Pacific Journal of Mathematics

DUAL SPACES AND ISOMORPHISMS OF SOME DIFFERENTIAL BANACH $*$-ALGEBRAS OF OPERATORS

Edward Kissin and Victor S. Shulman

Volume 190 No. 2 October 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 190, No. 2, 1999

DUAL SPACES AND ISOMORPHISMS OF SOME DIFFERENTIAL BANACH $*$-ALGEBRAS OF OPERATORS

Edward Kissin and Victor S. Shulman

The paper continues the study of differential Banach *-algebras $\mathcal{A}_S$ and $\mathcal{F}_S$ of operators associated with symmetric operators S on Hilbert spaces H. The algebra $\mathcal{A}_S$ is the domain of the largest *-derivation δ_S of $B(H)$ implemented by S and the algebra $\mathcal{F}_S$ is the closure of the set of all finite rank operators in $\mathcal{A}_S$ with respect to the norm $\|A\| = \|A\| + \|\delta_S(A)\|$. When S is selfadjoint, $\mathcal{F}_S$ is the domain of the largest *-derivation of the algebra $C(H)$ implemented by S. If S is bounded, $\mathcal{F}_S = C(H)$ and $\mathcal{A}_S = B(H)$, so $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. For unbounded selfadjoint operators S the paper establishes the full analogy with the bounded case: $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. The paper also classifies the algebras $\mathcal{A}_S$ and $\mathcal{F}_S$ up to isometrical *-isomorphism and obtains some partial results about bounded but not necessarily isometrical *-isomorphisms of the algebras $\mathcal{F}_S$.

1. Introduction and preliminaries.

Extensive development of non-commutative geometry requires elaborating of the theory of differential Banach *-algebras, that is, dense *-subalgebras of C^*-algebras whose properties in many respects are analogous to the properties of algebras of differentiable functions.

Blackadar and Cuntz [**2**] and the authors [**12**] introduced and studied various classes of differential Banach *-algebras; the most interesting class consists of **D**-algebras, that is, dense *-subalgebras $\mathcal{A}$ of C^*-algebras $(\mathfrak{U}, \|\cdot\|)$ which, in turn, are Banach *-algebras with respect to another norm $\|\cdot\|_1$ and the norms $\|\cdot\|$ and $\|\cdot\|_1$ on $\mathcal{A}$ satisfy the inequality:

$$\|xy\| \leq D(\|x\|\,\|y\|_1 + \|x\|_1\|y\|), \quad \text{for } x, y \in \mathcal{A}, \tag{1.1}$$

for some $D > 0$. This class contains, for example, the domains $D(\delta)$ of closed unbounded *-derivations δ of C^*-algebras $\mathfrak{U}$ where the norm $\|\cdot\|_1$ on $D(\delta)$ is defined, as usual, by the formula

$$\|A\|_1 = \|A\| + \|\delta(A)\|, \quad \text{for } A \in D(\delta).$$

Much work has been done on the investigation of properties of the differential Banach *-algebras (see Blackadar and Cuntz [**2**] and Kissin and Shulman [**12**, **13**]) and the algebras $D(\delta)$ in particular (see, for example, Bratteli and Robinson [**3**] and Sakai [**16**]).

In many cases closed *-derivations of C^*-algebras $\mathfrak{U}$ of operators on Hilbert spaces are implemented by closed symmetric operators. In particular, Bratteli and Robinson [**3**] showed that if $\mathfrak{U}$ contains the ideal of all compact operators then any closed *-derivation of $\mathfrak{U}$ is implemented by a symmetric operator.

Any closed symmetric operator S on a Hilbert space H implements closed *-derivations of various C^*-algebras of operators on H. Among all these derivations there is the largest one - δ_S with domain $D(\delta_S)$ (which we denote by $\mathcal{A}_S$) containing the domains of all derivations implemented by S:

$$\mathcal{A}_S = \Big\{A \in B(H) : AD(S) \subseteq D(S),\ A^*D(S) \subseteq D(S) \quad \text{and}$$
$$(SA - AS)|_{D(S)} \ \text{extends to a bounded operator} \ \ A_S\Big\}$$
$$\text{and} \ \ \delta_S(A) = i\,\mathrm{Closure}\,(SA - AS), \quad \text{for} \ \ A \in \mathcal{A}_S.$$

The closure of $\mathcal{A}_S$ with respect to the norm $\|\cdot\|$ in $B(H)$ is the enveloping C^*-algebra which we denote by $\mathfrak{U}_S$.

The algebra $\mathcal{A}_S$ is a unital Banach *-algebra with respect to the norm

$$\|A\|_S = \|A\| + \|A_S\|. \tag{1.2}$$

If S implements a *-derivation δ of a C^*-algebra $\mathfrak{U}$ of operators on H then

$$D(\delta) \subseteq \mathcal{A}_S, \quad \mathfrak{U} \subseteq \mathfrak{U}_S \ \text{ and } \ \delta = \delta_S|\mathfrak{U}.$$

By $C(H)$ we denote the algebra of all compact operators on H. The *-algebras

$$\mathcal{K}_S = \mathcal{A}_S \cap C(H) \quad \text{and} \quad \mathcal{J}_S = \{A \in \mathcal{K}_S : \delta_S(A) \in C(H)\}$$

are dense in $C(H)$ and are the domains of the largest closed *-derivations from $C(H)$ into $B(H)$ and $C(H)$, respectively, implemented by S.

By $\mathcal{F}_S$ we denote the closure with respect to the norm $\|\cdot\|_S$ of the subalgebra of all finite rank operators in $\mathcal{A}_S$.

It was shown in [**13**] that $(\mathcal{K}_S, \|\cdot\|_S)$ and $(\mathcal{J}_S, \|\cdot\|_S)$ are semisimple Banach *-algebras, that $(\mathcal{F}_S, \|\cdot\|_S)$ is a simple Banach *-algebra and

$$\mathcal{F}_S \subseteq \mathcal{J}_S \subseteq \mathcal{K}_S \subseteq \mathcal{A}_S.$$

Furthermore, $\mathcal{F}_S$, $\mathcal{J}_S$ and $\mathcal{K}_S$ are closed two-sided ideals of $(\mathcal{A}_S, \|\cdot\|_S)$ and $\mathcal{F}_S$ is contained in any closed two-sided ideal of $(\mathcal{A}_S, \|\cdot\|_S)$. The relation between the ideals $\mathcal{F}_S$, $\mathcal{J}_S$ and $\mathcal{K}_S$ and the question of how the properties of the operator S are reflected in the structure of $\mathcal{K}_S$, $\mathcal{J}_S$ and $\mathcal{F}_S$ were investigated in [**13**]. In particular, it was established that $\overline{(\mathcal{K}_S)^2} = \overline{(\mathcal{J}_S)^2} = \mathcal{F}_S$, for all symmetric S, and that the ideals $\mathcal{J}_S$ and $\mathcal{F}_S$ have a bounded approximate identity if and only if S is selfadjoint. For selfadjoint S, it was also proved that $\mathcal{K}_S \neq \mathcal{J}_S = \mathcal{F}_S$.

In spite of the fact that the structure of the algebras $\mathcal{F}_S$, $\mathcal{J}_S$, $\mathcal{K}_S$, $\mathcal{A}_S$ and $\mathfrak{U}_S$ is comparatively simple, many important questions still remain open. In Section 2 we mainly study the structure of the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ in the case when S is a selfadjoint operator. However, we also consider the case when S is a symmetric operator with at least one finite deficiency index and show that the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ contain closed ideals of finite codimension.

If S is a bounded symmetric operator on H then $\mathcal{F}_S = C(H)$ and $\mathcal{A}_S = B(H)$, so $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. In Section 3 we investigate the structure of the dual and the second dual spaces of the algebras $\mathcal{F}_S$ for unbounded symmetric operators S. In the case when S is selfadjoint we establish the full analogy with the bounded case: The algebra $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$.

In Section 4 we study the problem of classification of the algebras $\mathcal{F}_S$ and $\mathcal{A}_S$ up to *-isomorphism. For isometrical *-isomorphism this problem is completely solved in Theorem 4.4. For bounded but not necessarily isometrical *-isomorphism we obtain some interesting partial results in the case when S is selfadjoint.

2. Structure of the algebras $\mathcal{A}_S$ and the enveloping C^*-algebras $\mathfrak{U}_S$.

The main purpose of this section is to study the structure of the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ in the case when S is a selfadjoint operator. However, we start the section by considering the case when S is a symmetric operator with at least one finite deficiency index. Making use of the existence of a J-symmetric representation of $\mathcal{A}_S$ on the deficiency space of S, we will show that the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ contain closed ideals of finite codimension.

Let S be symmetric, S^* be the adjoint operator, let $N_-(S)$ and $N_+(S)$ be the deficiency spaces of S and

$$n_\pm(S) = \dim\left(N_\pm(S)\right)$$

be the deficiency indices of S. It is well known that $D(S^*)$ is a Hilbert space with respect to the scalar product

$$\langle x, y\rangle = (x, y) + (S^*x, S^*y), \quad \text{for } \ x, y \in D(S^*),$$

and it is the orthogonal sum of the closed subspaces $D(S), N_-(S)$ and $N_+(S)$:

$$D(S^*) = D(S)_{\langle + \rangle} N_-(S)_{\langle + \rangle} N_+(S).$$

Set $N(S) = N_-(S)_{\langle + \rangle} N_+(S)$ and let Q be the projection on $N(S)$ in $D(S^*)$. It was shown in [**7**] and [**8**] that

$$[x, y] = i(x, S^*y) - i(S^*x, y), \quad \text{for} \;\; x, y \in N(S),$$

is an indefinite non-degenerate sesquilinear form on $N(S)$, that

$$\pi_S(A) = QA|_{N(S)}, \quad \text{for} \;\; A \in \mathcal{A}_S,$$

is a bounded representation of $(\mathcal{A}_S, \|\cdot\|_S)$ on $N(S)$ and that it is J-*symmetric*:

$$[\pi_S(A)x, y] = [x, \pi_S(A^*)y], \quad \text{for} \;\; x, y \in N(S).$$

A subspace L in $N(S)$ is *neutral* if

$$[x, y] = 0, \quad \text{for all} \;\; x, y \in L.$$

The operator S is *well-behaved* if the representation π_S has no neutral invariant subspace.

Let $\kappa_S = \min(n_-(S), n_+(S))$ and assume that $0 < \kappa_S < \infty$. It was proved in [**10**] that the representation π_S has a κ_S-dimensional subrepresentation σ. Let ρ be an irreducible subrepresentation of σ. It was shown in [**11**] that ρ is bounded with respect to the operator norm $\|\cdot\|$ in $\mathcal{A}_S$ and, therefore, extends to a bounded *-representation of the enveloping C^*-algebra $\mathfrak{U}_S$. If S is well-behaved, it follows from Theorem 28.13 [**14**] that $\mathcal{K}_S \subseteq \mathrm{Ker}(\rho)$. This yields

Theorem 2.1. *Let S be a symmetric unbounded operator and $0 < \kappa_S < \infty$.*

(i) *There exists a closed two-sided ideal J in the Banach *-algebra $(\mathcal{A}_S, \|\cdot\|)$ such that the quotient algebra $\mathcal{A}_S/J$ is isomorphic to the full matrix algebra $M_n(\mathbb{C})$ with $0 < n \le \kappa_S$.*

(ii) *The uniform closure $\overline{J}$ of J in $\mathfrak{U}_S$ is a closed two-sided ideal and the quotient algebra $\mathfrak{U}_S/\overline{J}$ is isomorphic to the full matrix algebra $M_n(\mathbb{C})$.*

(iii) *If S is well-behaved then $\mathcal{K}_S \subseteq J$ and $C(H) \subseteq \overline{J}$.*

Example 2.2. Let $H = L^2(0,1)$ and $S = i\frac{d}{dt}$ with domain $D(S)$ consisting of all absolutely continuous functions h such that $h' \in L^2(0,1)$ and $h(0) = h(1) = 0$. Then S is a symmetric operator and $n_-(S) = n_+(S) = 1$.

It was proved in [**9**] that S is well-behaved. Therefore it follows from Theorem 2.1 that there exists a closed two-sided ideal J in $(\mathcal{A}_S, \|\cdot\|)$ containing $\mathcal{K}_S$ such that $\dim(\mathcal{A}_S/J) = 1$ and that the uniform closure of J in $\mathfrak{U}_S$ is an ideal of codimension 1. $\square$

Let S be the same as in Example 2.2 and let $\mathrm{Lip}\,(0,1)$ be the algebra of all functions on $[0,1]$ satisfying a Lipshitz condition: $|g(t) - g(s)| \le K_g|t - s|$ for some $K_g > 0$ and all $t, s \in [0,1]$. For $g \in \mathrm{Lip}\,(0,1)$, denote by M_g the operator of multiplication by g on $L^2(0,1)$ and set $\mathcal{B} = \{M_g : g \in \mathrm{Lip}\,(0,1)\}$. Then $M_gD(S) \subseteq D(S)$, $(M_g)^*D(S) = M_{\overline{g}}D(S) \subseteq D(S)$ and

$SM_g - M_gS$ extends to the operator $iM_{g'}$ which is bounded, since g' is essentially bounded on $[0,1]$. Thus $\mathcal{B} \subset \mathcal{A}_S$.

(The authors are grateful to the referee of the paper for pointing out an error in the definition of the algebra $\mathcal{B}$ in the first version of the paper.)

Problem 2.3. Is $\mathcal{A}_S = \mathcal{B} + \mathcal{K}_S$?

The assumption that a symmetric operator S is selfadjoint makes the task of studying the structure of the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ easier. First of all, the structure of the ideals $\mathcal{K}_S$, $\mathcal{J}_S$ and $\mathcal{F}_S$ is simpler. While for arbitrary symmetric operators S it is only known (see [**13**]) that $\overline{(\mathcal{K}_S)^2} = \overline{(\mathcal{J}_S)^2} = \mathcal{F}_S$, where the closure is taken with respect to the norm $\|\cdot\|_S$, for selfadjoint operators S it was shown in [**13**] that $\mathcal{F}_S = \mathcal{J}_S \neq \mathcal{K}_S$. Secondly, in the selfadjoint case we can employ the Spectral Theorem to establish the structure of $\mathcal{A}_S$ and $\mathfrak{U}_S$.

Let
$$S = \int_{-\infty}^{\infty} \lambda \, dE_S(\lambda)$$
be the spectral decomposition of S. For every integer n, set

$$P_S(n) = E_S(n+1) - E_S(n) \quad \text{and} \quad [S] = \sum_{-\infty}^{\infty} nP_S(n). \tag{2.1}$$

Then $[S]$ is a selfadjoint operator, $\mathrm{Sp}\,([S]) \subseteq \mathbb{Z}$ and the operator $S - [S]$ is bounded. Therefore it follows that

$$\mathcal{A}_S = \mathcal{A}_{[S]}, \quad \mathcal{K}_S = \mathcal{K}_{[S]} \quad \text{and} \quad \mathcal{F}_S = \mathcal{F}_{[S]}$$

and the norms $\|\cdot\|_S$ and $\|\cdot\|_{[S]}$ are equivalent on $\mathcal{A}_S$. This reduces the problem of the description of the structure of the algebras $\mathcal{A}_S$ and $\mathfrak{U}_S$ to the case when $\mathrm{Sp}\,(S) \subseteq \mathbb{Z}$.

