DUAL SPACES AND ISOMORPHISMS OF SOME DIFFERENTIAL BANACH ∗-ALGEBRAS OF OPERATORS

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The paper continues the study of differential Banach ∗-algebras $\mathcal{A}_S$ and $\mathcal{F}_S$ of operators associated with symmetric operators $S$ on Hilbert spaces $H$. The algebra $\mathcal{A}_S$ is the domain of the largest ∗-derivation $\delta_S$ of $B(H)$ implemented by $S$ and the algebra $\mathcal{F}_S$ is the closure of the set of all finite rank operators in $\mathcal{A}_S$ with respect to the norm $\|A\| = \|A\| + \|\delta_S(A)\|$. When $S$ is selfadjoint, $\mathcal{F}_S$ is the domain of the largest ∗-derivation of the algebra $C(H)$ implemented by $S$. If $S$ is bounded, $\mathcal{F}_S = C(H)$ and $\mathcal{A}_S = B(H)$, so $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. For unbounded selfadjoint operators $S$ the paper establishes the full analogy with the bounded case: $\mathcal{A}_S$ is isometrically isomorphic to the second dual of $\mathcal{F}_S$. The paper also classifies the algebras $\mathcal{A}_S$ and $\mathcal{F}_S$ up to isometrical ∗-isomorphism and obtains some partial results about bounded but not necessarily isometrical ∗-isomorphisms of the algebras $\mathcal{F}_S$.

1. Introduction and preliminaries.

Extensive development of non-commutative geometry requires elaborating of the theory of differential Banach ∗-algebras, that is, dense ∗-subalgebras of $C^*$-algebras whose properties in many respects are analogous to the properties of algebras of differentiable functions.

Blackadar and Cuntz [2] and the authors [12] introduced and studied various classes of differential Banach ∗-algebras; the most interesting class consists of $D$-algebras, that is, dense ∗-subalgebras $\mathcal{A}$ of $C^*$-algebras $(\mathcal{U}, \|\cdot\|)$ which, in turn, are Banach ∗-algebras with respect to another norm $\|\cdot\|_1$ and the norms $\|\cdot\|$ and $\|\cdot\|_1$ on $\mathcal{A}$ satisfy the inequality:

\[
\|xy\| \leq D(\|x\|_1\|y\| + \|x\|\|y\|), \quad \text{for} \quad x, y \in \mathcal{A},
\]

for some $D > 0$. This class contains, for example, the domains $D(\delta)$ of closed unbounded ∗-derivations $\delta$ of $C^*$-algebras $\mathcal{U}$ where the norm $\|\cdot\|_1$ on $D(\delta)$ is defined, as usual, by the formula

\[
\|A\|_1 = \|A\| + \|\delta(A)\|, \quad \text{for} \quad A \in D(\delta).
\]
Much work has been done on the investigation of properties of the differential Banach *-algebras (see Blackadar and Cuntz [2] and Kissin and Shulman [12, 13]) and the algebras $D(\delta)$ in particular (see, for example, Bratteli and Robinson [3] and Sakai [16]).

In many cases closed *-derivations of $C^*$-algebras $U$ of operators on Hilbert spaces are implemented by closed symmetric operators. In particular, Bratteli and Robinson [3] showed that if $U$ contains the ideal of all compact operators then any closed *-derivation of $U$ is implemented by a symmetric operator.

Any closed symmetric operator $S$ on a Hilbert space $H$ implements closed *-derivations of various $C^*$-algebras of operators on $H$. Among all these derivations there is the largest one - $\delta_S$ with domain $D(\delta_S)$ (which we denote by $A_S$) containing the domains of all derivations implemented by $S$:

$$A_S = \left\{ A \in B(H) : AD(S) \subseteq D(S), \ A^*D(S) \subseteq D(S) \quad \text{and} \quad (SA - AS)_{|D(S)} \quad \text{extends to a bounded operator} \quad A_S \right\},$$

and $\delta_S(A) = i \text{Closure} (SA - AS)$, for $A \in A_S$.

The closure of $A_S$ with respect to the norm $\| \cdot \|$ in $B(H)$ is the enveloping $C^*$-algebra which we denote by $U_S$.

The algebra $A_S$ is a unital Banach *-algebra with respect to the norm

$$\|A\|_S = \|A\| + \|A_S\|.$$  

If $S$ implements a *-derivation $\delta$ of a $C^*$-algebra $U$ of operators on $H$ then

$$D(\delta) \subseteq A_S, \quad U \subseteq U_S \quad \text{and} \quad \delta = \delta_S|U.$$  

By $C(H)$ we denote the algebra of all compact operators on $H$. The *-algebras

$$K_S = A_S \cap C(H) \quad \text{and} \quad J_S = \left\{ A \in K_S : \delta_S(A) \in C(H) \right\}$$

are dense in $C(H)$ and are the domains of the largest closed *-derivations from $C(H)$ into $B(H)$ and $C(H)$, respectively, implemented by $S$.

By $F_S$ we denote the closure with respect to the norm $\| \cdot \|_S$ of the subalgebra of all finite rank operators in $A_S$.

It was shown in [13] that $(K_S, \| \cdot \|_S)$ and $(J_S, \| \cdot \|_S)$ are semisimple Banach *-algebras, that $(F_S, \| \cdot \|_S)$ is a simple Banach *-algebra and

$$F_S \subseteq J_S \subseteq K_S \subseteq A_S.$$  

Furthermore, $F_S$, $J_S$ and $K_S$ are closed two-sided ideals of $(A_S, \| \cdot \|_S)$ and $F_S$ is contained in any closed two-sided ideal of $(A_S, \| \cdot \|_S)$. The relation between the ideals $F_S$, $J_S$ and $K_S$ and the question of how the properties of the operator $S$ are reflected in the structure of $K_S$, $J_S$ and $F_S$ were investigated in [13]. In particular, it was established that $(K_S)^2 = (J_S)^2 = F_S$, for all
symmetric $S$, and that the ideals $J_S$ and $F_S$ have a bounded approximate identity if and only if $S$ is selfadjoint. For selfadjoint $S$, it was also proved that $K_S \neq J_S = F_S$.

In spite of the fact that the structure of the algebras $F_S$, $J_S$, $K_S$, $A_S$ and $\mathcal{U}_S$ is comparatively simple, many important questions still remain open. In Section 2 we mainly study the structure of the algebras $A_S$ and $\mathcal{U}_S$ in the case when $S$ is a selfadjoint operator. However, we also consider the case when $S$ is a symmetric operator with at least one finite deficiency index and show that the algebras $A_S$ and $\mathcal{U}_S$ contain closed ideals of finite codimension.

If $S$ is a bounded symmetric operator on $H$ then $F_S = C(H)$ and $A_S = B(H)$, so $A_S$ is isometrically isomorphic to the second dual of $F_S$. In Section 3 we investigate the structure of the dual and the second dual spaces of the algebras $F_S$ for unbounded symmetric operators $S$. In the case when $S$ is selfadjoint we establish the full analogy with the bounded case: The algebra $A_S$ is isometrically isomorphic to the second dual of $F_S$.

In Section 4 we study the problem of classification of the algebras $F_S$ and $A_S$ up to *-isomorphism. For isometrical *-isomorphism this problem is completely solved in Theorem 4.4. For bounded but not necessarily isometrical *-isomorphism we obtain some interesting partial results in the case when $S$ is selfadjoint.

2. Structure of the algebras $A_S$ and the enveloping $C^*$-algebras $\mathcal{U}_S$.

The main purpose of this section is to study the structure of the algebras $A_S$ and $\mathcal{U}_S$ in the case when $S$ is a selfadjoint operator. However, we start the section by considering the case when $S$ is a symmetric operator with at least one finite deficiency index. Making use of the existence of a $J$-symmetric representation of $A_S$ on the deficiency space of $S$, we will show that the algebras $A_S$ and $\mathcal{U}_S$ contain closed ideals of finite codimension.

Let $S$ be symmetric, $S^*$ be the adjoint operator, let $N_-(S)$ and $N_+(S)$ be the deficiency spaces of $S$ and

$$n_{\pm}(S) = \dim (N_{\pm}(S))$$

be the deficiency indices of $S$. It is well known that $D(S^*)$ is a Hilbert space with respect to the scalar product

$$\langle x, y \rangle = (x, y) + (S^* x, S^* y), \quad \text{for } x, y \in D(S^*),$$

and it is the orthogonal sum of the closed subspaces $D(S), N_-(S)$ and $N_+(S)$:

$$D(S^*) = D(S)_{\langle + \rangle} N_-(S)_{\langle + \rangle} N_+(S).$$
Set \( N(S) = N_-(S) \cap N_+(S) \) and let \( Q \) be the projection on \( N(S) \) in \( D(S^\ast) \). It was shown in [7] and [8] that
\[
[x, y] = i(x, S^\ast y) - i(S^\ast x, y), \quad \text{for } x, y \in N(S),
\]
is an indefinite non-degenerate sesquilinear form on \( N(S) \), that
\[
\pi_S(A) = QA|_{N(S)}, \quad \text{for } A \in \mathcal{A}_S,
\]
is a bounded representation of \((\mathcal{A}_S, \| \cdot \|_S)\) on \( N(S) \) and that it is \( J \)-symmetric:
\[
[\pi_S(A)x, y] = [x, \pi_S(A^\ast)y], \quad \text{for } x, y \in N(S).
\]
A subspace \( L \) in \( N(S) \) is neutral if
\[
[x, y] = 0, \quad \text{for all } x, y \in L.
\]
The operator \( S \) is well-behaved if the representation \( \pi_S \) has no neutral invariant subspace.

Let \( \kappa_S = \min(n_-(S), n_+(S)) \) and assume that \( 0 < \kappa_S < \infty \). It was proved in [10] that the representation \( \pi_S \) has a \( \kappa_S \)-dimensional subrepresentation \( \sigma \). Let \( \rho \) be an irreducible subrepresentation of \( \sigma \). It was shown in [11] that \( \rho \) is bounded with respect to the operator norm \( \| \cdot \| \) in \( \mathcal{A}_S \) and, therefore, extends to a bounded \(*\)-representation of the enveloping \( C^* \)-algebra \( \mathcal{U}_S \). If \( S \) is well-behaved, it follows from Theorem 28.13 [14] that \( K_S \subseteq \text{Ker}(\rho) \). This yields

**Theorem 2.1.** Let \( S \) be a symmetric unbounded operator and \( 0 < \kappa_S < \infty \).

(i) There exists a closed two-sided ideal \( J \) in the Banach \(*\)-algebra \((\mathcal{A}_S, \| \cdot \|)\) such that the quotient algebra \( \mathcal{A}_S/J \) is isomorphic to the full matrix algebra \( M_n(\mathbb{C}) \) with \( 0 < n \leq \kappa_S \).

(ii) The uniform closure \( \overline{J} \) of \( J \) in \( \mathcal{U}_S \) is a closed two-sided ideal and the quotient algebra \( \mathcal{U}_S/\overline{J} \) is isomorphic to the full matrix algebra \( M_n(\mathbb{C}) \).

(iii) If \( S \) is well-behaved then \( K_S \subseteq J \) and \( C(H) \subseteq \overline{J} \).

**Example 2.2.** Let \( H = L^2(0, 1) \) and \( S = i \frac{d}{dt} \) with domain \( D(S) \) consisting of all absolutely continuous functions \( h \) such that \( h' \in L^2(0, 1) \) and \( h(0) = h(1) = 0 \). Then \( S \) is a symmetric operator and \( n_-(S) = n_+(S) = 1 \).

It was proved in [9] that \( S \) is well-behaved. Therefore it follows from Theorem 2.1 that there exists a closed two-sided ideal \( J \) in \((\mathcal{A}_S, \| \cdot \|)\) containing \( K_S \) such that \( \dim(\mathcal{A}_S/J) = 1 \) and that the uniform closure of \( J \) in \( \mathcal{U}_S \) is an ideal of codimension 1.

Let \( S \) be the same as in Example 2.2 and let \( \text{Lip}(0, 1) \) be the algebra of all functions on \([0, 1]\) satisfying a Lipshitz condition: \(|g(t) - g(s)| \leq K_g|t - s|\) for some \( K_g > 0 \) and all \( t, s \in [0, 1] \). For \( g \in \text{Lip}(0, 1) \), denote by \( M_g \) the operator of multiplication by \( g \) on \( L^2(0, 1) \) and set \( \mathcal{B} = \{ M_g : g \in \text{Lip}(0, 1) \} \). Then \( M_g D(S) \subseteq D(S) \), \( (M_g)^* D(S) = M_{\overline{g}} D(S) \subseteq D(S) \) and
SM\_g - M\_gS extends to the operator iM\_g' which is bounded, since g' is essentially bounded on [0, 1]. Thus \( \mathcal{B} \subset \mathcal{A}_S \).

(The authors are grateful to the referee of the paper for pointing out an error in the definition of the algebra \( \mathcal{B} \) in the first version of the paper.)

**Problem 2.3.** Is \( \mathcal{A}_S = \mathcal{B} + \mathcal{K}_S \)?

The assumption that a symmetric operator \( S \) is selfadjoint makes the task of studying the structure of the algebras \( \mathcal{A}_S \) and \( \mathcal{U}_S \) easier. First of all, the structure of the ideals \( \mathcal{K}_S \), \( \mathcal{J}_S \) and \( \mathcal{F}_S \) is simpler. While for arbitrary symmetric operators \( S \) it is only known (see [13]) that \( (\mathcal{K}_S)^2 = (\mathcal{J}_S)^2 = \mathcal{F}_S \), where the closure is taken with respect to the norm \( \| \cdot \|_S \), for selfadjoint operators \( S \) it was shown in [13] that \( \mathcal{F}_S = \mathcal{J}_S \neq \mathcal{K}_S \). Secondly, in the selfadjoint case we can employ the Spectral Theorem to establish the structure of \( \mathcal{A}_S \) and \( \mathcal{U}_S \).

