EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF $E_7$

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Let $\mathfrak{g}$ be a noncompact real form of the simple complex Lie algebra $\mathfrak{g}^c$ of type $E_7$. Up to isomorphism, there are exactly three such algebras: EV, EVI, and EVII in Cartan notations. For each of these algebras we obtain a list of representatives of the adjoint orbits of standard triples $(E, H, F)$, i.e., triples $\{E, H, F\} \subset \mathfrak{g}$ spanning a subalgebra isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and such that $[H, E] = 2E$, $[H, F] = -2F$, and $[F, E] = H$. These representative standard triples are chosen to be Cayley triples with respect to a fixed Cartan decomposition of $\mathfrak{g}$.

1. Introduction.

The nilpotent adjoint orbits in noncompact real forms of exceptional simple Lie algebras have been classified in our papers [4, 5]. This classification is based on the so-called Kostant-Sekiguchi bijection for which we refer to [2] and [7]. See Section 5 for a more detailed discussion of this bijection. In our first two papers mentioned above, we did not compute the representatives of the nilpotent orbits.

This paper is a sequel to [7] and we shall freely use the notations introduced there. In that paper we have compiled a list of representatives of $G$-orbits of standard triples $(E, H, F)$ in $\mathfrak{g}$ where $\mathfrak{g}$ is a noncompact real form of $\mathfrak{g}^c$, and $\mathfrak{g}^c$ is a simple complex Lie algebra of type $G_2, F_4$, or $E_6$. In fact these representative triples were chosen to be real Cayley triples with respect to a fixed Cartan decomposition of $\mathfrak{g}$. In the present paper we accomplish the same objective for noncompact real forms $\mathfrak{g}$ of $\mathfrak{g}^c$ when the latter is of the type $E_7$. Up to $G^c$-conjugacy, there are exactly 3 such real forms. They are denoted by EV, EVI, EVII or $E_7(7), E_7(-5), E_7(-25)$, respectively. The nilpositive elements $E$ of these representative Cayley triples are representatives of the nonzero nilpotent adjoint orbits.
By using a result from our recent note [6] it is easy to determine which complex nilpotent adjoint orbits possess real points. It is more delicate to determine the number of real nilpotent orbits that are contained in a given complex orbit.

We record here that D.R. King [8, p. 254] has detected an error in [5, Table VIII]. Namely the last entry for orbit 5 of that table should be \(\mathfrak{sl}(3, \mathbb{C})\) instead of \(2\mathfrak{su}(2, 1)\).

Several misprints in our paper [4] have been mentioned in [7]. There is one more: Namely on p. 515, in Table XII, the labels “020220 0” of the orbit No. 31 (given in the second column) should be replaced by “020220 2”. Consequently these labels should also be corrected in [2, p. 158].

2. The root system of \(E_7\).

We denote by \(\mathfrak{h}\) a maximally split Cartan subalgebra of \(\mathfrak{g}\) which is stable under the Cartan involution \(\theta\), and by \(\mathfrak{h}^c\) its complexification. The number of positive roots of \(\mathfrak{g}^c = E_7\) is \(N = 63\). The positive roots are enumerated as \(\alpha_1, \alpha_2, \ldots, \alpha_N\) with \(\Pi = \{\alpha_1, \ldots, \alpha_7\}\) as a base. The enumeration is chosen so that the heights increase, i.e., \(\text{ht}(\alpha_i) \leq \text{ht}(\alpha_j)\) for \(i < j\). The negative root \(\alpha_i\) is written also as \(\alpha_{-i}\). The extended Dynkin diagram of \(E_7\) is given in Fig. 1.

\[
\begin{align*}
\alpha_{-63} & \quad \alpha_1 & \quad \alpha_3 & \quad \alpha_4 & \quad \alpha_5 & \quad \alpha_6 & \quad \alpha_7 \\
\alpha_2
\end{align*}
\]

\textbf{Figure 1.}

Since \(E_7\) is simply laced, if \(\alpha_i = k_1\alpha_1 + \cdots + k_7\alpha_7\) is a positive root, then the corresponding coroot \(H_i\) is given by \(H_i = k_1H_1 + \cdots + k_7H_7\). Our enumeration of positive roots \(\alpha_i\) (and coroots \(H_i\)) is given in Table 1.
Table 1.
Positive roots of $E_7$.

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3. The structure constants of $E_7$.

As in [7], we use an algorithm of J. Kurtzke [9] to fix the choice of a Chevalley basis of $\mathfrak{g}_c$:

$$H_i, \quad 1 \leq i \leq 7; \quad X_i, X_{-i}, \quad 1 \leq i \leq 63.$$  

If $\alpha_i + \alpha_j = \alpha_k$, then

$$[X_i, X_j] = N(i, j)X_k.$$  

As all roots of $E_7$ have the same length, the nonzero structure constants are $\pm 1$, i.e., $N(i, j) = \varepsilon(i, j)$.

We specify that $N(1, 3) = +1$. Then, by Kurtzke's algorithm,

$$N(3, 4) = N(5, 6) = -1, \quad N(4, 2) = N(4, 5) = N(6, 7) = +1,$$

and $N(i, j) = +1$ whenever $\alpha_i + \alpha_j$ is a root and $1 \leq i \leq 7 < j \leq 63$. Furthermore, all other $N(i, j)$'s are uniquely determined.

For the convenience of the reader, we list in the Appendix, the nonzero structure constants $N(i, j)$ for all $i > 0$. For $i < 0$ one can use the formula

$$N(-i, -j) = N(i, j).$$
4. The conjugation $\sigma$.

We recall that $\sigma$ denotes the conjugation of $g^c$ with respect to $g$, and that $h^c$ is $\sigma$-invariant. The action of $\sigma$ on $h^c$ induces naturally an action on the dual space of $h^c$ which preserves $\Phi$. As $\sigma$ acts on $\Phi$ as an automorphism, it suffices to know the action of $\sigma$ on $\Pi$. If $\sigma(\alpha_i) = \alpha_j$ we also write $\sigma(i) = j$.

Note that $\sigma(i) = j$ implies that $\sigma(-i) = -j$.

One can further assume that the Chevalley basis has been chosen so that, in addition to the properties mentioned earlier, the action of $\sigma$ on the $X_i$'s is given by

$$\sigma(X_i) = \xi_i X_{\sigma(i)}$$

where $\xi_i = \pm 1$. We recall that $\xi_i = 1$ whenever $\alpha_i \in \Phi_0$, and $\xi_{-i} = \xi_i$ for all $i$. For all three noncompact real forms $g$ of $g^c$ we may choose $\xi_i = 1$ for $1 \leq i \leq 7$. With this information, one can compute the coefficients $\xi_i$ for arbitrary $i$.

For reader’s convenience we list the vectors $\sigma(X_i)$, $i > 0$, in Table 2 for EVI and Table 3 for EVII. When $g$ is of type EV, i.e., $g$ is the split real form of $g^c$, then the action of $\sigma$ on $\Phi$ is trivial. Hence in that case we have $\sigma(X_i) = X_i$ for all $i$.

**Table 2.**

EVI = $E_7(-5)$ : Action of $\sigma$ on the $X_i$'s.

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Explicit Cayley Triples in Real Forms of $E_7$

Table 3.

$\text{EVII} = E_7(-25)$ : Action of $\sigma$ on the $X_i$'s.

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5. The Kostant-Sekiguchi bijection.

We refer the reader to our previous paper [7] for definition of $G, K, G^c, K^c$, standard triples, normal triples, real and complex Cayley triples, etc.

In the following diagram we exhibit several important sets on which some of the above groups act and some natural maps between these sets.

Set: Group

- Nonzero nilpotent elements in $\mathfrak{g}$: $G$
  - $\uparrow \alpha$
    - Standard triples in $\mathfrak{g}$: $G$
      - $\uparrow \beta$
        - Real Cayley triples in $\mathfrak{g}$: $K$
          - $\downarrow \gamma$
            - Complex Cayley triples in $\mathfrak{g}^c$: $K$
              - $\downarrow \delta$
                - Normal triples in $\mathfrak{g}^c$: $K^c$
                  - $\downarrow \varepsilon$
                    - Nonzero nilpotent elements in $\mathfrak{p}^c$: $K^c$
The map \( \alpha \) (resp. \( \varepsilon \)) sends the standard (resp. normal) triple \((E, H, F)\) to its nilpositive part \(E\). The maps \( \beta \) and \( \delta \) are the inclusion maps. The map \( \gamma \) is the Cayley transformation. Clearly \( \gamma \) is bijective and \( K \)-equivariant. The maps \( \beta \) and \( \delta \) are also \( K \)-equivariant. The map \( \alpha \) is \( G \)-equivariant while \( \varepsilon \) is \( K^c \)-equivariant. We shall prove below that, on the level of orbits, each of these maps induces a bijection.

Since \( \alpha \) is \( G \)-equivariant, it induces a map \( \alpha^* \) from the set of \( G \)-orbits of standard triples in \( g \) to the set of \( G \)-orbits of nonzero nilpotent elements of \( g \). One defines similarly the maps \( \beta^*, \gamma^*, \delta^* \), and \( \varepsilon^* \).

**Proposition 1.** Each of the maps \( \alpha^*, \ldots, \varepsilon^* \) is a bijection.

**Proof.** All the references in this proof are to the book [2]. Since \( \gamma \) is bijective, so is \( \gamma^* \). The map \( \alpha^* \) is surjective by Theorem 9.2.1, and injective by Theorem 9.2.3. The map \( \varepsilon^* \) is surjective by Theorem 9.4.2, and injective by Theorem 9.4.3. The map \( \beta^* \) is surjective by Theorem 9.4.1. The fact that \( \delta^* \) is surjective is shown in the proof of Theorem 9.5.1.

We shall prove now that \( \beta^* \) is injective. Thus, if \((E, H, F)\) and \((E', H', F')\) are real Cayley triples which are \( G \)-conjugate, we have to show that they are also \( K \)-conjugate. As \( H, H' \in p \), Lemma 9.4.5 shows that \( H \) and \( H' \) are \( K \)-conjugate. Hence without any loss of generality we may assume that \( H' = H \). By our hypothesis, there exists \( g \in G \) such that \( g \cdot (E, H, F) = (E', H, F') \). In particular, \( g \in Z_G(H) \) and \( g \cdot (E + F) = E' + F' \). Since \( H \) is semisimple, the centralizer \( Z_G(H) \) is reductive. By applying the proof of Lemma 9.4.5 and observing that \( E + F, E' + F' \in \mathfrak{t} \), we conclude that there exists \( k \in Z_K(H) \) such that \( k \cdot (E + F) = E' + F' \). The formula \([H, E + F] = 2(E - F)\) now implies that \( k \cdot (E - F) = E' - F' \). It follows that \( k \cdot E = E' \) and \( k \cdot F = F' \), i.e., the real Cayley triples \((E, H, F)\) and \((E', H', F')\) are \( K \)-conjugate.

By Theorem 9.5.1 the composite map \( \varepsilon^* \circ \delta^* \circ \gamma^* \circ (\beta^*)^{-1} \circ (\alpha^*)^{-1} \) is a bijection. It follows that \( \delta^* \) must be also injective. \( \square \)

The composite map \( \alpha^* \circ \beta^* \circ (\gamma^*)^{-1} \circ (\delta^*)^{-1} \circ (\varepsilon^*)^{-1} \) is the Kostant–Sekiguchi bijection from the set of nonzero nilpotent \( K^c \)-orbits in \( p^c \) to the set of nonzero nilpotent \( G \)-orbits in \( g \). Explicitly, if \((E, H, F)\) is a real Cayley triple and \((E', H', F')\) its Cayley transform, then the orbit \( G \cdot E \) corresponds to the orbit \( K^c \cdot E' \).

Define a partial order in the set of nilpotent \( K^c \)-orbits in \( p^c \) by setting \( O_1 \geq O_2 \) if \( O_2 \) is contained in the closure of the orbit \( O_1 \). Define similarly the partial order in the set of nilpotent \( G \)-orbits in \( g \). It was shown very recently [1] that the Kostant–Sekiguchi bijection preserves these partial orders.
6. Two invariants.
Let \((E, H, F)\) be a real Cayley triple and \((E', H', F')\) its Cayley transform. In order to distinguish between various \(G\)-orbits in \(\mathfrak{g}\) which are contained in the same nonzero nilpotent \(G^c\)-orbit in \(\mathfrak{g}^c\), we use two invariants:
\[
\text{tr} := \text{trace} \left( \text{ad}(H')^2 \right)
\]
and
\[
\text{inv} := \dim Z_{\mathfrak{k}^c}(H').
\]
The second one was used in our previous paper [7], while the first one is easier to compute. It is evident from our tables that in some instances \(\text{tr}\) fails to distinguish two orbits and we have to use \(\text{inv}\). In a few instances \(\text{inv}\) fails, while \(\text{tr}\) succeeds to distinguish two orbits. Our method for computing the representative real Cayley triples \((E, H, F)\) in \(\mathfrak{g}\) is described in detail in [7]. In a relatively few cases the method fails and we had to do extensive computations to find the desired representatives. We shall describe the difficulties that arise on one such example in Section 9 (the most difficult case).

7. Pairs of orbits \(G \cdot E\) and \(-G \cdot E\).
If \(E \neq 0\) is a nilpotent element in \(\mathfrak{g}\), then there exists an automorphism of \(\mathfrak{g}\) which maps \(E\) to \(-E\) (see [3]). In general, this automorphism is not inner and so \(E\) and \(-E\) may belong to different \(G\)-orbits. One can decide whether or not \(G \cdot E = -G \cdot E\) by means of the following proposition.

**Proposition 2.** Let \((E, H, F)\) be a real Cayley triple and \((E', H', F')\) its Cayley transform. Then \(G \cdot E = -G \cdot E\) if and only if \(K^c \cdot H' = -K^c \cdot H'\).

**Proof.** The triple \((-E, H, -F)\) is another real Cayley triple, and its Cayley transform is \((-E', -H', -F')\). We have \(G \cdot E = -G \cdot E\) if and only if the real Cayley triples \((E, H, F)\) and \((-E, H, -F)\) are \(G\)-conjugate. By the properties of the Sekiguchi bijection, this is the case if and only if the corresponding complex Cayley triples, namely \((E', H', F')\) and \((-F', -H', -E')\) are \(K^c\)-conjugate. The latter condition is equivalent to the \(K^c\)-conjugacy of \(H'\) and \(-H'\).

The characteristics \(H'\) of the nonzero nilpotent \(K^c\)-orbits in \(\mathfrak{p}^c\) are known [4]. Since \(H' \in \mathfrak{h}^c\), \(H'\) and \(-H'\) are \(K^c\)-conjugate if and only if they belong to the same orbit of the Weyl group \(W(\mathfrak{t}^c, \mathfrak{g}^c)\). Hence it is easy to check whether or not \(G \cdot E = -G \cdot E\).

When \(\mathfrak{g}\) is of Cartan type EVI, then \(\text{Aut} (\mathfrak{g}) = G\), and so \(G \cdot E = -G \cdot E\) for all nonzero nilpotent elements \(E \in \mathfrak{g}\). In the other two cases, when \(\mathfrak{g}\) is of Cartan type EV or EVII, there exist nonzero nilpotent elements \(E \in \mathfrak{g}\).
such that $G \cdot E \neq -G \cdot E$. Such pairs of orbits are easily recognizable from Tables 4 and 6 because we give their representatives jointly as $\pm E$.

8. Tables of real Cayley triples.

We give here lists of representatives $(E, H, F)$ for $K$-orbits of real Cayley triples in $\mathfrak{g}$, a noncompact real form of the simple complex Lie algebra $\mathfrak{g}^c$ of type $E_7$.

We record only the elements $E$ and $H$ because $F$ can be easily computed by using

$$F = \theta(E) = \theta \sigma(E) = \sigma_u(E)$$

and $\sigma_u(X_i) = X_{-i}$ (for all $i$).

For the neutral element $H$, we list both the labels $\alpha_i(H)$ for $1 \leq i \leq 7$ and the coefficients $k_i$ in the linear combination $H = k_1H_1 + \cdots + k_7H_7$.

The nilpositive element $E$ is written explicitly as a linear combination of the root vectors $X_i, i > 0$.

Note that the element $H$ is always the characteristic of the nilpotent orbit $G^c \cdot E$. On the other hand there is no natural choice for the nilpositive element $E$. Our preference was to choose $E$ so that its support (i.e. the number of nonzero coefficients) is minimal even though this may have a drawback of introducing irrational coefficients.
Table 4. Cayley triples in EV = $E_7(7)$. 