We denote by $\mathcal{S}_{\mathbb{Z}}$ the set of all selfadjoint operators S on H such that $\mathrm{Sp}\,(S) \subseteq \mathbb{Z}$ and set

$$H_S(n) = P_S(n)H, \quad \text{for } n \in \mathrm{Sp}\,(S). \tag{2.2}$$

Then

$$H = \sum_{n \in \mathrm{Sp}(S)} \oplus H_S(n). \tag{2.3}$$

We omit the proof of the following simple result.

Proposition 2.4. *Let $S, T \in \mathcal{S}_{\mathbb{Z}}$. If there exists a one-to-one mapping φ from* $\mathrm{Sp}\,(T)$ *onto* $\mathrm{Sp}\,(S)$ *such that* $\dim(H_T(n)) = \dim(H_S(\varphi(n))$, *for* $n \in \mathrm{Sp}\,(T)$, *and*

$$\sup_{n \in \mathrm{Sp}(T)} |\varphi(n) - n| < \infty$$

then there exists a unitary operator U such that $\mathcal{A}_T = U\mathcal{A}_SU^*$.

Let $S \in \mathcal{S}_{\mathbb{Z}}$. Every operator A in $B(H)$ has a block-matrix form $A = (A_{ij})$, $i, j \in \mathrm{Sp}(S)$, with respect to decomposition (2.3). We denote by $\mathcal{D}_S$ the C^*-algebra of all block-diagonal operators $A = (A_{ij})$ in $B(H)$, that is, $A_{ij} = 0$ if $i \neq j$. By $\mathcal{R}$ we denote the subalgebra of all operators $A = (A_{ij})$ in $B(H)$ with only finite number of non-zero entries A_{ij}. Then, clearly,

$$\mathcal{D}_S \subseteq \mathcal{A}_S \quad \text{and} \quad \mathcal{R}_S \subseteq \mathcal{A}_S.$$

Let $\overline{\mathcal{R}}_S$ be the closure of $\mathcal{R}_S$ in $(\mathcal{A}_S, \|\cdot\|_S)$ and let $C_S(H)$ be the uniform closure of $\mathcal{R}_S$ in $B(H)$.

Lemma 2.5. *$\mathcal{D}_S + C_S(H)$ is a C^*-subalgebra of $\mathfrak{U}_S$ and $\mathcal{D}_S + \overline{\mathcal{R}}_S$ is a closed *-subalgebra of $(\mathcal{A}_S, \|\cdot\|_S)$.*

Proof. Let $\mathcal{L}$ be the uniform closure of $\mathcal{D}_S + \mathcal{R}_S$ in $B(H)$. Then $\mathcal{L}$ is a C^*-subalgebra of $\mathfrak{U}_S$. Since $\mathcal{R}_S$ is a two-sided ideal of the algebra $\mathcal{D}_S + \mathcal{R}_S$, the C^*-algebra $C_S(H)$ is a two-sided ideal of $\mathcal{L}$. Therefore it follows from Corollary 1.8.4 [**4**] that $\mathcal{D}_S + C_S(H)$ is a C^*-algebra, so $\mathcal{L} = \mathcal{D}_S + C_S(H)$.

For $A \in B(H)$, set

$$\phi(A) = \sum_{n \in \mathrm{Sp}(S)} P_S(n) A P_S(n) \quad \text{and} \quad \widetilde{A} = A - \phi(A).$$

Then ϕ is a conditional expectation from $B(H)$ onto $\mathcal{D}_S$ and

$$\|\phi(A)\| \leq \|A\| \quad \text{and} \quad \|\widetilde{A}\| \leq 2\|A\|. \tag{2.4}$$

If $A \in \mathcal{A}_S$ then $\widetilde{A} \in \mathcal{A}_S$ and $\mathrm{Closure}\,(SA - AS) = \mathrm{Closure}\,(S\widetilde{A} - \widetilde{A}S)$.

Assume that $\{A_n\}$ converge to A in $\mathcal{A}_S$ with respect to $\|\cdot\|_S$. Then

$$\|A - A_n\| \to 0 \ \text{ and } \ \|\mathrm{Closure}\,(S(A - A_n) - (A - A_n)S)\| \to 0, \ \text{ as } \ n \to \infty,$$

and therefore, by (1.2) and (2.4),

$$\begin{aligned} \|\widetilde{A} - \widetilde{A}_n\|_S &= \|\widetilde{A} - \widetilde{A}_n\| + \|\mathrm{Closure}\,(S(\widetilde{A} - \widetilde{A}_n) - (\widetilde{A} - \widetilde{A}_n)S)\| \\ &\leq 2\|A - A_n\| + \|\mathrm{Closure}\,(S(A - A_n) - (A - A_n)S)\| \to 0, \\ &\qquad \text{as } \ n \to \infty. \end{aligned} \tag{2.5}$$

Hence $\widetilde{A}_n$ converge to $\widetilde{A}$ with respect to $\|\cdot\|_S$.

Suppose now that $B \in \overline{\mathcal{R}}_S$. Then there are $\{B_n\}$ in $\mathcal{R}_S$ converging to B with respect to $\|\cdot\|_S$. It follows from (2.5) that $\widetilde{B}_n$ converge to $\widetilde{B}$ with respect to $\|\cdot\|_S$ and, since $\widetilde{B}_n$ belong to $\mathcal{R}_S$, we obtain that $\widetilde{B} \in \overline{\mathcal{R}}_S$.

Finally, let $C_n = A_n + B_n$ converge to C in $\mathcal{A}_S$ with respect to $\|\cdot\|_S$ where $A_n \in \mathcal{D}_S$ and $B_n \in \overline{\mathcal{R}}_S$. Then $\widetilde{C}_n = \widetilde{B}_n$ and, by (2.5), $\widetilde{B}_n$ converge to $\widetilde{C}$ with respect to $\|\cdot\|_S$. Since, by the above argument, all $\widetilde{B}_n$ belong to $\overline{\mathcal{R}}_S$, the operator $\widetilde{C}$ also belong to $\overline{\mathcal{R}}_S$. Hence $C \in \mathcal{D}_S + \overline{\mathcal{R}}_S$ and $\mathcal{D}_S + \overline{\mathcal{R}}_S$ is a closed *-subalgebra of $(\mathcal{A}_S, \|\cdot\|_S)$. □

Let $S \in \mathcal{S}_{\mathbb{Z}}$. We number the elements of $\mathrm{Sp}(S)$ in such a way that $\mathrm{Sp}(S) = \{n_i\}_{i \in I}$ is an increasing sequence,

$$0 \le n_i, \quad \text{for } 0 \le i, \quad \text{and } 0 > n_i, \quad \text{for } 0 > i.$$

Then $|i| \le |n_i|$ and, depending on S, the set I is either the set $\mathbb{Z}$ of all integers, or the set of all integers from $-\infty$ to some m, or from m to ∞. We consider the case when $I = \mathbb{Z}$. Two other cases can be considered similarly.

Set

$$\rho_S(k) = \left(\inf_{i \in \mathbb{Z}} |n_{i+k} - n_i| \right)^{-1}, \quad \text{for } k \ne 0, \text{ and } \rho_S(0) = 0.$$

Since $\inf_{i \in \mathbb{Z}} |n_{i+k} - n_i| \ge |k|$,

$$0 < \rho_S(k) \le \frac{1}{|k|}, \quad \text{for } k \ne 0.$$

Proposition 2.6. *If*

$$\lim_{|i| \to \infty} (n_{i+1} - n_i) = \infty \tag{2.6}$$

$$\textit{and} \sum_{k \in \mathbb{Z}} \rho(k) \textit{ converges} \tag{2.7}$$

then $\mathfrak{U}_S = \mathcal{D}_S + C_S(H)$.

Proof. Let $A = (A_{ij}) \in \mathcal{A}_S$, where A_{ij} are bounded operators from $H_S(n_j)$ into $H_S(n_i)$. Then the operator

$$B = SA - AS = (B_{ij}), \quad \text{where } B_{ij} = (n_i - n_j) A_{ij},$$

is bounded. Set $b = \|B\|$. Since $\|B_{ij}\| \le \|B\|$, for all $i, j \in \mathbb{Z}$,

$$\|A_{ij}\| \le \frac{b}{|n_i - n_j|}, \quad \text{for } i \ne j. \tag{2.8}$$

For $k \in \mathbb{Z} \setminus 0$ and $m > 0$, let

$$G_{ij}^{km} = A_{ij}, \text{ if } j = i + k \text{ and } -m \le i \le m, \text{ and } G_{ij}^{km} = 0 \text{ otherwise.}$$

Then the operator $G^{km} = (G_{ij}^{km})$ belongs to $\mathcal{R}_S$. Taking into account (2.6) and (2.8), we obtain that the operators G^{km} converge uniformly in $B(H)$ to a bounded operator $G^k = (G_{ij}^k)$, as $m \to \infty$, where

$$G_{ij}^k = A_{ij}, \text{ if } j = i + k, \text{ and } G_{ij}^k = 0 \text{ otherwise.}$$

Therefore $G^k \in C_S(H)$ and, by (2.8),

$$\|G^k\| = \sup_i \|A_{ii+k}\| \le b \rho_S(k).$$

It follows from (2.7) that the operator $G = \sum_{k \in \mathbb{Z} \setminus 0} G^k$ belongs to $C_S(H)$. Since $A - G \in \mathcal{D}_S$, we obtain that $A \in \mathcal{D}_S + C_S(H)$, so that $\mathcal{A}_S \subseteq \mathcal{D}_S + C_S(H)$. It follows from Lemma 2.5 that $\mathfrak{U}_S = \mathcal{D}_S + C_S(H)$. □

Corollary 2.7. *If there are $a > 0$, $c > 0$ and an integer N such that*

$$c|i|^a \leq n_{i+1} - n_i \quad \text{for} \ \ N \leq |i|$$

then $\mathfrak{U}_S = \mathcal{D}_S + C_S(H)$.

Proof. Condition (2.6), clearly, holds. Let $k > 4N$. Then

$$\begin{aligned}
\rho_S(k)^{-1} &= \inf_{i\in\mathbb{Z}} |n_{i+k} - n_i| = \inf_{i\in\mathbb{Z}} \sum_{p=1}^{k} (n_{i+p} - n_{i+p-1}) \\
&\geq c \sum_{m=N}^{[\frac{k}{2}]} m^a \geq \frac{c}{a+1} \left(\left[\frac{k}{2}\right]^{a+1} - (N-1)^{a+1} \right) \\
&\geq \frac{c}{a+1} \left(\frac{k}{4}\right)^{a+1}.
\end{aligned}$$

Similarly, if $k < -2N$ then $\rho_S(k)^{-1} \geq \frac{c}{a+1} \left(\frac{|k|}{4}\right)^{a+1}$. Therefore condition (2.7) also holds and the result follows from Proposition 2.6. □

Suppose now that $\dim(H_S(n)) = \infty$ for all $n \in \mathrm{Sp}(S)$ and let $n_0 \in \mathrm{Sp}(S)$. Set $K = H_S(n_0)$. Then there exists a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) = \infty$ such that the C^*-algebra $C_S(H)$ is isomorphic to the tensor product $B(K) \otimes C(\mathcal{H})$ where $C(\mathcal{H})$ is the C^*-algebra of all compact operators on $\mathcal{H}$. Choosing a basis $\{e_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, we obtain that the algebra $\mathcal{D}_S$ is isomorphic to the von Neumann algebra tensor product $B(K)\overline{\otimes}\mathcal{L}$ of $B(K)$ and the W^*-algebra $\mathcal{L}$ of all operators on $\mathcal{H}$ diagonal with respect to $\{e_n\}_{n=1}^{\infty}$. From this and from Proposition 2.6 we obtain the following result.

Corollary 2.8. *Let $S \in \mathcal{S}_{\mathbb{Z}}$. If $\dim(H_S(n)) = \infty$ for all $n \in \mathrm{Sp}(S)$ and conditions (2.6) and (2.7) hold then there exist Hilbert spaces K and $\mathcal{H}$ such that $\mathfrak{U}_S$ is isomorphic to $B(K)\overline{\otimes}\mathcal{L} + B(K)\otimes C(\mathcal{H})$, where $\mathcal{L}$ is the W^*-algebra of all operators on $\mathcal{H}$ diagonal with respect to some basis.*

Assume now that $\dim(H_S(n)) < \infty$ for all $n \in \mathrm{Sp}(S)$. Then $C_S(H)$ coincides with the algebra $C(H)$ of all compact operators on H. Taking into account the definition of the ideal $\mathcal{K}_S$ and applying Proposition 2.6 we obtain the following result.

Corollary 2.9. *Let $S \in \mathcal{S}_{\mathbb{Z}}$ and $\dim(H_S(n)) < \infty$ for all $n \in \mathrm{Sp}(S)$. If conditions (2.6) and (2.7) hold then $\mathfrak{U}_S = \mathcal{D}_S + C(H)$ and $\mathcal{A}_S = \mathcal{D}_S + \mathcal{K}_S$.*

Example 2.10. Let $\{e_i\}_{i=-\infty}^{\infty}$ be an orthonormal basis in H and let

$$Se_i = \mathrm{sgn}\,(i)|i|^{1+a} e_i, \quad \text{where} \ \ a > 0.$$

Then $S \in \mathcal{S}_{\mathbb{Z}}$ and $n_i = \mathrm{sgn}\,(i)|i|^{1+a}$, so that

$$\lim_{|i|\to\infty} \frac{n_{i+1} - n_i}{\mathrm{sgn}\,(i)|i|^a} = 1 + a.$$

Therefore, by Corollaries 2.7 and 2.9, $\mathfrak{U}_S = \mathcal{D}_S + C(H)$ and $\mathcal{A}_S = \mathcal{D}_S + \mathcal{K}_S$ where $\mathcal{D}_S$ is the algebra of all operators diagonal with respect to $\{e_i\}_{i=-\infty}^{\infty}$. Thus the quotient algebra $\mathcal{A}_S/\mathcal{K}_S$ is isomorphic to the commutative C^*-algebra $\mathcal{D}_S/\mathfrak{L}$ where $\mathfrak{L}$ is the algebra of all compact diagonal operators on H. $\square$

Let $\{e_i\}_{i=-\infty}^{\infty}$ be an orthonormal basis in H and let

$$Se_i = ie_i \text{ and } Ue_i = e_{i+1}, \quad \text{for all } i \in \mathbb{Z}.$$

Then $S \in \mathcal{S}_{\mathbb{Z}}$ and U is the shift operator. We have that

$$UD(S) \subseteq D(S),\ U^*D(S) \subseteq D(S) \text{ and } (SU - US)|_{D(S)} \text{ extends to } U,$$

so that $U \in \mathcal{A}_S$. Hence $\mathfrak{U}_S$ contains the C^*-algebra $C(\mathcal{D}_S, U)$ generated by U and by the commutative algebra $\mathcal{D}_S$ of all operators diagonal with respect to $\{e_i\}_{i=-\infty}^{\infty}$.