Let

\[
S = \int_{-\infty}^{\infty} \lambda dE_S(\lambda)
\]

be the spectral decomposition of \( S \). For every integer \( n \), set

\[
P_S(n) = E_S(n+1) - E_S(n) \quad \text{and} \quad [S] = \sum_{-\infty}^{\infty} n P_S(n).
\]

Then \([S]\) is a selfadjoint operator, \( \text{Sp}([S]) \subseteq \mathbb{Z} \) and the operator \( S - [S] \) is bounded. Therefore it follows that

\[
\mathcal{A}_S = \mathcal{A}_{[S]}, \quad \mathcal{K}_S = \mathcal{K}_{[S]} \quad \text{and} \quad \mathcal{F}_S = \mathcal{F}_{[S]}
\]

and the norms \( \| \cdot \|_S \) and \( \| \cdot \|_{[S]} \) are equivalent on \( \mathcal{A}_S \). This reduces the problem of the description of the structure of the algebras \( \mathcal{A}_S \) and \( \mathcal{U}_S \) to the case when \( \text{Sp}(S) \subseteq \mathbb{Z} \).

We denote by \( \mathcal{S}_Z \) the set of all selfadjoint operators \( S \) on \( H \) such that \( \text{Sp}(S) \subseteq \mathbb{Z} \) and set

\[
H_S(n) = P_S(n)H, \quad \text{for} \quad n \in \text{Sp}(S).
\]

Then

\[
H = \sum_{n \in \text{Sp}(S)} \oplus H_S(n).
\]

We omit the proof of the following simple result.

**Proposition 2.4.** Let \( S, T \in \mathcal{S}_Z \). If there exists a one-to-one mapping \( \varphi \) from \( \text{Sp}(T) \) onto \( \text{Sp}(S) \) such that \( \dim(H_T(n)) = \dim(H_S(\varphi(n)), \text{ for } n \in \text{Sp}(T), \) and

\[
\sup_{n \in \text{Sp}(T)} |\varphi(n) - n| < \infty
\]

then there exists a unitary operator \( U \) such that \( \mathcal{A}_T = U \mathcal{A}_S U^* \).
Let $S \in \mathcal{S}_S$. Every operator $A$ in $B(H)$ has a block-matrix form $A = (A_{ij})$, $i, j \in \text{Sp}(S)$, with respect to decomposition (2.3). We denote by $\mathcal{D}_S$ the $C^*$-algebra of all block-diagonal operators $A = (A_{ij})$ in $B(H)$, that is, $A_{ij} = 0$ if $i \neq j$. By $\mathcal{R}$ we denote the subalgebra of all operators $A = (A_{ij})$ in $B(H)$ with only finite number of non-zero entries $A_{ij}$. Then, clearly, $\mathcal{D}_S \subseteq A_S$ and $\mathcal{R}_S \subseteq A_S$.

Let $\overline{\mathcal{R}}_S$ be the closure of $\mathcal{R}_S$ in $(A_S, \| \cdot \|_S)$ and let $C_S(H)$ be the uniform closure of $\mathcal{R}_S$ in $B(H)$.

**Lemma 2.5.** $\mathcal{D}_S + C_S(H)$ is a $C^*$-subalgebra of $\mathfrak{U}_S$ and $\mathcal{D}_S + \overline{\mathcal{R}}_S$ is a closed $*$-subalgebra of $(A_S, \| \cdot \|_S)$.

**Proof.** Let $\mathcal{L}$ be the uniform closure of $\mathcal{D}_S + \mathcal{R}_S$ in $B(H)$. Then $\mathcal{L}$ is a $C^*$-subalgebra of $\mathfrak{U}_S$. Since $\mathcal{R}_S$ is a two-sided ideal of the algebra $\mathcal{D}_S + \mathcal{R}_S$, the $C^*$-algebra $C_S(H)$ is a two-sided ideal of $\mathcal{L}$. Therefore it follows from Corollary 1.8.4 [4] that $\mathcal{D}_S + C_S(H)$ is a $C^*$-algebra, so $\mathcal{L} = \mathcal{D}_S + C_S(H)$.

For $A \in B(H)$, set

$$\phi(A) = \sum_{n \in \text{Sp}(S)} P_S(n)AP_S(n) \quad \text{and} \quad \tilde{A} = A - \phi(A).$$

Then $\phi$ is a conditional expectation from $B(H)$ onto $\mathcal{D}_S$ and

$$\|\phi(A)\| \leq \|A\| \quad \text{and} \quad \|\tilde{A}\| \leq 2\|A\|.$$  

(2.4)

If $A \in A_S$ then $\tilde{A} \in A_S$ and $\text{Closure}(SA - AS) = \text{Closure}(S\tilde{A} - \tilde{A}S)$.

Assume that $\{A_n\}$ converge to $A$ in $A_S$ with respect to $\| \cdot \|_S$. Then

$$\|A - A_n\| \to 0 \quad \text{and} \quad \|\text{Closure}(S(A - A_n) - (A - A_n)S)\| \to 0, \quad \text{as} \quad n \to \infty,$$

and therefore, by (1.2) and (2.4),

$$\|\tilde{A} - \tilde{A}_n\|_S = \|\tilde{A} - \tilde{A}_n\| + \|\text{Closure}(S(\tilde{A} - \tilde{A}_n) - (\tilde{A} - \tilde{A}_n)S)\| \leq 2\|A - A_n\| + \|\text{Closure}(S(A - A_n) - (A - A_n)S)\| \to 0,$$

(2.5) as $n \to \infty$.

Hence $\tilde{A}_n$ converge to $\tilde{A}$ with respect to $\| \cdot \|_S$.

Suppose now that $B \in \overline{\mathcal{R}}_S$. Then there are $\{B_n\}$ in $\mathcal{R}_S$ converging to $B$ with respect to $\| \cdot \|_S$. It follows from (2.5) that $\tilde{B}_n$ converge to $\tilde{B}$ with respect to $\| \cdot \|_S$ and, since $\tilde{B}_n$ belong to $\mathcal{R}_S$, we obtain that $\tilde{B} \in \overline{\mathcal{R}}_S$.

Finally, let $C_n = A_n + B_n$ converge to $C$ in $A_S$ with respect to $\| \cdot \|_S$ where $A_n \in \mathcal{D}_S$ and $B_n \in \overline{\mathcal{R}}_S$. Then $\tilde{C}_n = \tilde{B}_n$ and, by (2.5), $\tilde{B}_n$ converge to $\tilde{C}$ with respect to $\| \cdot \|_S$. Since, by the above argument, all $\tilde{B}_n$ belong to $\overline{\mathcal{R}}_S$, the operator $\tilde{C}$ also belong to $\overline{\mathcal{R}}_S$. Hence $C \in \mathcal{D}_S + \overline{\mathcal{R}}_S$ and $\mathcal{D}_S + \overline{\mathcal{R}}_S$ is a closed $*$-subalgebra of $(A_S, \| \cdot \|_S)$. □
Let \( S \in \mathcal{S}_Z \). We number the elements of \( \text{Sp}(S) \) in such a way that \( \text{Sp}(S) = \{n_i\}_{i \in I} \) is an increasing sequence,

\[
0 \leq n_i, \text{ for } 0 \leq i, \text{ and } 0 > n_i, \text{ for } 0 > i.
\]

Then \(|i| \leq |n_i|\) and, depending on \( S \), the set \( I \) is either the set \( \mathbb{Z} \) of all integers, or the set of all integers from \(-\infty \) to some \( m \), or from \( m \) to \( \infty \). We consider the case when \( I = \mathbb{Z} \). Two other cases can be considered similarly.

Set

\[
\rho_S(k) = \left( \inf_{i \in \mathbb{Z}} |n_{i+k} - n_i| \right)^{-1}, \quad \text{for } k \neq 0, \text{ and } \rho_S(0) = 0.
\]

Since \( \inf_{i \in \mathbb{Z}} |n_{i+k} - n_i| \geq |k| \),

\[
0 < \rho_S(k) \leq \frac{1}{|k|}, \quad \text{for } k \neq 0.
\]

**Proposition 2.6.** If

\[
\lim_{|i| \to \infty} (n_{i+1} - n_i) = \infty \quad (2.6)
\]

and

\[
\sum_{k \in \mathbb{Z}} \rho(k) \text{ converges} \quad (2.7)
\]

then \( \mathcal{U}_S = \mathcal{D}_S + C_S(H) \).

**Proof.** Let \( A = (A_{ij}) \in \mathcal{A}_S \), where \( A_{ij} \) are bounded operators from \( H_S(n_j) \) into \( H_S(n_i) \). Then the operator

\[
B = SA - AS = (B_{ij}), \quad \text{where } B_{ij} = (n_i - n_j)A_{ij},
\]

is bounded. Set \( b = \|B\|. \) Since \( \|B_{ij}\| \leq \|B\| \), for all \( i, j \in \mathbb{Z} \),

\[
\|A_{ij}\| \leq \frac{b}{|n_i - n_j|}, \quad \text{for } i \neq j. \quad (2.8)
\]

For \( k \in \mathbb{Z} \setminus \{0\} \) and \( m > 0 \), let

\[
G_{ij}^{km} = A_{ij}, \quad \text{if } j = i + k \text{ and } -m \leq i \leq m, \text{ and } G_{ij}^{km} = 0 \text{ otherwise.}
\]

Then the operator \( G^{km} = (G_{ij}^{km}) \) belongs to \( \mathcal{R}_S \). Taking into account (2.6) and (2.8), we obtain that the operators \( G^{km} \) converge uniformly in \( B(H) \) to a bounded operator \( G^k = (G_{ij}^k) \), as \( m \to \infty \), where

\[
G_{ij}^k = A_{ij}, \quad \text{if } j = i + k, \text{ and } G_{ij}^k = 0 \text{ otherwise.}
\]

Therefore \( G^k \in C_S(H) \) and, by (2.8),

\[
\|G^k\| = \sup_i \|A_{ii+k}\| \leq b\rho_S(k).
\]

It follows from (2.7) that the operator \( G = \sum_{k \in \mathbb{Z} \setminus \{0\}} G^k \) belongs to \( C_S(H) \). Since \( A - G \in \mathcal{D}_S \), we obtain that \( A \in \mathcal{D}_S + C_S(H) \), so that \( \mathcal{A}_S \subseteq \mathcal{D}_S + C_S(H) \). It follows from Lemma 2.5 that \( \mathcal{U}_S = \mathcal{D}_S + C_S(H) \). \( \square \)
Corollary 2.7. If there are $a > 0$, $c > 0$ and an integer $N$ such that
\[ c |i|^a \leq n_{i+1} - n_i \quad \text{for } N \leq |i| \]
then $\Delta_S = D_S + C_S(H)$.

Proof. Condition (2.6), clearly, holds. Let $k < 4N$. Then
\[
\rho_S(k)^{-1} = \inf_{i \in \mathbb{Z}} |n_{i+k} - n_i| = \inf_{i \in \mathbb{Z}} \sum_{p=1}^{k} (n_{i+p} - n_{i+p-1}) \geq c \sum_{m=N}^{\lfloor k/2 \rfloor} m^a \geq c \frac{(k^{a+1})}{a+1} - (N-1)^{a+1}) \\
\geq \frac{c}{a+1} \left( \frac{k}{4} \right)^{a+1}. 
\]
Similarly, if $k < -2N$ then $\rho_S(k)^{-1} \geq \frac{c}{a+1} \left( \frac{|k|}{4} \right)^{a+1}$. Therefore condition (2.7) also holds and the result follows from Proposition 2.6. \qed

Suppose now that $\dim(H_S(n)) = 1$ for all $n \in \text{Sp}(S)$ and let $n_0 \in \text{Sp}(S)$. Set $K = H_S(n_0)$. Then there exists a Hilbert space $\mathcal{H}$ with $\dim(\mathcal{H}) = 1$ such that the $C^*$-algebra $C_S(H)$ is isomorphic to the tensor product $B(K) \otimes C(\mathcal{H})$ where $C(\mathcal{H})$ is the $C^*$-algebra of all compact operators on $\mathcal{H}$. Choosing a basis $\{e_n\}_{n=1}^{\infty}$ in $\mathcal{H}$, we obtain that the algebra $D_S$ is isomorphic to the von Neumann algebra tensor product $B(K) \otimes \mathcal{L}$ of $B(K)$ and the $W^*$-algebra $\mathcal{L}$ of all operators on $\mathcal{H}$ diagonal with respect to $\{e_n\}_{n=1}^{\infty}$. From this and from Proposition 2.6 we obtain the following result.

Corollary 2.8. Let $S \in \mathcal{S}_2$. If $\dim(H_S(n)) = 1$ for all $n \in \text{Sp}(S)$ and conditions (2.6) and (2.7) hold then there exist Hilbert spaces $K$ and $\mathcal{H}$ such that $\Delta_S$ is isomorphic to $B(K) \otimes \mathcal{L} + B(K) \otimes C(\mathcal{H})$, where $\mathcal{L}$ is the $W^*$-algebra of all operators on $\mathcal{H}$ diagonal with respect to some basis.

Assume now that $\dim(H_S(n)) < 1$ for all $n \in \text{Sp}(S)$. Then $C_S(H)$ coincides with the algebra $C(\mathcal{H})$ of all compact operators on $\mathcal{H}$. Taking into account the definition of the ideal $K_S$ and applying Proposition 2.6 we obtain the following result.

Corollary 2.9. Let $S \in \mathcal{S}_2$ and $\dim(H_S(n)) < 1$ for all $n \in \text{Sp}(S)$. If conditions (2.6) and (2.7) hold then $\Delta_S = D_S + C_S(H)$ and $A_S = D_S + K_S$.

Example 2.10. Let $\{e_i\}_{i=-\infty}^{\infty}$ be an orthonormal basis in $H$ and let
\[ Se_i = \text{sgn} \,(i)|i|^{1+a} e_i, \quad \text{where } a > 0. \]
Then $S \in \mathcal{S}_2$ and $n_i = \text{sgn} \,(i)|i|^{1+a}$, so that
\[ \lim_{|i| \to \infty} \frac{n_{i+1} - n_i}{\text{sgn} \,(i)|i|^{a}} = 1 + a. \]
Therefore, by Corollaries 2.7 and 2.9, \( \mathcal{U}_S = \mathcal{D}_S + C(H) \) and \( \mathcal{A}_S = \mathcal{D}_S + \mathcal{K}_S \) where \( \mathcal{D}_S \) is the algebra of all operators diagonal with respect to \( \{e_i\}_{i=-\infty}^{\infty} \). Thus the quotient algebra \( \mathcal{A}_S/\mathcal{K}_S \) is isomorphic to the commutative \( C^* \)-algebra \( \mathcal{D}_S/\mathcal{L} \) where \( \mathcal{L} \) is the algebra of all compact diagonal operators on \( H \).