<table>
<thead>
<tr>
<th>$\alpha_i(H)$</th>
<th>$k_i$</th>
<th>$E$</th>
<th>tr</th>
<th>inv</th>
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<td>1</td>
<td>1000000</td>
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<td>2</td>
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<td>64</td>
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<tr>
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<td>2, 3, 4, 6, 5, 4, 3</td>
<td>$\pm (X_7 + X_{49} + X_{63})$</td>
<td>96</td>
</tr>
<tr>
<td>5</td>
<td>0010000</td>
<td>3, 4, 6, 8, 6, 4, 2</td>
<td>$X_{37} + X_{55} + X_{61}$</td>
<td>96</td>
</tr>
<tr>
<td>6</td>
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<td>$X_1 + X_{37} + X_{55} + X_{61}$</td>
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<tr>
<td>7</td>
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</tr>
<tr>
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<td>3, 5, 6, 9, 7, 5, 3</td>
<td>$\pm (X_{30} + X_{47} + X_{53} + X_{59})$</td>
<td>128</td>
</tr>
<tr>
<td>9</td>
<td>1000010</td>
<td>4, 5, 7, 10, 8, 6, 3</td>
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<td>11</td>
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<td>4, 6, 8, 12, 9, 6, 3</td>
<td>$X_{28} + X_{38} + X_{46} + X_{47} + X_{48} + X_{49}$</td>
<td>192</td>
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<tr>
<td>13</td>
<td>14</td>
<td>0001000</td>
<td>4, 6, 8, 12, 9, 6, 3</td>
<td>$\pm (X_2 + X_{28} - X_{38} + X_{46} + X_{47} + X_{48} + X_{49})$</td>
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<td>17</td>
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<td>$\pm [X_{19} + X_{40} + 2X_{41} + \sqrt{3}(X_{21} + X_{33})]$</td>
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<td>25</td>
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<td>$\pm [X_{15} + X_{30} + 2X_{32} + X_{33} + \sqrt{3}(X_{29} - X_{31})]$</td>
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</table>
Table 4. (continued)

<table>
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<tr>
<th>$\alpha_i(H)$</th>
<th>$k_i$</th>
<th>$E$</th>
<th>tr</th>
<th>inv</th>
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<td>$a(M)$</td>
<td>$h_k$</td>
<td>$E$</td>
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<td>$\pm\sqrt{2}(x_3 + x_5) + \sqrt{3}(x_4 + x_6)$</td>
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<td>$\pm\sqrt{2}(x_3 - x_5) + \sqrt{3}(x_4 - x_6)$</td>
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</table>
Table 4. (continued)

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<tr>
<th>$\alpha_i(H)$</th>
<th>$k_i$</th>
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<th>$\text{inv}$</th>
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<td>$+2\sqrt{7}X_{12} + \sqrt{10}X_{22}$</td>
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<td>$+\sqrt{15}(X_7 + X_9)$]</td>
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<td>$+\sqrt{30}(X_9 + X_{18}) + \sqrt{42}X_3$]</td>
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<td>$+\frac{1}{\sqrt{5}}(\sqrt{33}X_5 - \sqrt{77}X_2$</td>
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<td></td>
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<td></td>
<td>$-6\sqrt{3}X_9 - 6\sqrt{7}X_{11})$]</td>
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<td>$\pm \sqrt{34}X_1 + 7X_2 + \sqrt{66}X_3$</td>
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<td>$+4\sqrt{6}X_4 + \sqrt{3}(5X_5 + 3X_7$</td>
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<td>$+2\sqrt{13}X_6$]</td>
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Table 5.

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<td>( X_{49} - X_{63} )</td>
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<td>33</td>
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<td>( X_{37} - X_{55} + X_{61} )</td>
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<td>37</td>
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<td>19</td>
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<td>( X_{28} + X_{38} + X_{46} + X_{47} - X_{48} + X_{49} )</td>
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<td>19</td>
</tr>
<tr>
<td>11 000000</td>
<td>4, 6, 8, 12, 9, 6, 3</td>
<td>( -X_{28} + X_{38} + X_{46} + X_{47} - X_{48} + X_{49} )</td>
<td>208</td>
<td>15</td>
</tr>
<tr>
<td>12 2000010</td>
<td>6, 7, 10, 14, 11, 8, 4</td>
<td>( \sqrt{5}(X_1 - X_{37}) + 2X_{49} )</td>
<td>272</td>
<td>21</td>
</tr>
<tr>
<td>13 2000010</td>
<td>6, 7, 10, 14, 11, 8, 4</td>
<td>( \sqrt{5}(X_1 + X_{37}) + 2X_{49} )</td>
<td>368</td>
<td>21</td>
</tr>
<tr>
<td>14 0000020</td>
<td>4, 6, 8, 12, 10, 8, 4</td>
<td>( \frac{1}{\sqrt{2}}[X_6 + X_{12} + X_{19}] )</td>
<td>320</td>
<td>49</td>
</tr>
<tr>
<td>15 0000020</td>
<td>4, 6, 8, 12, 10, 8, 4</td>
<td>( \sqrt{2}(X_6 + X_{12} + X_{50} + X_{56}) )</td>
<td>320</td>
<td>25</td>
</tr>
<tr>
<td>16 0010010</td>
<td>5, 7, 10, 14, 11, 8, 4</td>
<td>( \sqrt{2}(X_{24} + X_{36} + X_{38} - X_{48}) )</td>
<td>352</td>
<td>13</td>
</tr>
<tr>
<td>17 1001000</td>
<td>6, 8, 11, 16, 12, 8, 4</td>
<td>( X_{28} + 2X_{49} + \sqrt{3}(X_{14} - X_{26}) )</td>
<td>432</td>
<td>21</td>
</tr>
<tr>
<td>18 1001000</td>
<td>6, 8, 11, 16, 12, 8, 4</td>
<td>( X_{28} - 2X_{49} + \sqrt{3}(X_{14} - X_{26}) )</td>
<td>368</td>
<td>13</td>
</tr>
<tr>
<td>19 0020000</td>
<td>6, 8, 12, 16, 12, 8, 4</td>
<td>( -X_3 + X_{28} + 2X_{49} )</td>
<td>480</td>
<td>39</td>
</tr>
<tr>
<td>20 0020000</td>
<td>6, 8, 12, 16, 12, 8, 4</td>
<td>( \sqrt{3}(X_{14} - X_{26}) )</td>
<td>352</td>
<td>23</td>
</tr>
<tr>
<td>21 0020000</td>
<td>6, 8, 12, 16, 12, 8, 4</td>
<td>( X_3 + X_{28} + 2X_{49} )</td>
<td>448</td>
<td>21</td>
</tr>
<tr>
<td>22 2020000</td>
<td>10, 12, 18, 24, 18, 12, 6</td>
<td>( \sqrt{10}X_1 + \sqrt{6}(X_3 - X_{28} + X_{49}) )</td>
<td>608</td>
<td>37</td>
</tr>
<tr>
<td>23 2020000</td>
<td>10, 12, 18, 24, 18, 12, 6</td>
<td>( \sqrt{10}X_1 + \sqrt{6}(X_3 + X_{28} + X_{49}) )</td>
<td>992</td>
<td>21</td>
</tr>
<tr>
<td>24 0001010</td>
<td>6, 9, 12, 18, 14, 10, 5</td>
<td>( X_{29} - X_{31} + 2X_{37} + iX_{27} )</td>
<td>528</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( -X_{39} + \sqrt{3}(X_{18} + X_{30}) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 5.  
(continued)

<table>
<thead>
<tr>
<th>$\alpha_i(H)$</th>
<th>$k_i$</th>
<th>$E$</th>
<th>$\text{tr}$</th>
<th>$\text{inv}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 200020</td>
<td>8, 10, 14, 20, 16, 12, 6</td>
<td>$-\sqrt{3}(X_6 + X_{19} + X_{40} + X_{41}) + 2(X_1 + X_{37})$</td>
<td>672</td>
<td>33</td>
</tr>
<tr>
<td>26 200020</td>
<td>8, 10, 14, 20, 16, 12, 6</td>
<td>$-\sqrt{3}(X_6 + X_{19} + X_{40} + X_{41}) + 2(X_1 + X_{37})$</td>
<td>800</td>
<td>17</td>
</tr>
<tr>
<td>27 1001010</td>
<td>8, 11, 15, 22, 17, 12, 6</td>
<td>$\frac{1}{3}(X_{18} + X_{29} + X_{30} - X_{31}) + X_{28} + 2(X_{14} - X_{26})$</td>
<td>832</td>
<td>9</td>
</tr>
<tr>
<td>28 2001010</td>
<td>10, 13, 18, 26, 20, 14, 7</td>
<td>$\sqrt{2}X_1 + X_{18} + X_{30}$</td>
<td>1072</td>
<td>11</td>
</tr>
<tr>
<td>29 0002000</td>
<td>8, 12, 16, 24, 18, 12, 6</td>
<td>$X_4 + X_{15} + X_{16} + X_{17} + X_{18} + X_{29} + X_{30} - X_{31}$</td>
<td>960</td>
<td>27</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$+\sqrt{2}(X_9 + X_{10} + X_{11} - X_{22}) + 2i(X_{27} - X_{39})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 1001020</td>
<td>10, 14, 19, 28, 22, 16, 8</td>
<td>$3X_{28} + 2\sqrt{2}(X_6 + X_{19}) + \sqrt{5}(X_{14} - X_{26})$</td>
<td>1328</td>
<td>13</td>
</tr>
<tr>
<td>31 0020020</td>
<td>10, 14, 20, 28, 22, 16, 8</td>
<td>$X_3 + 2\sqrt{2}(X_6 + X_{19}) + 3X_{28} + \sqrt{5}(X_{14} - X_{26})$</td>
<td>1408</td>
<td>13</td>
</tr>
<tr>
<td>32 0020020</td>
<td>10, 14, 20, 28, 22, 16, 8</td>
<td>$-X_3 + 2\sqrt{2}(X_6 + X_{19}) + 3X_{28} + \sqrt{5}(X_{14} - X_{26})$</td>
<td>1312</td>
<td>23</td>
</tr>
<tr>
<td>33 2020020</td>
<td>14, 18, 26, 36, 28, 20, 10</td>
<td>$\sqrt{14}X_1 + \sqrt{2}(2X_4 + 3X_{28}) + \sqrt{10}(X_{14} + X_{19})$</td>
<td>2272</td>
<td>13</td>
</tr>
<tr>
<td>34 2020020</td>
<td>14, 18, 26, 36, 28, 20, 10</td>
<td>$\sqrt{14}X_1 + \sqrt{2}(3X_{28} - 2X_4) + \sqrt{10}(X_6 + X_{19})$</td>
<td>1888</td>
<td>21</td>
</tr>
<tr>
<td>35 002020</td>
<td>12, 18, 24, 36, 28, 20, 10</td>
<td>$\sqrt{6}(X_4 + X_{15} + X_{16}) + X_17 + X_{20} - iX_{21} + \sqrt{10}(X_6 + X_{19})$</td>
<td>2240</td>
<td>19</td>
</tr>
<tr>
<td>36 202020</td>
<td>16, 22, 30, 44, 34, 24, 12</td>
<td>$4X_1 + 2\sqrt{3}(X_6 + X_{19}) + \sqrt{7}(X_4 + X_{16}) + \sqrt{15}(X_{15} + X_{17})$</td>
<td>3232</td>
<td>15</td>
</tr>
<tr>
<td>37 202200</td>
<td>22, 30, 42, 60, 46, 32, 16</td>
<td>$4(X_6 + X_{19}) + \sqrt{22}X_1 + \sqrt{42}X_3 + \sqrt{30}(X_4 + X_{16})$</td>
<td>5728</td>
<td>13</td>
</tr>
</tbody>
</table>
Let $\mathfrak{g}$ be of type $E_{7(-5)}$ and $\mathcal{O}^c$ the complex nilpotent orbit of $\mathfrak{g}^c$ with characteristic 0000020, i.e., $H = 2H_6$. Then $\mathcal{O}^c \cap \mathfrak{g}$ is the union of two $G$-orbits, namely orbits No. 14 and 15 in Table 5.

The characteristic $H$ defines a gradation of $\mathfrak{g}^c$ (and $\mathfrak{g}$) such that

$$\mathfrak{g}^c = \bigoplus_{k=-2}^{2} \mathfrak{g}(2k)^c.$$  

The subspace $\mathfrak{g}(0)^c$ is a reductive subalgebra with 1-dimensional center and the derived subalgebra of type $A_1 + D_5$. The derived subgroup of $G(0)^c$ is isomorphic to

$$(\text{Spin}_{10} \times \text{SL}_2)/\mathbb{Z}_2.$$  

The dimensions of the spaces $\mathfrak{g}(2k)^c$ are as follows:

$$\dim \mathfrak{g}(0)^c = 49,$$
$$\dim \mathfrak{g}(-2)^c = \dim \mathfrak{g}(2)^c = 32,$$
$$\dim \mathfrak{g}(-4)^c = \dim \mathfrak{g}(4)^c = 10.$$  

---

### Table 6.

Cayley triples in $\text{EVII} = E_{7(-25)}$.

<table>
<thead>
<tr>
<th>$\alpha_i(H)$</th>
<th>$k_i$</th>
<th>$E$</th>
<th>$\text{tr}$</th>
<th>$\text{inv}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 2</td>
<td>2, 2, 3, 4, 3, 2, 1</td>
<td>$\pm X_{63}$</td>
<td>32</td>
<td>47</td>
</tr>
<tr>
<td>3, 4</td>
<td>2, 3, 4, 6, 5, 4, 2</td>
<td>$\pm (X_{49} - X_{63})$</td>
<td>32</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>2, 3, 4, 6, 5, 4, 2</td>
<td>$X_{49} + X_{63}$</td>
<td>96</td>
<td>31</td>
</tr>
<tr>
<td>6, 7</td>
<td>2, 3, 4, 6, 5, 4, 3</td>
<td>$\pm (X_7 - X_{49} - X_{63})$</td>
<td>0</td>
<td>79</td>
</tr>
<tr>
<td>8, 9</td>
<td>2, 3, 4, 6, 5, 4, 3</td>
<td>$\pm (X_7 + X_{49} + X_{63})$</td>
<td>128</td>
<td>47</td>
</tr>
<tr>
<td>10</td>
<td>4, 4, 6, 8, 6, 4, 2</td>
<td>$X_1 + X_{37} + X_{55} + X_{61}$</td>
<td>192</td>
<td>37</td>
</tr>
<tr>
<td>11, 12</td>
<td>4, 5, 7, 10, 8, 6</td>
<td>$\pm (X_{27} + X_{39} + X_{49} + X_{53} - X_{54})$</td>
<td>224</td>
<td>21</td>
</tr>
<tr>
<td>13, 14</td>
<td>6, 7, 10, 14</td>
<td>$\pm [2X_{49} + \sqrt{3}(X_1 + X_{37})]$</td>
<td>416</td>
<td>31</td>
</tr>
<tr>
<td>15</td>
<td>4, 6, 8, 12, 10, 8, 4</td>
<td>$\sqrt{2}(X_6 + X_{34} + X_{40} + X_{56})$</td>
<td>384</td>
<td>31</td>
</tr>
<tr>
<td>16, 19</td>
<td>6, 7, 10, 14, 11, 8, 5</td>
<td>$\pm [-X_7 + 2X_{49} + \sqrt{3}(X_1 + X_{37})]$</td>
<td>384</td>
<td>31</td>
</tr>
<tr>
<td>17, 18</td>
<td>6, 7, 10, 14</td>
<td>$\pm [X_7 + 2X_{49} + \sqrt{3}(X_1 + X_{37})]$</td>
<td>512</td>
<td>47</td>
</tr>
<tr>
<td>20</td>
<td>8, 10, 14, 20</td>
<td>$2(X_1 + X_{37})$</td>
<td>960</td>
<td>19</td>
</tr>
<tr>
<td>21, 22</td>
<td>10, 13, 18, 26</td>
<td>$\pm [\sqrt{5}(X_1 + X_{37}) + 2\sqrt{2}(X_6 + X_{40}) + 3X_7]$</td>
<td>1536</td>
<td>31</td>
</tr>
</tbody>
</table>
The subspace $g(2)^c$ has a basis consisting of the root vectors $X_k$, $k \in I$, where $I = I_1 \cup I_2$ and

\[ I_1 = \{6, 12, 18, 23, 24, 27, 29, 33, 35, 38, 40, 42, 43, 46, 50, 53\}, \]
\[ I_2 = \{13, 19, 25, 30, 31, 34, 36, 39, 41, 44, 45, 47, 48, 51, 54, 56\}. \]

As a $G(0)^c$-module, $g(2)^c$ is the tensor product of a half-spin module of $Spin_{10}$ and the 2-dimensional simple module of $SL_2$. The subspace of $g(2)^c$ spanned by the vectors $X_k$ with $k \in I_1$ (or $k \in I_2$) is a half-spin module of $Spin_{10}$.

