changing variables and by using Koebe's distortion theorem, (1) is equivalent to

$$\iint_{\mathbb{R}_2^+} |f'|^2 |\Phi'| y\, dx\, dy \le c \int_{\mathbb{R}} |f|^2 |\Phi'|\, dx \tag{4}$$

where f is a holomorphic function on $\mathbb{R}_2^+$ with $f(z) = O\left(\frac{1}{z}\right)$ at ∞.

Consider now $g = f(\Phi')^{1/2}$. Then applying Green's Theorem as in the remark of the introduction, we get

$$\int_{\mathbb{R}} |g|^2\, dx = 4 \iint_{\mathbb{R}_2^+} |g'|^2 y\, dx\, dy.$$

Since $f'(\Phi')^{1/2} = g' - \frac{1}{2}g\frac{\Phi''}{\Phi'}$

$$\begin{aligned}\iint_{\mathbb{R}_2^+} |f'|^2|\Phi'|y\,dx\,dy &= \iint_{\mathbb{R}_2^+} \left|g' - \frac{1}{2}g\frac{\Phi''}{\Phi'}\right|^2 y\,dx\,dy \\ &\le 2\iint_{\mathbb{R}_2^+} \left(|g'|^2 + \frac{1}{4}|g|^2\left|\frac{\Phi''}{\Phi'}\right|^2\right) y\,dx\,dy \\ &\le \frac{1}{2}\left(\int_{\mathbb{R}} |g|^2\,dx + \iint_{\mathbb{R}_2^+} |g|^2\frac{|\Phi''|^2}{|\Phi'|^2}y\,dx\,dy\right).\end{aligned}$$

By the remark at the end of the introduction if $\log\Phi' \in \mathrm{BMOA}(\mathbb{R})$, then $\frac{|\Phi''|^2}{|\Phi'|}y$ is a Carleson measure and (4) holds.

On the other hand, if (4) holds then

$$\iint_{\mathbb{R}_2^+} |g|^2\left|\frac{\Phi''}{\Phi'}\right|^2 y\,dx\,dy = 4\iint_{\mathbb{R}_2^+} |g' - (f')^2\Phi'|^2 y\,dx\,dy \le 4c\int_{\mathbb{R}} |g|^2\,dx$$

which is equivalent to $\frac{|\Phi''|^2}{|\Phi'|^2}y$ being a Carleson measure ([**G**, p. 33]). $\square$

Proof of Theorem 2. Let I be an arc on Γ with length $l(I)$ and endpoints α, β.

Set $f = \chi_I$ and consider the functions $C_\pm f(z)$ defined in the introduction. Since $f = f_+ - f_-$, (2) implies

$$\begin{aligned}\int_\Gamma |f|^2\,ds &\le c\left(\iint_{\Omega_+} |(C_+f)'|^2\delta(z)\,dx\,dy + \iint_{\Omega_-} |(C_-f)'|^2\delta(z)\,dx\,dy\right) \\ &= c\iint_{\mathbb{C}\setminus\Gamma} |(Cf)'|^2\delta(z)\,dx\,dy\end{aligned}$$

that is

$$l(I) \le C\iint_{\mathbb{C}\setminus\Gamma} \delta(z)\left|\int_I \frac{d\zeta}{(\zeta - z)^2}\right|\,dx\,dy.$$

Let $\zeta(s)$, $s \in [a, b]$ be a parameterization of I by arclength, then

$$\int_I \frac{d\zeta}{(\zeta - z)^2} = \int_a^b \frac{\zeta'(s)}{(\zeta(s) - z)^2}\,ds = \frac{1}{\zeta(a) - z} - \frac{1}{\zeta(b) - z} = \frac{\beta - \alpha}{(\alpha - z)(\beta - z)}.$$

Therefore

$$l(I) \le |\beta - \alpha|^2\iint_{\mathbb{C}\setminus\Gamma} \frac{\delta(z)}{|z - \alpha|^2|z - \beta|^2}\,dx\,dy.$$

It only remains to estimate the last integral. To do so we split it into three integrals. Let B_1 be the ball centered at α with radius $\frac{|\beta-\alpha|}{2}$ and let B_2 be the corresponding one centered at β. Then

$$\iint_{B_1} \frac{\delta(z)}{|z-\alpha|^2|z-\beta|^2}\,dx\,dy \le \frac{4}{|\beta-\alpha|^2}\iint_{B_1}\frac{dx\,dy}{|z-\alpha|} = \frac{c}{|\beta-\alpha|}.$$