Let \( \{e_i\}_{i=-\infty}^{\infty} \) be an orthonormal basis in \( H \) and let
\[
S e_i = i e_i \text{ and } U e_i = e_{i+1}, \text{ for all } i \in \mathbb{Z}.
\]
Then \( S \in \mathcal{S} \) and \( U \) is the shift operator. We have that
\[
UD(S) \subseteq D(S), \quad (SU-US)|_{D(S)} \text{ extends to } U,
\]
so that \( U \in \mathcal{A}_S \). Hence \( \mathcal{U}_S \) contains the \( C^* \)-algebra \( C(\mathcal{D}_S,U) \) generated by \( U \) and by the commutative algebra \( \mathcal{D}_S \) of all operators diagonal with respect to \( \{e_i\}_{i=-\infty}^{\infty} \).

**Problem 2.11.** Is \( \mathcal{U}_S = C(\mathcal{D}_S,U) \)?

### 3. Dual and second dual spaces of the algebras \( \mathcal{F}_S \).

Let \( S \) be a closed symmetric operator. Recall that \( \mathcal{F}_S \) is the closure with respect to the norm \( \| \cdot \|_S \) (see (1.2)) of the subalgebra of all finite rank operators in \( \mathcal{A}_S \). If \( S \) is a bounded symmetric operator on \( H \), it follows that \( \mathcal{F}_S = C(H) \) and \( \mathcal{A}_S = B(H) \), so that \( \mathcal{A}_S \) is isometrically isomorphic to the second dual of \( \mathcal{F}_S \). In this section we study the structure of the dual and the second dual spaces of the algebra \( \mathcal{F}_S \) for unbounded symmetric operators \( S \). In the case when \( S \) is selfadjoint we establish the full analogy with the bounded case: The algebra \( \mathcal{A}_S \) is isometrically isomorphic to the second dual of \( \mathcal{F}_S \).

By \( T(H) \) we denote the Banach \( * \)-algebra of trace class operators on \( H \) with the norm
\[
|A| = \sum_{i=1}^{\infty} s_i(A) = \text{Tr} \left( (A^*A)^{1/2} \right),
\]
where \( \{s_i(A)\}_{i=1}^{\infty} \) is the set of all eigenvalues of the positive compact operator \( (A^*A)^{1/2} \).

It is well known that \( T(H) \) can be identified with the dual space of the algebra \( C(H) \): For any \( T \in T(H) \),
\[
F_T(A) = \text{Tr}(AT), \quad A \in C(H),
\]
is a bounded linear functional on \( C(H) \) and \( \|F_T\| = |T| \); and that \( B(H) \) can be identified with the dual space of \( T(H) \): For any \( B \in B(H) \),
\[
\theta_B(T) = \text{Tr}(BT), \quad T \in T(H),
\]
is a bounded linear functional on \( T(H) \) and \( \|	heta\| = \|B\| \).
Set \( \hat{B}(H) = B(H) \oplus B(H) \) and \( \hat{C}(H) = C(H) \oplus C(H) \). Then \( \hat{B}(H) \) and \( \hat{C}(H) \) are Banach spaces with the norm
\[
\|A \oplus B\| = \|A\| + \|B\|.
\]
Set \( \hat{T}(H) = T(H) \oplus T(H) \). It is a Banach space with the norm
\[
|R \oplus T| = \max(|R|, |T|), \quad T, R \in T(H),
\]
and it can be identified with the dual space of \( \hat{C}(H) \): For \( R, T \in T(H) \),
\[
(3.1) \quad F_{R \oplus T}(A \oplus B) = \text{Tr}(AR) + \text{Tr}(BT), \quad A \oplus B \in \hat{C}(H),
\]
is a bounded linear functional on \( \hat{C}(H) \) and \( \|F_{R \oplus T}\| = |R \oplus T| \). Similarly, \( \hat{B}(H) \) can be identified with the dual space of \( \hat{T}(H) \): For \( A, B \in B(H) \),
\[
(3.2) \quad \theta_{A \oplus B}(R \oplus T) = \text{Tr}(AR) + \text{Tr}(BT), \quad R \oplus T \in \hat{T}(H),
\]
is a bounded linear functional on \( \hat{T}(H) \) and \( \|\theta_{A \oplus B}\| = \|A \oplus B\| \).

Set
\[
\hat{A}_S = \{A \oplus A_S : A \in A_S\} \quad \text{and} \quad \hat{F}_S = \{A \oplus A_S : A \in F_S\},
\]
where \( A_S = \text{Closure}(SA - AS) \). Then \( (A_S, \| \cdot \|_S) \) and \( (\hat{A}_S, \| \cdot \|) \) and \( (F_S, \| \cdot \|) \) are isometrically isomorphic, since
\[
\|A\|_S = \|A\| + \|A_S\| = \|A \oplus A_S\|.
\]
Therefore \( \hat{A}_S \) is a closed subspace of \( \hat{B}(H) \) and \( \hat{F}_S \) is a closed subspace of \( \hat{C}(H) \), since \( A \in F_S \) implies \( A_S \in C(H) \).

Set
\[
\mathcal{S}_S = \left\{ T \in T(H) : TD(S) \subseteq D(S^*), T^*D(S) \subseteq D(S^*) \right\}
\]
and the operator
\[
(S^*T - TS)|_{D(S)} \text{ extends to a bounded trace class operator } T\}
\]
is a linear subspace in \( \hat{T}(H) \). For \( T \in \mathcal{S}_S \) and \( z, u \in D(S) \),
\[
-(T^*S)z = -(z, Tsu) = -(z, (S^*T - TS)u) = ((S^*T - T^*S)z, u),
\]
so that
\[
(3.3) \quad -(T^*S)|_{D(S)} = (S^*T^* - T^*S)|_{D(S)} = (T^*)S|_{D(S)}.
\]
Therefore \( T^* \in \mathcal{S}_S \).

For \( x, y \in H \), the rank one operator \( x \otimes y \) on \( H \) is defined by the formula
\[
(3.4) \quad (x \otimes y)z = (z, x)y.
\]
It is easy to check that
\begin{equation}
\|x \otimes y\| = \|x\| \|y\|,
\end{equation}
\begin{equation}
(x \otimes y)^* = y \otimes x, \ (x \otimes y)(u \otimes v) = (v,x)(u \otimes y),
\end{equation}
\begin{equation}
R(x \otimes y) = x \otimes Ry, \text{ and } (x \otimes y)R \text{ extends to } (R^*x) \otimes y,
\end{equation}
if $R$ is a densely defined operator, $y \in D(R)$ and $x \in D(R^*)$. Let $\{e_j\}_{j=1}^\infty$ be a basis in $H$. Then
\begin{equation}
\text{Tr}(x \otimes y) = \sum_{j=1}^\infty ((x \otimes y)e_j, e_j) = \sum_{j=1}^\infty (e_j, x)(y, e_j)
\end{equation}
\begin{equation}
= \left( y, \sum_{j=1}^\infty (x, e_j)e_j \right) = (y, x).
\end{equation}

Let $x, y \in D(S^*)$ and $T = x \otimes y$. By (3.4) and (3.5),
\begin{equation}
Tz = (z, x)y \in D(S^*) \quad T^*z = (y \otimes x)z = (z, y)x \in D(S^*), \text{ for } z \in H,
\end{equation}
and
\begin{equation}
T_S = S^*T - TS = x \otimes S^*y - (S^*x) \otimes y \in T(H),
\end{equation}
so that $T \in \mathfrak{F}_S$. By $\Phi_S$ we denote the set of all linear combinations of the operators $x \otimes y$, for $x, y \in D(S^*)$. Clearly, $\Phi \subset \mathfrak{F}_S$ and
\begin{equation}
\mathfrak{F}_S = \{T_S \oplus T : T \in \Phi_S\}
\end{equation}
is a linear subspace of $\mathfrak{F}_S$. Let $X^*$ be the dual space of a Banach space $X$ and $Y$ be a linear subspace of $X$. The annihilator
\begin{equation}
Y^\perp = \{F \in X^* : F(y) = 0, \text{ for all } y \in Y\}
\end{equation}
of $Y$ in $X^*$ is a closed subspace of $X^*$ and from the general theory of Banach spaces (see [5] II.4.18 and [15] III, Problem 30) we have the following lemma.

**Lemma 3.1.** The dual space $Y^*$ of a closed subspace $Y$ of $X$ is isometrically isomorphic to the quotient space $X^*/Y^\perp$ and the second dual $Y^{**}$ of $Y$ is isometrically isomorphic to $Y^{** \perp}$ where
\begin{equation}
Y^{** \perp} = \{\theta \in X^{**} : \theta(F) = 0, \text{ for all } F \in Y^\perp\}.
\end{equation}

Since $\mathfrak{F}_S \subseteq \hat{C}(H)$, the annihilator $(\mathfrak{F}_S)\perp$ is a closed subspace of the dual space $\hat{C}(H)^* = \hat{T}(H)$ and, since $\Phi_S \subseteq \mathfrak{F}_S \subseteq \hat{T}(H)$, the annihilator $(\Phi_S)\perp$ is a closed subspace of the dual space $\hat{T}(H)^* = \hat{B}(H)$.

**Theorem 3.2.** (i) $\mathfrak{F}_S$ is a closed subspace in $\hat{T}(H)$ and $(\mathfrak{F}_S)\perp = \mathfrak{F}_S$.
(ii) $(\mathfrak{F}_S)\perp \subseteq (\Phi_S)\perp = \{A \oplus A_S : A \in \mathcal{A}_S \text{ and } AD(S^*) \subseteq D(S)\} \subseteq \hat{\mathcal{A}}_S$. 

Hence and \( x, y \in T \), therefore

\[
A_S = S(x \otimes y) - (x \otimes y)S = x \otimes Sy - (Sx) \otimes y,
\]
\[
A_ST = (x \otimes Sy)T - ((Sx) \otimes y)T = (T^*x) \otimes Sy - (T^*Sx) \otimes y,
\]
\[
AT_S = (x \otimes y)T_S = ((T_S)^*x) \otimes y = ((T^*S - S^*T^*)x) \otimes y.
\]

Therefore, by (3.1), (3.6) and (3.8),

\[
F_{T_S \oplus T}(A \oplus A_S) = \text{Tr}(AT_S) + \text{Tr}(A_ST)
= (y, (T^*S - S^*T^*)x) + (Sy, T^*x) - (y, T^*Sx) = 0.
\]

It follows from Lemma 3.1 [13] that any finite rank operator \( A \) in \( \mathcal{F}_S \) has the form \( A = \sum_{i=1}^n x_i \otimes y_i \) where \( x_i, y_i \in D(S) \). Hence \( F_{T_S \oplus T}(A \oplus A_S) = 0 \) for any finite rank operator \( A \in \mathcal{F}_S \). Since, by definition of \( \mathcal{F}_S \), finite rank operators are dense in \( (\mathcal{F}_S, \| \cdot \|_S) \) and since \( (\mathcal{F}_S, \| \cdot \|_S) \) and \( (\hat{\mathcal{F}}_S, \| \cdot \|) \) are isometrically isomorphic, the operators \( A \oplus A_S \), where \( A \) are finite rank operators, are dense in \( \hat{\mathcal{F}}_S \). Since \( F_{T_S \oplus T} \) is continuous on \( \hat{\mathcal{C}}(H) \), \( F_{T_S \oplus T}(A \oplus A_S) = 0 \), for all \( A \in \mathcal{F}_S \). Therefore \( F_{T_S \oplus T} \in (\hat{\mathcal{F}}_S)^\perp \), so that \( \hat{\mathcal{F}}_S \subseteq (\hat{\mathcal{F}}_S)^\perp \).

Conversely, let \( R \oplus T \in (\hat{\mathcal{F}}_S)^\perp \subseteq \hat{T}(H) \) and let \( A = x \otimes y \in \mathcal{F}_S \), where \( x, y \in D(S) \). From (3.1), (3.5), (3.6) and (3.8) it follows that

\[
0 = F_{R \oplus T}(A \oplus A_S) = \text{Tr}(AR) + \text{Tr}(A_ST)
= \text{Tr}((R^*x) \otimes y) + \text{Tr}((T^*x) \otimes Sy - (T^*Sx) \otimes y)
= (y, R^*x) + (Sy, T^*x) - (y, T^*Sx).
\]

Hence

\[
(Sy, T^*x) = (y, (T^*S - R^*)x), \quad \text{for all } x, y \in D(S).
\]

Therefore \( T^*x \in D(S^*) \) and \( S^*T^*x = (T^*S - R^*)x \). Thus \( T^*D(S) \subseteq D(S^*) \) and

\[
(Sx, Ty) = (T^*Sx, y) = (S^*T^*x, y) + (R^*x, y) = (x, TSy) + (x, Ry).
\]

From this it follows that \( Ty \in D(S^*) \) and \( S^*Ty = TSy + Ry \). Hence

\[
TD(S) \subseteq D(S^*) \quad \text{and} \quad R|_{D(S)} = S^*T|_{D(S)} - TS|_{D(S)}.
\]

Therefore \( T \in \mathcal{I}_S \) and \( R = T_S \). Thus \( (\hat{\mathcal{F}}_S)^\perp \subseteq \mathcal{I}_S \), so that \( (\hat{\mathcal{F}}_S)^\perp = \mathcal{I}_S \) and

\[
\text{from this we also obtain that } \mathcal{I}_S \text{ is a closed subspace of } \hat{T}(H). \text{ Part (i) is proved.}
\]

Since \( \hat{\Phi}_S \subseteq \mathcal{I}_S \), we have \( (\hat{\Phi}_S)^\perp \subseteq (\hat{\Phi}_S)^\perp \). Let now \( A \oplus A_S \in \hat{A}_S \) and \( AD(S^*) \subseteq D(S) \). It was shown in Lemma 3.1 [13] that

\[
A_S|_{D(S^*)} = (S^*A - AS^*)|_{D(S^*)}.
\]
For \( x, y \in D(S^*) \), the operator \( T = x \otimes y \) belongs to \( \Phi_S \) and, taking the above equality into account, we obtain from (3.5) and (3.7) that
\[
A_S T = x \otimes A_S y = x \otimes (S^* A - A S^*) y \quad \text{and}
\]
\[
A T_S = A (x \otimes S^* y - (S^* x) \otimes y) = x \otimes A S^* y - (S^* x) \otimes Ay.
\]
Therefore, by (3.2) and (3.6),
\[
\theta_{A \oplus A_S}(T_S \oplus T) = \text{Tr}(A T_S) + \text{Tr}(A_S T)
\]
\[
= (A S^* y, x) - (A y, S^* x) + (S^* A y, x) - (A S^* y, x)
\]
\[
= (S^* A y, x) - (A y, S^* x).
\]
Since \( AD(S^*) \subseteq D(S) \), it follows that \( S^* A y = S A y \) and \( (A y, S^* x) = (S A y, x) \). Hence \( \theta_{A \oplus A_S}(T_S \oplus T) = 0 \) and, by linearity, it holds for all \( T \in \Phi_S \).