We find several subsets $J \subset I$ such that the subspace of $g(2)^c$ spanned by $X_k$ with $k \in J$ is $\sigma$-stable, for $k, j \in J$ the difference $\alpha_k - \alpha_j$ is not a root, and $H$ belongs to the subspace spanned by all coroots $H_k$ with $k \in J$. All of them have size 4. For instance, the set $J = \{6, 19, 50, 56\}$ satisfies all the conditions mentioned above. As $\sigma(X_6) = X_{19}$ and $\sigma(X_{50}) = X_{56}$ (see Table 2), the vector

\[ E = (aX_6 + aX_{19}) + (bX_{50} + bX_{56}) \]

is real in the sense that $\sigma(E) = E$, i.e., $E \in g(2)$.

The equation $[F, E] = H$, where $F = \theta(E) = \sigma_u(E)$, implies that $|a|^2 = 2$. All possible choices for $a$ and $b$ produce an element $E$ belonging to the orbit 15. This is established by computing the invariant inv which turns out to be 25 in all cases. The other sets $J$ also produce only representatives for the orbit 15.

In order to find a representative for the orbit 14 we had to undertake the following tedious calculation. Since $E \in g(2)$, i.e., $E \in g(2)^c$ and $\sigma(E) = E$, the representative $E$ must have the form

\[ E = (aX_6 + aX_{19}) + (bX_{12} + bX_{13}) + (cX_{18} + cX_{30}) + (dX_{23} - dX_{25}) + (eX_{24} + eX_{36}) + (fX_{27} + fX_{39}) + (gX_{29} - gX_{31}) + (hX_{33} - hX_{34}) + (\alphaX_{35} - \alphaX_{45}) + (\betaX_{38} - \betaX_{48}) + (\gammaX_{40} + \gammaX_{41}) + (\deltaX_{42} - \deltaX_{51}) + (\epsilonX_{43} + \epsilonX_{44}) + (\zetaX_{46} + \zetaX_{47}) + (\etaX_{50} + \etaX_{56}) + (\thetaX_{53} - \thetaX_{54}) \]

where $a, b, \ldots, \theta$ are some complex numbers.

Since $F = \theta(E) = \sigma_u(E)$, we must have

\[ F = (\bar{a}X_{-6} + aX_{-19}) + (\bar{b}X_{-12} + bX_{-13}) + (\bar{c}X_{-18} + cX_{-30}) + (\bar{d}X_{-23} - dX_{-25}) + (\bar{e}X_{-24} + eX_{-36}) + (\bar{f}X_{-27} + fX_{-39}) + (\bar{g}X_{-29} - gX_{-31}) + (\bar{h}X_{-33} - hX_{-34}) + (\bar{\alpha}X_{-35} - \alphaX_{-45}) + (\bar{\beta}X_{-38} - \betaX_{-48}) + (\bar{\gamma}X_{-40} + \gammaX_{-41}) + (\bar{\delta}X_{-42} - \deltaX_{-51}) + (\bar{\epsilon}X_{-43} + \epsilonX_{-44}) + (\bar{\zeta}X_{-46} + \zetaX_{-47}) + (\bar{\eta}X_{-50} + \etaX_{-56}) + (\bar{\theta}X_{-53} - \thetaX_{-54}). \]
Next we have computed the bracket $[F, E]$ by using the above expressions for $E$ and $F$ and the list of the structure constants given in the Appendix. As $[F, E] = H$ where

$$H = 4H_1 + 6H_2 + 8H_3 + 12H_4 + 10H_5 + 8H_6 + 4H_7,$$

we obtain the following system of equations.

$\begin{align*}
\Re (e \tilde{f} + g \tilde{h} + \alpha \beta + \gamma \bar{\varepsilon}) &= 0 \\
\Re (ae + d\bar{g} + \beta \delta + \varepsilon \bar{\zeta}) &= 0 \\
\Re (ef + dh - \alpha \delta - \gamma \bar{\zeta}) &= 0 \\
\Re (-a\bar{a} + b\bar{\gamma} + f\bar{\eta} + h\bar{\theta}) &= 0 \\
\Re (a\bar{b} - b\bar{\varepsilon} + c\bar{\eta} + g\bar{\theta}) &= 0 \\
\Re (-a\bar{\delta} + b\bar{\zeta} + c\bar{\eta} + d\bar{\theta}) &= 0
\end{align*}$

\[
\begin{align*}
\bar{b}c - a\bar{d} + \bar{g}\alpha + \bar{h}\beta + e\bar{\gamma} + f\bar{\varepsilon} + \bar{\zeta}\eta + \delta\bar{\theta} &= 0 \\
a\bar{c} + b\bar{d} - \bar{e}\alpha - \tilde{f}\beta + g\bar{\gamma} + h\bar{\varepsilon} + \bar{\zeta}\theta - \delta\bar{\eta} &= 0 \\
a\bar{g} - \bar{b}e + c\bar{\gamma} + d\alpha - \tilde{f}\bar{\zeta} - h\bar{\delta} + \beta\bar{\theta} + \varepsilon\bar{\eta} &= 0 \\
a\bar{h} - bf + c\bar{\varepsilon} + d\bar{\beta} + e\bar{\zeta} + \bar{g}\delta - \alpha\theta - \gamma\bar{\eta} &= 0 \\
a\bar{e} + bg + c\bar{\alpha} - d\bar{\gamma} - \tilde{f}\bar{\delta} + h\bar{\zeta} + \beta\bar{\eta} - \varepsilon\theta &= 0 \\
a\bar{f} + bh + c\bar{\beta} - d\bar{\varepsilon} + \bar{e}\delta + g\bar{\zeta} - \alpha\bar{\eta} + \gamma\theta &= 0
\end{align*}
\]

\[
\begin{align*}
|f|^2 + |h|^2 + |\beta|^2 + |\delta|^2 + |\varepsilon|^2 + |\bar{\zeta}|^2 + |\theta|^2 + |\eta|^2 &= 2 \\
|\alpha|^2 + |b|^2 + |c|^2 + |d|^2 + |e|^2 + |g|^2 + |\alpha|^2 + |\gamma|^2 &= 2 \\
|\bar{e}|^2 + |g|^2 + |\alpha|^2 + |\gamma|^2 + |\delta|^2 + |\bar{\zeta}|^2 + |\theta|^2 + |\eta|^2 &= 2 \\
|c|^2 + |d|^2 + |e|^2 + |g|^2 + |\theta|^2 + |\eta|^2 + 2|\alpha|^2 + 2|\gamma|^2 &= 2 + |f|^2 + |h|^2.
\end{align*}
\]

After some judicious specializations and experimentations using the Maple’s \texttt{solve} routine, we have found the following solution of the above system:

\[
\begin{align*}
a &= b = 1/\sqrt{2}, & c &= d = e = f = g = h = 0, \\
\alpha &= \gamma = 1/\sqrt{2}, & \beta &= \delta = \varepsilon = i/\sqrt{2}, \\
\zeta &= -i/\sqrt{2}, & \eta &= \theta = 0.
\end{align*}
\]

This means that we have a real Cayley triple $(E, H, F)$ with

\[
E = \frac{1}{\sqrt{2}} [X_6 + X_{12} + X_{13} + X_{19} + X_{35} + X_{40} + X_{41} - X_{45} \\
+i(X_{38} + X_{42} + X_{43} - X_{44} - X_{46} + X_{47} + X_{48} + X_{51})].
\]

For this triple we find that \text{inv} = 49, and so $E$ is a representative of the orbit 14.

10. Appendix.

We list here the nonzero structure constants $N(i, j)$ of $E_7$ for $i > 0$ and $j$ arbitrary. The $i$-th entry in this list contains two sequences separated by a semicolon. The first (resp., second) sequence consists of those $j$ for which $N(i, j)$ is $+1$ (resp., $-1$). Note that each of these sequences has length 16.
The nonzero structure constants of $E_7$

1. $3, 10, 15, 17, 22, 24, 28, 29, 31, 35, 36, 40, 41, 45, 49, 62$;
   $-8, -14, -20, -21, -26, -27, -32, -33, -34, -38, -39, -43, -44, -48, -52, -63$

2. $10, 11, 14, 17, 18, 21, 24, 25, 27, 31, 34, 50, 54, 57, 59, -9$;

3. $9, 11, 16, 18, 23, 25, 30, 32, 38, 43, 44, 48, 52, 61, -8, -10$;
   $1, 4, -15, -17, -22, -24, -29, -31, -36, -37, -42, -46, -47, -51, -55, -62$

4. $2, 3, 5, 8, 12, 19, 22, 26, 29, 33, 36, 39, 46, 51, 55, 60$;
   $-9, -10, -11, -14, -18, -25, -28, -32, -35, -38, -41, -44, -50, -54, -57, -61$

5. $9, 10, 13, 14, 15, 20, 35, 38, 41, 42, 44, 47, 57, 58, -11, -12$;
   $4, -16, -17, -19, -21, -22, -26, -40, -43, -45, -46, -48, -51, -59, -60$

6. $5, 7, 11, 16, 17, 21, 22, 26, 28, 32, 37, 45, 48, 51, 54, 56$;
   $-12, -13, -18, -23, -24, -27, -29, -33, -35, -38, -42, -49, -52, -55, -57, -58$

7. $12, 18, 23, 24, 27, 29, 33, 35, 38, 40, 42, 43, 46, 50, 53, -13$;
   $6, -19, -25, -30, -31, -34, -36, -39, -41, -44, -45, -47, -48, -51, -54, -56$

8. $9, 11, 16, 18, 23, 25, 30, 61, -1, -14, -37, -42, -46, -47, -51, -55$;
   $4, 28, 35, 40, 41, 45, 49, -3, -20, -21, -26, -27, -33, -34, -39, -63$

   $3, 5, 8, 12, 19, 46, 51, 55, -2, -28, -32, -35, -38, -41, -44, -61$

10. $16, 23, 30, 43, 48, 52, -4, -14, -15, -17, -24, -31, -37, -42, -47, -62$;
    $1, 2, 5, 12, 19, 26, 33, 39, 60, -3, -28, -35, -41, -50, -54, -57$

11. $13, 15, 20, 42, 47, 58, -4, -16, -17, -18, -21, -40, -43, -45, -48, -59$;
    $2, 3, 6, 8, 29, 33, 36, 39, 55, -5, -25, -28, -32, -50, -54, -61$

12. $9, 10, 14, 15, 20, 41, 44, 47, -6, -18, -19, -40, -43, -46, -59, -60$;
    $4, 7, 28, 32, 37, 54, 56, -5, -23, -24, -27, -29, -33, -49, -52, -55$
EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF $E_7$

13  $40,43,46,50,53,-6,-19,-25,-30,-31,-34,-36,-39,-41,-44,-47;
     5,11,16,17,21,22,26,28,32,37,-7,-49,-52,-55,-57,-58$

14  $16,22,23,29,30,36,-1,-4,-20,-21,-27,-34,-50,-54,-57,-63;
     2,5,12,19,40,45,49,60,-8,-10,-32,-37,-38,-42,-44,-47$

15  $21,27,34,43,48,52,59,-2,-3,-20,-22,-28,-29,-35,-36,-41;
     1,5,11,12,18,19,25,-9,-10,-37,-42,-47,-53,-56,-58,-62$

16  $13,24,27,31,34,42,47,-2,-5,-22,-23,-26,-28,-32,-60,-61;
     3,6,8,10,14,55,57,-9,-11,-30,-40,-43,-45,-48,-53,-56$

17  $13,20,23,30,52,58,-3,-5,-21,-22,-24,-28,-46,-50,-51,-54;
     1,2,6,9,33,38,39,44,-10,-11,-31,-37,-40,-45,-59,-62$

18  $15,20,22,26,47,51,-4,-6,-23,-24,-27,-49,-50,-52,-61;
     2,3,7,8,36,37,39,56,-11,-12,-35,-38,-40,-43,-57,-59$

19  $9,10,14,15,20,50,53,-5,-7,-25,-45,-48,-49,-51,-52,-55;
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20  $59,-1,-2,-8,-26,-32,-33,-37,-38,-39,-42,-44,-47,-53,-56,-58;
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22  $13,27,34,52,-2,-3,-5,-26,-28,-29,-37,-40,-45,-53,-56,-62;
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23  $31,34,47,51,54,-2,-6,-12,-29,-30,-33,-35,-38,-40,-43,-53;
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25  $15,20,22,26,29,33,53,-4,-7,-11,-30,-31,-34,-54,-57,-59;
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### Explicit Cayley Triples in Real Forms of $E_7$

| 40  | 8,14,20,34,39,44,47,-5,-16,-17,-18,-19,-28,-29,-33,-43,-45,-49; |
|     | 1,7,13,-11,-12,-22,-23,-24,-35,-46,-50,-53,-59,-60,-61,-62     |
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|     | 1,6,27,33,38,42,-11,-19,-22,-30,-31,-40,-41,-51,-54,-56     |
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| 48  | 24,29,35,-1,-5,-7,-16,-21,-25,-32,-39,-51,-52,-54,-56,-63;     |
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57 16, 22, 26, -4, -6, -10, -13, -18, -27, -31, -35, -42, -44, -49, -58, -59;
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63 -1, -8, -20, -21, -32, -33, -34, -42, -43, -44, -50, -51, -52, -56, -57, -60;
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EXPLICIT CAYLEY TRIPLES IN REAL FORMS OF $E_7$
The index formula for elliptic pseudodifferential operators on a two-dimensional manifold with conical points contains the Atiyah-Singer integral as well as two additional terms. One of the two is the ‘eta’ invariant defined by the conormal symbol, and the other term is explicitly expressed via the principal and subprincipal symbols of the operator at conical points. The aim of this paper is an explicit description of the contribution of a conical point for higher-order differential operators. We show that changing the origin in the complex plane reduces the entire contribution of the conical point to the shifted ‘eta’ invariant. In turn this latter is expressed in terms of the monodromy matrix for an ordinary differential equation defined by the conormal symbol.

Introduction.

In [FST97] we proved the following index formula for elliptic pseudodifferential operators on a two-dimensional manifold with a conical point:

\[
\text{ind } A = \int_{S^1 \times \mathbb{R}} \text{AS}(A) - \frac{1}{2} \eta(A_c) + \frac{i}{4\pi^2} \int_{S^1} \text{tr} \sigma_0^{-1} \sigma_{\text{sub}} |_{\tau=1} d\xi dx,
\]

where \( M \) is the manifold in question whose cross-section close to the conical point is identified with the unit circle \( S^1 \).

The index is evaluated for \( A \) acting on weighted Sobolev spaces on \( M \) as \( H^{s,\gamma}(M, E^0) \rightarrow H^{s-m,\gamma}(M, E^1) \), where \( E^0 \) and \( E^1 \) are \( C^\infty \) vector bundles over the smooth part of \( M \) which behave properly when approaching the conical point.

The first term on the right-hand side of this formula is the Atiyah-Singer integral derived from the principal interior symbol \( \sigma_0 \) of \( A \) and the curvature forms \( \Omega^0 \) and \( \Omega^1 \) of the bundles \( E^0 \) and \( E^1 \), respectively. We have

\[
\text{AS}(A) = \frac{1}{4\pi^2} \left( \frac{1}{6} \text{tr} (\sigma_0^{-1} \partial \sigma_0)^3 - \frac{1}{2} \text{tr} (\Omega^0 \sigma_0^{-1} \partial \sigma_0 + \Omega^1 \partial \sigma_0 \sigma_0^{-1}) \right).
\]

The weight exponent \( \gamma \) enters only the second term on the right side of (0.1) which is known as the ‘eta’ invariant of the conormal symbol \( A_c \) of \( A \).
at the conical point. More precisely, 

\[ \eta(A_c) = -\frac{1}{\pi i} \text{Tr} \left( A_c^{-1}(\tau + i\gamma)A'_c(\tau + i\gamma) - i\gamma \frac{d}{d\tau} (A_c^{-1}(\tau + i\gamma)A'_c(\tau + i\gamma)) \right) , \]

\( \text{Tr} \) being a regularised trace (cf. Melrose [Mel95]).