Problem 2.11. Is $\mathfrak{U}_S = C(\mathcal{D}_S, U)$?

3. Dual and second dual spaces of the algebras $\mathcal{F}_S$.

Let S be a closed symmetric operator. Recall that $\mathcal{F}_S$ is the closure with respect to the norm $\|\cdot\|_S$ (see (1.2)) of the subalgebra of all finite rank operators in $\mathcal{A}_S$. If S is a bounded symmetric operator on H, it follows that $\mathcal{F}_S = C(H)$ and $\mathcal{A}_S = B(H)$, so that $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. In this section we study the structure of the dual and the second dual spaces of the algebra $\mathcal{F}_S$ for unbounded symmetric operators S. In the case when S is selfadjoint we establish the full analogy with the bounded case: The algebra $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$.

By $T(H)$ we denote the Banach *-algebra of trace class operators on H with the norm

$$|A| = \sum_{i=1}^{\infty} s_i(A) = \mathrm{Tr}\left((A^*A)^{1/2}\right),$$

where $\{s_i(A)\}_{i=1}^{\infty}$ is the set of all eigenvalues of the positive compact operator $(A^*A)^{1/2}$.

It is well known that $T(H)$ can be identified with the dual space of the algebra $C(H)$: For any $T \in T(H)$,

$$F_T(A) = \mathrm{Tr}(AT), \quad A \in C(H),$$

is a bounded linear functional on $C(H)$ and $\|F_T\| = |T|$; and that $B(H)$ can be identified with the dual space of $T(H)$: For any $B \in B(H)$,

$$\theta_B(T) = \mathrm{Tr}\,(BT), \quad T \in T(H),$$

is a bounded linear functional on $T(H)$ and $\|\theta\| = \|B\|$.

Set $\widehat{B}(H) = B(H) \oplus B(H)$ and $\widehat{C}(H) = C(H) \oplus C(H)$. Then $\widehat{B}(H)$ and $\widehat{C}(H)$ are Banach spaces with the norm

$$\|A \oplus B\| = \|A\| + \|B\|.$$

Set $\widehat{T}(H) = T(H) \oplus T(H)$. It is a Banach space with the norm

$$|R \oplus T| = \max(|R|, |T|), \quad T, R \in T(H),$$

and it can be identified with the dual space of $\widehat{C}(H)$: For $R, T \in T(H)$,

$$F_{R\oplus T}(A \oplus B) = \mathrm{Tr}\,(AR) + \mathrm{Tr}\,(BT), \quad A \oplus B \in \widehat{C}(H), \tag{3.1}$$

is a bounded linear functional on $\widehat{C}(H)$ and $\|F_{R\oplus T}\| = |R \oplus T|$. Similarly, $\widehat{B}(H)$ can be identified with the dual space of $\widehat{T}(H)$: For $A, B \in B(H)$,

$$\theta_{A\oplus B}(R \oplus T) = \mathrm{Tr}\,(AR) + \mathrm{Tr}\,(BT), \quad R \oplus T \in \widehat{T}(H), \tag{3.2}$$

is a bounded linear functional on $\widehat{T}(H)$ and $\|\theta_{A\oplus B}\| = \|A \oplus B\|$.

Set

$$\widehat{\mathcal{A}}_S = \{A \oplus A_S : A \in \mathcal{A}_S\} \quad \text{and} \quad \widehat{\mathcal{F}}_S = \{A \oplus A_S : A \in \mathcal{F}_S\},$$

where $A_S = \mathrm{Closure}\,(SA - AS)$. Then $(\mathcal{A}_S, \|\cdot\|_S)$ and $(\widehat{\mathcal{A}}_S, \|\cdot\|)$, $(\mathcal{F}_S, \|\cdot\|_S)$ and $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ are isometrically isomorphic, since

$$\|A\|_S = \|A\| + \|A_S\| = \|A \oplus A_S\|.$$

Therefore $\widehat{\mathcal{A}}_S$ is a closed subspace of $\widehat{B}(H)$ and $\widehat{\mathcal{F}}_S$ is a closed subspace of $\widehat{C}(H)$, since $A \in \mathcal{F}_S$ implies $A_S \in C(H)$.

Set

$$\mathfrak{T}_S = \Big\{T \in T(H) : TD(S) \subseteq D(S^*),\ T^*D(S) \subseteq D(S^*) \ \text{ and the operator } (S^*T - TS)|_{D(S)} \ \text{ extends to a bounded trace class operator } \mathbb{T}\Big\}.$$

If $T \in \mathfrak{T}_S \cap \mathcal{A}_S$ then $\mathbb{T}_S = T_S$. In particular, if S is selfadjoint then $\mathbb{T}_S = T_S$ for all $T \in \mathfrak{T}_S$. Clearly, $\mathfrak{T}_S$ is a linear subspace in $T(H)$ and

$$\check{\mathfrak{T}}_S = \{\mathbb{T}_S \oplus T : T \in \mathfrak{T}_S\}$$

is a linear subspace in $\widehat{T}(H)$. For $T \in \mathfrak{T}_S$ and $z, u \in D(S)$,

$$-((\mathbb{T}_S)^* z, u) = -(z, \mathbb{T}_S u) = -(z, (S^*T - TS)u) = ((S^*T^* - T^*S)z, u),$$

so that

$$-(\mathbb{T}_S)^*|_{D(S)} = (S^*T^* - T^*S)|_{D(S)} = (\mathbb{T}^*)_S|_{D(S)}. \tag{3.3}$$

Therefore $T^* \in \mathfrak{T}_S$.

For $x, y \in H$, the rank one operator $x \otimes y$ on H is defined by the formula

$$(x \otimes y)z = (z, x)y. \tag{3.4}$$

It is easy to check that

$$\|x \otimes y\| = \|x\|\,\|y\|, \tag{3.5}$$
$$(x \otimes y)^* = y \otimes x,\ (x \otimes y)(u \otimes v) = (v, x)(u \otimes y),$$
$$R(x \otimes y) = x \otimes Ry, \quad \text{and} \quad (x \otimes y)R \ \text{ extends to } \ (R^* x) \otimes y,$$

if R is a densely defined operator, $y \in D(R)$ and $x \in D(R^*)$. Let $\{e_j\}_{j=1}^{\infty}$ be a basis in H. Then

$$\begin{aligned} \operatorname{Tr}(x \otimes y) &= \sum_{j=1}^{\infty}((x \otimes y)e_j, e_j) = \sum_{j=1}^{\infty}(e_j, x)(y, e_j) \\ &= \left(y, \sum_{j=1}^{\infty}(x, e_j)e_j\right) = (y, x). \end{aligned} \tag{3.6}$$

Let $x, y \in D(S^*)$ and $T = x \otimes y$. By (3.4) and (3.5),

$$Tz = (z, x)y \in D(S^*) \tag{3.7}$$
$$T^* z = (y \otimes x)z = (z, y)x \in D(S^*), \quad \text{for } \ z \in H,$$
$$\text{and } \ \mathbb{T}_S = S^* T - TS = x \otimes S^* y - (S^* x) \otimes y \in T(H),$$

so that $T \in \mathfrak{T}_S$. By Φ_S we denote the set of all linear combinations of the operators $x \otimes y$, for $x, y \in D(S^*)$. Clearly, $\Phi \subset \mathfrak{T}_S$ and

$$\check{\Phi}_S = \{\mathbb{T}_S \oplus T : T \in \Phi_S\}$$

is a linear subspace of $\check{\mathfrak{T}}_S$.

Let X^* be the dual space of a Banach space X and Y be a linear subspace of X. The *annihilator*

$$Y^{\perp} = \{F \in X^* : F(y) = 0, \quad \text{for all } \ y \in Y\}$$

of Y in X^* is a closed subspace of X^* and from the general theory of Banach spaces (see [**5**] II.4.18 and [**15**] III, Problem 30) we have the following lemma.

Lemma 3.1. *The dual space Y^* of a closed subspace Y of X is isometrically isomorphic to the quotient space $X^*/Y^{\perp}$ and the second dual Y^{**} of Y is isometrically isomorphic to $Y^{\perp\perp}$ where*

$$Y^{\perp\perp} = \{\theta \in X^{**} : \theta(F) = 0, \quad \textit{for all } \ F \in Y^{\perp}\}.$$

Since $\widehat{\mathcal{F}}_S \subseteq \widehat{C}(H)$, the annihilator $(\widehat{\mathcal{F}}_S)^{\perp}$ is a closed subspace of the dual space $\widehat{C}(H)^* = \widehat{T}(H)$ and, since $\check{\Phi}_S \subseteq \check{\mathfrak{T}}_S \subseteq \widehat{T}(H)$, the annihilator $(\check{\Phi}_S)^{\perp}$ is a closed subspace of the dual space $\widehat{T}(H)^* = \widehat{B}(H)$.

Theorem 3.2. (i) *$\check{\mathfrak{T}}_S$ is a closed subspace in $\widehat{T}(H)$ and $(\widehat{\mathcal{F}}_S)^{\perp} = \check{\mathfrak{T}}_S$.*
(ii) $(\check{\mathfrak{T}}_S)^{\perp} \subseteq (\check{\Phi}_S)^{\perp} = \{A \oplus A_S : A \in \mathcal{A}_S$ *and* $AD(S^*) \subseteq D(S)\} \subseteq \widehat{\mathcal{A}}_S$.

Proof. Let $\mathbb{T}_S \oplus T \in \check{\mathfrak{T}}_S$ and $x, y \in D(S)$. Then $A = x \otimes y \in \mathcal{F}_S$ and, by (3.3) and (3.5),

$$\begin{aligned} A_S &= S(x \otimes y) - (x \otimes y)S = x \otimes Sy - (Sx) \otimes y, \\ A_S T &= (x \otimes Sy)T - ((Sx) \otimes y)T = (T^*x) \otimes Sy - (T^*Sx) \otimes y, \\ A\mathbb{T}_S &= (x \otimes y)\mathbb{T}_S = ((\mathbb{T}_S)^*x) \otimes y = ((T^*S - S^*T^*)x) \otimes y. \end{aligned} \tag{3.8}$$

Therefore, by (3.1), (3.6) and (3.8),

$$\begin{aligned} F_{\mathbb{T}_S \oplus T}(A \oplus A_S) &= \mathrm{Tr}\,(A\mathbb{T}_S) + \mathrm{Tr}(A_S T) \\ &= (y, (T^*S - S^*T^*)x) + (Sy, T^*x) - (y, T^*Sx) = 0. \end{aligned}$$

It follows from Lemma 3.1 [**13**] that any finite rank operator A in $\mathcal{F}_S$ has the form $A = \sum_{i=1}^n x_i \otimes y_i$ where $x_i, y_i \in D(S)$. Hence $F_{\mathbb{T}_S \oplus T}(A \oplus A_S) = 0$ for any finite rank operator $A \in \mathcal{F}_S$. Since, by definition of $\mathcal{F}_S$, finite rank operators are dense in $(\mathcal{F}_S, \|\cdot\|_S)$ and since $(\mathcal{F}_S, \|\cdot\|_S)$ and $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ are isometrically isomorphic, the operators $A \oplus A_S$, where A are finite rank operators, are dense in $\widehat{\mathcal{F}}_S$. Since $F_{\mathbb{T}_S \oplus T}$ is continuous on $\widehat{C}(H)$, $F_{\mathbb{T}_S \oplus T}(A \oplus A_S) = 0$, for all $A \in \mathcal{F}_S$. Therefore $F_{\mathbb{T}_S \oplus T} \in (\widehat{\mathcal{F}}_S)^\perp$, so that $\check{\mathfrak{T}}_S \subseteq (\widehat{\mathcal{F}}_S)^\perp$.

Conversely, let $R \oplus T \in (\widehat{\mathcal{F}}_S)^\perp \subseteq \widehat{T}(H)$ and let $A = x \otimes y \in \mathcal{F}_S$, where $x, y \in D(S)$. From (3.1), (3.5), (3.6) and (3.8) it follows that

$$\begin{aligned} 0 = F_{R \oplus T}(A \oplus A_S) &= \mathrm{Tr}(AR) + \mathrm{Tr}(A_S T) \\ &= \mathrm{Tr}((R^*x) \otimes y) + \mathrm{Tr}[(T^*x) \otimes Sy - (T^*Sx) \otimes y] \\ &= (y, R^*x) + (Sy, T^*x) - (y, T^*Sx). \end{aligned}$$

Hence

$$(Sy, T^*x) = (y, (T^*S - R^*)x), \quad \text{for all } x, y \in D(S).$$

Therefore $T^*x \in D(S^*)$ and $S^*T^*x = (T^*S - R^*)x$. Thus $T^*D(S) \subseteq D(S^*)$ and

$$(Sx, Ty) = (T^*Sx, y) = (S^*T^*x, y) + (R^*x, y) = (x, TSy) + (x, Ry).$$

From this it follows that $Ty \in D(S^*)$ and $S^*Ty = TSy + Ry$. Hence

$$TD(S) \subseteq D(S^*) \quad \text{and} \quad R|_{D(S)} = S^*T|_{D(S)} - TS|_{D(S)}.$$

Therefore $T \in \mathfrak{T}_S$ and $R = \mathbb{T}_S$. Thus $(\widehat{\mathcal{F}}_S)^\perp \subseteq \check{\mathfrak{T}}_S$, so that $(\widehat{\mathcal{F}}_S)^\perp = \check{\mathfrak{T}}_S$. From this we also obtain that $\check{\mathfrak{T}}_S$ is a closed subspace of $\widehat{T}(H)$. Part (i) is proved.