Therefore
\[
\{ A \oplus A_S : A \in A_S \quad \text{and} \quad AD(S^*) \subseteq D(S) \} \subseteq (\Phi_S)^{\perp}.
\]
Conversely, let \( A \oplus B \in (\Phi_S)^{\perp} \). Then, for every \( x, y \in D(S^*) \), \( T = x \otimes y \in \Phi_S \) and
\[
\theta_{A \oplus B}(T_S \oplus T) = \text{Tr}(A T_S) + \text{Tr}(B T) = 0.
\]
By (3.5), \( B T = x \otimes B y \) and, as above, \( A T_S = x \otimes A S^* y - (S^* x) \otimes Ay \). Hence, by (3.6),
\[
0 = (A S^* y, x) - (A y, S^* x) + (B y, x).
\]
Thus
\[
(A y, S^* x) = (A S^* y, x) + (B y, x), \quad \text{for all} \quad x, y \in D(S^*).
\]
Therefore \( A y \in D(S^{**}) \) and \( S^{**} A y = A S^* y + B y \). Since \( S \) is closed, \( S^{**} = S \) and we obtain that
\[
(3.10) \quad \text{AD}(S^*) \subseteq D(S) \quad \text{and} \quad B|_{D(S^*)} = (S A - A S^*)|_{D(S^*)}.
\]
Restricting (3.10) to \( D(S) \), we have
\[
\text{AD}(S) \subseteq D(S) \quad \text{and} \quad B|_{D(S)} = (S A - A S)|_{D(S)}.
\]
Making use of (3.10), we obtain that for \( z \in D(S) \) and \( u \in D(S^*) \),
\[
(A^* z, S^* u) = (z, A S^* u) = (z, S A u) - (z, B u) = (A^* S z, u) - (B^* z, u).
\]
Therefore \( A^* z \in D(S^{**}). \) Since \( S^{**} = S \), we have \( A^* D(S) \subseteq D(S) \). Thus \( A \in A_S \) and \( B = A_S \), so \( A \oplus B = A \oplus A_S \in \hat{A}_S \). Taking into account that \( AD(S^*) \subseteq D(S) \), we obtain that
\[
(\Phi_S)^{\perp} \subseteq \{ A \oplus A_S : A \in A_S \quad \text{and} \quad AD(S^*) \subseteq D(S) \}.
\]
Combining this with (3.9), we complete the proof of the theorem. \( \square \)

Since the Banach spaces \( (F, \| \cdot \|) \) and \( (\hat{F}, \| \cdot \|) \) and the Banach spaces \( (A, \| \cdot \|) \) and \( (\hat{A}, \| \cdot \|) \) are isometrically isomorphic and since \( (\hat{F}, \| \cdot \|) \) is a closed subspace of \( \hat{C}(H) \), Lemma 3.1 and Theorem 3.2 yield:
Corollary 3.3. The dual space of the Banach $^*$-algebra $(\mathcal{F}_S, \| \cdot \|_S)$ is isometrically isomorphic to the quotient space $\hat{T}(H)/\tilde{\mathcal{F}}_S$ and the second dual space of $(\mathcal{F}_S, \| \cdot \|_S)$ is isometrically isomorphic to a closed subspace of $(\mathcal{A}_S, \| \cdot \|_S)$.

The following example shows that if $S$ is not selfadjoint then, generally speaking, $(\hat{\Phi}_S)^\perp \neq \hat{\mathcal{A}}_S$, so that $(\mathcal{F}_S)^{\perp\perp} \neq \hat{\mathcal{A}}_S$ and the second dual space of $(\mathcal{F}_S, \| \cdot \|_S)$ is isometrically isomorphic to a proper subspace of $(\mathcal{A}_S, \| \cdot \|_S)$.

Example 3.4. Let, as in Example 2.2, $H = L^2(0, 1)$ and the operator $S = \frac{d}{dt}$ with domain $D(S) = \{ h(t) : h, h' \in L^2(0, 1) \text{ and } h(0) = h(1) = 0 \}$. Then $S$ is a symmetric operator, non-selfadjoint and

$$D(S^*) = \{ h(t) : h, h' \in L^2(0, 1) \}.$$ 

Let $g(t)$ be a differentiable function on $[0, 1]$ such that $g(0) \neq 0$ and let $M_g$ be the bounded operator of multiplication by $g(t)$ on $H$. Then $M_g \in \mathcal{A}_S$. If $h(t) \in D(S^*)$ and $h(0) \neq 0$ then $(M_g h)(0) = g(0) h(0) \neq 0$, so that $M_g h \notin D(S)$. Thus $M_g \oplus (M_g)_S \notin \{ A \oplus \mathcal{A}_S : A \in \mathcal{A}_S \text{ and } AD(S^*) \subseteq D(S) \}$. Hence $(\hat{\Phi}_S)^\perp \neq \hat{\mathcal{A}}_S$.

Assume now that $S$ is selfadjoint. Then $D(S^*) = D(S)$, $\mathcal{A}_S = T_S$, for $T \in \mathcal{A}_S$, and

$$\mathcal{A}_S = \{ T \in T(H) \cap \mathcal{A}_S : T_S \in T(H) \} \subseteq \mathcal{A}_S.$$ 

It is well known (see, for example, [5] and [6]) that the algebra $T(H)$ is a two-sided ideal of $B(H)$ and if $A \in B(H)$ and $B \in T(H)$ then

$$|AB| \leq \| A \| \| B \|, \quad |B^*| = |B| \quad \text{and} \quad \| B \| \leq |B|.$$ 

We consider now two equivalent norms on $\mathcal{A}_S$:

$$|T|_1 = |T| + |T_S| \quad \text{and} \quad |T|_2 = \max(|T|, |T_S|), \quad \text{for } T \in \mathcal{A}_S.$$ 

Since

$$T_S = T \quad \text{and} \quad |T|_2 = \max(|T|, |T_S|) = |T_S| \oplus T, \quad \text{for } T \in \mathcal{A}_S,$$

$$(\mathcal{A}_S, | \cdot |_2)$$ is isometrically isomorphic to $\mathcal{A}_S$.

Proposition 3.5. Let $S$ be selfadjoint. Then:

(i) $\mathcal{A}_S \subset \mathcal{F}_S$ and $(\mathcal{A}_S, | \cdot |_2)$ is a two-sided Banach $\mathcal{A}_S$-module;

(ii) $(\mathcal{A}_S, | \cdot |_1)$ is a Banach $^*$-algebra and a $\mathcal{D}$-subalgebra of $C(H)$ (see (1.1)) with $D = 1$.

Proof. It was shown in [13] that if $S$ is selfadjoint then $\mathcal{F}_S$ coincides with the algebra $\mathcal{J}_S = \{ A \in \mathcal{A}_S : A \text{ and } \mathcal{A}_S \text{ belong to } C(H) \}$. Since $\mathcal{A}_S \subset \mathcal{J}_S$, we obtain that $\mathcal{A}_S \subset \mathcal{F}_S$.

In Theorem 3.2(i) it was shown that $\mathcal{A}_S$ is a closed subspace of $\hat{T}(H)$. Since $(\mathcal{A}_S, | \cdot |_2)$ is isometrically isomorphic to $\mathcal{A}_S$, it is a Banach space.
Since the norms $| \cdot |_1$ and $| \cdot |_2$ are equivalent, $(\mathfrak{T}_S, | \cdot |_1)$ is also a Banach space.

For $A, B \in \mathcal{A}_S$,
\[ (AB)_S|_{D(S)} = (SAB - ABS)|_{D(S)} = [(SA - AS)B + A(SB - BS)]|_{D(S)} = (ASB + AB_S)|_{D(S)}, \]
so that
\[ (AB)_S = ASB + AB_S. \]

Let $T \in \mathfrak{T}_S$ and $A \in \mathcal{A}_S$. Then $T, T_S \in T(H)$. Since $\mathfrak{T}_S \subset \mathcal{A}_S$ and $T(H)$ is a two-sided ideal of $B(H)$, it follows that $AT \in T(H) \cap \mathcal{A}_S$ and, by (3.12),
\[ (AT)_S = AS T + AT_S \in T(H). \]
Therefore $AT \in \mathfrak{T}_S$. Making use of (3.11), we obtain that
\[ |AT|_2 = \max(|AT|, |(AT)_S|) \leq \max(\|A\| |T|, \|A_S\| |T| + \|A\| |T_S|) \leq (\|A\| + \|A_S\|) \max(\|T\|, |T_S|) = \|A\| |s|^{\|T\|}_2. \]

Similarly, $TA \in \mathfrak{T}_S$ and $|TA|_2 \leq \|A\| |s|^{\|T\|}_2$. Thus $(\mathfrak{T}_S, | \cdot |_2)$ is a two-sided Banach $\mathcal{A}_S$-module. Part (i) is proved.

From (i) and from the fact that $\mathfrak{T}_S \subseteq \mathcal{A}_S$, we have that $\mathfrak{T}_S$ is an algebra. We also have that $T^* \in \mathfrak{T}_S$ and, since $\mathfrak{T}_S = T_S$, it follows from (3.3) that $(T^*)_S = -(T_S)^* \in T(H)$. Taking this and (3.11) into account, we obtain that
\[ |T^*|_1 = |T^*| + |(T^*)_S| = |T^*| + |-(T_S)^*| = |T| + |T_S| = |T|_1 \]
and
\[ |TR|_1 = |TR| + |(TR)_S| = |TR| + |T_SR + TR_S| \leq \|T\| |R| + |T_S| |R| + \|T\| |R_S| \leq |T| |R| + |T_S| |R| + |T| |R_S| \leq |T|_1 |R|_1, \]
for $T, R \in \mathfrak{T}_S$. Hence $(\mathfrak{T}_S, | \cdot |_1)$ is a Banach $^*$-algebra.

Clearly, $\mathfrak{T}_S$ is dense in $C(H)$. For $T, R \in \mathfrak{T}_S$, it follows from (3.11) that
\[ |TR|_1 = |TR| + |(TR)_S| = |TR| + |T_SR + TR_S| \leq \|T\| |R| + |T_S| |R| + \|T\| |R_S| \leq \|T\| (|R| + |R_S|) + (|T| + |T_S|) \|R\| = \|T\| |R|_1 + |T|_1 |R|_1. \]
Thus $(\mathfrak{T}_S, | \cdot |_1)$ is a $D$-subalgebra of $C(H)$ with the constant $D = 1$. \hfill $\Box$

If $S$ is selfadjoint, it follows from Theorem 3.2 that $(\Phi_S)^\perp = \mathfrak{A}_S$ and
\[ (\mathfrak{F}_S)^\perp = (\mathfrak{T}_S)^\perp \subseteq (\Phi_S)^\perp = \mathfrak{A}_S. \]
In order to prove that \((\mathcal{F}_S)^{\perp} = \hat{A}_S\) it suffices to show that \(\Phi_S\) is dense in \(\mathfrak{T}_S\). For this we need the following lemma which is a partial case of the general result obtained by Golberg and Krein \cite[Theorem 6.3]{6} for symmetrically normable ideals.

**Lemma 3.6.** Let \(T \in T(H)\) and let \(Q_n\) be finite rank projections which converge to \(1_H\) in the strong operator topology. Then

\[
|T - Q_nT| \to 0 \quad \text{and} \quad |T - TQ_n| \to 0, \quad \text{as} \quad n \to \infty.
\]

**Proof.** Let \(A = x \otimes y, x, y \in H\). By \((3.5)\), \(A^*A = \|y\|^2(x \otimes x)\) and the operator \((A^*A)^{1/2} = \frac{y}{\|y\|}(x \otimes x)\) has only one non-zero eigenvalue \(\lambda = \|x\| \|y\|\). Hence

\[
(3.13) \quad |x \otimes y| = |A| = \text{Tr}(A^*A)^{1/2} = \|x\| \|y\|.
\]

If \(T = \sum_{i=1}^k x_i \otimes y_i\) is a finite rank operator then, by \((3.5)\) and \((3.13)\),

\[
|T - Q_nT| = \left| \sum_{i=1}^k x_i \otimes (y_i - Q_ny_i) \right| \leq \sum_{i=1}^k |x \otimes (y_i - Q_ny_i)|
\]

\[
= \sum_{i=1}^k \|x_i\| \|y_i - Q_ny_i\| \to 0,
\]

as \(n \to \infty\). For any \(T \in T(H)\) and any \(\varepsilon > 0\), there is a finite rank operator \(T_\varepsilon\) such that \(|T - T_\varepsilon| < \varepsilon\). Making use of the inequality \((3.11)\), we obtain that

\[
|T - Q_nT| \leq |T - T_\varepsilon| + |T_\varepsilon - Q_nT_\varepsilon| + |Q_n(T - T_\varepsilon)|
\]

\[
\leq \varepsilon + |T_\varepsilon - Q_nT_\varepsilon| + \|Q_n\| |T - T_\varepsilon|
\]

\[
\leq 2\varepsilon + |T_\varepsilon - Q_nT_\varepsilon|.
\]

Since \(T_\varepsilon\) is a finite rank operator, by the above argument, there is \(n_\varepsilon\) such that \(|T_\varepsilon - Q_nT_\varepsilon| \leq \varepsilon\), for \(n > n_\varepsilon\). Hence \(|T - Q_nT| \leq 3\varepsilon\) and \(|T - Q_nT| \to 0\), as \(n \to \infty\). Similarly, one can prove that \(|T - TQ_n| \to 0\), as \(n \to \infty\). \(\Box\)

**Proposition 3.7.** Let \(S\) be selfadjoint. Then \(\Phi_S\) is dense in \((\mathfrak{T}_S, | \cdot |_1)\).

**Proof.** Let \([S]\) be the selfadjoint operator constructed in Section 2. Then \(D(S) = D([S])\), so that \(\Phi_S = \Phi_{[S]}\). Since \(B = S - [S]\) is a bounded operator, \(BT - TB \in T(H)\), for \(T \in T(H)\). Therefore, taking into account that

\[
(ST - TS)_{D(S)} = ([S][T - T[S]] + (BT - TB)_{D(S)})_{D(S)},
\]

we conclude that \(\mathfrak{T}_S = \mathfrak{T}_{[S]}\) and \(T_S = T_{[S]} + BT - TB\).