Both these terms occur in the Atiyah-Patodi-Singer formula for the index of Dirac operators (cf. [APS75]). In contrast to this latter formula, (0.1) contains the additional third term which does not vanish even for the Cauchy-Riemann operator on the plane. This summand also depends on the conormal symbol \( A_c(\tau) \) only because the principal symbol \( \sigma_0 \) and the so-called subprincipal symbol

\[ \sigma_{\text{sub}} = \sigma_1 + \frac{i}{2} \frac{\partial^2 \sigma_0}{\partial \tau_0} \]

are evaluated at the conical point. Here \( \sigma_1 \) means the homogeneous component of degree \( \deg \sigma_0 - 1 \) of the complete symbol of \( A_c(\tau) \).

Of course, formula (0.1) is still true for manifolds with several conical points. A slight change we have to do is that the ‘eta’ invariant and the additional terms should be summed up over all conical points of \( M \).

The aim of this paper is an explicit description of the contribution of a conical point for elliptic differential operators. To this end we show first that by changing the origin in the complex \( \tau \)-plane we can make the third term to vanish reducing the whole contribution of the conical point to the shifted ‘eta’ invariant. The new origin \( \tau_0 \) which we refer to as the centre is the root of the linear equation

\[ \int_{S^1 \times \mathbb{R}} \text{tr} \sigma_0^{-1} \left( \frac{\partial \sigma_0}{\partial \tau_\tau_0} + \sigma_{\text{sub}} \right) \bigg|_{\tau=1}^{\tau=-1} d\xi d\tau = 0. \]

The next goal is to express the ‘eta’ invariant in terms of the monodromy matrix \( M(\tau) \) for an ordinary differential equation defined by the conormal symbol \( A_c(\tau) \). We introduce a phase function

\[ \varphi(\tau) = \frac{1}{2} \log \det \left( M(\tau) + M^{-1}(\tau) - 2 \right) \]

which is an analytic function of \( \tau \) with logarithmic ramification points. Then our final index theorem reads

\[ \text{ind} A = \int_{S^1 \times \mathbb{M}} \text{AS}(A) + \frac{1}{2\pi i} \Delta_{\Gamma,\tau_0} \varphi(\tau) \]

where \( \Delta_{\Gamma,\tau_0} \varphi(\tau) \) denotes the variation of the phase function along a suitable contour defined by the weight line \( \Gamma \) and the centre \( \tau_0 \) (Theorem 3.1).

In some particular cases we may say more about the variation \( \Delta_{\Gamma,\tau_0} \varphi(\tau) \). For example, if the function \( f(\tau) = \det(M(\tau) + M^{-1}(\tau) - 2) \) is even with
respect to $\tau_0$, that is
\[ f(\tau_0 - T) = f(\tau_0 + T), \]
then the second term in (0.3) may be calculated in terms of zeros of $f(\tau)$ and turns out to be half-integer (Theorem 3.2). Thinking over these properties we have come to a generalisation of the symmetry conditions used in [SSS97]. A detailed treatment of this symmetry in the higher-dimensional case will be given in a forthcoming paper.

Finally, we show that the above integrality of $\frac{1}{\pi i} \Delta_{\Gamma,\tau_0} \varphi(\tau)$ holds for any first-order elliptic system, no matter whether the symmetry condition is fulfilled or not. To this end we investigate the asymptotical behaviour of solutions and the monodromy matrix when $\Re \tau \to \pm \infty$ and $\Im \tau$ remains bounded. Although there exists vast literature on this topic, we have not found the desired facts and were forced to prove them. The proof uses the ideas of Faddeev and Takhtajan [FT87] for the non-linear Schrödinger equation.

**Acknowledgments.** The authors wish to express their gratitude to M. Lesch for drawing the authors’ attention to the paper [BFK91] where the monodromy matrix appeared in an expression for the determinant of an elliptic differential operator on a circle.

### 1. The existence of the centre.

Recall that the neighbourhood of a conical point is treated as a cylindrical end with coordinates $t \in \mathbb{R}_+$ and $x \in \mathbb{R} \mod (2\pi)$. Since any complex vector bundle over a circle is trivial, we may assume that $E^0 \cong E^1 \cong \mathbb{C}^r$ over the cylindrical end and, for given trivialisations, the connection one-forms $\Gamma^0$, $\Gamma^1$ are equal to 0.

The conormal symbol of an $m$th order differential operator has the form
\[
A_c(\tau) = a_m(x) \frac{\partial^m}{\partial x^m} + a_{m-1}(x, \tau) \frac{\partial^{m-1}}{\partial x^{m-1}} + \ldots + a_0(x, \tau).
\]
(1.1)

So, it is an ordinary differential operator on a circle whose coefficients
\[
a_k(x, \tau) = \sum_{l=0}^{m-k} a_{k,l}(x) \tau^l
\]
are polynomials in $\tau$ of degree $m - k$. Thus, the principal symbol of the operator $A$ restricted to the boundary is
\[
\sigma_0(A) = \sum_{k=0}^{m} a_{m-k,k}(x) \tau^k (i\xi)^{m-k}
\]
and for the lower-order term we have

\[ \sigma_1(A) = \sum_{k=0}^{m-1} a_{m-1-k,k}(x) \tau^k (i\xi)^{m-1-k}. \]

The interior ellipticity means that \( \sigma_0(A) \) is an invertible matrix for any real \((\xi, \tau) \neq (0,0)\); in particular, the coefficient \( a_m(x) \) in (1.1) is an invertible matrix-valued function on a circle. Without loss of generality we assume that \( a_m(x) \equiv 1 \), otherwise we change the frame in \( E^1 \) using \( a_m \) as a transition matrix.

Replacing \( \tau \) by \( \tau + \tau_0 \) in (1.1), we see that the shift by \( \tau_0 \) in the complex \( \tau \)-plane does not change the principal symbol \( \sigma_0 \), while for \( \sigma_1 \) we have a new expression

\[ \tilde{\sigma}_1 = \sigma_1 + \frac{\partial \sigma_0}{\partial \tau} \tau_0. \]

The subprincipal symbol \( \sigma_{\text{sub}} \) obeys the same rule

\[ \tilde{\sigma}_{\text{sub}} = \sigma_{\text{sub}} + \frac{\partial \sigma_0}{\partial \tau} \tau_0. \]

Thus, after shifting we obtain a new additional term in (0.1) proportional to the left-hand side of (0.2). The following theorem guaranties a unique solvability of the linear equation (0.2).

**Theorem 1.1.** For any elliptic differential operator \( A \),

\[ \int_{S^1 \times \mathbb{R}} \text{tr} \sigma_0^{-1} \frac{\partial \sigma_0}{\partial \tau} \bigg| \begin{array}{c} \tau = 1 \\ \tau = -1 \end{array} \quad dx d\xi \neq 0. \]  

**Proof.** Denoting \( \det \sigma_0 \) by \( f(x, \tau, \xi) \), we have

\[ \text{tr} \sigma_0^{-1} \frac{\partial \sigma_0}{\partial \tau} = f^{-1} \frac{\partial f}{\partial \tau}. \]

From ellipticity we deduce that the roots of the polynomial \( f = f(\tau, \xi) \) for fixed real \( \tau \) form two disjoint sets corresponding to the upper and lower half-planes. The integrand in (1.2) is a rational function in \( \xi \) decaying as \( O(|\xi|^{-2}) \) when \( \xi \to \infty \). Thus, integrating over \( \xi \), we may replace the real axis by a closed contour \( c_{\pm} \) consisting of a large semicircle in the upper (lower) half-plane and its diameter and surrounding all the poles in the corresponding half-plane.

By the Euler theorem for homogeneous functions,

\[ f^{-1} \frac{\partial f}{\partial \tau} = \frac{1}{\tau} \left( m \tau - \xi f^{-1} \frac{\partial f}{\partial \xi} \right), \]

so that

\[ f^{-1} \frac{\partial f}{\partial \tau} \bigg| \begin{array}{c} \tau = 1 \\ \tau = -1 \end{array} = 2m - \xi f^{-1} \frac{\partial f}{\partial \xi} \bigg|_{\tau = 1} - \xi f^{-1} \frac{\partial f}{\partial \xi} \bigg|_{\tau = -1} \]
and the residue theorem yields
\[
\int_{\mathbb{R}} f^{-1} \frac{\partial f}{\partial \tau} \bigg|_{\tau = -1} \, d\xi = -2\pi i \left( \sum \xi_k^+ + \sum \xi_k^- \right)
\]
where \(\xi_k^\pm\) are the roots of the equation \(f(\pm 1, \xi) = 0\) in the upper half-plane. Since \(f(\pm 1, \xi) = 0\) is equivalent to \(f(1, \pm \xi) = 0\), we see that at least one set \(\xi^+\) or \(\xi^-\) is not empty. Thus,
\[
\Im \left( \sum \xi_k^+ + \sum \xi_k^- \right) > 0,
\]
proving the theorem. \(\square\)

**Remark 1.2.** Our proof uses essentially the fact that \(f(\tau, \xi)\) is a homogeneous polynomial. Clearly, for rational homogeneous functions \(f(\tau, \xi)\) having no zeros and poles on the real axis \(\Im \xi = 0\) the theorem is not true.

### 2. The Green function and the monodromy matrix.

The operator
\[
\frac{d}{d\tau} \left( A_c^{-1} \frac{d}{d\tau} A_c(\tau) \right)
\]
is a pseudodifferential operator of order \(-2\) on the circle, thus it belongs to the trace class. Its trace may be explicitly calculated in terms of the so-called monodromy matrix.

Consider the ordinary differential equation
\[
(2.1) \quad A_c(\tau) u = u^{(m)}(x) + a_{m-1}(x, \tau) u^{(m-1)}(x) + \ldots + a_0(x, \tau) u(x) = 0.
\]
Its solutions form a linear space of dimension \(mr\). Since the coefficients are \(2\pi\)-periodic functions, the shift \(u(x) \mapsto u(x + 2\pi)\) defines a linear transformation \(M\) of the space of solutions called the monodromy.

**Theorem 2.1.** The monodromy transformations \(M(\tau), M^{-1}(\tau)\) are entire functions in \(\tau\), and
\[
\text{Tr} \frac{d}{d\tau} \left( A_c^{-1} \frac{d}{d\tau} A_c(\tau) \right) = \frac{1}{2} \frac{d^2}{d\tau^2} \log \det \left( M(\tau) + M^{-1}(\tau) - 2 \right).
\]

**Proof.** Any solution \(u(x)\) is uniquely defined by the vector of its Cauchy data
\[
(2.2) \quad \vec{u}(x) = \begin{pmatrix} u(x) \\ u'(x) \\ \ldots \\ u^{(m-1)}(x) \end{pmatrix}
\]
at some point \(x_0\). The monodromy carries the vector \(\vec{u}(x_0)\) to \(\vec{u}(x_0 + 2\pi)\) and we may calculate the monodromy matrix as follows. Consider the Wronsky
matrix $U(x, x_0, \tau)$ consisting of linearly independent vector-valued functions (2.2) normalised by the initial condition

\begin{equation}
U |_{x=x_0} = U(x_0, x_0, \tau) = 1.
\end{equation}

Then,

$$
M(\tau) = U(x_0 + 2\pi, x_0, \tau),
$$
$$
M^{-1}(\tau) = U(x_0 - 2\pi, x_0, \tau).
$$

The Wronsky matrix satisfies a first-order differential equation

\begin{equation}
A(\tau) U = 0
\end{equation}

where $A(\tau)$ is given by the block matrix

\begin{equation}
A(\tau) = \begin{pmatrix}
d/dx & -1 & \ldots & 0 \\
0 & d/dx & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
a_0(x, \tau) & a_1(x, \tau) & \ldots & a_{m-1}(x, \tau) + d/dx
\end{pmatrix}.
\end{equation}

Since the coefficients are polynomials in $\tau$, the solution $U(x, x_0, \tau)$ is a holomorphic function in $\tau \in \mathbb{C}$ and so are $M(\tau)$ and $M^{-1}(\tau)$. We will express the operator $A'(\tau)A_c^{-1}(\tau)$ through the first order operator (2.5).

Introduce the notion $\sum \mathcal{B}$ for the sum of diagonal blocks of a block matrix $\mathcal{B}$.

**Lemma 2.2.** The following equality holds

$$
A'_c(\tau)A_c^{-1}(\tau) = \sum A'(\tau)A^{-1}(\tau).
$$

**Proof.** To find $A^{-1}(\tau)$, write the equation $A\vec{u}(\tau) = \vec{v}$ for $\vec{u}$ in components

$$
\frac{du_0}{dx} - u_1 = v_0,
$$
$$
\frac{du_1}{dx} - u_2 = v_1,
$$

$$
\frac{du_{m-1}}{dx} + a_{m-1}u_{m-1} + \ldots + a_0u_0 = v_{m-1}.
$$

Eliminating

$$
u_1 = \frac{du_0}{dx} - v_0,
$$
$$
u_2 = \frac{d}{dx} \left( \frac{du_0}{dx} - v_0 \right) - v_1
$$

and so on, we obtain an equation for $u_0$ of the form

$$
(d/dx)^m u_0 + a_{m-1} (d/dx)^{m-1} u_0 + \ldots + a_0 u_0 = w
$$
where \( w \) is a known function, namely a linear combination of \( v_0, v_1, \ldots, v_{m-1} \) and their derivatives. Thus, \( u_0 = A_c^{-1}(\tau)w \), and moving backward we find successively \( u_1, u_2, \ldots, u_{m-1} \). The most simple expression we have in the case when \( v_0 = v_1 = \ldots = v_{m-2} = 0 \). Then
\[
\begin{align*}
  u_0 &= A_c^{-1}(\tau)v_{m-1}, \\
  u_1 &= (d/dx)A_c^{-1}(\tau)v_{m-1}, \\
  \ldots \ldots \ldots \\
  u_{m-1} &= (d/dx)^{m-1}A_c^{-1}(\tau)v_{m-1}.
\end{align*}
\]
It follows that \( A^{-1} \) exists exactly when \( A_c^{-1} \) does and
\[
A^{-1}(\tau) = \begin{pmatrix} & & & A_c^{-1}(\tau) \\ & & & (d/dx)A_c^{-1}(\tau) \\ & & & \ldots \ldots \ldots \\ & & & (d/dx)^{m-1}A_c^{-1}(\tau) \end{pmatrix}
\]
where * means any expression whose explicit form is irrelevant. Next,
\[(2.6) \quad A'(\tau) = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \\ a'_0 & a'_1 & \ldots & a'_{m-1} \end{pmatrix},
\]
the prime meaning the derivation in \( \tau \), so that
\[
A'(\tau)A^{-1}(\tau) = \begin{pmatrix} & & & 0 \\ & & & \ldots \ldots \ldots \\ & & & 0 \\ & & & A_c'(\tau)A_c^{-1}(\tau) \end{pmatrix}
\]
proving the lemma.