By a similar argument one can show that the same estimate holds on B_2 and outside $B_1 \cup B_2$. Therefore

$$l(I) \le c|\beta-\alpha|.$$

□

Proof of Theorem 3. Suppose first that Γ is a regular curve. Fix a point $w \notin \Gamma$, choose $z_0 \in \Gamma$ such that $\delta(w) = |w - z_0|$ and consider the ball B centered at z_0 with radius $2\delta(w)$. So:

$$\int_\Gamma \frac{|dz|}{|z-w|^2} = \int_{\Gamma\cap B}\frac{|dz|}{|z-w|^2} + \int_{\Gamma\setminus B}\frac{|dz|}{|z-w|^2}.$$

If $z \in \Gamma \cap B$, then $|z-w| \cong \delta(w)$. Also, since Γ is regular $l(\Gamma\cap B) \cong \delta(w)$. Therefore, trivially

$$\int_{\Gamma\cap B}\frac{|dz|}{|z-w|^2} \cong \frac{1}{\delta(w)}.$$

On the other hand

$$\int_{\Gamma\setminus B}\frac{|dz|}{|z-w|^2} = \sum_{k=1}^{\infty}\int_{A_k}\frac{|dz|}{|z-w|^2}$$

where $A_k = \{z \in \Gamma : 2^k\delta(w) \le |z-z_0| \le 2^{k+1}\delta(w)\}$.

If $z \in A_k$, $|z-w| \cong 2^k\delta(w)$. Since $l(A_k) \cong 2^k\delta(w)$ we get

$$\int_{\Gamma\setminus B}\frac{|dz|}{|z-w|^2} \cong \frac{1}{\delta(w)}$$

which proves the first part of the theorem.

Suppose now that (3) holds. Choose any $r > 0$ and any point $z_0 \in \mathbb{C}$ and consider the ball B centered at z_0 of radius r. Let A be the annulus $A = \{2r < |z-z_0| < 3r\}$ and let $w \in A$ be a point with the property that

$$\delta(w) = \sup_{z\in A}\delta(z).$$

We claim that there is a constant c depending only on c_1, c_2 such that $\delta(w) \ge cr$. Assuming the claim let us finish the proof of the theorem:

$$\frac{l(\Gamma\cap B)}{r^2} \le \int_{\Gamma\cap B}\frac{|dz|}{|z-w|^2} \le \int_\Gamma \frac{|dz|}{|z-w|^2} \cong \frac{1}{\delta(w)} \le \frac{c}{r}.$$

Therefore $l(\Gamma\cap B)\le cr$, i.e. Γ is regular. To prove the claim consider a grid on A of size $\delta(w)$. Then, because of the choice of w, any square of the grid contains points of Γ. So, letting $N\cong r/\delta(w)$, we have

$$\frac{c_2}{\delta(w)}\ge\int_\Gamma\frac{|dz|}{|z-w|^2}\ge\sum_{k=1}^N\int_{\substack{\{z\in\Gamma:\\ k\delta(w)<|z-w|<(k+1)\delta(w)\}}}\frac{|dz|}{|z-w|^2}$$

$$\ge c\sum_{k=1}^N\frac{1}{k\delta(w)}\cong\frac{c}{\delta(w)}\log r/\delta(w).$$

Therefore $r/\delta(w)\le c$ which proves the claim. $\square$

Note that the same result holds if we replace $\frac{1}{|z-w|^2}$ by $\frac{1}{|z-w|^\alpha}$ for any $\alpha>1$. Then instead of (3) we get

$$\int_\Gamma\frac{|dz|}{|z-w|^\alpha}\cong(\delta(w))^{-\alpha+1}.$$

The proof is the same.

2. Further remarks.

Let $w(x)>0$ be locally integrable on $\mathbb{R}$.

Set $w(E)=\int_E w(x)\,dx$, and let $|E|$ denote the Lebesgue measure of E. We say that w is an A_∞ weight if for every $\varepsilon>0$, there is a $\delta>0$ such that if I is any interval and $E\subseteq I$, then

$$\frac{|E|}{|I|}<\delta\Rightarrow\frac{w(E)}{w(I)}<\varepsilon.$$

If ω is an A_∞ weight, then $\log w\in$ BMO. For a proof of this fact and some related ones see [**S**].

As before, given an unbounded simply connected domain Ω other than the plane itself, Φ will denote the conformal mapping from $\mathbb{R}^+_2$ onto Ω fixing ∞.

There is a theorem due to Laurentiev which states that if Ω is a domain bounded by a chord-arc curve, then arc-length and harmonic measure on $\partial\Omega$ are A_∞-equivalent. That is, $|\Phi'|$ is an A_∞-weight.

A version of a converse is given in [**J-K**]. Before stating it we need some more definitions.

A Jordan curve Γ that passes through ∞ is called a quasi-circle if it satisfies the three-point condition, that is there is a constant c such that for any three points $z_1,z_2\in\Gamma$ and z_3 on the arc joining z_1 and z_2, $|z_1-z_3|\le c|z_1-z_2|$. Obviously a chord-arc curve is a quasicircle.