Making use of \((3.11)\), we obtain that for any \(T \in \mathfrak{T}_S\),

\[
|T| + |T_S| = |T| + |T_{[S]} + BT - TB|
\]

\[
\leq |T| + |T_{[S]}| + 2\|B\| |T|
\]

\[
\leq (1 + 2\|B\|) (|T| + |T_{[S]}|).
\]
Similarly, $|T| + |T[S]| \leq (1 + 2\|B\|)(|T| + |T[S]|)$. Thus the norms $| \cdot |_1$ generated by the operators $S$ and $[S]$ on $\mathfrak{F}_S$ are equivalent. Hence to obtain the proof we only have to show that $\Phi_{[S]}$ is dense in $(\mathfrak{F}[S], | \cdot |_1)$.

In every subspace $H_S(n)$ (see (2.2)) we choose an increasing sequence of finite-dimensional projections $\{Q^n\}_{n=1}^\infty$ converging to the projection $P_S(n)$ (see (2.1)) in the strong operator topology as $k \to \infty$. Set

$$Q^k = \sum_{n=-k}^{k} \oplus Q^k_n.$$ 

Then $Q^k$ are finite-dimensional projections commuting with $[S]$. Hence $Q^k \in \Phi_{[S]}$. The projections $Q^k$ converge to $1_H$ in the strong operator topology. Let $T \in \mathfrak{F}[S]$. Then $Q^n T \in \Phi_{[S]}$ and


Therefore $(Q^k T)[S] = Q^k T[S]$.

Since $T, T[S] \in T(H)$, we obtain from Lemma 3.6 that

$$|T - Q^k T| \to 0 \text{ and } |T[S] - (Q^k T)[S]| = |T[S] - Q^k T[S]| \to 0, \text{ as } k \to \infty.$$ 

Hence

$$|T - Q^k T|_1 = |T - Q^k T| + |T[S] - (Q^k T)[S]| \to 0$$

as $k \to \infty$, so that $\Phi_{[S]}$ is dense in $(\mathfrak{F}[S], | \cdot |_1)$.

\begin{corollary}
Let $S$ be a selfadjoint operator. Then:

(i) the Banach $^*$-algebra $(\mathfrak{F}_S, | \cdot |_1)$ is simple;
(ii) $(\mathfrak{F}_S)^\perp = (\Phi_S)^\perp = \mathcal{A}_S$;
(iii) the dual space of $(\mathfrak{F}_S, | \cdot |_2)$ is isometrically isomorphic to the quotient space $\overline{B(H)}/\mathcal{A}_S$.
\end{corollary}

\begin{proof}
Let $I$ be a closed two-sided ideal of $(\mathfrak{F}_S, | \cdot |_1)$ and $0 \neq T \in I$. Since $D(S)$ is dense in $H$, there is $x \in D(S)$ such that $Tx \neq 0$. Since $S$ is selfadjoint, it follows from the definition of $\mathfrak{F}_S$ that $Tx \in D(S)$. From this and from the discussion before Lemma 3.1 we obtain that the rank one operators $y \otimes x$ and $Tx \otimes z$ belong to $\mathfrak{F}_S$ for any $y, z \in D(S)$. By (3.5),

$$T(y \otimes x) = (y \otimes Tx) \in I \text{ and }$$(

$$Tx \otimes z)(y \otimes Tx) = \|Tx\|^2(y \otimes z) \in I.$$ 

Thus $y \otimes z \in I$ and, therefore, $\Phi_S \subseteq I$. Since $I$ is closed, we obtain from Proposition 3.7 that $I = \mathfrak{F}_S$. Part (i) is proved.

Since the norms $| \cdot |_1$ and $| \cdot |_2$ are equivalent on $\mathfrak{F}_S$, it follows from Proposition 3.7 that $\Phi_S$ is dense in $(\mathfrak{F}_S, | \cdot |_2)$. Taking into account that $(\mathfrak{F}_S, | \cdot |_2)$ is isometrically isomorphic to the closed subspace $\mathfrak{F}_S$ of $\overline{T(H)}$,
we obtain that the linear subspace $\Phi_S$ is dense in $\check{\Phi}_S$. From this and from Theorem 3.2(ii) we obtain $(\check{\Phi}_S)^\perp = (\Phi_S)^\perp = \check{A}_S$. Part (ii) is proved.

The dual space of $(\check{\Phi}_S, | \cdot |_2)$ is isometrically isomorphic to the dual space of the closed subspace $\check{\Phi}_S$ of $\check{T}(H)$. Since $\check{T}(H)^* = \check{B}(H)$, part (iii) follows from (ii) and from Lemma 3.1.\hfill $\Box$

**Theorem 3.9.** If $S$ is a selfadjoint operator then $(\check{\Phi}_S)^{\perp\perp} = \check{A}_S$ and the second dual space of the algebra $(A, \| \cdot \|_S)$ is isometrically isomorphic to the algebra $(\check{A}_S, \| \cdot \|_S)$.

**Proof.** Combining Theorem 3.2(i) and Corollary 3.8(ii) yields $(\check{\Phi}_S)^{\perp\perp} = \check{A}_S$.

Therefore it follows from Lemma 3.1 that the second dual space of $(\check{\Phi}_S, \| \cdot \|)$ is isometrically isomorphic to $(\check{A}_S, \| \cdot \|)$. Taking into account that $(A, \| \cdot \|_S)$ isometrically isomorphic to $(\check{\Phi}_S, \| \cdot \|)$ and that $(\check{A}_S, \| \cdot \|_S)$ isometrically isomorphic to $(\check{A}_S, \| \cdot \|)$, we complete the proof. \hfill $\Box$

### 4. Isomorphism of the algebras $F_S$ and $A_S$.

In this section we study the problem of classification of the algebras $F_S$ and $A_S$ up to *-isomorphism. For isometrical *-isomorphism this problem is completely solved in Theorem 4.4. As far as bounded but not necessarily isometrical *-isomorphism is concerned, we have obtained some partial results in Theorems 4.6 and 4.8 for the case when $S$ is selfadjoint.

Banach *-algebras $(A, \| \cdot \|_A)$ and $(B, \| \cdot \|_B)$ are *-isomorphic if there is a bounded *-isomorphism $\varphi$ from $A$ onto $B$. They are isometrically *-isomorphic if, in addition, $\|\varphi(A)\|_B = \|A\|_A$, for $A \in A$.

Let $(A, \| \cdot \|_A)$ and $(B, \| \cdot \|_B)$ be Banach *-algebras of operators on Hilbert spaces $H$ and $\mathcal{H}$ (the norms $\| \cdot \|_A$ and $\| \cdot \|_B$ do not, generally speaking, coincide with the operator norms in $B(H)$ and $B(\mathcal{H})$) and let $\varphi$ be a bounded *-isomorphism from $A$ onto $B$. An isometry operator $U$ from $H$ into $\mathcal{H}$ implements $\varphi$ if

$$
\varphi(A) = UA^*U^*, \quad A \in A.
$$

**Lemma 4.1.** Let $R$ and $T$ be symmetric operators on $\mathcal{H}$, $S$ be a symmetric operators on $H$, $U$ be an isometry operator from $\mathcal{H}$ onto $H$ and $t \in \mathbb{R}$.

(i) If $F_R = F_T$ then the norms $\| \cdot \|_R$ and $\| \cdot \|_T$ on this algebra are equivalent, so that the Banach *-algebras $(F_R, \| \cdot \|_R)$ and $(F_T, \| \cdot \|_T)$ are *-isomorphic.

(ii) If $R = \pm T + t1_H$ then $F_R = F_T$ and the norms $\| \cdot \|_R$ and $\| \cdot \|_T$ coincide.

(iii) If $S = \lambda U T U^* + B$, where $0 \neq \lambda \in R$ and $B$ is a bounded selfadjoint operator, then $A \rightarrow UAU^*$ is a bounded *-isomorphism from $(F_T, \| \cdot \|_T)$ onto $(F_S, \| \cdot \|_S)$. If $\lambda = \pm 1$ and $B = t1_H$ then $A \rightarrow UAU^*$ is an isometric *-isomorphism.
The same results hold for the algebras $A_S$.

Proof. By Proposition 3.2 [13], the algebras $F_R$ and $F_T$ are semisimple. Hence if $F_R = F_T$ then it follows from Johnson's uniqueness of norm theorem that the norms $\| \cdot \|_R$ and $\| \cdot \|_T$ on this algebra are equivalent. Therefore the identity mapping is a bounded *-isomorphism from $(F_R, \| \cdot \|_R)$ onto $(F_T, \| \cdot \|_T)$.

Let $R = x + t1_H$. Then $D(R) = D(T)$ and $A_T = A_R$ for any $A \in A_T$. Hence $\|A\|_R = \|A\|_T$ and $A_R = A_T$. The sets of finite rank operators in the algebras $F_R$ and $F_T$ coincide and, since these algebras are the closures of these sets with respect to the norm $\| \cdot \|_T$, we obtain that $F_S = F_T$.

If $S = \lambda UTU^* + B$ then $D(S) = UD(T)$ and, for $A \in A_T$,

$$ UA^*D(S) = UAD(T) \subseteq UD(T) = D(S) \quad \text{and} \quad SUAU^* - UAU^*S = \lambda U(TA - AT)U^* + (BA - AB), $$

so that $UA^* \in A_S$ and $(UA^*)_S = \lambda UA_TU^* + (BA - AB)$. Thus $A_S = U,A_TU^*$ and

$$ \|UA^*\|_S = \|UA^*\| + \|(UA^*)_S\| = \|A\| + \|\lambda UA_TU^* + (BA - AB)\| \leq \|A\| + \lambda \|A\| + 2\|B\| \|A\| \leq \max(\lambda, 1 + 2\|B\|) \|A\|_T, $$

so that $\psi(A) = UA^*$ is a bounded *-isomorphism from $(A_T, \| \cdot \|_T)$ onto $(A_S, \| \cdot \|_S)$. If $A$ is a finite rank operator in $A_T$ then $UA^*$ is a finite rank operator in $A_S$. Therefore $F_S = \psi(F_T)$. 

Let $S$ be a symmetric operator with domain $D(S)$. It was shown in Lemma 3.1 [13] that a finite rank operator $A$ belongs to $F_S$ if and only if

$$ A = \sum_{i=1}^n x_i \otimes y_i, \quad \text{where} \quad x_i, y_i \in D(S). $$

**Theorem 4.2.** Let $S$ and $T$ be symmetric operators on $H$ and $H$ and let $B$ and $C$ be closed *-subalgebras of $(A_S, \| \cdot \|_S)$ and $(A_T, \| \cdot \|_T)$, respectively, such that $F_S \subseteq B$ and $F_T \subseteq C$. Let $\psi$ be a bounded *-isomorphism from $C$ onto $B$ and let $\varphi = \psi|F_T$. Then:

1. $\varphi$ is a bounded *-isomorphism of $(F_T, \| \cdot \|_T)$ onto $(F_S, \| \cdot \|_S)$;
2. there is an isometry operator $U$ from $H$ onto $H$ implementing $\psi$:

$$ \psi(A) = UA^*, \quad \text{for} \quad A \in C, $$

and $D(S) = UD(T)$ and $F_{U^*TU} = F_S$.

**Proof.** For $x, y \in D(T)$, $x \neq 0$, $y \neq 0$, set $Y = \varphi(x \otimes y)$. If $Y$ is not a rank one operator, there are $z, u \in D(S)$ such that $Yz \neq 0$, $Yu \neq 0$, and $Yz \notperp Yu$. Since $Y \in A_S$, we have that $Yz, Yu \in D(S)$, so that $Yz \otimes z \in F_S$ and $u \otimes Yu \in F_S$. By (3.5)

$$ (Yz \otimes z)(u \otimes Yu) = (Yu, Yz)(u \otimes z) = 0. $$

(4.2)
Thus \( (z \otimes z)^* = z \otimes z \) and \( \varphi \) is a *-isomorphism, it follows from (3.5) that
\[
(\varphi^{-1}(z \otimes z)x) \otimes y = (x \otimes y) [\varphi^{-1}(z \otimes z)]^* = \varphi^{-1}(Y)\varphi^{-1}(z \otimes z) = \varphi^{-1}(z \otimes Yz) \neq 0.
\]
Thus \( \varphi^{-1}(z \otimes z)x \neq 0 \). Similarly, \( \varphi^{-1}(u \otimes u)x \neq 0 \). From this and from (3.5) and (4.2) it follows that
\[
0 = \varphi^{-1}((Yz \otimes z)(u \otimes Yu)) = \varphi^{-1}((z \otimes z)Y^*Y(u \otimes u)) = \varphi^{-1}(z \otimes z)\varphi^{-1}(Y^*)\varphi^{-1}(Y)\varphi^{-1}(u \otimes u)
= \varphi^{-1}(z \otimes z)(y \otimes x)(x \otimes y)\varphi^{-1}(u \otimes u) = \varphi^{-1}(z \otimes z) \|y\|^2(x \otimes x)\varphi^{-1}(u \otimes u)
= \|y\|^2([\varphi^{-1}(u \otimes u)x] \otimes [\varphi^{-1}(z \otimes z)x]) \neq 0.
\]
This contradiction shows that \( Y \) is a rank one operator. Hence \( Y \in \mathcal{F}_S \) and, by (4.1), \( \varphi \) maps all finite rank operators in \( \mathcal{F}_T \) into finite rank operators in \( \mathcal{F}_S \). Since \( \varphi \) is bounded \( \varphi(\mathcal{F}_T) \subseteq \mathcal{F}_S \). Similarly, \( \varphi^{-1}(\mathcal{F}_S) \subseteq \mathcal{F}_T \), so that \( \varphi \) is a bounded *-isomorphism from \( \mathcal{F}_T \) onto \( \mathcal{F} \). Part (i) is proved.