Now we find \( A^{-1}(\tau) \) in another way. Let \( x \) vary in the closed interval \([0, 2\pi]\) and let \( U(x, \tau) \) be the Wronsky matrix satisfying (2.3) at \( x_0 = 0 \). The operator \( A^{-1}(\tau) \), when considered on periodic functions on \([0, 2\pi]\), is an integral operator whose kernel \( G(x, y, \tau) \) (the Green function) is a periodic solution of the equation
\[
A(\tau)G(x, y, \tau) = \delta(x - y).
\]
We treat \( x \) as an argument while \( y \in [0, 2\pi] \) is considered as a parameter, \( \delta \) being the Dirac \( \delta \)-function. This equation means that \( G \) satisfies the homogeneous equation on \([0, y]\) and \((y, 2\pi]\), whence
\[
\begin{align*}
  G(x, y, \tau) &= U(x, \tau)C_-, \quad \text{for } x \in [0, y), \\
  G(x, y, \tau) &= U(x, \tau)C_+, \quad \text{for } x \in (y, 2\pi],
\end{align*}
\]
the matrices \( C_\pm \) being independent of \( x \). To produce the \( \delta \)-function, these matrices should satisfy the relation \( C_+ - C_- = U^{-1}(y, \tau) \) while periodicity yields \[ C_- = U(2\pi, \tau)C_+ = M(\tau)C_+. \]

Solving this system, we get a usual expression for the Green function, namely
\[
G(x, y, \tau) = \begin{cases} 
U(x, \tau)(1 - M(\tau))^{-1}U^{-1}(y, \tau), & x \in [0, y), \\
U(x, \tau)M(\tau)(1 - M(\tau))^{-1}U^{-1}(y, \tau), & x \in (y, 2\pi],
\end{cases}
\]
or equivalently
\[
G(x, y, \tau) = \frac{1}{2} U(x, \tau)(1 + M(\tau))(1 - M(\tau))^{-1}U^{-1}(y, \tau) + \frac{1}{2} \sgn(x - y)U(x, \tau)U^{-1}(y, \tau).
\]

Using Lemma 2.2, we conclude that the operator \((A'_c(\tau)A^{-1}_c(\tau))'\) has the kernel
\[
(2.7) \quad \frac{1}{2} \sum \frac{d}{d\tau} \left(A'(\tau)U(x, \tau)(1 + M(\tau))(1 - M(\tau))^{-1}U^{-1}(y, \tau)\right)
+ \frac{1}{2} \sgn(x - y) \sum \frac{d}{d\tau} \left(A'(\tau)U(x, \tau)U^{-1}(y, \tau)\right).
\]
The second term vanishes at \( x = y \) because from (2.6)
\[
\sum \frac{d}{d\tau} A'(\tau) = a''_{m-1}(\tau)
= 0
\]
since \( a_{m-1} \) is a linear function in \( \tau \). To calculate the trace of \((A'_c(\tau)A^{-1}_c(\tau))'\) (which belongs to the trace class), we put \( x = y \) in (2.7), take the matrix trace and integrate over \([0, 2\pi] \). The second term in (2.7) may be dropped and we obtain
\[
\text{Tr} \left(A'_c(\tau)A^{-1}_c(\tau)\right)' = \int_0^{2\pi} \frac{dG}{d\tau}(x, x, \tau) \, dx
= \frac{1}{2} \frac{d}{d\tau} \text{tr} (1 + M(\tau))(1 - M(\tau))^{-1} \int_0^{2\pi} U^{-1}(x, \tau)A'(x, \tau)U(x, \tau) \, dx.
\]

To complete the proof of Theorem 2.1, we need the following lemma.
Lemma 2.3. We have
\[ M^{-1}(\tau) M'(\tau) = -\int_0^{2\pi} U^{-1}(x, \tau) A'(x, \tau) U(x, \tau) \, d\tau. \]

Proof. Differentiating (2.4) in \( \tau \), we obtain
\[ A(\tau) U' + A'(\tau) U = 0 \tag{2.8} \]
with an initial condition
\[ U'(x, \tau) \big|_{x=0} = 0, \]
where the prime means derivation in \( \tau \). To find \( U' \), we apply a variation of constants to (2.8) looking for \( U' \) in the form \( U V \). Then (2.8) yields
\[ U \frac{dV}{dx} + A'(\tau) U = 0, \]
so that
\[ V(x) = -\int_0^x U^{-1}(y, \tau) A'(y, \tau) U(y, \tau) \, dy \]
and
\[ U'(x, \tau) = U(x, \tau) \int_0^x U^{-1}(y, \tau) A'(y, \tau) U(y, \tau) \, dy. \]
Taking \( x = 2\pi \) yields the desired identity. \( \square \)

Now
\[ \text{Tr} \left( A'_c(\tau) A_c^{-1}(\tau) \right)' = -\frac{1}{2} \frac{d}{d\tau} \text{tr} \left( 1 + M \right) (1 - M)^{-1} M^{-1} M' \]
\[ = \frac{d}{d\tau} \text{tr} \left( M - 1 \right)^{-1} M' - \frac{1}{2} \frac{d}{d\tau} \text{tr} M^{-1} M' \]
\[ = \frac{1}{2} \frac{d^2}{d\tau^2} \log \det(M - 1)^2 M^{-1} \]
which is precisely (2.1). \( \square \)

3. The index formula.

Combining the results of Sections 1 and 2, we obtain a simple interpretation of the boundary terms in the index formula (0.1). We also introduce a symmetry condition generalising that of [SSS97]. It allows one to simplify further the boundary term reducing it to the number of poles of \( A_c^{-1} \) in a strip.

Consider two horizontal lines \( \Gamma, \Gamma_0 \) in the complex \( \tau \)-plane, \( \Gamma_0 \) passing through the centre \( \tau_0 \). In the strip between these lines the operator \( A_c^{-1}(\tau) \) has a finite number of poles. In particular, for \( |\Re\tau| > T_0 \gg 1 \) there are no poles at all. Consider a contour starting at the point \( \tau_0 - T \) with \( T > 0 \)
large enough, so that \(|\Re(\tau_0 - T)| > T_0\), then going along \(\Gamma\) in the region where \(|\Re\tau| < T_0\), and terminating at the point \(\tau_0 + T\) (see Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Variation of \(\varphi(t)\) along \(\Gamma\).}
\end{figure}

The function
\[\varphi(\tau) = \frac{1}{2} \log \det(M + M^{-1} - 2)\]
is analytic with ramification points at zeros of
\[f(\tau) = e^{\varphi(\tau)} = \det^{1/2}(M + M^{-1} - 2) = \frac{\det(M - 1)}{\det^{1/2}M}.\]
Clearly, the zeros of \(\det(M(\tau) - 1)\) are the poles of \(A_c^{-1}(\tau)\). Denote by \(\varphi(\tau_0 + T) - \varphi(\tau_0 - T)\) the variation of \(\varphi(\tau)\) along the contour described above and set
\[(3.1) \quad \Delta_{\Gamma,\tau_0} \varphi(\tau) = \lim_{T \to \infty} (\varphi(\tau_0 + T) - \varphi(\tau_0 - T)).\]
With this notation we have the following theorem.

**Theorem 3.1.** Let \(\tau_0\) be the centre and \(M(\tau)\) be the monodromy matrix of the ordinary differential operator \(A_c(\tau)\) on the circle. Then
\[(3.2) \quad \text{ind} A = \int_{S^*M} AS(A) + \frac{1}{2\pi i} \Delta_{\Gamma,\tau_0} \varphi(\tau).\]

**Proof.** Let us consider the isomorphisms of the bundles \(E^0, E^1\) consisting in multiplication by \(\exp(i\tau_0 t)\). The local expressions for the operator \(A = A_c(-i\partial/\partial t)\) in cylindrical charts change to
\[e^{-i\tau_0 t}A e^{i\tau_0 t} = A_c \left( -i \frac{\partial}{\partial t} + \tau_0 \right).\]
So, its conormal symbol changes to \(A_c(\tau + \tau_0)\).
If $\tau_0$ is the centre, then according to Section 1 the boundary term consists of

$$-\frac{1}{2} \eta(A_c(\tau + \tau_0)) = \frac{1}{2\pi i} \text{Tr} A_c^{-1}(\tau + \tau_0) A_c'(\tau + \tau_0).$$

(3.3)

It is sufficient to compute (3.3) for $\tau_0 = 0$. Consider

$$Q(\tau) = \text{Tr} \frac{\partial}{\partial \tau} \left( A_c^{-1}(\tau) A_c'(\tau) - i\gamma \frac{\partial}{\partial \tau} A_c^{-1}(\tau) A_c'(\tau) \right).$$

By Theorem 2.1, this quantity is equal to

$$\frac{\partial^2}{\partial \tau^2} \left( \varphi(\tau) - i\gamma \frac{\partial}{\partial \tau} \varphi(\tau) \right).$$

According to the definition of $\text{Tr}$ (see Melrose [Mel95]) and $\eta(A_c)$, we obtain

$$-\frac{1}{2} \eta(A_c) = \lim_{T \to \infty} \int_{-T}^{T} d\tau_1 \int_{0}^{\tau_1} Q(\tau + i\gamma) d\tau,$$

the right-hand side being understood as a constant term in the asymptotic expansion when $T \to \infty$. Thus,

$$-\frac{1}{2} \eta(A_c) = \frac{1}{2\pi i} \lim_{T \to \infty} \left( \varphi(\tau) - i\gamma \frac{\partial}{\partial \tau} \varphi(\tau) \right)_{\tau = T + i\gamma}^{\tau = -T + i\gamma},$$

and the variation of $\varphi(\tau)$ is taken along the weight line $\Gamma$ (for $(\partial/\partial \tau) \varphi(\tau)$, the variation does not depend on the path). In the region $|\Re \tau| > T_0$ where $\varphi(\tau)$ is holomorphic in the strip between $\Gamma$ and the real axis, we may use the Taylor formula, thus obtaining

$$\varphi(\tau) - i\gamma \frac{\partial}{\partial \tau} \varphi(\tau) = \varphi(\tau - i\gamma) + R_2(\tau, \gamma)$$

where $R_2(\tau, \gamma)$ is a remainder term which tends to 0 for $\Re \tau \to \pm \infty$ and $|\Im \tau| \leq C$. Hence it follows that

$$-\frac{1}{2} \eta(A_c) = \frac{1}{2\pi i} \lim_{T \to \infty} (\varphi(T) - \varphi(-T)),$$

the variation is taken along the contour in Fig. 1 with $\tau_0 = 0$. This completes the proof. $\square$

There are important particular cases when the variation (3.1) can be calculated by the residue theorem.

**Theorem 3.2.** Let $f^2(\tau) = \det(M(\tau) + M^{-1}(\tau) - 2)$ be an even function with respect to the centre $\tau_0$, that is

$$f^2(\tau_0 + T) = f^2(\tau_0 - T)$$

(3.4)
for any \( T \). Then

\[
\text{ind} A = \int_{S^*M} AS(A) + \left( p + \frac{1}{2}q \right) \text{sgn}(\Im \tau_0 - \gamma)
\]

where \( p \) is the number of zeros of \( f(\tau) \) (counted along with their multiplicities) in the strip between \( \Gamma \) and \( \Gamma_0 \), and \( q \) is the number of zeros on the line \( \Gamma_0 \).

**Proof.** To be specific, let \( \Im \tau_0 < \gamma \). Consider a closed contour \( l = l_1 \cup l_2 \) where \( l_1 \) is the contour on Fig. 1 and \( l_2 \) goes along the line \( \Gamma_0 \) bypassing the zeros lying on \( \Gamma_0 \) along small semicircles (see Fig. 2).

\[
\begin{align*}
\int_{\Delta \tau_0} \Delta \varphi &= 1 \frac{1}{2\pi i} \Delta \varphi(\tau_0) \\
&= -p - \frac{1}{2\pi i} \Delta \varphi(\tau_0).
\end{align*}
\]

We next observe that the variation of \( \varphi(\tau) \) along \( l_2 \) is equal to the sum of variations along all the semicircles. Indeed, the variations along the segments of \( \Gamma_0 \) cancel because of (3.4). When the radii of the semicircles tend to 0, the variations along them tend to \( \pi i \) times the number \( q \) of zeros on \( \Gamma_0 \) counted together with their multiplicities. This is the desired conclusion. \( \square \)

Since the result is very simple, it is desirable to have simple sufficient conditions for (3.4) to be fulfilled. One of these is the symmetry condition of [SSS97] for the conormal symbol: there exist isomorphisms \( \varphi_0(x) \) and \( \varphi_1(x) \) of the bundles \( E^0 \) and \( E^1 \), such that

\[
A_c(\tau_0 - T) = \varphi_1(x) A_c(\tau_0 + T) \varphi_0(x)
\]

for each real \( T \). Roughly speaking (3.6) means that the symmetry transformation \( \tau \mapsto 2\tau_0 - \tau \) acts on \( A_c(\tau) \) by an automorphism of the algebra of differential operators on \( \mathbb{S}^1 \) induced by isomorphisms of the bundles \( E^0, E^1 \).
We introduce more general symmetry conditions including automorphisms generated by changes of variables.

**Definition 3.3.** The conormal symbol $A_c(\tau)$ is called symmetric (with respect to the centre $\tau_0$) if there exist a diffeomorphism $g : S^1 \to S^1$ and bundle isomorphisms

$$
v_0 : g^*E^0 \to E^0,
$$

$$
v_1 : g^*E^1 \to E^1
$$

such that

(3.7) \hspace{1cm} A_c(\tau_0 - T) = (g^{-1})^* v_1^{-1} A_c(\tau_0 + T) v_0 g^*.

The definition gains in interest if we realise that differential operators with symmetric conormal symbols meet the condition of Theorem 3.2.

**Proposition 3.4.** For symmetric conormal symbols (3.4) holds.

**Proof.** A diffeomorphism $g : S^1 \to S^1$ is defined by a monotone function $g(x), x \in \mathbb{R}^1$, such that

$$
g(x + 2\pi) = g(x) \pm 2\pi
$$

where the sign ‘+’ means that $g$ preserves the orientation while ‘−’ corresponds to diffeomorphisms reversing the orientation. If $U(x, \tau_0 - T)$ is the Wronsky matrix for $A_c(\tau_0 - T)$, then by (3.7) we have

$$
U(x, \tau_0 + T) = v_0(x) U(g(x), \tau_0 - T) v_0^{-1}(x).
$$

Taking $x = 2\pi$, we obtain

$$
M(\tau_0 + T) = v_0(0) \left( M_{\pm 1}(\tau_0 - T) \right) v_0^{-1}(0).
$$

Thus, in the case of orientation-preserving diffeomorphisms $g$ (in particular, under the symmetry condition (3.6)) we have

$$
M(\tau_0 + T) = v_0(0) M(\tau_0 - T) v_0^{-1}(0),
$$

while an orientation-reversing diffeomorphism $g$ yields

$$
M(\tau_0 + T) = v_0(0) M^{-1}(\tau_0 - T) v_0^{-1}(0).
$$

Both these properties imply (3.4) and thus (3.5). \qed

Consider some examples illustrating Proposition 3.4.

**Example 3.5.** Let $\tau_0 = 0$; $g : x \mapsto -x$ and $v_0 = 1, v_1 = (-1)^m$. Then (3.7) written for

$$
A_c(\tau) = \sum_{k+l \leq m} a_{k,l}(x) \tau^k \frac{\partial^l}{\partial x^l}
$$

just amounts to the fact that

$$
a_{k,l}(-x) = (-1)^{m+k+l} a_{k,l}(x).
$$
In other words, the coefficients are even matrix functions if \( k + l \) and \( m \) have the same parity, and odd functions otherwise. In particular, constant coefficients will do, provided that \( a_{k,l} = 0 \) for \( k + l \not\equiv m \pmod{2} \).

**Example 3.6.** For a first-order scalar differential operator

\[
A_c(\tau) = \frac{d}{dx} - a(x)\tau - b(x)
\]

Proposition 3.4 always holds. Indeed, the monodromy is given by a scalar factor

\[
M(\tau) = \exp\left(\int_0^{2\pi} (a(x)\tau + b(x)) \, dx\right)
\]

and the centre \( \tau_0 \) is the root of Equation (0.2) which in our case reduces to

\[
\int_0^{2\pi} (a(x)\tau_0 + b(x)) \, dx = 0.
\]

Clearly, \( M(\tau_0 - T) = M^{-1}(\tau_0 + T) \).

A particular case of this example is the Cauchy-Riemann operator on a Riemann surface with conical points.

**Remark 3.7.** It is interesting that the index formula in the form (3.5) under symmetry condition (3.7) is valid in the general setting of pseudodifferential operators on a higher-dimensional manifold with conical singularities. The proof using the ideas of [SSS97] and the machinery of [FST97] will be given in a forthcoming paper.

### 4. First-order operators.

Consider in more detail the case of a first-order matrix-valued operator

\[
A_c(\tau) = \frac{d}{dx} - A(x)\tau - B(x).
\]

We will show that, similarly to Example 3.6, the centre is completely determined by the monodromy matrix, or rather by its asymptotic behaviour when \( \Re \tau \to \pm \infty \) while \( \Im \tau \) remains bounded. The asymptotics implies that the boundary contribution in the index formula (3.2) is half-integer provided the frames in \( E^0, E^1 \) are chosen in an appropriate way. Consequently, the Atiyah-Singer term also has a half-integer value. The interpretation in terms of zeros as in (3.5) fails in general.