Since $\check{\Phi}_S \subseteq \check{\mathfrak{T}}_S$, we have $(\check{\mathfrak{T}}_S)^\perp \subseteq (\check{\Phi}_S)^\perp$. Let now $A \oplus A_S \in \widehat{\mathcal{A}}_S$ and $AD(S^*) \subseteq D(S)$. It was shown in Lemma 3.1 [**13**] that

$$A_S|_{D(S^*)} = (S^*A - AS^*)|_{D(S^*)}.$$

For $x, y \in D(S^*)$, the operator $T = x \otimes y$ belongs to Φ_S and, taking the above equality into account, we obtain from (3.5) and (3.7) that

$$A_S T = x \otimes A_S y = x \otimes (S^*A - AS^*)y \quad \text{and}$$
$$A\mathbb{T}_S = A(x \otimes S^*y - (S^*x) \otimes y) = x \otimes AS^*y - (S^*x) \otimes Ay.$$

Therefore, by (3.2) and (3.6),

$$\begin{aligned}\theta_{A\oplus A_S}(\mathbb{T}_S \oplus T) &= \mathrm{Tr}(A\mathbb{T}_S) + \mathrm{Tr}(A_S T)\\ &= (AS^*y, x) - (Ay, S^*x) + (S^*Ay, x) - (AS^*y, x)\\ &= (S^*Ay, x) - (Ay, S^*x).\end{aligned}$$

Since $AD(S^*) \subseteq D(S)$, it follows that $S^*Ay = SAy$ and $(Ay, S^*x) = (SAy, x)$. Hence $\theta_{A\oplus A_S}(\mathbb{T}_S \oplus T) = 0$ and, by linearity, it holds for all $T \in \Phi_S$. Therefore

$$\{A \oplus A_S : A \in \mathcal{A}_S \;\text{ and }\; AD(S^*) \subseteq D(S)\} \subseteq \left(\check{\Phi}_S\right)^{\perp}. \tag{3.9}$$

Conversely, let $A \oplus B \in (\check{\Phi}_S)^{\perp}$. Then, for every $x, y \in D(S^*)$, $T = x \otimes y \in \Phi_S$ and

$$\theta_{A\oplus B}(\mathbb{T}_S \oplus T) = \mathrm{Tr}(AT_S) + \mathrm{Tr}(BT) = 0.$$

By (3.5), $BT = x \otimes By$ and, as above, $A\mathbb{T}_S = x \otimes AS^*y - (S^*x) \otimes Ay$. Hence, by (3.6),

$$0 = (AS^*y, x) - (Ay, S^*x) + (By, x).$$

Thus

$$(Ay, S^*x) = (AS^*y, x) + (By, x), \quad \text{for all } x, y \in D(S^*).$$

Therefore $Ay \in D(S^{**})$ and $S^{**}Ay = AS^*y + By$. Since S is closed, $S^{**} = S$ and we obtain that

$$AD(S^*) \subseteq D(S) \quad \text{and} \quad B|_{D(S^*)} = (SA - AS^*)|_{D(S^*)}. \tag{3.10}$$

Restricting (3.10) to D(S), we have

$$AD(S) \subseteq D(S) \quad \text{and} \quad B|_{D(S)} = (SA - AS)|_{D(S)}.$$

Making use of (3.10), we obtain that for $z \in D(S)$ and $u \in D(S^*)$,

$$(A^*z, S^*u) = (z, AS^*u) = (z, SAu) - (z, Bu) = (A^*Sz, u) - (B^*z, u).$$

Therefore $A^*z \in D(S^{**})$. Since $S^{**} = S$, we have $A^*D(S) \subseteq D(S)$. Thus $A \in \mathcal{A}_S$ and $B = A_S$, so $A \oplus B = A \oplus A_S \in \widehat{\mathcal{A}}_S$. Taking into account that $AD(S^*) \subseteq D(S)$, we obtain that

$$(\check{\Phi}_S)^{\perp} \subseteq \{A \oplus A_S : A \in \mathcal{A}_S \;\text{ and }\; AD(S^*) \subseteq D(S)\}.$$

Combining this with (3.9), we complete the proof of the theorem. □

Since the Banach spaces $(\mathcal{F}_S, \|\cdot\|_S)$ and $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ and the Banach spaces $(\mathcal{A}_S, \|\cdot\|_S)$ and $(\widehat{\mathcal{A}}_S, \|\cdot\|)$ are isometrically isomorphic and since $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ is a closed subspace of $\widehat{C}(H)$, Lemma 3.1 and Theorem 3.2 yield:

Corollary 3.3. *The dual space of the Banach *-algebra* $(\mathcal{F}_S, \|\cdot\|_S)$ *is isometrically isomorphic to the quotient space* $\widehat{T}(H)/\check{\mathfrak{T}}_S$ *and the second dual space of* $(\mathcal{F}_S, \|\cdot\|_S)$ *is isometrically isomorphic to a closed subspace of* $(\mathcal{A}_S, \|\cdot\|_S)$.

The following example shows that if S is not selfadjoint then, generally speaking, $(\check{\Phi}_S)^{\perp} \neq \widehat{\mathcal{A}}_S$, so that $(\mathcal{F}_S)^{\perp\perp} \neq \widehat{\mathcal{A}}_S$ and the second dual space of $(\mathcal{F}_S, \|\cdot\|_S)$ is isometrically isomorphic to a proper subspace of $(\mathcal{A}_S, \|\cdot\|_S)$.

Example 3.4. Let, as in Example 2.2, $H = L^2(0,1)$ and the operator $S = i\frac{d}{dt}$ with domain $D(S) = \{h(t) : h, h' \in L_2(0,1) \text{ and } h(0) = h(1) = 0\}$. Then S is a symmetric operator, non-selfadjoint and

$$D(S^*) = \{h(t) : h, h' \in L^2(0,1)\}.$$

Let $g(t)$ be a differentiable function on $[0,1]$ such that $g(0) \neq 0$ and let M_g be the bounded operator of multiplication by $g(t)$ on H. Then $M_g \in \mathcal{A}_S$. If $h(t) \in D(S^*)$ and $h(0) \neq 0$ then $(M_g h)(0) = g(0)h(0) \neq 0$, so that $M_g h \notin D(S)$. Thus $M_g \oplus (M_g)_S \notin \{A \oplus A_S : A \in \mathcal{A}_S \text{ and } AD(S^*) \subseteq D(S)\}$. Hence $(\check{\Phi}_S)^{\perp} \neq \mathcal{A}_S$.

Assume now that S is selfadjoint. Then $D(S^*) = D(S)$, $\mathbb{T}_S = T_S$, for $T \in \mathfrak{T}_S$, and

$$\mathfrak{T}_S = \{T \in T(H) \cap \mathcal{A}_S : T_S \in T(H)\} \subseteq \mathcal{A}_S.$$

It is well known (see, for example, [**5**] and [**6**]) that the algebra $T(H)$ is a two-sided ideal of $B(H)$ and if $A \in B(H)$ and $B \in T(H)$ then

$$(3.11) \qquad |AB| \leq \|A\|\,|B|, \quad |B^*| = |B| \quad \text{and} \quad \|B\| \leq |B|.$$

We consider now two equivalent norms on $\mathfrak{T}_S$:

$$|T|_1 = |T| + |T_S| \ \text{ and } \ |T|_2 = \max(|T|, |T_S|), \ \text{ for } \ T \in \mathfrak{T}_S.$$

Since

$$\mathbb{T}_S = T \ \text{ and } \ |T|_2 = \max(|T|, |T_S|) = |\mathbb{T}_S \oplus T|, \ \text{ for } \ T \in \mathfrak{T}_S,$$

$(\mathfrak{T}_S, |\cdot|_2)$ is isometrically isomorphic to $\check{\mathfrak{T}}_S$.

Proposition 3.5. *Let S be selfadjoint. Then:*

(i) $\mathfrak{T}_S \subset \mathcal{F}_S$ *and* $(\mathfrak{T}_S, |\cdot|_2)$ *is a two-sided Banach* $\mathcal{A}_S$ *-module;*
(ii) $(\mathfrak{T}_S, |\cdot|_1)$ *is a Banach *-algebra and a* **D***-subalgebra of* $C(H)$ *(see* (1.1)*) with* $D = 1$.

Proof. It was shown in [**13**] that if S is selfadjoint then $\mathcal{F}_S$ coincides with the algebra $\mathcal{J}_S = \{A \in \mathcal{A}_S : A \text{ and } A_S \text{ belong to } C(H)\}$. Since $\mathfrak{T}_S \subset \mathcal{J}_S$, we obtain that $\mathfrak{T}_S \subset \mathcal{F}_S$.

In Theorem 3.2(i) it was shown that $\check{\mathfrak{T}}_S$ is a closed subspace of $\widehat{T}(H)$. Since $(\mathfrak{T}_S, |\cdot|_2)$ is isometrically isomorphic to $\check{\mathfrak{T}}_S$, it is a Banach space.

Since the norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent, $(\mathfrak{T}_S, |\cdot|_1)$ is also a Banach space.

For $A, B \in \mathcal{A}_S$,

$$\begin{aligned}(AB)_S|_{D(S)} &= (SAB - ABS)|_{D(S)} \\ &= [(SA - AS)B + A(SB - BS)]|_{D(S)} = (A_S B + AB_S)|_{D(S)},\end{aligned}$$

so that

$$(AB)_S = A_S B + AB_S. \tag{3.12}$$

Let $T \in \mathfrak{T}_S$ and $A \in \mathcal{A}_S$. Then $T, T_S \in T(H)$. Since $\mathfrak{T}_S \subset \mathcal{A}_S$ and $T(H)$ is a two-sided ideal of $B(H)$, it follows that $AT \in T(H) \cap \mathcal{A}_S$ and, by (3.12),

$$(AT)_S = A_S T + AT_S \in T(H).$$

Therefore $AT \in \mathfrak{T}_S$. Making use of (3.11), we obtain that

$$\begin{aligned}|AT|_2 &= \max(|AT|,\, |(AT)_S|) \leq \max(\|A\|\,|T|,\, \|A_S\|\,|T| + \|A\|\,|T_S|) \\ &\leq (\|A\| + \|A_S\|)\max(|T|,\, |T_S|) = \|A\|_S |T|_2.\end{aligned}$$

Similarly, $TA \in \mathfrak{T}_S$ and $|TA|_2 \leq \|A\|_S |T|_2$. Thus $(\mathfrak{T}_S, |\cdot|_2)$ is a two-sided Banach $\mathcal{A}_S$-module. Part (i) is proved.

From (i) and from the fact that $\mathfrak{T}_S \subseteq \mathcal{A}_S$, we have that $\mathfrak{T}_S$ is an algebra. We also have that $T^* \in \mathfrak{T}_S$ and, since $\mathbb{T}_S = T_S$, it follows from (3.3) that $(T^*)_S = -(T_S)^* \in T(H)$. Taking this and (3.11) into account, we obtain that

$$|T^*|_1 = |T^*| + |(T^*)_S| = |T^*| + |-(T_S)^*| = |T| + |T_S| = |T|_1$$

and

$$\begin{aligned}|TR|_1 &= |TR| + |(TR)_S| = |TR| + |T_S R + TR_S| \\ &\leq \|T\|\,|R| + |T_S|\,\|R\| + \|T\|\,|R_S| \\ &\leq |T|\,|R| + |T_S|\,|R| + |T|\,|R_S| \leq |T|_1\,|R|_1,\end{aligned}$$

for $T, R \in \mathfrak{T}_S$. Hence $(\mathfrak{T}_S, |\cdot|_1)$ is a Banach *-algebra.

Clearly, $\mathfrak{T}_S$ is dense in $C(H)$. For $T, R \in \mathfrak{T}_S$, it follows from (3.11) that

$$\begin{aligned}|TR|_1 &= |TR| + |(TR)_S| = |TR| + |T_S R + TR_S| \\ &\leq \|T\|\,|R| + |T_S|\,\|R\| + \|T\|\,|R_S| \\ &\leq \|T\|(|R| + |R_S|) + (|T| + |T_S|)\|R\| \\ &= \|T\|\,|R|_1 + |T|_1\|R\|.\end{aligned}$$

Thus $(\mathfrak{T}_S, |\cdot|_1)$ is a **D**-subalgebra of $C(H)$ with the constant $D = 1$. □

If S is selfadjoint, it follows from Theorem 3.2 that $(\check{\Phi}_S)^\perp = \widehat{\mathcal{A}}_S$ and

$$\left(\widehat{\mathcal{F}}_S\right)^{\perp\perp} = \left(\check{\mathfrak{T}}_S\right)^\perp \subseteq \left(\check{\Phi}_S\right)^\perp = \widehat{\mathcal{A}}_S.$$

In order to prove that $(\widehat{\mathcal{F}}_S)^{\perp\perp} = \widehat{\mathcal{A}}_S$ it suffices to show that $\check{\Phi}_S$ is dense in $\check{\mathfrak{T}}_S$. For this we need the following lemma which is a partial case of the general result obtained by Gohberg and Krein [**6**, Theorem 6.3] for symmetrically normable ideals.

Lemma 3.6. *Let $T \in T(H)$ and let Q_n be finite rank projections which converge to $\mathbf{1}_H$ in the strong operator topology. Then*

$$|T - Q_nT| \to 0 \quad and \quad |T - TQ_n| \to 0, \quad as \ n \to \infty.$$

Proof. Let $A = x \otimes y$, $x, y \in H$. By (3.5), $A^*A = \|y\|^2(x \otimes x)$ and the operator $(A^*A)^{1/2} = \frac{\|y\|}{\|x\|}(x \otimes x)$ has only one non-zero eigenvalue $\lambda = \|x\| \, \|y\|$. Hence

$$|x \otimes y| = |A| = \operatorname{Tr}(A^*A)^{1/2} = \lambda = \|x\| \, \|y\|. \tag{3.13}$$

If $T = \sum_{i=1}^k x_i \otimes y_i$ is a finite rank operator then, by (3.5) and (3.13),

$$\begin{aligned} |T - Q_nT| &= \left| \sum_{i=1}^k x_i \otimes (y_i - Q_ny_i) \right| \le \sum_{i=1}^k |x \otimes (y_i - Q_ny_i)| \\ &= \sum_{i=1}^k \|x_i\| \, \|y_i - Q_ny_i\| \to 0, \end{aligned}$$

as $n \to \infty$. For any T in $T(H)$ and any $\varepsilon > 0$, there is a finite rank operator T_ε such that $|T - T_\varepsilon| < \varepsilon$. Making use of the inequality (3.11), we obtain that

$$\begin{aligned} |T - Q_nT| &\le |T - T_\varepsilon| + |T_\varepsilon - Q_nT_\varepsilon| + |Q_n(T - T_\varepsilon)| \\ &\le \varepsilon + |T_\varepsilon - Q_nT_\varepsilon| + \|Q_n\| \, |T - T_\varepsilon| \\ &\le 2\varepsilon + |T_\varepsilon - Q_nT_\varepsilon|. \end{aligned}$$

Since T_ε is a finite rank operator, by the above argument, there is n_ε such that $|T_\varepsilon - Q_nT_\varepsilon| \le \varepsilon$, for $n > n_\varepsilon$. Hence $|T - Q_nT| \le 3\varepsilon$ and $|T - Q_nT| \to 0$, as $n \to \infty$. Similarly, one can prove that $|T - TQ_n| \to 0$, as $n \to \infty$. □

Proposition 3.7. *Let S be selfadjoint. Then Φ_S is dense in $(\mathfrak{T}_S, |\cdot|_1)$.*

Proof. Let $[S]$ be the selfadjoint operator constructed in Section 2. Then $D(S) = D([S])$, so that $\Phi_S = \Phi_{[S]}$. Since $B = S - [S]$ is a bounded operator, $BT - TB \in T(H)$, for $T \in T(H)$. Therefore, taking into account that

$$(ST - TS)_{D(S)} = ([S]T - T[S])_{D(S)} + (BT - TB)_{D(S)},$$

we conclude that $\mathfrak{T}_S = \mathfrak{T}_{[S]}$ and $T_S = T_{[S]} + BT - TB$.

Making use of (3.11), we obtain that for any $T \in \mathfrak{T}_S$,

$$\begin{aligned} |T| + |T_S| &= |T| + \left|T_{[S]} + BT - TB\right| \\ &\le |T| + \left|T_{[S]}\right| + 2\|B\| \, |T| \\ &\le (1 + 2\|B\|) \left(|T| + \left|T_{[S]}\right|\right). \end{aligned}$$

Similarly, $|T|+|T_{[S]}| \le (1+2\|B\|)(|T|+|T_S|)$. Thus the norms $|\cdot|_1$ generated by the operators S and $[S]$ on $\mathfrak{T}_S$ are equivalent. Hence to obtain the proof we only have to show that $\Phi_{[S]}$ is dense in $(\mathfrak{T}_{[S]}, |\cdot|_1)$.