Fix \( x_0 \in D(T), \|x_0\| = 1 \). Since \( x_0 \otimes x_0 \) is a projection, \( \varphi(x_0 \otimes x_0) \) is a one-dimensional projection in \( \mathcal{F}_S \). By (4.1), we can choose \( \xi_0 \) in \( D(S), \|\xi_0\| = 1 \), such that \( \varphi(x_0 \otimes x_0) = \xi_0 \otimes \xi_0 \). Let \( y \in D(T) \). Making use of the equality \( x_0 \otimes y = (x_0 \otimes y)(x_0 \otimes x_0) \), we obtain that
\[
\varphi(x_0 \otimes y) = \varphi(x_0 \otimes y)\varphi(x_0 \otimes x_0)
= \varphi(x_0 \otimes y)(\xi_0 \otimes \xi_0) = \xi_0 \otimes \varphi(x_0 \otimes y)\xi_0.
\]
Since \( \varphi(x_0 \otimes y) \in \mathcal{F}_S \), it follows from (4.1) that \( \varphi(x_0 \otimes y)\xi_0 \) belongs to \( D(S) \).

Now \( U : y \in D(T) \rightarrow \varphi(x_0 \otimes y)\xi_0 \) is a linear mapping from \( D(T) \) into \( D(S) \) and \( \varphi(x_0 \otimes y) = \xi_0 \otimes Uy \). Then
\[
\varphi((y \otimes x_0)(x_0 \otimes y)) = \|y\|^2\varphi(x_0 \otimes x_0) = \|y\|^2(\xi_0 \otimes \xi_0)
= \varphi((x_0 \otimes y)^*)\varphi(x_0 \otimes y)
= (Uy \otimes \xi_0)(\xi_0 \otimes Uy) = \|Uy\|^2(\xi_0 \otimes \xi_0).
\]
Thus \( \|Uy\|^2 = \|y\|^2 \), for \( y \in D(T) \), and \( U \) extends to an isometry operator from \( \mathcal{H} \) into \( H \) which we also denote by \( U \). We have that, for \( x, y \in D(T), \)
\[
(4.3) \quad \varphi(x \otimes y) = \varphi((x_0 \otimes y)(x \otimes x_0)) = (\xi_0 \otimes Uy)(\xi_0 \otimes Ux)^*
= Ux \otimes Uy = U(x \otimes y)U^*.
\]

Similarly, there is an isometry operator \( V \) which maps \( D(S) \) into \( D(T) \) such that \( \varphi^{-1}(\xi \otimes \eta) = V\xi \otimes V\eta \), for \( \xi, \eta \in D(S) \). Hence
\[
\xi \otimes \eta = \varphi(\varphi^{-1}(\xi \otimes \eta)) = \varphi(V\xi \otimes V\eta) = UV\xi \otimes UV\eta.
\]
Thus \( UV\xi = \lambda(\xi)\xi \) where \( \lambda \) is a function on \( D(S) \) such that \(|\lambda(\xi)| = 1\). Hence \( UD(T) = D(S) \). Since \( D(S) \) is dense in \( H \) and \( U \) is an isometry operator, we have \( UH = H \).

Let \( A \in \mathcal{C} \) and set \( R = U^*\psi(A)U \). Then \( x \otimes y \in \mathcal{F}_T \), for any \( x, y \in D(T) \), and, since \( \mathcal{F}_T \) is an ideal of \( \mathcal{A}_T \), we have \( A(x \otimes y) = x \otimes Ay \in \mathcal{F}_T \). By (4.3),

\[
R(x \otimes y) = U^*\psi(A)U(x \otimes y) = U^*\psi(A)U(x \otimes y)U^*U = U^*\psi(A)\varphi(x \otimes y)U = U^*\psi(A(x \otimes y))U = U^*\varphi(x \otimes Ay)U = x \otimes Ay.
\]

Therefore \( R(x \otimes y) = x \otimes Ry = x \otimes Ay \), so that \( Ry = Ay \). Thus \( R = A \) and

\[
\psi(A) = UAU^*.
\]

The operator \( F = UTU^* \) is symmetric and \( D(F) = UD(T) = D(S) \). By Lemma 4.1, \( \mathcal{F}_F = U\mathcal{F}_T U^* \) and \( A \rightarrow UAU^* \) is an isometric \(*\)-isomorphism from \( (\mathcal{F}_T, \| \cdot \|_T) \) onto \( (\mathcal{F}_F, \| \cdot \|_F) \). Hence

\[
\varphi(U^*BU) = U(U^*BU)U^* = B, \quad \text{for } B \in \mathcal{F}_F,
\]

is a bounded \(*\)-isomorphism from \( \mathcal{F}_F \) onto \( \mathcal{F}_S \). Therefore \( \mathcal{F}_F = \mathcal{F}_S \).

It was shown in Theorem 3.4 [13] that the algebra \( (\mathcal{F}_S, \| \cdot \|_S) \) has a bounded approximate identity if and only if \( S \) is selfadjoint. Making use of this and of Theorem 4.2, we obtain the following result.

**Corollary 4.3.** If the algebras \( \mathcal{F}_S \) and \( \mathcal{F}_T \) are \(*\)-isomorphic or the algebras \( \mathcal{A}_S \) and \( \mathcal{A}_T \) are \(*\)-isomorphic then the operators \( S \) and \( T \) are either selfadjoint or non-selfadjoint at the same time.

Apart from the sufficient conditions of Lemma 4.1 and the necessary conditions of Corollary 4.3 for two algebras \( \mathcal{F}_S \) and \( \mathcal{F}_T \) to be \(*\)-isomorphic we do not know any other sufficient or necessary condition in the case when \( S \) and \( T \) are arbitrary symmetric operators. Later, in Theorem 4.6 and Corollary 4.8 we consider a particular case when the operators \( S \) and \( T \) are selfadjoint.

It follows from Theorem 4.2 that if \( \mathcal{F}_S \) and \( \mathcal{F}_T \) are \(*\)-isomorphic, they are unitary isomorphic. This, however, does not necessarily imply that they are isometrically isomorphic. In the following theorem we obtain necessary and sufficient conditions for algebras \( \mathcal{F}_S \) and \( \mathcal{F}_T \) to be isometrically \(*\)-isomorphic.

**Theorem 4.4.** The algebras \( (\mathcal{F}_S, \| \cdot \|_S) \) and \( (\mathcal{F}_T, \| \cdot \|_T) \) are isometrically \(*\)-isomorphic if and only if there are \( \lambda \in \mathbb{R} \) and an isometry operator \( U \) such that \( S - \lambda 1_H = \pm UTU^* \). The same result holds for \( (\mathcal{A}_S, \| \cdot \|_S) \) and \( (\mathcal{A}_T, \| \cdot \|_T) \).

**Proof.** From Lemma 4.1 it follows that the conditions of the theorem are sufficient. From Theorem 4.2 it follows that if these conditions are necessary for the algebras \( (\mathcal{F}_S, \| \cdot \|_S) \) and \( (\mathcal{F}_T, \| \cdot \|_T) \) to be isometrically \(*\)-isomorphic, they are also necessary for the algebras \( (\mathcal{A}_S, \| \cdot \|_S) \) and \( (\mathcal{A}_T, \| \cdot \|_T) \).
Let $\varphi$ be an isometric $^*$-isomorphism from $(\mathcal{F}_T, \| \cdot \|_T)$ onto $(\mathcal{F}_S, \| \cdot \|_S)$ and let $U$ be the isometry operator as in Theorem 4.2 which implements $\varphi$:

$$\varphi(A) = UAU^*, \quad \text{for } A \in \mathcal{F}_T.$$ 

Set $F = UU^*$. Then $F$ is a symmetric operator on $H$, $D(S) = D(F) = UD(T)$ and $\mathcal{F}_S = \mathcal{F}_F$. Since $\varphi$ is isometric, the norms $\| \cdot \|_S$ and $\| \cdot \|_F$ coincide.

We will show that there is $\lambda \in \mathbb{R}$ such that either $S - \lambda 1_H = F$ or $S - \lambda 1_H = -F$.

**Step 1.** Suppose that $z \in D(S)$ is not an eigenvector of $S$ and $\|z\| = 1$. Set $s = (Sz, z), \quad t = (Fz, z), \quad R = S - s 1_H$ and $G = F - t 1_H$.

Since $S$ an $F$ are symmetric, $s, t \in \mathbb{R}$, the operators $R$ and $G$ are symmetric and

$$D(R) = D(G), \quad Rz \neq 0 \quad \text{and} \quad (Rz, z) = (Gz, z) = 0. \quad (4.4)$$

Set $D = D(R) = D(G)$. Since $\mathcal{F}_S = \mathcal{F}_F$ and the norms $\| \cdot \|_S$ and $\| \cdot \|_F$ coincide, it follows from Lemma 4.1 that $\mathcal{F}_R = \mathcal{F}_G$ and the norms $\| \cdot \|_R$ and $\| \cdot \|_G$ coincide.

Taking into account that $R$ and $G$ are symmetric, we obtain from (3.5) that

$$\|y \otimes x\|_R = \|y \otimes x\| + \|y \otimes Rx - (Ry) \otimes x\| = \|y \otimes x\|_G$$

$$= \|y \otimes x\| + \|y \otimes Gx - (Gy) \otimes x\|,$$

for $x, y \in D$. Therefore

$$\|y \otimes Rx - (Ry) \otimes x\| = \|y \otimes Gx - (Gy) \otimes x\|. \quad (4.5)$$

Represent the elements $Rx$ and $Gx$ in the form

$$Rx = \alpha(x)x + x_R \quad \text{and} \quad Gx = \beta(x)x + x_G,$$

where $x_R$ and $x_G$ are orthogonal to $x$. Then

$$\alpha(x)\|x\|^2 = (Rx, x) = (x, Rx) = \overline{\alpha(x)}\|x\|^2.$$

Thus $\alpha(x)$ is real, for $x \in D$. Therefore

$$x \otimes Rx - (Rx) \otimes x = \alpha(x)(x \otimes x) + x \otimes x_R - \alpha(x)(x \otimes x) - x_R \otimes x$$

$$= x \otimes x_R - x_R \otimes x.$$

Since $x$ and $x_R$ are orthogonal, any $u \in H$ can be represented in the form $u = \nu x + \tau x_R + \tilde{u}$, where $\nu, \tau \in \mathbb{C}$ and $\tilde{u}$ is orthogonal to $x$ and $x_R$. Therefore

$$\|u\| = |\nu|^2\|x\|^2 + |\tau|^2\|x\|^2 + \|\tilde{u}\|^2.$$
and, by (3.5),
\[
\|(x \otimes x_R + x_R \otimes x)u\|^2 = \|(u, x)x_R + (u, x_R)x\|^2
\]
\[
= \|\nu\|^2x_R^2 + \tau\|x_R\|^2x^2
\]
\[
= \|\nu\|^2\|x\|^4\|x_R\|^2 + \|\tau\|^2\|x_R\|^4\|x\|^2
\]
\[
= \|x\|^2\|x_R\|^2(\|\nu\|^2\|x\|^2 + \|\tau\|^2\|x_R\|^2).
\]
Consequently,
\[
\|x \otimes Rx - (Rx) \otimes x\|^2 = \|x \otimes x_R - x_R \otimes x\|^2 = \|x\|^2\|x_R\|^2.
\]
Similarly, \(\|x \otimes Gx - (Gx) \otimes x\|^2 = \|x\|^2\|x_G\|^2\) and it follows from (4.5) that
\[
\|x_R\| = \|x_G\|, \quad \text{for } x \in D.
\]
Therefore we obtain from (4.6) that for \(x \in D\)
\[
\|x\|^2\|Rx\|^2 - |(Rx, x)|^2 = \|x\|^2(\|\alpha(x)\|^2\|x\|^2 + \|x_R\|^2) - |\alpha(x)|^2\|x\|^4
\]
\[
= \|x\|^2\|x_R\|^2 = \|x\|^2\|x_G\|^2
\]
\[
= \|x\|^2\|Gx\|^2 - |(Gx, x)|^2.
\]
Hence
\[
(4.7) \quad \|x\|^2(|Rx|^2 - \|Gx\|^2) = |(Rx, x)|^2 - |(Gx, x)|^2.
\]
In particular, it follows from (4.4), (4.6) and (4.7) that
\[
(4.8) \quad Rz = z_R, \quad Gz = z_G \quad \text{and} \quad \|Rz\| = \|Gz\|.
\]
**Step 2.** Set \(D_\frac{1}{2} = \{y \in D : y \text{ is orthogonal to } z\}.\) Let \(y \in D_\frac{1}{2}\) and \(x = y + \mu z,\)
\(\mu \in \mathbb{C}.\) Then \(\|x\|^2 = \|y\|^2 + \|\mu z\|^2 = \|y\|^2 + |\mu|^2\) and, by (4.8),
\[
\|Rx\|^2 - \|Gx\|^2 = \|Ry\|^2 + \|\mu Rz\|^2 + 2\text{Re}[\mu (Rz, Ry)]
\]
\[
- \|Gy\|^2 - \|\mu Gz\|^2 - 2\text{Re}[\mu (Gz, Gy)]
\]
\[
= A + 2\text{Re}(\mu B),
\]
where
\[
A = \|Ry\|^2 - \|Gy\|^2 \quad \text{and} \quad B = (Rz, Ry) - (Gz, Gy).
\]
Since \(R\) is symmetric, it follows from (4.4) that
\[
(Rx, x) = (Ry, y) + (\mu Rz, y) + (Ry, \mu z) + (\mu Rz, \mu z)
\]
\[
= (Ry, y) + 2\text{Re}[\mu (Rz, y)].
\]
Similarly, \((Gx, x) = (Gy, y) + 2\text{Re}[\mu (Gz, y)].\)
Let \(\mu = re^{i\psi}.\) Substituting all this in (4.7), we obtain that
\[
(4.9) \quad (\|y\|^2 + r^2)[A + 2\text{Re}(e^{i\psi}B)]
\]
\[
= \{(Ry, y) + 2\text{Re}[e^{i\psi}(Rz, y)]\}^2 - \{(Gy, y) + 2\text{Re}[e^{i\psi}(Gz, y)]\}^2.
\]
Set
\[ C = (R_y, y)\Re[e^{i\psi}(R_z, y)] - (G_y, y)\Re[e^{i\psi}(G_z, y)] \quad \text{and} \]
\[ E = \{\Re[e^{i\psi}(R_z, y)]\}^2 - \{\Re[e^{i\psi}(G_z, y)]\}^2.\]
Since $R$ and $G$ are symmetric, $(R_y, y)$ and $(G_y, y)$ are real. Hence
\[ C = \Re\{e^{i\psi}[(R_y, y)(R_z, y) - (G_y, y)(G_z, y)]\}.\]
Comparing the coefficients of the same powers of $r$ in (4.9), we obtain that
\[ \Re(e^{i\omega}B) = 0, \quad A = 4E \quad \text{and} \quad C = 0.\]
Taking into account that $\Re(e^{i\psi}K) = 0$, for $0 \leq \psi < 2\pi$, implies $K = 0$, we obtain that $C = 0$ implies
\[ (4.10) \quad (R_y, y)(R_z, y) - (G_y, y)(G_z, y) = 0.\]
Set $(R_z, y) = ae^{ib}$ and $(G_z, y) = ce^{id}$. Then
\[ E = a^2 \left[\Re(e^{i(\psi+\beta)})\right]^2 - c^2 \left[\Re(e^{i(\psi+\delta)})\right]^2 \]
\[ = a^2 \cos^2(\psi + b) - c^2 \cos^2(\psi + d).\]
Since $A = 4E$ and since $A$ does not depend on $\psi$, neither does $E$. Hence
\[ a^2 = c^2 \quad \text{and} \quad d = b \quad \text{or} \quad d = b + \pi. \]
Since $a \geq 0$ and $c \geq 0$, $a = c$. Thus
\[ (4.11) \quad (R_z, y) = \pm(G_z, y), \quad \text{for} \quad y \in D_Z^\perp.\]
Since $D$ is dense in $\mathcal{H}$, $D_Z^\perp$ is dense in the subspace $\{\mathbb{C}z\}^\perp$. Hence (4.11) holds for all $y \in \{\mathbb{C}z\}^\perp$. From (4.9) it follows that $R_z = z_R \in \{\mathbb{C}z\}^\perp$.
Substituting $R_z$ for $y$ in (4.11), we obtain $\|R_z\| = (R_z, R_z) = \pm(G_z, R_z)$. Let $G_z = \nu R_z + u$, where $\nu \in \mathbb{C}$ and $u$ is orthogonal to $R_z$. Then
\[ \|R_z\|^2 = \pm(G_z, R_z) = \pm\nu\|R_z\|^2.\]
Since $R_z \neq 0$ (see (4.4)), $\nu = \pm 1$. Taking (4.9) into account, we obtain
\[ \|R_z\|^2 = \|G_z\|^2 = (\nu R_z + u, \nu R_z + u) \]
\[ = |\nu|^2\|R_z\|^2 + \|u\|^2 = \|R_z\|^2 + \|u\|^2.\]
Hence $u = 0$ and either $R_z = G_z$ or $R_z = -G_z$.