We begin with a choice of frames in \( E^0 \) and \( E^1 \). By the interior ellipticity, the spectrum of \( A(x) \) at any \( x \in S^1 \) does not intersect the imaginary axis,
so it consists of two disjoint parts in the right and left half-planes. The corresponding spectral projectors are given by the Cauchy integrals

\[
P_{\pm}(x) = \frac{1}{2\pi i} \int_{c_{\pm}} (\xi + iA(x))^{-1} \, d\xi
\]

where the contours \(c_{\pm}\) surround the spectrum in the corresponding half-planes. These projectors depend smoothly on \(x\) defining a splitting of the trivial bundle \(C^r \cong E^0 \cong E^1\) into a direct sum of two subbundles. Like any complex bundle over a circle, these subbundles are trivial. It follows that we may choose a frame in \(C^r\) with a transition matrix \(C(x)\), so that

\[
P_+(x) = C(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} C^{-1}(x),
\]

\[
P_-(x) = C(x) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} C^{-1}(x)
\]

and

\[
A_1^{-1}(x)A_2(x) = iC(x) \begin{pmatrix} a_+(x) & 0 \\ 0 & a_-(x) \end{pmatrix} C^{-1}(x),
\]

where \(a_\pm(x)\) are \((r_+ \times r_-)\)-matrices having the spectra in the right (left) half-plane. Passing to new frames in \(E^0, E^1\) with the same transition matrix \(C(x)\), we reduce the matrix \(A(x)\) to a block-diagonal form

\[
A(x) = \begin{pmatrix} a_+(x) & 0 \\ 0 & a_-(x) \end{pmatrix}.
\]

Here the matrices \(a_\pm\) have their spectra in the right (left) half-planes. The matrix \(B(x)\) changes to

\[
C^{-1}BC + C^{-1} \frac{dC}{dx}
\]

and may be written in a block form

\[
B(x) = \begin{pmatrix} b_{11}(x) & b_{12}(x) \\ b_{21}(x) & b_{22}(x) \end{pmatrix}
\]

corresponding to (4.3). Thus, we have reduced the conormal symbol \(A_c(\tau)\) to the canonical form (4.1), (4.3), (4.4) with a block-diagonal matrix \(A(x)\).

**Lemma 4.1.** The centre \(\tau_0\) is the root of the equation

\[
\int_0^{2\pi} \left( \text{tr} \left( a_+(x)\tau + b_{11}(x) \right) - \text{tr} \left( a_-(x)\tau + b_{22}(x) \right) \right) \, dx = 0.
\]

**Proof.** We have

\[
\sigma_0 = i\xi - A(x)\tau;
\]

\[
\sigma_{\text{sub}} = -B(x),
\]
so Equation (0.2) reduces to
\[ \text{tr} \int_0^{2\pi} dx \int_{-\infty}^{\infty} ((\xi + iA(x))^{-1} - (\xi - iA(x))^{-1}) (A(x)\tau_0 + B(x)) d\xi = 0. \]
Integrating over \( \xi \) and using (4.2), we obtain
\[ \text{tr} \int_0^{2\pi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (A(x)\tau_0 + B(x)) dx = 0, \]
which is precisely (4.5).

5. Lyapunov estimates.

In this section we consider the so-called stable case when the part \( a_+(x) \) in (4.3) is absent. So, all the eigenvalues of \( A(x) = a_-(x) \) have negative real parts. Such matrices will be called stable. The Wronsky matrix as a function of \( x \) is a solution of the Cauchy problem
\[
\frac{dU}{dx} = (A(x)\tau + B(x))U, \quad U |_{x=y} = 1.
\]
The following theorem gives an estimate for the fundamental solution \( U(x,y,\tau) = U(x,\tau)U^{-1}(y,\tau) \).

**Theorem 5.1.** Let \( A(x) \) be a stable matrix. Then there exist constants \( C, d > 0 \) such that, for \( \tau \gg 1 \),
\[
\|U(x,y,\tau)\| \leq C \exp(-d(x-y)\tau),
\]
provided \( x \geq y \), where \( \| \cdot \| \) means any matrix norm.

**Remark 5.2.** In the case of constant coefficients \( A \) and \( B \), estimate (5.1) is obvious, because the solutions can be expressed in terms of exponential functions. For variable coefficients it is not, however, so obvious (recall stable and unstable zones for the Schrödinger equation).

The following necessary and sufficient condition of stability is due to Lyapunov (see e.g. [Gan86]).

**Lemma 5.3.** A complex matrix \( A \) is stable if and only if there exists a Hermitian positive definite matrix \( X \) such that
\[
A^*X +XA = -1.
\]

**Proof.** If \( A \) is stable, so is \( A^* \). Hence both \( \exp At \) and \( A^*t \) are exponentially decaying as \( t \to +\infty \). The matrix \( X \) may be defined by an explicit expression, namely
\[
X = \int_0^{\infty} \exp(A^*t) \exp(At) dt.
\]
Indeed,
\[ A^*X +XA = \int_0^\infty \frac{\partial}{\partial t} (\exp(A^t) \exp(At)) dt \]
\[ = -1. \]

Conversely, from (5.2) it follows, for an eigenvector \( e \) of \( A \) with an eigenvalue \( \lambda \), that
\[ (e,e) = -(X Ae,e) - (Xe,Ae) \]
\[ = -(\lambda + \overline{\lambda})(Xe,e). \]

Hence \( \Re \lambda < 0 \), as desired. \( \square \)

**Proof of Theorem 5.1.** If \( A(x) \) is a smooth periodic function in \( x \), then (5.3) shows that \( X(x) \) is also a smooth periodic function. In particular, there are bounds independent of \( x \), for
\[ 0 < C_1 \leq X(x) \leq C_2 \]
in the sense of quadratic forms. Denoting the usual norm in \( \mathbb{C}^r \) by \( \|e\| = \sqrt{(e,e)} \), we define a new norm
\[ \|e\|_X = \sqrt{(Xe,e)} \]
which is equivalent to the usual one. Then, inequalities (5.4) give a precise form of the equivalence relations
\[ C_1 \|e\|^2 \leq \|e\|^2_X \leq C_2 \|e\|^2. \]

For a solution \( U(x,y,\tau) \), we consider the function
\[ f(x) = \|Ue\|_X^2 \]
\[ = (U^*(x,y,\tau)X(x)U(x,y,\tau)e,e). \]

Differentiating and using (5.2), we get
\[ \frac{\partial f}{\partial x} = \left( U^* \left( (A\tau + B)^*X + X(A\tau + B) + X' \right) Ue, e \right) \]
\[ = -\tau (Ue,Ue) + \left( (B^*X + XB + X') Ue, Ue \right) . \]

The matrix \( B^*X + XB + X' \) is Hermitian and, for \( \tau \) large enough, we have
\[ -\frac{\tau}{2} \leq B^*X + XB + X' \leq \frac{\tau}{2} \]
in the sense of quadratic forms. By (5.5), the norm \( \|Ue\|^2 \) may be replaced by \( \|Ue\|^2_X \); hence
\[ \frac{\partial f}{\partial x} \leq -d\tau f(x) \]
with some positive constant \( d \). Dividing by \( f(x) \) and integrating from \( y \) to \( x \), with \( x \geq y \), we obtain
\[
\log \frac{f(x)}{f(y)} \leq -d \tau (x - y)
\]
which means that
\[
\| U(x, \tau)e \|^2_{X(x)} \leq \exp(-d (x - y) \tau) \| e \|^2_{X(y)}.
\]
Since the norms \( \| \cdot \|_{X(x)} \) are equivalent to any fixed norm \( \| \cdot \| \), we come to (5.1), which completes the proof. \( \Box \)

This theorem has some obvious modifications. For example, an estimate
\[
\| U(x, \tau)U^{-1}(y, \tau) \| \leq C \exp(-d (x - y) \tau)
\]
holds if \( \tau \to -\infty \) and \( x \leq y \). Next, we may replace a stable matrix \( A = a_- \) by a matrix \( A = a_+ \) with a spectrum in the right half-plane. In this case we have
\[
\| U(x, \tau)U^{-1}(y, \tau) \| \leq C \exp(d (x - y) \tau)
\]
for \( \tau \to +\infty \) and \( x \leq y \) or \( \tau \to -\infty \) and \( x \geq y \), with some \( C, d > 0 \).

6. Asymptotics of solutions.

In this section we consider the general case of Equation (4.1) with a split matrix \( A(x) \). So, we write it in the form
\[
\frac{\partial U}{\partial x} = (\Lambda(x, \tau) + B(x)) U
\]
where
\[
\Lambda(x, \tau) = \begin{pmatrix}
\lambda_+(x, \tau) & 0 \\
0 & \lambda_-(x, \tau)
\end{pmatrix}
\]
(6.2)
is a block-diagonal part and
\[
B(x) = \begin{pmatrix}
0 & b_{12}(x) \\
b_{21}(x) & 0
\end{pmatrix}
\]
is an antidiagonal part of the coefficients. We assume that both \( a_-(x) \) and \( -a_+(x) \) are stable matrices.

Let us look for a solution of (6.1) in the form (cf. (4.5) in [FT87, Ch. 1])
\[
U(x, \tau) = (1 + W(x, \tau))Z(x, \tau),
\]
(6.3)
where $Z$ is a block-diagonal matrix and $W$ is an antidiagonal matrix. Substituting (6.3) into (6.1) and separating diagonal and antidiagonal parts, we obtain

$$
\frac{\partial W}{\partial x} Z + W \frac{\partial Z}{\partial x} = \Lambda W Z + B Z,
$$

$$
\frac{\partial Z}{\partial x} = (\Lambda + BW) Z.
$$

Eliminating $Z$, we arrive at a matrix Riccati equation for $W$

$$
\frac{\partial W}{\partial x} = \Lambda W - W \Lambda + B - WBW.
$$

Were $W$ a solution of (6.5), the second equation in (6.4) would give us an equation for $Z$ with a block-diagonal coefficient $\Lambda + BW$.

To find $W$, we observe that Equation (6.5) is equivalent to two separate equations for $w_{12}$ and $w_{21}$,

$$
\frac{\partial w_{12}}{\partial x} = \lambda_+ w_{12} - w_{12} \lambda_- + b_{12} - w_{12} b_{21} w_{12},
$$

$$
\frac{\partial w_{21}}{\partial x} = \lambda_- w_{21} - w_{21} \lambda_+ + b_{21} - w_{21} b_{12} w_{21}.
$$

Assuming $\lambda_\pm$ to be of the form (6.2), let us consider $\tau$ positive and large enough. We will look for solutions to (6.6) and (6.7) on the closed interval $x \in [0, 2\pi]$ with initial conditions

$$
w_{12}(2\pi) = 0,
$$

$$
w_{21}(0) = 0.
$$

**Lemma 6.1.** The solutions of (6.6), (6.8) and (6.7), (6.9) exist, for $\tau$ large enough, and satisfy the estimates

$$
\|w_{12}(x, \tau)\| = O\left(\frac{1}{\tau}\right),
$$

$$
\|w_{21}(x, \tau)\| = O\left(\frac{1}{\tau}\right)
$$

uniformly in $x \in [0, 2\pi]$.

**Proof.** Let us consider the case of $w_{12}$, the reasoning for $w_{21}$ is similar. First we reduce (6.6), (6.8) to an equivalent integral equation. To this end, let us treat $f = b_{12} - w_{12} b_{21} w_{12}$ as a known function and apply the variation of constants to the equation

$$
w_{12}' = \lambda_+ w_{12} - w_{12} \lambda_- + f.
$$

In other words, we look for a solution of the form

$$
w_{12}(x) = U_+(x)V(x)U_-^{-1}(x)
$$
where $U_\pm(x, \tau)$ are fundamental solutions to the Cauchy problems

$$
\frac{\partial U_\pm}{\partial x} = \lambda_\pm U_\pm, \\
U_\pm|_{x=0} = 1.
$$

Substituting, we obtain

$$
\frac{\partial V}{\partial x} = U_-^{-1} f U_-
$$

and taking into account (6.8),

$$
V(x) = -\int_x^{2\pi} U_+^{-1}(y) f(y) U_-(y) \, dy.
$$

Now, returning to (6.11) and replacing $f(y)$, we come to the integral equation

$$
w_{12}(x) = -\int_x^{2\pi} U_+(x) U_-^{-1}(y) (b_{12}(y) - w_{12}(y)b_{21}(y) w_{12}(y)) U_-(y) U_-^{-1}(x) \, dy.
$$

This equation may be solved by iterations. From Theorem 5.1 and what has been said at the end of Section 5, we deduce that

$$
\|U_+(x) U_-^{-1}(y)\| \leq C \exp(d(x - y) \tau),
$$

(6.12)

$$
\|U_-(y) U_-^{-1}(x)\| \leq C \exp(d(x - y) \tau)
$$

(6.13)

for $\tau \gg 1$ and $x \leq y$. In particular, these expressions are uniformly bounded for $\tau \gg 1$ and $0 \leq x \leq y \leq 2\pi$. The initial iteration

$$
-\int_x^{2\pi} U_+(x) U_-^{-1}(y) b_{12}(y) U_-(y) U_-^{-1}(x) \, dy
$$

may be estimated by means of (6.12), (6.13) as

$$
C \int_x^{2\pi} \exp(2d(x - y) \tau) \, dy \leq \frac{C}{2d\tau} \leq O\left(\frac{1}{\tau}\right).
$$

When combined with the boundedness of (6.12) and (6.13), this estimate implies the convergence of the iterations and the desired estimate (6.10).

Similarly, for $w_{21}$ we obtain an integral equation

$$
w_{21}(x) = \int_0^x U_-(x) U_-^{-1}(y) (b_{21}(y) - w_{21}(y)b_{12}(y) w_{21}(y)) U_+(y) U_+^{-1}(x) \, dy
$$

and then repeat the previous arguments. □
Turning to the block-diagonal part, we denote by $Z_\pm(x, \tau)$ the entries of $Z$. More precisely, we take them as solutions of the Cauchy problems

$$\frac{\partial Z_+}{\partial x} = (\lambda_+ + b_{12} w_{21}) Z_+, \quad Z_+|_{x=0} = 1$$

and

$$\frac{\partial Z_-}{\partial x} = (\lambda_- + b_{21} w_{12}) Z_-, \quad Z_-|_{x=0} = 1.$$  

The crucial property of the coefficients in (6.14) and (6.15) is that, for $\tau \gg 1$, the matrix $\lambda_- + b_{21} w_{12} = \lambda_- + O\left(\frac{1}{\tau}\right)$ is stable and so is $-(\lambda_+ + b_{12} w_{21})$.

In particular, this implies estimates (5.1), (5.6) for $Z_-$ and (5.7) for $Z_+$.