In every subspace $H_S(n)$ (see (2.2)) we choose an increasing sequence of finite-dimensional projections $\{Q_n^k\}_{k=1}^{\infty}$ converging to the projection $P_S(n)$ (see (2.1)) in the strong operator topology as $k \to \infty$. Set

$$Q^k = \sum_{n=-k}^{k} \oplus Q_n^k.$$

Then Q^k are finite-dimensional projections commuting with $[S]$. Hence $Q^k \in \Phi_{[S]}$. The projections Q^k converge to $\mathbf{1}_H$ in the strong operator topology. Let $T \in \mathfrak{T}_{[S]}$. Then $Q_n T \in \Phi_{[S]}$ and

$$[S]Q^kT - Q^kT[S] = Q^k[S]T - Q^kT[S] = Q^k([S]T - T[S]) = Q^kT_{[S]}.$$

Therefore $(Q^kT)_{[S]} = Q^kT_{[S]}$.

Since $T, T_{[S]} \in T(H)$, we obtain from Lemma 3.6 that

$$|T - Q^kT| \to 0 \quad \text{and} \quad \left|T_{[S]} - (Q^kT)_{[S]}\right| = \left|T_{[S]} - Q^kT_{[S]}\right| \to 0, \quad \text{as} \quad k \to \infty.$$

Hence

$$|T - Q^kT|_1 = |T - Q^kT| + \left|T_{[S]} - (Q^kT)_{[S]}\right| \to 0$$

as $k \to \infty$, so that $\Phi_{[S]}$ is dense in $(\mathfrak{T}_{[S]}, |\cdot|_1)$. □

Corollary 3.8. *Let S be a selfadjoint operator. Then:*

(i) *the Banach *-algebra $(\mathfrak{T}_S, |\cdot|_1)$ is simple;*
(ii) $(\check{\mathfrak{T}}_S)^{\perp} = (\check{\Phi}_S)^{\perp} = \widehat{\mathcal{A}}_S$;
(iii) *the dual space of $(\mathfrak{T}_S, |\cdot|_2)$ is isometrically isomorphic to the quotient space $\widehat{B}(H)/\widehat{\mathcal{A}}_S$.*

Proof. Let I be a closed two-sided ideal of $(\mathfrak{T}_S, |\cdot|_1)$ and $0 \ne T \in I$. Since $D(S)$ is dense in H, there is $x \in D(S)$ such that $Tx \ne 0$. Since S is selfadjoint, it follows from the definition of $\mathfrak{T}_S$ that $Tx \in D(S)$. From this and from the discussion before Lemma 3.1 we obtain that the rank one operators $y \otimes x$ and $Tx \otimes z$ belong to $\mathfrak{T}_S$ for any $y, z \in D(S)$. By (3.5), $T(y \otimes x) = (y \otimes Tx) \in I$ and

$$(Tx \otimes z)(y \otimes Tx) = \|Tx\|^2 (y \otimes z) \in I.$$

Thus $y \otimes z \in I$ and, therefore, $\Phi_S \subseteq I$. Since I is closed, we obtain from Proposition 3.7 that $I = \mathfrak{T}_S$. Part (i) is proved.

Since the norms $|\cdot|_1$ and $|\cdot|_2$ are equivalent on $\mathfrak{T}_S$, it follows from Proposition 3.7 that Φ_S is dense in $(\mathfrak{T}_S, |\cdot|_2)$. Taking into account that $(\mathfrak{T}_S, |\cdot|_2)$ is isometrically isomorphic to the closed subspace $\check{\mathfrak{T}}_S$ of $\widehat{T}(H)$, we obtain that the linear subspace $\check{\Phi}_S$ is dense in $\check{\mathfrak{T}}_S$. From this and from Theorem 3.2(ii) we obtain $(\check{\mathfrak{T}}_S)^\perp = (\check{\Phi}_S)^\perp = \widehat{\mathcal{A}}_S$. Part (ii) is proved.

The dual space of $(\mathfrak{T}_S, |\cdot|_2)$ is isometrically isomorphic to the dual space of the closed subspace $\check{\mathfrak{T}}_S$ of $\widehat{T}(H)$. Since $\widehat{T}(H)^* = \widehat{B}(H)$, part (iii) follows from (ii) and from Lemma 3.1. □

Theorem 3.9. *If S is a selfadjoint operator then $(\widehat{\mathcal{F}}_S)^{\perp\perp} = \widehat{\mathcal{A}}_S$ and the second dual space of the algebra $(\mathcal{F}_S, \|\cdot\|_S)$ is isometrically isomorphic to the algebra $(\mathcal{A}_S, \|\cdot\|_S)$.*

Proof. Combining Theorem 3.2(i) and Corollary 3.8(ii) yields $(\widehat{\mathcal{F}}_S)^{\perp\perp} = \widehat{\mathcal{A}}_S$. Therefore it follows from Lemma 3.1 that the second dual space of $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ is isometrically isomorphic to $(\widehat{\mathcal{A}}_S, \|\cdot\|)$. Taking into account that $(\mathcal{F}_S, \|\cdot\|_S)$ is isometrically isomorphic to $(\widehat{\mathcal{F}}_S, \|\cdot\|)$ and that $(\mathcal{A}_S, \|\cdot\|_S)$ is isometrically isomorphic to $(\widehat{\mathcal{A}}_S, \|\cdot\|)$, we complete the proof. □

4. Isomorphism of the algebras $\mathcal{F}_S$ and $\mathcal{A}_S$.

In this section we study the problem of classification of the algebras $\mathcal{F}_S$ and $\mathcal{A}_S$ up to *-isomorphism. For isometrical *-isomorphism this problem is completely solved in Theorem 4.4. As far as bounded but not necessarily isometrical *-isomorphism is concerned, we have obtained some partial results in Theorems 4.6 and 4.8 for the case when S is selfadjoint.

Banach *-algebras $(\mathcal{A}, \|\ \|_\mathcal{A})$ and $(\mathcal{B}, \|\ \|_\mathcal{B})$ are *-isomorphic if there is a bounded *-isomorphism φ from $\mathcal{A}$ onto $\mathcal{B}$. They are isometrically *-isomorphic if, in addition, $\|\varphi(A)\|_\mathcal{B} = \|A\|_\mathcal{A}$, for $A \in \mathcal{A}$.

Let $(\mathcal{A}, \|\ \|_\mathcal{A})$ and $(\mathcal{B}, \|\ \|_\mathcal{B})$ be Banach *-algebras of operators on Hilbert spaces H and $\mathcal{H}$ (the norms $\|\cdot\|_\mathcal{A}$ and $\|\cdot\|_\mathcal{B}$ do not, generally speaking, coincide with the operator norms in $B(H)$ and $B(\mathcal{H})$) and let φ be a bounded *-isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. An isometry operator U from H into $\mathcal{H}$ implements φ if

$$\varphi(A) = UAU^*, \quad A \in \mathcal{A}.$$

Lemma 4.1. *Let R and T be symmetric operators on $\mathcal{H}$, S be a symmetric operators on H, U be an isometry operator from $\mathcal{H}$ onto H and $t \in \mathbb{R}$.*

(i) *If $\mathcal{F}_R = \mathcal{F}_T$ then the norms $\|\cdot\|_R$ and $\|\cdot\|_T$ on this algebra are equivalent, so that the Banach *-algebras $(\mathcal{F}_R, \|\cdot\|_R)$ and $(\mathcal{F}_T, \|\cdot\|_T)$ are *-isomorphic.*

(ii) *If $R = \pm T + t\mathbf{1}_\mathcal{H}$ then $\mathcal{F}_R = \mathcal{F}_T$ and the norms $\|\cdot\|_R$ and $\|\cdot\|_T$ coincide.*

(iii) *If $S = \lambda UTU^* + B$, where $0 \neq \lambda \in R$ and B is a bounded selfadjoint operator, then $A \to UAU^*$ is a bounded *-isomorphism from $(\mathcal{F}_T, \|\cdot\|_T)$ onto $(\mathcal{F}_S, \|\cdot\|_S)$. If $\lambda = \pm 1$ and $B = t\mathbf{1}_H$ then $A \to UAU^*$ is an isometric *-isomorphism.*

The same results hold for the algebras $\mathcal{A}_S$.

Proof. By Proposition 3.2 [**13**], the algebras $\mathcal{F}_R$ and $\mathcal{F}_T$ are semisimple. Hence if $\mathcal{F}_R = \mathcal{F}_T$ then it follows from Johnson's uniqueness of norm theorem that the norms $\|\cdot\|_R$ and $\|\cdot\|_T$ on this algebra are equivalent. Therefore the identity mapping is a bounded *-isomorphism from $(\mathcal{F}_R, \|\cdot\|_R)$ onto $(\mathcal{F}_T, \|\cdot\|_T)$.

Let $R = \pm T + t\mathbf{1}_{\mathcal{H}}$. Then $D(R) = D(T)$ and $A_T = A_R$ for any $A \in \mathcal{A}_T$. Hence $\|A\|_R = \|A\|_T$ and $\mathcal{A}_R = \mathcal{A}_T$. The sets of finite rank operators in the algebras $\mathcal{F}_R$ and $\mathcal{F}_T$ coincide and, since these algebras are the closures of these sets with respect to the norm $\|\cdot\|_T$, we obtain that $\mathcal{F}_S = \mathcal{F}_T$.

If $S = \lambda UTU^* + B$ then $D(S) = UD(T)$ and, for $A \in \mathcal{A}_T$,

$$UAU^*D(S) = UAD(T) \subseteq UD(T) = D(S) \quad \text{and}$$
$$SUAU^* - UAU^*S = \lambda U(TA - AT)U^* + (BA - AB),$$

so that $UAU^* \in \mathcal{A}_S$ and $(UAU^*)_S = \lambda UA_TU^* + (BA - AB)$. Thus $\mathcal{A}_S = U\mathcal{A}_TU^*$ and

$$\begin{aligned}\|UAU^*\|_S &= \|UAU^*\| + \|(UAU^*)_S\| = \|A\| + \|\lambda UA_TU^* + (BA - AB)\| \\ &\le \|A\| + \lambda\|A\| + 2\|B\|\,\|A\| \le \max(\lambda, 1 + 2\|B\|)\|A\|_T,\end{aligned}$$

so that $\psi(A) = UAU^*$ is a bounded *-isomorphism from $(\mathcal{A}_T, \|\cdot\|_T)$ onto $(\mathcal{A}_S, \|\cdot\|_S)$. If A is a finite rank operator in $\mathcal{A}_T$ then UAU^* is a finite rank operator in $\mathcal{A}_S$. Therefore $\mathcal{F}_S = \psi(\mathcal{F}_T)$. □

Let S be a symmetric operator with domain $D(S)$. It was shown in Lemma 3.1 [**13**] that a finite rank operator A belongs to $\mathcal{F}_S$ if and only if

$$A = \sum_{i=1}^{n} x_i \otimes y_i, \quad \text{where } x_i, y_i \in D(S). \tag{4.1}$$

Theorem 4.2. *Let S and T be symmetric operators on H and $\mathcal{H}$ and let $\mathcal{B}$ and $\mathcal{C}$ be closed *-subalgebras of $(\mathcal{A}_S, \|\cdot\|_S)$ and $(\mathcal{A}_T, \|\cdot\|_T)$, respectively, such that $\mathcal{F}_S \subseteq \mathcal{B}$ and $\mathcal{F}_T \subseteq \mathcal{C}$. Let ψ be a bounded *-isomorphism from $\mathcal{C}$ onto $\mathcal{B}$ and let $\varphi = \psi|\mathcal{F}_T$. Then:*

(i) *φ is a bounded *-isomorphism of $(\mathcal{F}_T, \|\cdot\|_T)$ onto $(\mathcal{F}_S, \|\cdot\|_S)$;*
(ii) *there is an isometry operator U from $\mathcal{H}$ onto H implementing ψ:*

$$\psi(A) = UAU^*, \quad \text{for } A \in \mathcal{C},$$

and $D(S) = UD(T)$ and $\mathcal{F}_{UTU^*} = \mathcal{F}_S$.

Proof. For $x, y \in D(T)$, $x \neq 0$, $y \neq 0$, set $Y = \varphi(x \otimes y)$. If Y is not a rank one operator, there are $z, u \in D(S)$ such that $Yz \neq 0$, $Yu \neq 0$ and $Yz \perp Yu$. Since $Y \in \mathcal{A}_S$, we have that $Yz, Yu \in D(S)$, so that $Yz \otimes z \in \mathcal{F}_S$ and $u \otimes Yu \in \mathcal{F}_S$. By (3.5)

$$(Yz \otimes z)(u \otimes Yu) = (Yu, Yz)(u \otimes z) = 0. \tag{4.2}$$

Since $(z \otimes z)^* = z \otimes z$ and φ is a *-isomorphism, it follows from (3.5) that

$$\begin{aligned}\left(\varphi^{-1}(z \otimes z)x\right) \otimes y &= (x \otimes y)\left[\varphi^{-1}(z \otimes z)\right]^* \\ &= \varphi^{-1}(Y)\varphi^{-1}(z \otimes z) = \varphi^{-1}(z \otimes Yz) \neq 0.\end{aligned}$$

Thus $\varphi^{-1}(z \otimes z)x \neq 0$. Similarly, $\varphi^{-1}(u \otimes u)x \neq 0$. From this and from (3.5) and (4.2) it follows that

$$\begin{aligned}0 &= \varphi^{-1}((Yz \otimes z)(u \otimes Yu)) = \varphi^{-1}((z \otimes z)Y^*Y(u \otimes u)) \\ &= \varphi^{-1}(z \otimes z)\varphi^{-1}(Y^*)\varphi^{-1}(Y)\varphi^{-1}(u \otimes u) \\ &= \varphi^{-1}(z \otimes z)(y \otimes x)(x \otimes y)\varphi^{-1}(u \otimes u) \\ &= \varphi^{-1}(z \otimes z)\|y\|^2(x \otimes x)\varphi^{-1}(u \otimes u) \\ &= \|y\|^2([\varphi^{-1}(u \otimes u)x] \otimes [\varphi^{-1}(z \otimes z)x]) \neq 0.\end{aligned}$$

This contradiction shows that Y is a rank one operator. Hence $Y \in \mathcal{F}_S$ and, by (4.1), φ maps all finite rank operators in $\mathcal{F}_T$ into finite rank operators in $\mathcal{F}_S$. Since φ is bounded $\varphi(\mathcal{F}_T) \subseteq \mathcal{F}_S$. Similarly, $\varphi^{-1}(\mathcal{F}_S) \subseteq \mathcal{F}_T$, so that φ is a bounded *-isomorphism from $\mathcal{F}_T$ onto $\mathcal{F}$. Part (i) is proved.