A domain is called a Smirnov domain if $\log|\Phi'(z)|$ is represented by its Poisson integral. In particular domains bounded by regular curves are Smirnov [**Z**].

Theorem 4 ([**J-K**])**.** *Suppose Ω is a Smirnov domain, $\partial\Omega$ is a quasicircle and harmonic measure is A_∞-equivalent to arc length. Then $\partial\Omega$ is a chord-arc curve.*

Note that its converse is also true.

Using Theorem 2 we give another "converse" to Laurentiev's theorem which is very similar to [**J-K**].

Corollary 1. *Suppose that the two sides of a curve Γ are Smirnov domains and that on each domain harmonic measure is A_∞-equivalent to arc-length. Then Γ is a chord-arc curve.*

As before note that its converse is also true.

Proof. Let Ω be one of the sides of Γ and let $\Phi : \mathbb{R}_2^+ \to \Omega$ be its conformal mapping, $\Phi(\infty) = \infty$. We are assuming that $|\Phi'|$ is an A_∞ weight, therefore [**G-W**],

$$\int_{\mathbb{R}} |F(x)|^2 |\Phi'(x)|\, dx \le c \int_{\mathbb{R}} \left(\iint_{\Gamma_x} |F'(z)|^2\, dA(z) \right) |\Phi'(x)|\, dx$$

where $\Gamma(x)$ is a cone centered at x:

$$\Gamma_x = \{(s, y) : |x - s| < ay\} \text{ for some } a \text{ fixed}$$

and F is a holomorphic function on $\mathbb{R}_2^+$ vanishing at ∞ as before. The constant c depends only on the opening of the cone and the A_∞-constant.

By Fubini's theorem the integral on right-hand side is equivalent to

$$\iint_{\mathbb{R}_2^+} |F'(z)|^2 \sigma(I_z)\, dx\, dy$$

where $\sigma(I_z) = \int_{I_z} |\Phi'(t)|\, dt$ and I_z is the interval on $\mathbb{R}$ centered at x and length $2ay$.

Since $\log \Phi' \in \mathrm{BMO}$,

$$\left| P_y * \log|\Phi'| - \frac{1}{|I_z|} \int_{I_z} \log|\Phi'| \right| \le c$$

with c depending on the BMO-constant of $\log|\Phi'|$ [**G**].

On the other hand, Ω being a Smirnov domain implies that $P_y * \log|\Phi'| = \log|\Phi'(z)|$ and $|\Phi'| \in A_\infty$ is equivalent to saying that

$$\exp\left(\frac{1}{|I|} \int_I \log|\Phi'|\, dt \right) \cong \frac{1}{|I|} \int_I |\Phi'|\, dt$$

for any interval $I \subset \mathbb{R}$.

So,

$$|\Phi'(z)| \cong \frac{1}{|I_z|} \int_{I_z} |\Phi'(t)|\, dt = \sigma(I_z)/2ay.$$

Hence, there is a constant c such that

$$\int_{\mathbb{R}} |F(x)|^2 |\Phi'(x)|\, dx \leq c \iint_{\mathbb{R}^+_2} |F'(z)|^2 |\Phi'(z)| y\, dx\, dy.$$

Since this inequality holds on both sides of Γ, by changing variables and using Koebe's distortion theorem we get the hypothesis of Theorem 2. Therefore Γ is chord-arc. $\square$

Next corollary involves quasiconformal mappings. The result we need to use is the quasiconformal analogue of Koebe's distortion theorem [**A-G**]: Suppose that Ω and Ω' are domains in $\mathbb{R}^2$ and that $\rho : \Omega \to \Omega'$ is K-quasiconformal with Jacobian J_ρ. For each $z \in \Omega$, define

$$a_\rho(z) = \frac{1}{|B_z|} \iint_{B_z} (J_\rho(\zeta))^{1/2}\, d\zeta\, d\bar{\zeta}$$

where B_z is the disk of center z and radius $\delta(z)$. Then

$$\delta(\rho(z)) \cong a_\rho(z)\delta(z).$$

Using this fact and a change of variables in (2) we get the following:

Corollary 2. *Let Γ be a locally rectifiable quasicircle analytic at ∞ bounding the domain Ω, and let ρ be a quasiconformal mapping that sends $\mathbb{R}^+_2$ onto Ω. Then Γ is a chord-arc curve if and only if*

$$\int_{\mathbb{R}} |F(x)|^2 J_\rho^{1/2}(x)\, dx \cong \iint_{\mathbb{C}} JF(z) a_\rho(z) y\, dx\, dy$$

for any quasiregular mapping F satisfying $\bar{\partial} F = \mu \partial F$ where μ is the dilatation of ρ and $F(z) = O\left(\frac{1}{z}\right)$ at ∞.

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