We have thus constructed a solution of the form (6.3), with $W(x, \tau) = O\left(\frac{1}{\tau}\right)$ uniformly in $x$. It does not satisfy the initial condition $U(0, \tau) = 1$, but this drawback can be easily corrected. Indeed,

$$V(x, \tau) = U(x, \tau) U^{-1}(0, \tau) = (1 + W(x, \tau)) Z(x, \tau) (1 + W(0, \tau))^{-1}$$

is the desired solution. For the monodromy matrix, we obtain

$$M(\tau) = V(2\pi, \tau) = (1 + W(2\pi, \tau)) Z(2\pi, \tau) (1 + W(0, \tau))^{-1}$$

$$= \left(1 + O\left(\frac{1}{\tau}\right)\right) \begin{pmatrix} Z_+(2\pi, \tau) & 0 \\ 0 & Z_-(2\pi, \tau) \end{pmatrix} \left(1 + O\left(\frac{1}{\tau}\right)\right).$$

Finally, we apply (6.16) to compute the asymptotic expansion of the phase function

$$\varphi(\tau) = \frac{1}{2} \log \det (M(\tau) + M^{-1}(\tau) - 2)$$
for $\Re \tau \to \pm \infty$ and $|\Im \tau| \leq C$. All the calculations will be performed modulo $\pi i$. From (6.16) it follows that
\[
M(\tau) + M^{-1}(\tau) - 2 = \left(1 + O\left(\frac{1}{\tau}\right)\right) \left\{ Z(2\pi, \tau) \left(1 + O\left(\frac{1}{\tau}\right)\right) \right.
\]
\[
+ \left(1 + O\left(\frac{1}{\tau}\right)\right) Z^{-1}(2\pi, \tau) - 2 \left(1 + O\left(\frac{1}{\tau}\right)\right) \left\}\right. \left(1 + O\left(\frac{1}{\tau}\right)\right)
\]
implying
\[
\varphi(\tau) = \frac{1}{2} \log \det \left\{ Z(2\pi, \tau) \left(1 + O\left(\frac{1}{\tau}\right)\right) \right.
\]
\[
+ \left(1 + O\left(\frac{1}{\tau}\right)\right) Z^{-1}(2\pi, \tau) - 2 \left(1 + O\left(\frac{1}{\tau}\right)\right) \right\} + O\left(\frac{1}{\tau}\right).
\]

A straightforward computation shows that the expression in curly brackets transforms further to
\[
\left( Z_+(2\pi, \tau) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left(1 + O\left(\frac{1}{\tau}\right)\right) \left( Z_-^{-1}(2\pi, \tau) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
\[
+ \left( Z_+^{-1}(2\pi, \tau) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left(1 + O\left(\frac{1}{\tau}\right)\right) \left( Z_-^{-1}(2\pi, \tau) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
\[
- 2 \left( Z_+^{-1}(2\pi, \tau) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left(1 + O\left(\frac{1}{\tau}\right)\right) \left( Z_-^{-1}(2\pi, \tau) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right).
\]

Now, $Z_+^{-1}(2\pi, \tau)$ and $Z_-^{-1}(2\pi, \tau)$ decay exponentially for $\Re \tau \to +\infty$. Indeed, applying (5.1) for $x = 2\pi$ and $y = 0$, we get
\[
\|Z_-^{-1}(2\pi, \tau)\| \leq C \exp(-2\pi d\tau)
\]
\[
= O\left(\frac{1}{\tau}\right);
\]
the same is true for $Z_+^{-1}(2\pi, \tau)$, as may be seen from (5.7) for $x = 0$ and $y = 2\pi$. Hence the previous expression can be rewritten as
\[
\left( Z_+(2\pi, \tau) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) \left(1 + O\left(\frac{1}{\tau}\right)\right) \left( Z_-^{-1}(2\pi, \tau) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)
\]
so that
\[
\varphi(\tau) = \frac{1}{2} \log \det Z_+(2\pi, \tau) - \frac{1}{2} \log \det Z_-^{-1}(2\pi, \tau) + O\left(\frac{1}{\tau}\right).
\]
Finally, using the Liouville formula for $\det Z_+$ and $\det Z_-$, we arrive at (6.17)

$$
\varphi(\tau) = \frac{1}{2} \int_0^{2\pi} \text{tr} (\lambda_+ + b_{12}w_{21}) dx - \frac{1}{2} \int_0^{2\pi} \text{tr} (\lambda_- + b_{21}w_{12}) dx + O\left(\frac{1}{\tau}\right)
$$

$$
= \frac{1}{2} \int_0^{2\pi} (\text{tr} (a_+(x)\tau + b_{11}(x)) - \text{tr} (a_-(x)\tau + b_{22}(x))) dx + O\left(\frac{1}{\tau}\right).
$$

Similarly, an asymptotic formula for $\varphi(\tau)$ may be obtained as $\Re \tau \to -\infty$ and $|\Im \tau| \leq C$. The result will be given by (6.17) with the opposite sign. We summarize these results as follows.

**Theorem 6.2.** Let $\Re \tau \to \pm \infty$ and $|\Im \tau| \leq C$. Then the following asymptotic formulas hold:

(6.18)

$$
\varphi(\tau) = \pm \frac{1}{2} \int_0^{2\pi} (\text{tr} (a_+(x)\tau + b_{11}(x)) - \text{tr} (a_-(x)\tau + b_{22}(x))) dx + \pi i N_\pm
$$

$$
+ O\left(\frac{1}{\tau}\right).
$$

The integers $N_{\pm}$ remain undetermined. We may fix one of them, then the other will depend on the path to be used for analytic extension.

**Corollary 6.3.** The variation $\Delta_{\Gamma, \tau_0}\varphi(\tau)$ is an integer multiple of $\pi i$.

**Proof.** Using (6.18) we write

$$
\Delta_{\Gamma, \tau_0}\varphi(\tau) = \pi i (N_+ - N_-)
$$

$$
+ \frac{1}{2} \int_0^{2\pi} (\text{tr} (a_+(x)(\tau_0 + T) + b_{11}(x)) - \text{tr} (a_-(x)(\tau_0 + T) + b_{22}(x))) dx
$$

$$
+ \frac{1}{2} \int_0^{2\pi} (\text{tr} (a_+(x)(\tau_0 - T) + b_{11}(x)) - \text{tr} (a_-(x)(\tau_0 - T) + b_{22}(x))) dx.
$$

The two integral terms give

$$
\int_0^{2\pi} (\text{tr} (a_+(x)\tau_0 + b_{11}(x)) - \text{tr} (a_-(x)\tau_0 + b_{22}(x))) dx
$$

which is zero in virtue of Lemma 4.1. □

**References**


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ORDERING THE BRAID GROUPS

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We give an explicit geometric argument that Artin’s braid group $B_n$ is right-orderable. The construction is elementary, natural, and leads to a new, effectively computable, canonical form for braids which we call left-consistent canonical form. The left-consistent form of a braid which is positive (respectively negative) in our order has consistently positive (respectively negative) exponent in the smallest braid generator which occurs. It follows that our ordering is identical to that of Dehornoy (1995) constructed by very different means, and we recover Dehornoy’s main theorem that any braid can be put into such a form using either positive or negative exponent in the smallest generator but not both.

Our definition of order is strongly connected with Mosher’s (1995) normal form and this leads to an algorithm to decide whether a given braid is positive, trivial, or negative which is quadratic in the length of the braid word.

0. Introduction.

Dehornoy [5, 6, 7] has proved that the braid group is right-orderable. More precisely, there is a total order on the elements of the braid group $B_n$ which is right invariant in the following sense. Suppose that $\alpha, \beta, \gamma \in B_n$ and $\alpha < \beta$, then $\alpha \gamma < \beta \gamma$. This ordering is uniquely defined by the condition that a braid $\beta_0\sigma_i\beta_1$ is positive (i.e. greater than the identity braid), where $\beta_0, \beta_1$ are words in $\sigma_{i+1}, \ldots, \sigma_{n-1}$. Dehornoy’s proof is based on some highly complicated algebra connected with left-distributive systems. In this paper we construct this order geometrically using elementary arguments.

Our construction leads to a new, effectively computable, canonical form for braids which we call left-consistent canonical form. The left-consistent form of a positive braid has the general shape

$$\beta_0\sigma_i^e\beta_1\ldots\beta_{l-1}\sigma_i^e\beta_l$$

where the $\beta_i$ are words in $\sigma_{i+1}, \ldots, \sigma_{n-1}$ and their inverses, and $e = +1$. For a negative braid the form is similar but with $e = -1$. It follows at once that our ordering is identical to Dehornoy’s and we recover Dehornoy’s main theorem that any braid can be put into such a shape for $e = 1$ or $e = -1$ but not both.
Our definition of order is strongly connected with Mosher’s automatic structure [13] and this implies that the braid group is order automatic, i.e., the order can be detected from the automatic normal form by a finite state automaton. Furthermore the resulting algorithm to decide whether a given braid is positive, trivial, or negative is linear in the length of the Mosher normal form and hence quadratic in the length of the braid word (in contrast, Dehornoy’s algorithm [7], although apparently fast in practice is only known to be exponential).

The paper is organised as follows. Section 1 contains basic definitions and introduces the curve diagram associated to a braid. In Section 2 we prove that a curve diagram can be placed in a unique reduced form with respect to another and in Section 3 we define the order by comparing the two curve diagrams in reduced form, and prove that it is right-invariant. In Section 4 we construct the left-consistent canonical form of a braid, deduce that our order coincides with Dehornoy’s and recover Dehornoy’s results. In Section 5 we give some counterexamples connected with the order, and in Section 6 we make the connection with Mosher’s normal form and deduce the existence of the quadratic time algorithm to detect order. Finally, in an appendix, we use cutting sequences to give a formal algorithm to turn a braid into the new left-consistent canonical form; note that this algorithm is not quadratic time.

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1. Braids and curve diagrams.

Let $D^2$ be the closed unit disk in $\mathbb{C}$, and let $D_n$ be the disk $D^2$ with $n$ distinct points in the real interval $(-1,1)$ removed. We consider the group $B_n$ of self-homeomorphisms $\gamma: D_n \to D_n$ with $\gamma|_{\partial D_n} = id$, up to isotopy of $D_n$ fixed on $\partial D_n$. Multiplication in $B_n$ is defined by composition. The group $B_n$ is well-defined independently of the points removed; indeed if $D'_n$ is a disk with any $n$-tuple of points removed and $B'_n$ the corresponding group then there is an isomorphism $B'_n \cong B_n$; if these points also lie on $(-1,1)$ then this isomorphism is natural.

The group $B_n$ is isomorphic to the group $\tilde{B}_n$ of braids on $n$ strings, with multiplication given by concatenation: If $\alpha, \beta$ are braids (pictured vertically)
then $\alpha \cdot \beta$ is $\alpha$ above $\beta$. It is well known that this group has presentation
\[
\tilde{B}_n \cong \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| \geq 2, \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \rangle,
\]
where the generator $\sigma_i$ ($i \in \{1, \ldots, n-1\}$) is indicated in Figure 1.

![Figure 1. The standard generator $\sigma_i$ of the braid group on $n$ strings.](a) (b) (c)

The isomorphism $\tilde{B}_n \rightarrow B_n$ is given by ‘putting the braid in a solid cylinder and sliding $D_n$ once along it’. The inverse map is defined as follows: Extend a given homeomorphism $\gamma: D_n \rightarrow D_n$ to a homeomorphism $\gamma': D^2 \rightarrow D^2$, then find a boundary-fixing isotopy $\gamma_t: D^2 \rightarrow D^2$ with $\gamma_0 = id$ and $\gamma_1 = \gamma$. Then the flow of the $n$ holes of $D_n$ under $\gamma_t$ defines a braid on $n$ strings. For details see [2].

On $D_n$ we draw $n+1$ line segments as in Figure 2(a). If $\gamma$ is a homeomorphism of $D_n$ representing an element $[\gamma]$ of $B_n$, then $\gamma$ sends these line segments to $n+1$ disjoint embedded curves, and if $[\gamma_1] = [\gamma_2] \in B_n$ then $\gamma_1$ and $\gamma_2$ give rise to isotopic collections of curves. For instance, Figure 2 shows the effect of the braid $\sigma_1 \sigma_2^{-1} \in B_3$. Here the holes of $D_n$, as well as $\pm 1$ are indicated by black dots. We call such a diagram of $n+1$ disjoint simple curves in an $n$–punctured disk a curve diagram, and we number the curves in the diagram 1 to $n+1$, as in Figure 2.

![Figure 2. Curve diagrams of the braids $1$, $\sigma_1$, and $\sigma_1 \sigma_2^{-1}$.](a) (b) (c)

Conversely, from the curve diagram we can reconstruct the homeomorphism $\gamma$ up to boundary-fixing isotopy; that is, we can reconstruct the element of the braid group.
2. Reduced form.

Let \( \Gamma \) and \( \Delta \) be curve diagrams of two braids \( \gamma \) and \( \delta \), say. In order to compare \( \Gamma \) and \( \Delta \), we superimpose the two diagrams and reduce the situation by removing unnecessary intersections. This process is well known and often called “pulling tight” (see e.g. [13]).

We will denote the \( i \)th curve of a curve diagram such as \( \Gamma \) by \( \Gamma_i \). The curves \( \Gamma_i \) and the \( \Delta_j \) are called parallel if they connect the same pairs of points and are isotopic in \( D_n \). For instance, curve 3 of Figure 2(b) and curve 2 of Figure 2(c) are parallel. We define \( \Delta \) to be transverse to \( \Gamma \) if every curve of \( \Delta \) either coincides precisely with some (parallel) curve of \( \Gamma \), or intersects the curves of \( \Gamma \) transversely.

We define the intersection index of two transverse curve diagrams to be

\[
 n + 1 + \#(\text{transverse intersections}) - \#(\text{coincident curves}).
\]

(The geometric significance is that \( D_n \) cut along \( \Gamma \) has two components, and cutting in addition along \( \Delta \) increases the number of components by the intersection index.) For example, the diagrams in Figure 2(a) and 2(c) have intersection index 6, the diagrams in Figure 2(a) and 2(b) have intersection index 2 and the diagrams in Figure 2(b) and 2(c) have intersection index 5. The intersection index of two curve diagrams is 0 if and only if the diagrams are identical.

We now fix a curve-diagram \( \Gamma \) for \( \gamma \), and look at all possible curve-diagrams for \( \delta \). They are all isotopic in \( D_n \), but they may have very different intersection-indices with \( \Gamma \). We say \( \Delta \) and \( \Delta' \) are equivalent (with respect to \( \Gamma \)) if they are related by an isotopy of \( D_n \), which is fixed on curves of \( \Delta \) which coincide with curves of \( \Gamma \), and which leaves the diagrams transverse all the time. So coincident curves remain coincident, and the intersection index remains unchanged.

We define a \( D \)-disk\(^1 \) between \( \Delta \) and \( \Gamma \) to be a subset of \( D_n \) homeomorphic to an open disk, which is bounded by one open segment of some curve of \( \Delta \), one open segment of some curve of \( \Gamma \), and two points, each of which may be an intersection-point of the two curves or one of the ‘holes’ of \( D_n \), or \( \pm 1 \in D_n \). There are three types of \( D \)-disks (types (a), (b), and (c)), indicated in Figure 3, where the curve-diagram \( \Gamma \) is drawn with dashed, and \( \Delta \) with solid lines, and the dots denote holes or \( \pm 1 \).

If there are no \( D \)-disks between \( \Delta \) and \( \Gamma \) then we say \( \Delta \) and \( \Gamma \) are reduced. If the curve-diagrams \( \Delta \) and \( \Gamma \) are not reduced, i.e. if they have a \( D \)-disk, then we can isotope \( \Delta \) so as to reduce the intersection index with \( \Gamma \) (Figure 3). This isotopy consists of ‘sliding a segment of a curve of \( \Delta \) across the \( D \)-disk’ (for reduction moves (I) and (II)), and of ‘squashing a \( D \)-disk to a line’ (for reduction move (III)). The three moves reduce the intersection index.

\(^1\)\( D \)-disks are often called “bigons” in the literature.
index by 2, 1, 1, respectively. We observe that any curve-diagram \( \Delta \) with intersection index 0 with \( \Gamma \) is reduced. Thus we can reduce curve diagrams by a finite sequence of ‘isotopies across \( D \)-disks’ as in Figure 3.

**Lemma 2.1** (Triple reduction lemma). Suppose \( \Sigma, \Gamma \) and \( \Delta \) are three curve diagrams such that \( \Gamma \) and \( \Delta \) are both reduced with respect to \( \Sigma \). Then there exists an isotopy between \( \Delta \) and a curve diagram \( \Delta' \), which is an equivalence with respect to \( \Sigma \), such that \( \Sigma, \Gamma \) and \( \Delta' \) are pairwise reduced.

**Proof.** We consider a \( D \)-disk bounded by one segment of curve of \( \Gamma \) and one of \( \Delta \). This \( D \)-disk may have several intersections with \( \Sigma \). There are, a priori, three possibilities for the type of such an intersection — they are indicated in Figure 4, labelled (1), (2), and (3). (In this figure, the \( D \)-disk is of type (b), the cases of types (a) and (c) are similar.)

However, (1) and (2) are impossible, because \( \Gamma \) and \( \Delta \) are reduced with respect to \( \Sigma \). So all intersections are of type (3), and the \( D \)-disk can be removed without disturbing the reduction of \( \Sigma \) with respect to \( \Gamma \) or \( \Delta \), as indicated in Figure 4. The statement follows inductively. \( \square \)

**Figure 4.** The solid line is \( \Delta \), the dashed \( \Gamma \), and the dotted \( \Sigma \).

**Lemma 2.2.** If two isotopic curve diagrams \( \Gamma \) and \( \Delta \) are reduced with respect to each other, then they coincide.
Proof. Suppose that the first curve $\Gamma_1$ of $\Gamma$ does not coincide with the first curve $\Delta_1$ of $\Delta$. Consider the word obtained by reading the intersections of $\Gamma_1$ with the curves of $\Delta$ in order. Since $\Gamma_1$ is isotopic to $\Delta_1$, this word must cancel to the trivial word. It follows by a simple innermost disk argument that there must be a $D$-disk. Hence $\Gamma_1$ must coincide with $\Delta_1$. Similarly all curves of $\Gamma$ and $\Delta$ must coincide. □

Proposition 2.3. If two curve diagrams $\Delta$ and $\Delta'$ of a braid $\delta$ are reduced with respect to $\Gamma$ then they are equivalent with respect to $\Gamma$.