Fix $x_0 \in D(T)$, $\|x_0\| = 1$. Since $x_0 \otimes x_0$ is a projection, $\varphi(x_0 \otimes x_0)$ is a one-dimensional projection in $\mathcal{F}_S$. By (4.1), we can choose ξ_0 in $D(S)$, $\|\xi_0\| = 1$, such that $\varphi(x_0 \otimes x_0) = \xi_0 \otimes \xi_0$. Let $y \in D(T)$. Making use of the equality $x_0 \otimes y = (x_0 \otimes y)(x_0 \otimes x_0)$, we obtain that

$$\begin{aligned}\varphi(x_0 \otimes y) &= \varphi(x_0 \otimes y)\varphi(x_0 \otimes x_0) \\ &= \varphi(x_0 \otimes y)(\xi_0 \otimes \xi_0) = \xi_0 \otimes \varphi(x_0 \otimes y)\xi_0.\end{aligned}$$

Since $\varphi(x_0 \otimes y) \in \mathcal{F}_S$, it follows from (4.1) that $\varphi(x_0 \otimes y)\xi_0$ belongs to $D(S)$.

Now $U : y \in D(T) \to \varphi(x_0 \otimes y)\xi_0$ is a linear mapping from $D(T)$ into $D(S)$ and $\varphi(x_0 \otimes y) = \xi_0 \otimes Uy$. Then

$$\begin{aligned}\varphi((y \otimes x_0)(x_0 \otimes y)) &= \|y\|^2\varphi(x_0 \otimes x_0) = \|y\|^2(\xi_0 \otimes \xi_0) \\ &= \varphi((x_0 \otimes y)^*)\varphi(x_0 \otimes y) \\ &= (Uy \otimes \xi_0)(\xi_0 \otimes Uy) = \|Uy\|^2(\xi_0 \otimes \xi_0).\end{aligned}$$

Thus $\|Uy\|^2 = \|y\|^2$, for $y \in D(T)$, and U extends to an isometry operator from $\mathcal{H}$ into H which we also denote by U. We have that, for $x, y \in D(T)$,

$$\begin{aligned}\varphi(x \otimes y) &= \varphi((x_0 \otimes y)(x \otimes x_0)) = (\xi_0 \otimes Uy)(\xi_0 \otimes Ux)^* \\ &= Ux \otimes Uy = U(x \otimes y)U^*.\end{aligned} \tag{4.3}$$

Similarly, there is an isometry operator V which maps $D(S)$ into $D(T)$ such that $\varphi^{-1}(\xi \otimes \eta) = V\xi \otimes V\eta$, for $\xi, \eta \in D(S)$. Hence

$$\xi \otimes \eta = \varphi(\varphi^{-1}(\xi \otimes \eta)) = \varphi(V\xi \otimes V\eta) = UV\xi \otimes UV\eta.$$

Thus $UV\xi = \lambda(\xi)\xi$ where λ is a function on $D(S)$ such that $|\lambda(\xi)| = 1$. Hence $UD(T) = D(S)$. Since $D(S)$ is dense in H and U is an isometry operator, we have $U\mathcal{H} = H$.

Let $A \in \mathcal{C}$ and set $R = U^*\psi(A)U$. Then $x \otimes y \in \mathcal{F}_T$, for any $x, y \in D(T)$, and, since $\mathcal{F}_T$ is an ideal of $\mathcal{A}_T$, we have $A(x \otimes y) = x \otimes Ay \in \mathcal{F}_T$. By (4.3),

$$\begin{aligned} R(x \otimes y) &= U^*\psi(A)U(x \otimes y) = U^*\psi(A)U(x \otimes y)U^*U \\ &= U^*\psi(A)\varphi(x \otimes y)U = U^*\psi(A)\psi(x \otimes y)U \\ &= U^*\psi(A(x \otimes y))U = U^*\varphi(x \otimes Ay)U = x \otimes Ay. \end{aligned}$$

Therefore $R(x \otimes y) = x \otimes Ry = x \otimes Ay$, so that $Ry = Ay$. Thus $R = A$ and

$$\psi(A) = UAU^*.$$

The operator $F = UTU^*$ is symmetric and $D(F) = UD(T) = D(S)$. By Lemma 4.1, $\mathcal{F}_F = U\mathcal{F}_TU^*$ and $A \to UAU^*$ is an isometric *-isomorphism from $(\mathcal{F}_T, \|\cdot\|_T)$ onto $(\mathcal{F}_F, \|\cdot\|_F)$. Hence

$$\varphi(U^*BU) = U(U^*BU)U^* = B, \quad \text{for } B \in \mathcal{F}_F,$$

is a bounded *-isomorphism from $\mathcal{F}_F$ onto $\mathcal{F}_S$. Therefore $\mathcal{F}_F = \mathcal{F}_S$. □

It was shown in Theorem 3.4 [**13**] that the algebra $(\mathcal{F}_S, \|\cdot\|_S)$ has a bounded approximate identity if and only if S is selfadjoint. Making use of this and of Theorem 4.2, we obtain the following result.

Corollary 4.3. *If the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ are *-isomorphic or the algebras $\mathcal{A}_S$ and $\mathcal{A}_T$ are *-isomorphic then the operators S and T are either selfadjoint or non-selfadjoint at the same time.*

Apart from the sufficient conditions of Lemma 4.1 and the necessary conditions of Corollary 4.3 for two algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ to be *-isomorphic we do not know any other sufficient or necessary condition in the case when S and T are arbitrary symmetric operators. Later, in Theorem 4.6 and Corollary 4.8 we consider a particular case when the operators S and T are selfadjoint.

It follows from Theorem 4.2 that if $\mathcal{F}_S$ and $\mathcal{F}_T$ are *-isomorphic, they are unitary isomorphic. This, however, does not necessarily imply that they are isometrically isomorphic. In the following theorem we obtain necessary and sufficient conditions for algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ to be *isometrically* *-isomorphic.

Theorem 4.4. *The algebras $(\mathcal{F}_S, \|\cdot\|_S)$ and $(\mathcal{F}_T, \|\cdot\|_T)$ are isometrically *-isomorphic if and only if there are $\lambda \in \mathbb{R}$ and an isometry operator U such that $S - \lambda\mathbf{1}_H = \pm UTU^*$. The same result holds for $(\mathcal{A}_S, \|\cdot\|_S)$ and $(\mathcal{A}_T, \|\cdot\|_T)$.*

Proof. From Lemma 4.1 it follows that the conditions of the theorem are sufficient. From Theorem 4.2 it follows that if these conditions are necessary for the algebras $(\mathcal{F}_S, \|\cdot\|_S)$ and $(\mathcal{F}_T, \|\cdot\|_T)$ to be isometrically *-isomorphic, they are also necessary for the algebras $(\mathcal{A}_S, \|\cdot\|_S)$ and $(\mathcal{A}_T, \|\cdot\|_T)$.

Let φ be an isometric *-isomorphism from $(\mathcal{F}_T, \|\cdot\|_T)$ onto $(\mathcal{F}_S, \|\cdot\|_S)$ and let U be the isometry operator as in Theorem 4.2 which implements φ:

$$\varphi(A) = UAU^*, \quad \text{for } \; A \in \mathcal{F}_T.$$

Set $F = UTU^*$. Then F is a symmetric operator on H, $D(S) = D(F) = UD(T)$ and $\mathcal{F}_S = \mathcal{F}_F$. Since φ is isometric, the norms $\|\cdot\|_S$ and $\|\cdot\|_F$ coincide.

We will show that there is $\lambda \in \mathbb{R}$ such that either $S - \lambda\mathbf{1}_H = F$ or $S - \lambda\mathbf{1}_H = -F$.

Step 1. Suppose that $z \in D(S)$ is not an eigenvector of S and $\|z\| = 1$. Set

$$s = (Sz, z), \quad t = (Fz, z), \quad R = S - s\mathbf{1}_H \quad \text{and} \quad G = F - t\mathbf{1}_H.$$

Since S an F are symmetric, $s, t \in \mathbb{R}$, the operators R and G are symmetric and

$$D(R) = D(G), \quad Rz \neq 0 \quad \text{and} \;\; (Rz, z) = (Gz, z) = 0. \tag{4.4}$$

Set $D = D(R) = D(G)$. Since $\mathcal{F}_S = \mathcal{F}_F$ and the norms $\|\cdot\|_S$ and $\|\cdot\|_F$ coincide, it follows from Lemma 4.1 that $\mathcal{F}_R = \mathcal{F}_G$ and the norms $\|\cdot\|_R$ and $\|\cdot\|_G$ coincide.

Taking into account that R and G are symmetric, we obtain from (3.5) that

$$\begin{aligned}\|y \otimes x\|_R &= \|y \otimes x\| + \|y \otimes Rx - (Ry) \otimes x\| = \|y \otimes x\|_G \\ &= \|y \otimes x\| + \|y \otimes Gx - (Gy) \otimes x\|,\end{aligned}$$

for $x, y \in D$. Therefore

$$\|y \otimes Rx - (Ry) \otimes x\| = \|y \otimes Gx - (Gy) \otimes x\|. \tag{4.5}$$

Represent the elements Rx and Gx in the form

$$Rx = \alpha(x)x + x_R \quad \text{and} \;\; Gx = \beta(x)x + x_G, \tag{4.6}$$

where x_R and x_G are orthogonal to x. Then

$$\alpha(x)\|x\|^2 = (Rx, x) = (x, Rx) = \overline{\alpha(x)}\|x\|^2.$$

Thus $\alpha(x)$ is real, for $x \in D$. Therefore

$$\begin{aligned}x \otimes Rx - (Rx) \otimes x &= \alpha(x)(x \otimes x) + x \otimes x_R - \alpha(x)(x \otimes x) - x_R \otimes x \\ &= x \otimes x_R - x_R \otimes x.\end{aligned}$$

Since x and x_R are orthogonal, any $u \in H$ can be represented in the form $u = \nu x + \tau x_R + \widetilde{u}$, where $\nu, \tau \in \mathbb{C}$ and $\widetilde{u}$ is orthogonal to x and x_R. Therefore

$$\|u\| = |\nu|^2\|x\|^2 + |\tau|^2\|x\|^2 + \|\widetilde{u}\|^2$$

and, by (3.5),

$$\begin{aligned}\|(x\otimes x_R + x_R\otimes x)u\|^2 &= \|(u,x)x_R + (u,x_R)\,x\|^2\\ &= \big\|\nu\,\|x\|^2 x_R + \tau\|x_R\|^2 x\big\|^2\\ &= |\nu|^2\|x\|^4\|x_R\|^2 + |\tau|^2\|x_R\|^4\|x\|^2\\ &= \|x\|^2\|x_R\|^2(|\nu|^2\|x\|^2 + |\tau|^2\|x_R\|^2).\end{aligned}$$

Consequently,

$$\|x\otimes Rx - (Rx)\otimes x\|^2 = \|x\otimes x_R - x_R\otimes x\|^2 = \|x\|^2\|x_R\|^2.$$

Similarly, $\|x\otimes Gx - (Gx)\otimes x\|^2 = \|x\|^2\|x_G\|^2$ and it follows from (4.5) that

$$\|x_R\| = \|x_G\|, \quad \text{for } \ x\in D.$$

Therefore we obtain from (4.6) that for $x\in D$

$$\begin{aligned}\|x\|^2\|Rx\|^2 - |(Rx,x)|^2 &= \|x\|^2(|\alpha(x)|^2\|x\|^2 + \|x_R\|^2) - |\alpha(x)|^2\|x\|^4\\ &= \|x\|^2\|x_R\|^2 = \|x\|^2\|x_G\|^2\\ &= \|x\|^2\|Gx\|^2 - |(Gx,x)|^2.\end{aligned}$$

Hence

$$\|x\|^2(\|Rx\|^2 - \|Gx\|^2) = |(Rx,x)|^2 - |(Gx,x)|^2. \tag{4.7}$$

In particular, it follows from (4.4), (4.6) and (4.7) that

$$Rz = z_R, \quad Gz = z_G \quad \text{and} \quad \|Rz\| = \|Gz\|. \tag{4.8}$$

Step 2. Set $D_Z^{\perp} = \{y\in D : y \text{ is orthogonal to } z\}$. Let $y\in D_Z^{\perp}$ and $x = y+\mu z$, $\mu\in\mathbb{C}$. Then $\|x\|^2 = \|y\|^2 + \|\mu z\|^2 = \|y\|^2 + |\mu|^2$ and, by (4.8),

$$\begin{aligned}\|Rx\|^2 - \|Gx\|^2 &= \|Ry\|^2 + \|\mu Rz\|^2 + 2\mathrm{Re}[\mu(Rz,Ry)]\\ &\qquad - \|Gy\|^2 - \|\mu Gz\|^2 - 2\mathrm{Re}[\mu(Gz,Gy)]\\ &= A + 2\mathrm{Re}(\mu B),\end{aligned}$$

where

$$A = \|Ry\|^2 - \|Gy\|^2 \quad \text{and} \quad B = (Rz,Ry) - (Gz,Gy).$$

Since R is symmetric, it follows from (4.4) that

$$\begin{aligned}(Rx,x) &= (Ry,y) + (\mu Rz,y) + (Ry,\mu z) + (\mu Rz,\mu z)\\ &= (Ry,y) + 2\mathrm{Re}[\mu(Rz,y)].\end{aligned}$$

Similarly, $(Gx,x) = (Gy,y) + 2\mathrm{Re}[\mu(Gz,y)]$.

Let $\mu = re^{i\psi}$. Substituting all this in (4.7), we obtain that

$$\begin{aligned}&(\|y\|^2 + r^2)[A + 2r\mathrm{Re}(e^{i\psi}B)]\\ &\quad = \{(Ry,y) + 2r\mathrm{Re}[e^{i\psi}(Rz,y)]\}^2 - \{(Gy,y) + 2r\mathrm{Re}[e^{i\psi}(Gz,y)]\}^2.\end{aligned} \tag{4.9}$$

Set

$$C = (Ry, y)\mathrm{Re}[e^{i\psi}(Rz, y)] - (Gy, y)\mathrm{Re}[e^{i\psi}(Gz, y)] \quad \text{and}$$
$$E = \{\mathrm{Re}[e^{i\psi}(Rz, y)]\}^2 - \{\mathrm{Re}[e^{i\psi}(Gz, y)]\}^2.$$

Since R and G are symmetric, (Ry, y) and (Gy, y) are real. Hence

$$C = \mathrm{Re}\{e^{i\psi}[(Ry, y)(Rz, y) - (Gy, y)(Gz, y)]\}.$$

Comparing the coefficients of the same powers of r in (4.9), we obtain that

$$\mathrm{Re}(e^{i\psi}B) = 0, \quad A = 4E \quad \text{and} \quad C = 0.$$

Taking into account that $\mathrm{Re}\,(e^{i\psi}K) = 0$, for $0 \le \psi < 2\pi$, implies $K = 0$, we obtain that $C = 0$ implies

$$(Ry, y)(Rz, y) - (Gy, y)(Gz, y) = 0. \tag{4.10}$$

Set $(Rz, y) = ae^{ib}$ and $(Gz, y) = ce^{id}$. Then

$$E = a^2\left[\mathrm{Re}\left(e^{i(\psi+b)}\right)\right]^2 - c^2\left[\mathrm{Re}\left(e^{i(\psi+d)}\right)\right]^2$$
$$= a^2\cos^2(\psi + b) - c^2\cos^2(\psi + d).$$

Since $A = 4E$ and since A does not depend on ψ, neither does E. Hence $a^2 = c^2$ and $d = b$ or $d = b + \pi$. Since $a \ge 0$ and $c \ge 0$, $a = c$. Thus

$$(Rz, y) = \pm(Gz, y), \quad \text{for} \ \ y \in D_Z^{\perp}. \tag{4.11}$$

Since D is dense in $\mathcal{H}$, $D_Z^{\perp}$ is dense in the subspace $\{\mathbb{C}z\}^{\perp}$. Hence (4.11) holds for all $y \in \{\mathbb{C}z\}^{\perp}$. From (4.9) it follows that $Rz = z_R \in \{\mathbb{C}z\}^{\perp}$. Substituting Rz for y in (4.11), we obtain $\|Rz\| = (Rz, Rz) = \pm(Gz, Rz)$. Let $Gz = \nu Rz + u$, where $\nu \in \mathbb{C}$ and u is orthogonal to Rz. Then

$$\|Rz\|^2 = \pm(Gz, Rz) = \pm\nu\|Rz\|^2.$$

Since $Rz \neq 0$ (see (4.4)), $\nu = \pm 1$. Taking (4.9) into account, we obtain

$$\|Rz\|^2 = \|Gz\|^2 = (\nu Rz + u, \nu Rz + u)$$
$$= |\nu|^2\|Rz\|^2 + \|u\|^2 = \|Rz\|^2 + \|u\|^2.$$

Hence $u = 0$ and either $Rz = Gz$ or $Rz = -Gz$.