Step 3. Let $R_z = G_z$. Set $W = R - G$. Then $W$ is symmetric, $Wz = 0$ and it follows from (4.10) that
\[ [(R_y, y) - (G_y, y)](R_z, y) = (W_y, y)(R_z, y) = 0, \quad \text{for} \quad y \in D_Z^\perp.\]
Any $x \in D$ can be represented in the form $x = y + \mu z$ where $\mu \in \mathbb{C}$ and $y \in D_Z^\perp$. Then $Wx = W_y$ and, since $(R_z, z) = 0$, we have $(R_z, x) = (R_z, y)$. Since $Wz = 0$,
\[ (Wx, x)(R_z, x) = (W_y, y + \mu z)(R_z, y) \]
\[ = [(W_y, y) + (y, \mu Wz)](R_z, y) = (W_y, y)(R_z, y) = 0.\]
Therefore

\[(4.12)\quad (Wx,x)(Rz,x) = 0, \quad \text{for } x \in D.\]

Let \(X = \{x \in H : (Rz,x) = 0\}\) be the orthogonal complement of the subspace \(\mathbb{C}Rz\) in \(H\). By (4.4), \(Rz \neq 0\), so \(X\) has codimension 1. Set \(D = \{x \in D : x \notin X\}\). Since \(D\) is dense in \(H\), \(D\) is also dense in \(H\). For \(x \in D\), we have \((Rz,x) \neq 0\). Hence, by (4.12),

\[(Wx,x) = 0.\]

If \(x, y \in D\), there is \(r > 0\) such that \(x + re^{iy}y \in D\), for all \(0 \leq \psi < 2\pi\). Taking into account that \(W\) is symmetric, we obtain that

\[
0 = (W(x + re^{iy}y), x + re^{iy}y) = (Wx,x) + 2r \text{Re}[e^{iy}(Wy, x)] + r^2(Wy, y) = 2r \text{Re}[e^{iy}(Wy, x)].
\]

Hence \((Wy,x) = 0\). Since \(D\) is dense in \(H\), we have \(Wy = 0\), for \(y \in D\).

Let \(u \in D \cap X\), so that \((Rz,u) = 0\). For \(y \in D\), \((Rz,y + u) = (Rz,y) \neq 0\). Hence \(y + u \in D\) and \(0 = W(y + u) = Wy + Wu = Wu\). Thus \(Wx = 0\), for all \(x \in D\), so that \(R = G\). Hence \(S - s1_H = F - t1_H\). Setting \(\lambda = s - t\), we obtain that

\[S - \lambda1_H = F = UTU^*.\]

Similarly, in the case when \(Rz = -Gz\) we obtain that \(S - \lambda1_H = -F = -UTU^*\) which concludes the proof of the theorem. \(\square\)

In the rest of this section we study conditions for the algebras \(\mathcal{F}_S\) and \(\mathcal{F}_T\) to be \(*\)-isomorphic but not necessarily isometrically \(*\)-isomorphic in the case when \(S\) and \(T\) are selfadjoint operators. Taking Theorem 4.2(ii) into account, we may assume, without loss of generality, that \(\mathcal{F}_S = \mathcal{F}_T\) and \(D(S) = D(T)\).

In Example 4.7 we show that the coincidence of the domains of selfadjoint operators \(S\) and \(T\) even in the case when \(\text{Sp}(S) \subseteq \mathbb{Z}\), \(\text{Sp}(T) \subseteq \mathbb{Z}\) and \(S\) and \(T\) have the same sets of eigenvectors is not sufficient for \(\mathcal{F}_S = \mathcal{F}_T\). In other words, the algebras \(\mathcal{F}_S\) and \(\mathcal{F}_T\) may be the closures of the same set of finite rank operators and, nevertheless, be non-isomorphic. Necessary and sufficient conditions for these algebras to be \(*\)-isomorphic will be obtained in Theorem 4.6.

Let \(\mathcal{H}\) be a Hilbert space with an orthogonal basis \(\{e_i\}_{i=-\infty}^{\infty}\). Every operator \(T\) in \(B(\mathcal{H})\) has a matrix representation \(T = (t_{ij})\), \(-\infty < i,j < \infty\), where \(t_{ij} = (Te_j,e_i)\). A matrix \(M = (m_{ij})\), \(-\infty < i,j < \infty\), is called a Schur multiplier, if, for any \(T = (t_{ij}) \in B(\mathcal{H})\), the matrix \(M \circ T = (m_{ij}t_{ij})\) belongs to \(B(\mathcal{H})\). Then \(T \rightarrow M \circ T\) is a bounded map of \(B(\mathcal{H})\) into itself; it will also be denoted by \(M\) and its norm by \(|M|_{B(\mathcal{H})}\).
Let $H = \sum_{i=-\infty}^{\infty} \oplus H_i$ be an orthogonal sum of Hilbert spaces $H_i$. Every operator $A$ in $B(H)$ has a block-matrix representation $A = (A_{ij})$, $-\infty < i, j < \infty$, where $A_{ij}$ are bounded operators from $H_j$ into $H_i$.

**Lemma 4.5.** Let $M = (m_{ij})$ be a Schur multiplier on $\mathcal{H}$. It defines a bounded operator $\mathcal{M}$ on $B(H)$ by the formula

$$\mathcal{M} \times A = (m_{ij} A_{ij}), \quad \text{where} \quad A = (A_{ij}) \in B(H),$$

and $|\mathcal{M}|_{B(H)} = |M|_{B(\mathcal{H})}$.

**Proof.** Let $G = \{g_j\}_{j=-\infty}^{\infty}$ and $F = \{f_j\}_{j=-\infty}^{\infty}$ be sequences of elements such that $g_j, f_j \in H_j$ and $\|g_j\| = \|f_j\| = 1$. For $A = (A_{ij}) \in B(H)$, let $T^{G,F}(A) = (a_{ij}^{GF}), -\infty < i, j < \infty$, be the matrix such that

$$a_{ij}^{GF} = (A_{ij}g_j, f_i) \in \mathbb{C}.$$  

For $\alpha = \sum_{j=-\infty}^{\infty} \oplus \alpha_j e_j \in \mathcal{H}$ and $\beta = \sum_{j=-\infty}^{\infty} \oplus \beta_j e_j \in \mathcal{H}$, set

$$x_\alpha^G = \sum_{j=-\infty}^{\infty} \oplus \alpha_j g_j \quad \text{and} \quad y_\beta^F = \sum_{j=-\infty}^{\infty} \oplus \beta_j f_j.$$  

Then $x_\alpha^G, y_\beta^F \in H, \|x_\alpha^G\| = \|\alpha\|, \|y_\beta^F\| = \|\beta\|$ and

$$(Ax_\alpha^G, y_\beta^F) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j \beta_i (A_{ij} g_j, f_i) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \alpha_j \beta_i a_{ij}^{GF} = (T^{G,F}(A) \alpha, \beta).$$

Therefore $T^{G,F}(A) \in B(\mathcal{H})$ and

$$\|A\| = \sup_{\alpha, \beta, G, F} \left( \frac{\|Ax_\alpha^G, y_\beta^F\|}{\|x_\alpha^G\| \|y_\beta^F\|} \right) = \sup_{G, F} \left( \sup_{\alpha, \beta} \frac{|(T^{G,F}(A) \alpha, \beta)|}{\|\alpha\| \|\beta\|} \right) = \sup_{G, F} \|T^{G,F}(A)\|.$$  

It follows from (4.13) that $T^{G,F}(\mathcal{M} \times A) = M \circ T^{G,F}(A)$. Since $M$ is a Schur multiplier, $M \circ T^{G,F}(A) \in B(\mathcal{H})$ and, by (4.14),

$$\|\mathcal{M} \times A\| = \sup_{G, F} \|T^{G,F}(\mathcal{M} \times A)\| = \sup_{G, F} \|M \circ T^{G,F}(A)\| \leq \sup_{G, F} \|M|_{B(\mathcal{H})}\| T^{G,F}(A)\| = |M|_{B(\mathcal{H})} \sup_{G, F} \|T^{G,F}(A)\| = |M|_{B(\mathcal{H})} \|A\|. $$
Hence $|\mathcal{M}|_{B(H)} \leq |\mathcal{M}|_{B(\mathbb{H})}$. On the other hand, it is easy to see that $|\mathcal{M}|_{B(\mathbb{S})} \leq |\mathcal{M}|_{B(H)}$. Thus $|\mathcal{M}|_{B(H)} = |\mathcal{M}|_{B(\mathbb{S})}$.

Let $S$ and $T$ be selfadjoint operators on $H$ and assume that $\text{Sp}(S) \subseteq \mathbb{Z}$, $\text{Sp}(T) \subseteq \mathbb{Z}$, and that

$$H = \sum_{i=-\infty}^{\infty} \oplus H_i \text{ where } S|_{H_i} = s_i 1_{H_i}, \ T|_{H_i} = t_i 1_{H_i},$$

$s_i \neq s_j$ and $t_i \neq t_j$ if $i \neq j$.

Set

$$M = (m_{ij}) \text{ where } m_{ij} = \frac{s_i - s_j}{t_i - t_j}, \text{ for } i \neq j, \text{ and } m_{ii} = 0, \text{ and}$$

$$N = (n_{ij}) \text{ where } n_{ij} = \frac{t_i - t_j}{s_i - s_j}, \text{ for } i \neq j, \text{ and } n_{ii} = 0.$$

**Theorem 4.6.** $\mathcal{F}_S = \mathcal{F}_T$ if and only if $M$ and $N$ are Schur multipliers.

**Proof.** In every $H_i$ we choose a non-decreasing sequence of finite-dimensional projections $\{Q^p_i\}_{p=1}^\infty$ which converge to $1_{H_i}$ in the strong operator topology as $p \to \infty$. Set

$$Q_p = \sum_{i=-p}^{p} \oplus Q^p_i.$$

The finite-dimensional projections $Q_p$ commute with $S$ and $T$, belong to $\mathcal{F}_S \cap \mathcal{F}_T$ and converge to $1_H$ in the strong operator topology. Therefore $\|Q_p\|_S = \|Q_p\|_T = \|Q_p\| = 1$.

For any $A = (A_{ij}) \in \mathcal{A}_S \cap \mathcal{A}_T$,

$$A_S = SA - AS = (A^S_{ij}) \text{ and } A_T = TA - AT = (A^T_{ij}),$$

where $A^S_{ij} = (s_i - s_j)A_{ij}$ and $A^T_{ij} = (t_i - t_j)A_{ij}$. Set $B = A_T$. Then $A_S = \mathcal{M} \times B$.

(4.15) $\|A\|_S = \|A\| + \|A_S\| = \|A\| + \|\mathcal{M} \times B\|$ and

$$\|A\|_T = \|A\| + \|A_T\| = \|A\| + \|B\|.$$

We assume now that $M$ and $N$ are Schur multipliers and show that $\mathcal{F}_S = \mathcal{F}_T$. By Lemma 4.5 and (4.15),

(4.16) $\|A\|_S \leq \|A\| + \|M\|_B \|$ 

$$\leq \|A\| + \|M\| (\|A\|_T - \|A\|) \leq (\|M\| + 1)\|A\|_T.$$

Similarly,

(4.17) $\|A\|_T \leq (\|N\| + 1)\|A\|_S$.

Let $A \in \mathcal{F}_S$. Then $Q_pA \in \mathcal{F}_S$ and, since $Q_p$ commute with $S$,

$$(Q_pA)_S = \text{Closure} (S Q_p A - Q_p A S) = \text{Closure} Q_p(S A - AS) = Q_pA_S.$$

Since $A$ and $A_S$ are compact and since $Q_p$ converge to $1_H$ in the strong operator topology,

$$\|A - Q_pA\| \to 0 \text{ and } \|A_S - (Q_pA)_S\| = \|A_S - Q_pA_S\| \to 0, \text{ as } p \to \infty.$$
Hence \( \| A - Q_p A \|_S \to 0 \), so that \( \{ Q_p \} \) is a bounded approximate identity in \( \mathcal{F}_S \). Similarly, it is a bounded approximate identity in \( \mathcal{F}_T \).

Let \( A \in \mathcal{F}_S \). For any \( p \), \( Q_p T = Q_p T Q_p = T Q_p \) is a finite rank operator. Hence

\[
(Q_p A Q_p)_T = T(Q_p A Q_p) - (Q_p A Q_p)T = (T Q_p)A Q_p - Q_p A (Q_p T)
\]

is a finite rank operator. Therefore \( Q_p A Q_p \in \mathcal{F}_S \cap \mathcal{F}_T \) and, by (4.17),

\[
\| Q_{p+k} A Q_{p+k} - Q_p A Q_p \|_T \leq (|N| + 1)\| Q_{p+k} A Q_{p+k} - Q_p A Q_p \|_S.
\]

Since \( \{ Q_p \} \) is a bounded approximate identity in \( \mathcal{F}_S \), the operators \( Q_p A Q_p \) converge to \( A \) with respect to \( \| \cdot \|_S \). From the above inequality it follows that \( \{ Q_p A Q_p \} \) is a fundamental sequence with respect to \( \| \cdot \|_T \). Hence there is \( A_1 \in \mathcal{F}_T \) such that \( \| A_1 - Q_p A Q_p \|_T \to 0 \), as \( p \to \infty \). Since \( \| A - Q_p A Q_p \|_S \to 0 \) and \( \| A_1 - Q_p A Q_p \| \leq \| A_1 - Q_p A Q_p \| \to 0 \), as \( p \to \infty \), we obtain that \( A = A_1 \), so \( \mathcal{F}_S \subseteq \mathcal{F}_T \). Similarly, \( \mathcal{F}_T \subseteq \mathcal{F}_S \). Thus we conclude that \( \mathcal{F}_S = \mathcal{F}_T \).

Suppose now that \( \mathcal{F}_S = \mathcal{F}_T \). Choose elements \( e_i \in H_i \) such that \( \| e_i \| = 1 \) and let \( \mathfrak{H} \) be the subspace of \( H \) generated by all \( e_i \), \( -\infty < i < \infty \). Then \( \mathfrak{H} \) is invariant for \( S \) and \( T \), \( Se_i = s_i e_i \) and \( Te_i = t_i e_i \). By \( S_{\mathfrak{H}} \) and \( T_{\mathfrak{H}} \) we denote the restrictions of \( S \) and \( T \) to \( \mathfrak{H} \). Since \( \mathcal{F}_S = \mathcal{F}_T \),

\[
\mathcal{F}_{S_{\mathfrak{H}}} = \mathcal{F}_{T_{\mathfrak{H}}}.
\]

We shall show now that \( M \) and \( N \) are Schur multipliers on \( \mathfrak{H} \).

The function \( f(t) = i(\pi - t) \) on \( [0, 2\pi] \) has Fourier coefficients \( c_0 = 0 \) and \( c_n = \frac{1}{n} \) for \( n = \pm 1, \pm 2, \ldots \). Let \( \mathcal{H} \) be a Hilbert space with an orthonormal basis \( \{ h_k \}_{k=\infty}^{\infty} \) and \( R = (r_{kl}) \), \( -\infty < k, l < \infty \), be a Toeplitz matrix such that \( r_{kk} = 0 \) and \( r_{kl} = c_{k-l} = \frac{1}{1-t_l}, \ k \neq l \). Then \( R \in B(\mathcal{H}) \) and it follows from Theorem 8.1 [1] that \( R \) is a Schur multiplier and \( |R| = \sup |f(t)| = \pi \).

Identifying \( e_i \in \mathfrak{H} \) with \( h_i \in \mathcal{H} \), we can consider \( \mathfrak{H} \) as a subspace of \( \mathcal{H} \). For \( \bar{B} = (b_{km}) \in B(\mathfrak{H}) \), where \( b_{km} = (Be_m, e_k) \), let \( \bar{B} = (\bar{b}_{ij}) \in B(\mathcal{H}) \) be such that \( \bar{B}|_{\mathfrak{H}} = B \) and \( \bar{B}|_{\mathfrak{H}^\perp} = 0 \). Then \( \| \bar{B} \| = \| B \| \),

\[
\bar{b}_{tk,t_m} = (\bar{B}h_{t_m}, h_{t_k}) = (Be_m, e_k) = b_{km}, \quad \text{and}
\]

\[
\bar{b}_{ij} = (\bar{B}h_j, h_i) = 0 \quad \text{if either} \quad i \neq t_k \text{ or } j \neq t_m.
\]

Since \( R \) is a Schur multiplier, the operator \( \bar{C} = (\bar{c}_{ij}) = R \circ \bar{B} \) belongs to \( B(\mathcal{H}) \), where

\[
\bar{c}_{tk,t_m} = r_{tk,t_m} \bar{b}_{tk,t_m} = (t_k - t_m)^{-1} b_{km}, \quad \text{if} \quad k \neq m, \quad \text{and}
\]

\[
\bar{c}_{ij} = 0 \quad \text{if either} \quad i \neq t_k \text{ or } j \neq t_m \quad \text{or} \quad i = j = t_k.
\]

Setting \( C = \bar{C}|_{\mathfrak{H}} \), we obtain that \( C = (c_{km}) \in B(\mathfrak{H}) \), where

\[
c_{km} = \bar{c}_{tk,t_m} = (t_k - t_m)^{-1} b_{km}, \quad \text{if} \quad k \neq m, \quad \text{and} \quad c_{kk} = 0,
\]
that \( \|C\| = \|B\|\) and that \( C = W \circ B \), where \( W = (w_{km}) \) is a matrix such that

\[
  w_{km} = (t_k - t_m)^{-1}, \quad k \neq m, \quad \text{and} \quad w_{kk} = 0.
\]

From this it follows that \( W \) is a Schur multiplier on \( \mathcal{F} \) and

\[
  \|W \circ B\| = \|C\| = \|\tilde{C}\| = \|R \circ \tilde{B}\| \leq |R| \|\tilde{B}\| = |R| \|B\|.
\]

Thus \( |W| \leq |R| = \pi \).

Let \( P_n \) be the orthoprojections in \( \mathcal{F} \) on the subspaces \( \sum_{j=-n}^n \mathbb{C} e_j \).
Then \( P_n \) are finite rank operators commuting with operators \( S_\mathcal{F} \) and \( T_\mathcal{F} \) and \( P_n S_\mathcal{F} \subseteq D(S_\mathcal{F}) \). Hence \( P_n \in \mathcal{F}_{S_\mathcal{F}} \). For every \( B \in \mathcal{B}(\mathcal{F}) \), \( P_n B P_n \) are finite rank operators preserving \( D(S_\mathcal{F}) \) and their adjoints \( P_n B^* P_n \) also preserve \( D(S_\mathcal{F}) \). Therefore

\[
  (4.18) \quad P_n B P_n \in \mathcal{F}_{S_\mathcal{F}}.
\]

Any \( B = (b_{km}) \in \mathcal{B}(\mathcal{F}) \) can be represented in the form \( B = B_d + B_0 \), where \( B_d \) is the diagonal operator such that \( (B_d) = b_{kk} \). Then

\[
  (4.19) \quad \|B_d\| \leq \|B\| \quad \text{and} \quad \|B_0\| = \|B - B_d\| \leq 2\|B\|.
\]

We have that

\[
  (4.20) \quad M \circ (P_n B P_n) = P_n (M \circ B) P_n.
\]

Since \( m_{kk} = 0 \) in the matrix \( M = (m_{km}) \),

\[
  (4.21) \quad M \circ (P_n B P_n) = M \circ (P_n B_0 P_n).
\]

Set \( A = W \circ B \). Since \( W \) is a Schur multiplier, \( A \in \mathcal{B}(\mathcal{F}) \) and, by (4.18), \( P_n A P_n \in \mathcal{F}_{S_\mathcal{F}} \). It is easy to check that

\[
  (4.22) \quad P_n B_0 P_n = T_\mathcal{F}(P_n A P_n) - (P_n A P_n) T_\mathcal{F} = (P_n A P_n) T_\mathcal{F}, \quad \text{and} \quad M \circ (P_n B_0 P_n) = S_\mathcal{F}(P_n A P_n) - (P_n A P_n) S_\mathcal{F} = (P_n A P_n) S_\mathcal{F}.
\]

Since \( \mathcal{F}_{S_\mathcal{F}} = \mathcal{F}_{T_\mathcal{F}} \), it follows from Lemma 4.1(i) that the norms \( \| \cdot \|_{S_\mathcal{F}} \) and \( \| \cdot \|_{T_\mathcal{F}} \) are equivalent. Therefore there exists \( D > 0 \) such that

\[
  \|P_n A P_n\|_{S_\mathcal{F}} \leq D \|P_n A P_n\|_{T_\mathcal{F}}.
\]

Hence we obtain from (4.19), (4.21) and (4.22) that

\[
  \|M \circ (P_n B P_n)\| = \|M \circ (P_n B_0 P_n)\| = \|(P_n A P_n) S_\mathcal{F}\|
\]

\[
  \leq \|P_n A P_n\|_{S_\mathcal{F}} \leq D \|P_n A P_n\|_{T_\mathcal{F}}
\]

\[
  = D (\|P_n A P_n\| + \|(P_n A P_n) T_\mathcal{F}\|)
\]

\[
  \leq D (\|A\| + \|P_n B_0 P_n\|) \leq D (\|A\| + \|B_0\|)
\]

\[
  = D (\|W \circ B\| + \|B_0\|) \leq D (|R| \|B\| + 2\|B\|) = \rho.
\]

Thus all operators \( M \circ (P_n B P_n) \), \( 1 \leq n < \infty \), lie in the ball \( B_\rho \) of \( B(\mathcal{F}) \) of radius \( \rho \). Compactness of \( B_\rho \) in the weak operator topology implies that the
sequence \( \{ M \circ (P_n BP_n) \}_{n=1}^{\infty} \) has a cluster point \( K \in B(\mathcal{S}) \). Therefore there is a subsequence \( \{ M \circ (P_{n_j} BP_{n_j}) \} \) such that for all \( e_k \) and \( e_m \),

\[
(K e_k, e_m) = \lim_{j \to \infty} (M \circ (P_{n_j} BP_{n_j}) e_k, e_m).
\]

If \( n_j \geq \max(|k|, |m|) \) then \( P_{n_j} e_k = e_k \) and \( P_{n_j} e_m = e_m \) and, by (4.20),

\[
(M \circ (P_{n_j} BP_{n_j}) e_k, e_m) = (P_{n_j} (M \circ B) P_{n_j} e_k, e_m) = (M \circ B e_k, e_m).
\]

Hence \( (K e_k, e_m) = ((M \circ B) e_k, e_m) \), \( -\infty < k, m < \infty \). Thus \( K = M \circ B \), so \( M \) is a Schur multiplier. Similarly, we obtain that \( N \) is also a Schur multiplier. 

\[\square\]

**Example 4.7.** Let

\[ s_i = i \quad \text{and} \quad t_i = (-1)^i \]

in Theorem 4.6. If \( \mathcal{F}_S = \mathcal{F}_T \) then, by Theorem 4.6, \( M \) is a Schur multiplier and we have that \( |m_{ij}| \leq |M| \) for all \( i \) and \( j \). Let \( i = 2k \) and \( j = -2k + 1 \). Then \( s_i = t_i = 2k \) and \( s_j = -t_j = -2k + 1 \). Hence

\[
m_{ij} = \frac{s_i - s_j}{t_i - t_j} = 4k - 1 \to \infty, \quad \text{as} \quad k \to \infty.
\]

This shows that \( M \) is not a Schur multiplier and, therefore, \( \mathcal{F}_S \neq \mathcal{F}_T \).

Making use of Theorem 4.6, we obtain the following result of a more general character.

**Theorem 4.8.** Let \( S \) and \( T \) be selfadjoint operators on \( H \) and \( \mathcal{H} \) respectively. If there exists a bijection \( \varphi \) of \( \mathbb{Z} \) onto \( \mathbb{Z} \) such that

\[
\dim(\mathcal{H}_T(\varphi(i))) = \dim(H_S(i)), \quad \text{for all} \quad i \in \mathbb{Z},
\]

(see (2.2) for definition of \( \mathcal{H}_T(i) \) and \( H_S(i) \)) and if

\[
M = (m_{ij}) \quad \text{where} \quad m_{ij} = \frac{\varphi(i) - \varphi(j)}{i - j}, \quad \text{for} \; i \neq j, \; \text{and} \; m_{ij} = 0, \; \text{and}
\]

\[
N = (n_{ij}) \quad \text{where} \quad n_{ij} = \frac{i - j}{\varphi(i) - \varphi(j)}, \quad \text{for} \; i \neq j, \; \text{and} \; n_{ij} = 0
\]

are Schur multipliers then the algebras \( \mathcal{F}_S \) and \( \mathcal{F}_T \) are *-isomorphic.

**Proof.** Consider the operators \([S]\) and \([T]\) (see (2.1)) and the corresponding decompositions

\[
H = \sum_{i \in \mathbb{Z}} \oplus H_S(i) \quad \text{and} \quad \mathcal{H} = \sum_{i \in \mathbb{Z}} \oplus \mathcal{H}_T(i)
\]

where \( H_S(i) = P_S(i) H \) and \( \mathcal{H}_T(i) = P_T(i) \mathcal{H} \) (see (2.3)). The operators \( S - [S] \) and \( T - [T] \) are bounded, so \( \mathcal{F}_S = \mathcal{F}[S] \) and \( \mathcal{F}_T = \mathcal{F}[T] \).
Consider the selfadjoint operator $R$ on $H$ such that all subspaces $H_S(i)$ are invariant for $R$ and $R|_{H_S(i)} = \varphi(i)1_{H_S(i)}$. Since $M$ and $N$ are Schur multipliers, it follows from Theorem 4.6 that $\mathcal{F}_R = \mathcal{F}_S$.

On the other hand, since $\dim(\mathcal{H}_T(\varphi(i))) = \dim(H_S(i))$, for all $i \in \mathbb{Z}$, there exists an isometry operator $U$ from $H$ onto $\mathcal{H}$ which maps $H_S(i)$ onto $\mathcal{H}_T(\varphi(i))$. Then $U^*TUU = R$. By Lemma 4.1, the algebras $\mathcal{F}_R$ and $\mathcal{F}_T$ are $^*$-isomorphic. Hence the algebras $\mathcal{F}_S$ and $\mathcal{F}_T$ are $^*$-isomorphic. $\square$

References


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