Step 3. Let $Rz = Gz$. Set $W = R - G$. Then W is symmetric, $Wz = 0$ and it follows from (4.10) that

$$[(Ry, y) - (Gy, y)](Rz, y) = (Wy, y)(Rz, y) = 0, \quad \text{for} \ \ y \in D_Z^{\perp}.$$

Any $x \in D$ can be represented in the form $x = y + \mu z$ where $\mu \in \mathbb{C}$ and $y \in D_Z^{\perp}$. Then $Wx = Wy$ and, since $(Rz, z) = 0$, we have $(Rz, x) = (Rz, y)$. Since $Wz = 0$,

$$(Wx, x)(Rz, x) = (Wy, y + \mu z)(Rz, y)$$
$$= [(Wy, y) + (y, \mu Wz)](Rz, y) = (Wy, y)(Rz, y) = 0.$$

Therefore

$$(Wx,x)(Rz,x) = 0, \quad \text{for } x \in D. \tag{4.12}$$

Let $X = \{x \in H : (Rz,x) = 0\}$ be the orthogonal complement of the subspace $\{\mathbb{C}Rz\}$ in H. By (4.4), $Rz \neq 0$, so X has codimension 1. Set $\mathcal{D} = \{x \in D : x \notin X\}$. Since D is dense in H, $\mathcal{D}$ is also dense in H. For $x \in \mathcal{D}$, we have $(Rz,x) \neq 0$. Hence, by (4.12),

$$(Wx,x) = 0.$$

If $x, y \in \mathcal{D}$, there is $r > 0$ such that $x + re^{i\psi}y \in \mathcal{D}$, for all $0 \leq \psi < 2\pi$. Taking into account that W is symmetric, we obtain that

$$\begin{aligned} 0 &= (W(x+re^{i\psi}y), x+re^{i\psi}y) = (Wx,x) + 2r\mathrm{Re}[e^{i\psi}(Wy,x)] + r^2(Wy,y) \\ &= 2r\mathrm{Re}[e^{i\psi}(Wy,x)]. \end{aligned}$$

Hence $(Wy,x) = 0$. Since $\mathcal{D}$ is dense in H, we have $Wy = 0$, for $y \in \mathcal{D}$.

Let $u \in D \cap X$, so that $(Rz,u) = 0$. For $y \in \mathcal{D}$, $(Rz, y+u) = (Rz,y) \neq 0$. Hence $y + u \in \mathcal{D}$ and $0 = W(y+u) = Wy + Wu = Wu$. Thus $Wx = 0$, for all $x \in D$, so that $R = G$. Hence $S - s\mathbf{1}_H = F - t\mathbf{1}_H$. Setting $\lambda = s - t$, we obtain that

$$S - \lambda \mathbf{1}_H = F = UTU^*.$$

Similarly, in the case when $Rz = -Gz$ we obtain that $S - \lambda\mathbf{1}_H = -F = -UTU^*$ which concludes the proof of the theorem. □

In the rest of this section we study conditions for the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ to be *-isomorphic but not necessarily isometrically *-isomorphic in the case when S and T are selfadjoint operators. Taking Theorem 4.2(ii) into account, we may assume, without loss of generality, that $\mathcal{F}_S = \mathcal{F}_T$ and $D(S) = D(T)$.

In Example 4.7 we show that the coincidence of the domains of selfadjoint operators S and T even in the case when $\mathrm{Sp}(S) \subseteq \mathbb{Z}$, $\mathrm{Sp}(T) \subseteq \mathbb{Z}$ and S and T have the same sets of eigenvectors is not sufficient for $\mathcal{F}_S = \mathcal{F}_T$. In other words, the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ may be the closures of the same set of finite rank operators and, nevertheless, be non-isomorphic. Necessary and sufficient conditions for these algebras to be *-isomorphic will be obtained in Theorem 4.6.

Let $\mathfrak{H}$ be a Hilbert space with an orthogonal basis $\{e_i\}_{i=-\infty}^{\infty}$. Every operator T in $B(\mathfrak{H})$ has a matrix representation $T = (t_{ij})$, $-\infty < i, j < \infty$, where $t_{ij} = (Te_j, e_i)$. A matrix $M = (m_{ij})$, $-\infty < i, j < \infty$, is called a *Schur multiplier*, if, for any $T = (t_{ij}) \in B(\mathfrak{H})$, the matrix $M \circ T = (m_{ij}t_{ij})$ belongs to $B(\mathfrak{H})$. Then $T \to M \circ T$ is a bounded map of $B(\mathfrak{H})$ into itself; it will also be denoted by M and its norm by $|M|_{B(\mathfrak{H})}$.

Let $H = \sum_{i=-\infty}^{\infty} \oplus H_i$ be an orthogonal sum of Hilbert spaces H_i. Every operator A in $B(H)$ has a block-matrix representation $A = (A_{ij})$, $-\infty < i, j < \infty$, where A_{ij} are bounded operators from H_j into H_i.

Lemma 4.5. *Let $M = (m_{ij})$ be a Schur multiplier on $\mathfrak{H}$. It defines a bounded operator $\mathcal{M}$ on $B(H)$ by the formula*

$$\mathcal{M} \times A = (m_{ij} A_{ij}), \quad \text{where } A = (A_{ij}) \in B(H),$$

and $|\mathcal{M}|_{B(H)} = |M|_{B(\mathcal{H})}$.

Proof. Let $G = \{g_j\}_{j=-\infty}^{\infty}$ and $F = \{f_j\}_{j=-\infty}^{\infty}$ be sequences of elements such that $g_j, f_j \in H_j$ and $\|g_j\| = \|f_j\| = 1$. For $A = (A_{ij}) \in B(H)$, let $T^{G,F}(A) = \left(a_{ij}^{GF}\right)$, $-\infty < i, j < \infty$, be the matrix such that

$$a_{ij}^{GF} = (A_{ij} g_j, f_i) \in \mathbb{C}. \tag{4.13}$$

For $\alpha = \sum_{j=-\infty}^{\infty} \oplus \alpha_j e_j \in \mathfrak{H}$ and $\beta = \sum_{j=-\infty}^{\infty} \oplus \beta_j e_j \in \mathfrak{H}$, set

$$x_\alpha^G = \sum_{j=-\infty}^{\infty} \oplus \alpha_j g_j \quad \text{and} \quad y_\beta^F = \sum_{j=-\infty}^{\infty} \oplus \beta_j f_j.$$

Then $x_\alpha^G, y_\beta^F \in H$, $\left\|x_\alpha^G\right\| = \|\alpha\|$, $\left\|y_\beta^F\right\| = \|\beta\|$ and

$$\begin{aligned}\left(A x_\alpha^G, y_\beta^F\right) &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j \bar{\beta}_i (A_{ij} g_j, f_i) \\ &= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j \bar{\beta}_i a_{ij}^{GF} = \left(T^{G,F}(A)\alpha, \beta\right).\end{aligned}$$

Therefore $T^{G,F}(A) \in B(\mathfrak{H})$ and

$$\begin{aligned}\|A\| &= \sup_{\alpha,\beta,G,F} \frac{\left|\left(A x_\alpha^G, y_\beta^F\right)\right|}{\|x_\alpha^G\| \left\|y_\beta^F\right\|} \\ &= \sup_{G,F} \left(\sup_{\alpha,\beta} \frac{|(T^{G,F}(A)\alpha, \beta)|}{\|\alpha\| \, \|\beta\|} \right) = \sup_{G,F} \left\|T^{G,F}(A)\right\|.\end{aligned} \tag{4.14}$$

It follows from (4.13) that $T^{G,F}(\mathcal{M} \times A) = M \circ T^{G,F}(A)$. Since M is a Schur multiplier, $M \circ T^{G,F}(A) \in B(\mathfrak{H})$ and, by (4.14),

$$\begin{aligned}\|\mathcal{M} \times A\| &= \sup_{G,F} \left\|T^{G,F}(\mathcal{M} \times A)\right\| = \sup_{G,F} \left\|M \circ T^{G,F}(A)\right\| \\ &\le \sup_{G,F} |M|_{B(\mathfrak{H})} \left\|T^{G,F}(A)\right\| = |M|_{B(\mathfrak{H})} \sup_{G,F} \left\|T^{G,F}(A)\right\| \\ &= |M|_{B(\mathfrak{H})} \|A\|.\end{aligned}$$

Hence $|\mathcal{M}|_{B(H)} \leq |M|_{B(\mathfrak{H})}$. On the other hand, it is easy to see that $|M|_{B(\mathfrak{H})} \leq |\mathcal{M}|_{B(H)}$. Thus $|\mathcal{M}|_{B(H)} = |M|_{B(\mathfrak{H})}$. □

Let S and T be selfadjoint operators on H and assume that $\mathrm{Sp}(S) \subseteq \mathbb{Z}$, $\mathrm{Sp}(T) \subseteq \mathbb{Z}$ and that

$$H = \sum_{i=-\infty}^{\infty} \oplus H_i \quad \text{where } S|_{H_i} = s_i \mathbf{1}_{H_i}, \quad T|_{H_i} = t_i \mathbf{1}_{H_i},$$
$$s_i \neq s_j \text{ and } t_i \neq t_j \text{ if } i \neq j.$$

Set

$$M = (m_{ij}) \text{ where } m_{ij} = \frac{s_i - s_j}{t_i - t_j}, \text{ for } i \neq j, \text{ and } m_{ii} = 0, \quad \text{and}$$
$$N = (n_{ij}) \text{ where } n_{ij} = \frac{t_i - t_j}{s_i - s_j}, \text{ for } i \neq j, \text{ and } n_{ii} = 0.$$

Theorem 4.6. *$\mathcal{F}_S = \mathcal{F}_T$ if and only if M and N are Schur multipliers.*

Proof. In every H_i we choose a non-decreasing sequence of finite-dimensional projections $\{Q_i^p\}_{p=1}^{\infty}$ which converge to $\mathbf{1}_{H_i}$ in the strong operator topology as $p \to \infty$. Set $Q_p = \sum_{i=-p}^{p} \oplus Q_i^p$. The finite-dimensional projections Q_p commute with S and T, belong to $\mathcal{F}_S \cap \mathcal{F}_T$ and converge to $\mathbf{1}_H$ in the strong operator topology. Therefore $\|Q_p\|_S = \|Q_p\|_T = \|Q_p\| = 1$.

For any $A = (A_{ij}) \in \mathcal{A}_S \cap \mathcal{A}_T$,

$$A_S = SA - AS = \left(A_{ij}^S\right) \text{ and } A_T = TA - AT = \left(A_{ij}^T\right),$$

where $A_{ij}^S = (s_i - s_j)A_{ij}$ and $A_{ij}^T = (t_i - t_j)A_{ij}$. Set $B = A_T$. Then $A_S = \mathcal{M} \times B$,

$$\begin{aligned} \|A\|_S &= \|A\| + \|A_S\| = \|A\| + \|\mathcal{M} \times B\| \qquad \text{and} \\ \|A\|_T &= \|A\| + \|A_T\| = \|A\| + \|B\|. \end{aligned} \tag{4.15}$$

We assume now that M and N are Schur multipliers and show that $\mathcal{F}_S = \mathcal{F}_T$. By Lemma 4.5 and (4.15),

$$\begin{aligned} \|A\|_S &\leq \|A\| + |M|\,\|B\| \\ &\leq \|A\| + |M|\,(\|A\|_T - \|A\|) \leq (|M| + 1)\|A\|_T. \end{aligned} \tag{4.16}$$

Similarly,

$$\|A\|_T \leq (|N| + 1)\|A\|_S. \tag{4.17}$$

Let $A \in \mathcal{F}_S$. Then $Q_pA \in \mathcal{F}_S$ and, since Q_p commute with S,

$$(Q_pA)_S = \text{Closure}\,(SQ_pA - Q_pAS) = \text{Closure}\,Q_p(SA - AS) = Q_pA_S.$$

Since A and A_S are compact and since Q_p converge to $\mathbf{1}_H$ in the strong operator topology,

$$\|A - Q_pA\| \to 0 \text{ and } \|A_S - (Q_pA)_S\| = \|A_S - Q_pA_S\| \to 0, \text{ as } p \to \infty.$$

Hence $\|A - Q_pA\|_S \to 0$, so that $\{Q_p\}$ is a bounded approximate identity in $\mathcal{F}_S$. Similarly, it is a bounded approximate identity in $\mathcal{F}_T$.

Let $A \in \mathcal{F}_S$. For any p, $Q_pT = Q_pTQ_p = TQ_p$ is a finite rank operator. Hence

$$(Q_pAQ_p)_T = T(Q_pAQ_p) - (Q_pAQ_p)T = (TQ_p)AQ_p - Q_pA(Q_pT)$$

is a finite rank operator. Therefore $Q_pAQ_p \in \mathcal{F}_S \cap \mathcal{F}_T$ and, by (4.17),

$$\|Q_{p+k}AQ_{p+k} - Q_pAQ_p\|_T \leq (|N| + 1)\|Q_{p+k}AQ_{p+k} - Q_pAQ_p\|_S.$$

Since $\{Q_p\}$ is a bounded approximate identity in $\mathcal{F}_S$, the operators Q_pAQ_p converge to A with respect to $\|\cdot\|_S$. From the above inequality it follows that $\{Q_pAQ_p\}$ is a fundamental sequence with respect to $\|\cdot\|_T$. Hence there is $A_1 \in \mathcal{F}_T$ such that $\|A_1 - Q_pAQ_p\|_T \to 0$, as $p \to \infty$. Since $\|A - Q_pAQ_p\| \leq \|A - Q_pAQ_p\|_S \to 0$ and $\|A_1 - Q_pAQ_p\| \leq \|A_1 - Q_pAQ_p\|_T \to 0$, as $p \to \infty$, we obtain that $A = A_1$, so $\mathcal{F}_S \subseteq \mathcal{F}_T$. Similarly, $\mathcal{F}_T \subseteq \mathcal{F}_S$. Thus we conclude that $\mathcal{F}_S = \mathcal{F}_T$.

Suppose now that $\mathcal{F}_S = \mathcal{F}_T$. Choose elements $e_i \in H_i$ such that $\|e_i\| = 1$ and let $\mathfrak{H}$ be the subspace of H generated by all e_i, $-\infty < i < \infty$. Then $\mathfrak{H}$ is invariant for S and T, $Se_i = s_ie_i$ and $Te_i = t_ie_i$. By $S_{\mathfrak{H}}$ and $T_{\mathfrak{H}}$ we denote the restrictions of S and T to $\mathfrak{H}$. Since $\mathcal{F}_S = \mathcal{F}_T$,

$$\mathcal{F}_{S_{\mathfrak{H}}} = \mathcal{F}_{T_{\mathfrak{H}}}.$$

We shall show now that M and N are Schur multipliers on $\mathfrak{H}$.

The function $f(t) = i(\pi - t)$ on $[0, 2\pi]$ has Fourier coefficients $c_0 = 0$ and $c_n = \frac{1}{n}$, for $n = \pm 1, \pm 2, \ldots$. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{h_k\}_{k=-\infty}^{\infty}$ and $R = (r_{kl})$, $-\infty < k, l < \infty$, be a *Toeplitz* matrix such that $r_{kk} = 0$ and $r_{kl} = c_{k-l} = \frac{1}{k-l}$, $k \neq l$. Then $R \in B(\mathcal{H})$ and it follows from Theorem 8.1 [**1**] that R is a Schur multiplier and $|R| = \sup |f(t)| = \pi$.

Identifying e_i in $\mathfrak{H}$ with h_{t_i} in $\mathcal{H}$, we can consider $\mathfrak{H}$ as a subspace of $\mathcal{H}$. For $B = (b_{km}) \in B(\mathfrak{H})$, where $b_{km} = (Be_m, e_k)$, let $\widetilde{B} = (\widetilde{b}_{ij}) \in B(\mathcal{H})$ be such that $\widetilde{B}|_{\mathfrak{H}} = B$ and $\widetilde{B}|_{\mathfrak{H}^\perp} = 0$. Then $\|\widetilde{B}\| = \|B\|$,

$$\widetilde{b}_{t_kt_m} = \left(\widetilde{B}h_{t_m}, h_{t_k}\right) = (Be_m, e_k) = b_{km}, \qquad \text{and}$$
$$\widetilde{b}_{ij} = \left(\widetilde{B}h_j, h_i\right) = 0 \ \text{ if either } \ i \neq t_k \ \text{ or } \ j \neq t_m.$$

Since R is a Schur multiplier, the operator $\widetilde{C} = (\widetilde{c}_{ij}) = R \circ \widetilde{B}$ belongs to $B(\mathcal{H})$, where

$$\widetilde{c}_{t_kt_m} = r_{t_kt_m}\widetilde{b}_{t_kt_m} = (t_k - t_m)^{-1}b_{km}, \quad \text{if } \ k \neq m, \quad \text{and}$$
$$\widetilde{c}_{ij} = 0 \ \text{ if either } \ i \neq t_k \ \text{ or } \ j \neq t_m \ \text{ or } \ i = j = t_k.$$

Setting $C = \widetilde{C}|_{\mathfrak{H}}$, we obtain that $C = (c_{km}) \in B(\mathfrak{H})$, where

$$c_{km} = \widetilde{c}_{t_kt_m} = (t_k - t_m)^{-1}b_{km}, \ \text{ if } \ k \neq m, \ \text{ and } \ c_{kk} = 0,$$

that $\|\widetilde{C}\| = \|C\|$ and that $C = W \circ B$, where $W = (w_{km})$ is a matrix such that

$$w_{km} = (t_k - t_m)^{-1}, \quad k \neq m, \text{ and } w_{kk} = 0.$$

From this it follows that W is a Schur multiplier on $\mathfrak{H}$ and

$$\|W \circ B\| = \|C\| = \left\|\widetilde{C}\right\| = \left\|R \circ \widetilde{B}\right\| \leq |R| \left\|\widetilde{B}\right\| = |R| \, \|B\|.$$

Thus $|W| \leq |R| = \pi$.

Let P_n be the orthoprojections in $\mathfrak{H}$ on the subspaces $\sum_{j=-n}^{n} \oplus \{\mathbb{C}e_j\}$. Then P_n are finite rank operators commuting with operators $S_{\mathfrak{H}}$ and $T_{\mathfrak{H}}$ and $P_n\mathfrak{H} \subseteq D(S_{\mathfrak{H}})$. Hence $P_n \in \mathcal{F}_{S_{\mathfrak{H}}}$. For every $B \in B(\mathfrak{H})$, P_nBP_n are finite rank operators preserving $D(S_{\mathfrak{H}})$ and their adjoints $P_nB^*P_n$ also preserve $D(S_{\mathfrak{H}})$. Therefore

$$P_nBP_n \in \mathcal{F}_{S_{\mathfrak{H}}}. \tag{4.18}$$

Any $B = (b_{km}) \in B(\mathfrak{H})$ can be represented in the form $B = B_d + B_0$, where B_d is the diagonal operator such that $(B_d) = b_{kk}$. Then

$$\|B_d\| \leq \|B\| \quad \text{and} \quad \|B_0\| = \|B - B_d\| \leq 2\|B\|. \tag{4.19}$$

We have that

$$M \circ (P_nBP_n) = P_n(M \circ B)P_n. \tag{4.20}$$

Since $m_{kk} = 0$ in the matrix $M = (m_{km})$,

$$M \circ (P_nBP_n) = M \circ (P_nB_0P_n). \tag{4.21}$$

Set $A = W \circ B$. Since W is a Schur multiplier, $A \in B(\mathfrak{H})$ and, by (4.18), $P_nAP_n \in \mathcal{F}_{S_{\mathfrak{H}}}$. It is easy to check that

$$\begin{gathered} P_nB_0P_n = T_{\mathfrak{H}}(P_nAP_n) - (P_nAP_n)T_{\mathfrak{H}} = (P_nAP_n)_{T_{\mathfrak{H}}}, \quad \text{and} \\ M \circ (P_nB_0P_n) = S_{\mathfrak{H}}(P_nAP_n) - (P_nAP_n)S_{\mathfrak{H}} = (P_nAP_n)_{S_{\mathfrak{H}}}. \end{gathered} \tag{4.22}$$

Since $\mathcal{F}_{S_{\mathfrak{H}}} = \mathcal{F}_{T_{\mathfrak{H}}}$, it follows from Lemma 4.1(i) that the norms $\|\cdot\|_{S_{\mathfrak{H}}}$ and $\|\cdot\|_{T_{\mathfrak{H}}}$ are equivalent. Therefore there exists $D > 0$ such that $\|P_nAP_n\|_{S_{\mathfrak{H}}} \leq D\|P_nAP_n\|_{T_{\mathfrak{H}}}$. Hence we obtain from (4.19), (4.21) and (4.22) that

$$\begin{aligned} \|M \circ (P_nBP_n)\| &= \|M \circ (P_nB_0P_n)\| = \left\|(P_nAP_n)_{S_{\mathfrak{H}}}\right\| \\ &\leq \|P_nAP_n\|_{S_{\mathfrak{H}}} \leq D\|P_nAP_n\|_{T_{\mathfrak{H}}} \\ &= D\left(\|P_nAP_n\| + \left\|(P_nAP_n)_{T_{\mathfrak{H}}}\right\|\right) \\ &\leq D(\|A\| + \|P_nB_0P_n\|) \leq D(\|A\| + \|B_0\|) \\ &= D(\|W \circ B\| + \|B_0\|) \leq D(|R| \, \|B\| + 2\|B\|) = \rho. \end{aligned}$$

Thus all operators $M \circ (P_nBP_n)$, $1 \leq n < \infty$, lie in the ball $\mathbf{B}_\rho$ of $B(\mathfrak{H})$ of radius ρ. Compactness of $\mathbf{B}_\rho$ in the weak operator topology implies that the sequence $\{M \circ (P_n B P_n)\}_{n=1}^{\infty}$ has a cluster point $K \in B(\mathfrak{H})$. Therefore there is a subsequence $\{M \circ (P_{n_j} B P_{n_j})\}$ such that for all e_k and e_m,

$$(Ke_k, e_m) = \lim_{j \to \infty} (M \circ (P_{n_j} B P_{n_j}) e_k, e_m).$$

If $n_j \geq \max(|k|, |m|)$ then $P_{n_j} e_k = e_k$ and $P_{n_j} e_m = e_m$ and, by (4.20),

$$\left(M \circ \left(P_{n_j} B P_{n_j}\right) e_k, e_m\right) = \left(P_{n_j} (M \circ B) P_{n_j} e_k, e_m\right) = (M \circ B e_k, e_m).$$

Hence $(Ke_k, e_m) = ((M \circ B)e_k, e_m)$, $-\infty < k, m < \infty$. Thus $K = M \circ B$, so M is a Schur multiplier. Similarly, we obtain that N is also a Schur multiplier. □

Example 4.7. Let

$$s_i = i \quad \text{and} \quad t_i = (-1)^i i$$

in Theorem 4.6. If $\mathcal{F}_S = \mathcal{F}_T$ then, by Theorem 4.6, M is a Schur multiplier and we have that $|m_{ij}| \leq |M|$ for all i and j. Let $i = 2k$ and $j = -2k + 1$. Then $s_i = t_i = 2k$ and $s_j = -t_j = -2k + 1$. Hence

$$m_{ij} = \frac{s_i - s_j}{t_i - t_j} = 4k - 1 \to \infty, \quad \text{as} \quad k \to \infty.$$

This shows that M is not a Schur multiplier and, therefore, $\mathcal{F}_S \neq \mathcal{F}_T$.

Making use of Theorem 4.6, we obtain the following result of a more general character.

Theorem 4.8. *Let S and T be selfadjoint operators on H and $\mathcal{H}$ respectively. If there exists a bijection φ of $\mathbb{Z}$ onto $\mathbb{Z}$ such that*

$$\dim(\mathcal{H}_T(\varphi(i))) = \dim(H_S(i)), \quad \textit{for all} \;\; i \in \mathbb{Z},$$

(*see* (2.2) *for definition of* $\mathcal{H}_T(i)$ *and* $H_S(i)$) *and if*

$$M = (m_{ij}) \;\; \textit{where} \;\; m_{ij} = \frac{\varphi(i) - \varphi(j)}{i - j}, \;\; \textit{for} \;\; i \neq j, \;\; \textit{and} \;\; m_{ij} = 0, \quad \textit{and}$$

$$N = (n_{ij}) \;\; \textit{where} \;\; n_{ij} = \frac{i - j}{\varphi(i) - \varphi(j)}, \;\; \textit{for} \;\; i \neq j, \;\; \textit{and} \;\; n_{ij} = 0$$

*are Schur multipliers then the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ are *-isomorphic.*

Proof. Consider the operators $[S]$ and $[T]$ (see (2.1)) and the corresponding decompositions

$$H = \sum_{i \in \mathbb{Z}} \oplus H_S(i) \quad \text{and} \quad \mathcal{H} = \sum_{i \in \mathbb{Z}} \oplus \mathcal{H}_T(i)$$

where $H_S(i) = P_S(i)H$ and $\mathcal{H}_T(i) = P_T(i)\mathcal{H}$ (see (2.3)). The operators $S - [S]$ and $T - [T]$ are bounded, so $\mathcal{F}_S = \mathcal{F}_{[S]}$ and $\mathcal{F}_T = \mathcal{F}_{[T]}$.

Consider the selfadjoint operator R on H such that all subspaces $H_S(i)$ are invariant for R and $R|_{H_S(i)} = \varphi(i)\mathbf{1}_{H_S(i)}$. Since M and N are Schur multipliers, it follows from Theorem 4.6 that $\mathcal{F}_R = \mathcal{F}_{[S]}$.

On the other hand, since $\dim(\mathcal{H}_T(\varphi(i))) = \dim(H_S(i))$, for all $i \in \mathbb{Z}$, there exists an isometry operator U from H onto $\mathcal{H}$ which maps $H_S(i)$ onto $\mathcal{H}_T(\varphi(i))$. Then $U^*[T]U = R$. By Lemma 4.1, the algebras $\mathcal{F}_R$ and $\mathcal{F}_{[T]}$ are *-isomorphic. Hence the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ are *-isomorphic. □

References

[1] G. Bennett, *Schur multipliers,* Duke Math. J., **44** (1977), 603-639.

[2] B. Blackadar and J. Cuntz, *Differential Banach algebra norms and smooth subalgebras of C^*-algebras*, J. Oper. Theory, **26** (1991), 255-282.

[3] O. Bratteli and D.W. Robinson, *Unbounded derivations of C^*-algebras*, I, Comm. Math. Phys., **42** (1975), 253-268.

[4] J. Dixmier, *Les C^*-algebres et leurs Representations*, Gauthier-Villars, Paris, 1969.

[5] N. Dunford and J.T. Schwartz, *Linear Operators*, Vol. I+II, Interscience Inc., New York, 1988.

[6] I.Ts. Gohberg and M.G. Krein, *Introduction to the theory of linear non-selfadjoint operators in Hilbert spaces*, Nauka, Moscow, 1965.

[7] P.E.T. Jorgensen and G.L. Price, *Index theory and second quantization of boundary value problems*, J. Funct. Anal., **104** (1992), 243-290.

[8] E. Kissin, *Symmetric operator extensions of unbounded derivations of C^*-algebras*, J. Funct. Anal., **81** (1988), 38-53.

[9] ______, *Indices of unbounded derivations of C^*-algebras*, Pacific J. Math., **152** (1992), 125-150.

[10] E. Kissin, A.I. Loginov and V.S. Shulman, *Derivations of C^*-algebras and almost Hermitian representations on Π_k-spaces*, Pacific J. Math., **174** (1996), 411-430.

[11] E. Kissin and V.S. Shulman, *Dense Q-subalgebras of Banach and C^*-algebras and unbounded derivations of Banach and C^*-algebras*, Proc. Edinburgh Math. Soc., **36** (1993), 261-276.

[12] ______, *Differential properties of some dense subalgebras of C^*-algebras*, Proc. Edinburgh Math. Soc., **37** (1994), 399-422.

[13] ______, *Differential Banach *-algebras of compact operators associated with symmetric operators*, J. Funct. Anal., **156** (1998), 1-29.

[14] ______, *Representations on Krein Spaces and Derivations of C^*-algebras*, Addison Wesley Longman, London, 1997.

[15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Academic Press, New York London, 1972.

[16] S. Sakai, *Operator Algebras in Dynamical Systems*, CUP, Cambridge, 1991.

Received December 10, 1997 and revised September 16, 1998.

University of North London
Great Britain
E-mail address: 11kissine@unl.ac.uk

Polytechnic Institute of Vologda
Vologda
Russia