Proof. By the triple reduction lemma we may reduce $\Delta$ with respect to $\Delta'$ by an isotopy of $\Delta$ which is an equivalence with respect to $\Gamma$. After this reduction $\Delta$ and $\Delta'$ coincide by Lemma 2.2. □

We have proved that by reducing a curve diagram $\Delta$ with respect to a curve diagram $\Gamma$ we can bring $\Delta$ into a uniquely defined standard form with respect to $\Gamma$. In particular reduction of $\Delta$ with respect to the trivial curve diagram representing $1 \in B_n$ (Figure 2(a)) leads to a canonical representation of braids in terms of cutting sequences, which will be discussed in detail in the appendix.

Remark 2.4. The following observation will be crucial at a later point. Let $\Gamma$ and $\Delta$ be transverse curve diagrams. Suppose the curve $\Delta_i$ on its own is reduced with respect to $\Gamma$. Then we can reduce $\Delta$ with respect to $\Gamma$ by an isotopy of $\Delta$ which is fixed on $\Delta_i$.

3. The right-invariant order on $B_n$.

We define a total ordering on the braid group $B_n$ as follows. Suppose $\gamma$ and $\delta$ are two braids on $n$ strings. We let $\Gamma$ be a curve diagram for $\gamma$, and $\Delta$ be a curve diagram for $\delta$ which is reduced with respect to $\Gamma$. The collection of curves of $\Gamma$ cuts $D_n$ into two components, which we call the upper and the lower component, containing the points $\sqrt{-1}$ respectively $-\sqrt{-1}$ in $D_n \subseteq \mathbb{C}$. We orient the curves of $\Delta$ coherently such that we obtain a path starting at $-1 \in D_n$ and ending at $1 \in D_n$.

If all curves of $\Delta$ coincide with the corresponding curves of $\Gamma$ then the braids $\gamma$ and $\delta$ are equal. Suppose that curves $1, 2, \ldots, i - 1$ of $\Delta$ agree with the corresponding curves of $\Gamma$, and the $i$th is the first transverse one, $1 \leq i \leq n + 1$. This oriented curve has the same startpoint as the $i$th curve of $\Gamma$, and first branches off $\Gamma$ either into the upper or the lower component. In the first case we define $\delta > \gamma$, in the second $\delta < \gamma$. This is well-defined, by Proposition 2.3.

Example. All the diagrams in Figure 2 are reduced with respect to each other, and we observe that $1 < \sigma_1 \cdot \sigma_2^{-1} < \sigma_1$. 
Proposition 3.1. The relation ‘<’ is an ordering, i.e., if \( \sigma, \gamma, \delta \) are braids with \( \sigma < \gamma < \delta \) then \( \sigma < \delta \).

Proof. By the triple reduction Lemma 2.1 we can find curve diagrams \( \Sigma, \Gamma, \Delta \) of these braids which are all pairwise reduced. The statement of the proposition follows immediately: If \( \Gamma \) branches off \( \Sigma \) to the left and \( \Delta \) branches off \( \Gamma \) to the left, then \( \Delta \) branches off \( \Sigma \) to the left. \( \square \)

Proposition 3.2. The ordering ‘<’ is right invariant.

Proof. Suppose we have two braids \( \delta \) and \( \gamma \) with \( \delta < \gamma \), and with reduced curve diagrams \( \Delta \) and \( \Gamma \). Let \( \sigma \) be a further braid, i.e. a homeomorphism of \( D_n \) fixing \( \partial D_n \). We obtain the curve diagrams for \( \delta \cdot \sigma \) and \( \gamma \cdot \sigma \) by applying \( \sigma \) to \( \Delta \) and \( \Gamma \). The resulting curve diagrams \( \sigma(\Delta) \) and \( \sigma(\Gamma) \) are still reduced, and \( \sigma(\Delta) \) still branches off \( \sigma(\Gamma) \) into the lower component of \( D_n \setminus \sigma(\Gamma) \), so \( \delta \cdot \sigma < \gamma \cdot \sigma \). \( \square \)

Let \( \epsilon \) be the trivial braid, with standard curve diagram \( E \) (see Figure 2(a)). We call a braid \( \gamma \) positive if \( \gamma > \epsilon \), and negative if \( \gamma < \epsilon \). If we want to stress that the first \( i-1 \) curves of \( \Gamma \) are parallel to the corresponding curves of \( E \), and the \( i \)th is the first non-parallel one, then we say \( \gamma \) is \( i \)-positive respectively \( i \)-negative. Since there is a very similar concept of \( \sigma_i \)-positive (see the next section) we shall often say geometrically \( i \)-positive or negative. Given two braids \( \gamma \) and \( \delta \) such that \( \gamma > \delta \) we say \( \gamma \) is (geometrically) \( i \)-greater than \( \delta \) if the \( i \)th curves are the first non-parallel ones. Any curve diagram in which the first \( i-1 \) curves are parallel to the corresponding curves of \( E \) is called \( (i-1) \)-neutral.

We note some simple consequences of right invariance. We have \( \gamma > \epsilon \) if and only if \( \epsilon > \gamma^{-1} \), so the inverse of a positive braid is negative. If \( \gamma > \epsilon \) and \( \delta \) is any braid, then \( \gamma \delta > \delta \). (Warning: it need not be true that \( \delta \gamma > \delta \) — see the next section.) In particular, the product of positive braids is positive.

4. Left-consistent canonical form.

In this section we connect our ordering with Dehornoy’s [5]. The following definition is taken from [5]. A word of the form

\[ \beta_0 \sigma_i \beta_1 \sigma_i \ldots \sigma_i \beta_k, \]

where \( i \in \{1, \ldots, n-1\} \), and \( \beta_0, \ldots, \beta_k \) are words in the letters \( \sigma_i^{\pm 1} \ldots \sigma_{i+1}^{\pm 1} \) is called a \( \sigma_i \)-positive word. A braid is \( \sigma_i \)-positive if it can be represented by a \( \sigma_i \)-positive word. A braid is called \( \sigma_i \)-negative if its inverse is \( \sigma_i \)-positive. We shall say that a braid is \( \sigma \)-positive or negative if it is \( \sigma_i \)-positive or negative for some \( i \). The following is the main result from [5]:

**Dehornoy’s theorem.** Every braid is precisely one of the following three: \( \sigma \)-positive, or \( \sigma \)-negative, or trivial.
Dehornoy uses this theorem to define a right-invariant order by $\alpha > \beta \iff \alpha\beta^{-1}$ is $\sigma$-positive. We shall prove that this order coincides with the order we defined in the last section by showing that the concepts of geometrically $i$-positive and $\sigma_i$-positive coincide. One way is easy.

**Proposition 4.1.** A braid which is $\sigma_i$-positive is geometrically $i$-positive.

**Proof.** A braid which can be represented by a word $\beta\sigma_i\beta'$, where $\beta, \beta'$ are words in the letters $\sigma_{i+1}, \ldots, \sigma_{n-1}$, is geometrically $i$-positive. To see this think of the homeomorphism determined by the braid word as a sequence of twists of adjacent holes around each other: $\beta$ leaves the first $i$ curves untouched and then $\sigma_i$ twists the $i$th hole around the $(i+1)$st producing a curve diagram in which the $i$th curve moves into the upper half of $D_n$; finally $\beta'$ leaves the start of the $i$-th curve untouched. Now by definition, every Dehornoy positive braid is a product of such words. The proposition now follows from the fact that the product of two geometrically $i$-positive braids is again geometrically $i$-positive. \hfill $\Box$

The proposition immediately implies part of Dehornoy’s theorem: every braid can take at most one of the three possible forms. To complete the proof that the concepts of geometrically $i$-positive and $\sigma_i$-positive coincide and to recover the remainder of Dehornoy’s theorem we shall construct a canonical $\sigma_i$-positive form for a given geometrically $i$-positive braid. This is the left-consistent canonical form of the braid:

**Theorem 4.2** (Left-consistent canonical form). Let $\gamma$ be a geometrically $i$-positive braid. Then there is a canonically defined $\sigma_i$-positive word which represents the same element of $B_n$.

We define the complexity of a braid $\gamma$ as follows. Take a curve diagram $\Gamma$ for $\gamma$ which is reduced with respect to the trivial curve diagram $E$. Suppose that the first $j - 1$ curves of $\Gamma$ coincide with the first $j - 1$ curves of $E$ and that the $j$th curve does not. Let $m \geq 0$ be the number of transverse intersections of $\Gamma$ with $j$th curve of $E$. The complexity of $\gamma$ is the pair $(j,m)$. We order complexity lexicographically with $j$ in reverse order. Thus $(1,m)$ is more complex than $(2,n)$ for any $m, n$ whilst $(j,m)$ is more complex than $(j,n)$ if and only if $m > n$. The main step in the proof of Theorem 4.2 is the following:

**Proposition 4.3.** Suppose that $\gamma$ is a geometrically $i$-positive braid. Then there is a word $\beta$ in the braid generators $\sigma_{i}, \ldots, \sigma_{n-1}$ and their inverses such that

1. $\beta$ contains $\sigma_i^{-1}$ exactly once
2. $\beta$ does not contain $\sigma_i$
3. $\gamma\beta$ is either geometrically $i$-positive or geometrically $i$-neutral
4. $\gamma\beta$ has smaller complexity than $\gamma$. 

Furthermore there is a canonical choice for $\beta$.

Theorem 4.2 follows from Proposition 4.3 by induction on complexity because, by (4) and induction, $\gamma' := \gamma \beta$ has a canonical form which by (3) is either $\sigma_i$-positive or $\sigma_j$-positive or negative for $j > i$ and then $\gamma' \beta^{-1}$ is the canonical form for $\gamma$.

Proof of Proposition 4.3. For definiteness we shall deal with the case $i = 1$ first. (We shall see that the general case is essentially the same as this case.) So let $\gamma$ be a geometrically 1-positive braid and $\Gamma$ a curve diagram for $\gamma$ which is reduced with respect to the trivial curve diagram. We shall define $\beta$ geometrically by sliding one particular hole of $D_n$ along a useful arc.

Let $E_1 \subseteq D_n$ be the 1st curve of $E$, i.e. a straight line from $-1$ to the leftmost hole of $D_n$, excluding this hole. We define a useful arc to be a segment $b$ of some curve of $\Gamma$ starting at some point of $E_1$ (possibly $-1$), and ending at some hole of $D_n$ other than the leftmost one such that

- the interior of $b$ does not intersect $E_1$,
- an initial segment of the arc $b$ lies in the upper half of the disk, i.e. the intersection of a neighbourhood of $E_1$ with the interior of $b$ consists of a line segment in the upper component of $D_n \setminus E$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Slide of a hole along a useful arc, followed by a reduction.}
\end{figure}

Suppose that $\Gamma$ contains useful arcs. Then each of them has precisely one point of intersection with $E_1$ and we call the one whose intersection point is leftmost the leftmost useful arc. Let $b$ be the leftmost useful arc. If $b$ starts in the interior of $E_1$, then we can slide the hole of $D_n$ at the endpoint of
b along b and back into $E_1$. If b starts at $-1$, then we push a small initial segment of b into $E_1$, and then perform the slide of the hole of $D_n$ (see Figure 5, where b is dotted). In either case we obtain a curve diagram $\Gamma'$ representing a braid $\gamma'$. Now $\Gamma'$ need not be reduced with respect to $E$. But notice that $\gamma'$ has lower complexity than $\gamma$ since the new $E_1$ now stops at the intersection of b with the old $E_1$ and hence there are fewer intersections with $\Gamma'$ even before reduction.

The movement of the hole of $D_n$ along b defines a braid $\beta$ on n strings, with $\gamma' = \gamma \beta$. Furthermore we can decompose $\beta$ as a canonical word in the generators $\sigma_j$ by writing down the appropriate $\sigma_j$ or $\sigma_j^{-1}$ whenever the hole passes over or under another hole. But, by definition of useful arc, the hole only passes once over or under the leftmost hole and it passes over and to the left and hence the word that we read contains $\sigma_1^{-1}$ only once and does not contain $\sigma_1$.

Therefore to prove case $i = 1$ of Proposition 4.3 it remains to prove the following two claims:

Claim 1. The diagram contains a useful arc.

Claim 2. The diagram $\Gamma'$ obtained by sliding a hole of $D_n$ along the leftmost useful arc is either 1-positive or 1-neutral, but not 1-negative.

To prove Claim 1, we consider the first curve of $\Gamma$ starting at $-1$. If it ends in a hole other than the leftmost one and does not intersect $E_1$ then it is a useful arc (Figure 6(a)). Otherwise we consider the closed curve in $D^2$ starting at $-1$, along the first curve of $\Gamma$, up to its first intersection with the closure of $E_1$ in $D^2$, and then back in a straight line to the point $-1$. This curve bounds a disk $S$ in $D^2$, which may be of three different types: $\Gamma$ hits $E_1$ either from above, or from below, or in the leftmost hole of $D_n$ (see Figure 6(b),(d),(c)). In cases (b) and (c) we note that since $\Gamma$ and $E$ are reduced, at least one hole of $D_n$ must lie in the interior of $S$. Moreover, all holes of $D_n$ are connected by curves of $\Gamma$, so there exists a curve of $\Gamma$ connecting one of the holes in $S$ to one of the holes outside $S$ or the point $1 \in D_n$. The first component of the intersection of this curve with $S$ is a useful arc.

In case (d) we walk along the oriented curve in $D^2$ starting at $-1$, along the curves of $\Gamma$. We write down the symbol + whenever we hit $E_1$ from below and ~ if we hit $E_1$ from above or in the leftmost hole of $D_n$. The sequence starts with a +, and since the curve has to leave the disk $S$ it must contain a ~. It follows that the string +~ must occur in the sequence; it represents an arc which, together with a segment of $E_1$, bounds a disk $S'$ in $D^2$. See Figure 6(d): $S'$ is bounded by part of the dotted arc between two intersections with $E_1$ and part of $E_1$. Since $\Gamma$ and $E$ are reduced, $S'$ contains a hole other than the leftmost one in its boundary or in its interior. In the first case, a segment of top (dotted) boundary of $S'$ is a useful arc; in
the second case the disk $S'$ is of the type indicated in Figure 6(b) or (c), so there is a useful arc inside $S'$. This finishes the proof of Claim 1.

To prove Claim 2, we distinguish two cases: Either the leftmost useful arc $b$ starts at the point $-1$, or it starts at some point in the interior of $E_1$. In the first case (e.g. Figure 5(b)), the curve diagram $\Gamma'$ obtained by sliding a hole along $b$ to near $-1$ is 1-neutral.

In the second case (Figure 5(a)) the curve diagram $\Gamma'$ is 1-positive, as we now prove. We recall that we had $\gamma' = \gamma \beta$, where $\beta$ represents the slide of a hole along the leftmost useful arc $b$. We observe that we can construct a curve diagram of the braid $\beta^{-1}$ such that the first curve $b_1$ of the diagram is a line segment in $E_1$ from $-1$ almost all the way to $E_1 \cap b$, followed by an arc parallel and close to the arc $b$, and finally running into the same hole as $b$. The construction of the arc $b_1$ is illustrated in Figure 7(a).

Next we examine the possible reductions of $\Gamma$ with respect to this arc $b_1$. If there was a $D$-disk of type (b) whose boundary contained the arc $b$, (i.e. to the right of $b_1$ in Figure 7) then cutting off the strip bounded by $b$, $b_1$ and $E_1$ would yield a $D$-disk of type (a) of $\Gamma$ with respect to $E$ (see Figure 7(a)). This is impossible by hypothesis. If there was a $D$-disk of type (b) whose boundary contained a final segment of the arc $b_1$ and a segment other than $b$ of a curve of $\Gamma$, (i.e. to the left of $b_1$ in Figure 7) then this segment would be a useful arc intersecting $E_1$ more to the left than $b$ (Figure 7(b)), which is also impossible. Finally, any $D$-disk of type (a) of $\Gamma$ with respect to $b_1$ would also be a $D$-disk of $\Gamma$ with respect to $E_1$. So there are no $D$-disks between $b_1$ and $\Gamma$. By Remark 2.4 it follows that we can reduce the curve diagram of $\beta^{-1}$ with respect to $\Gamma$ without touching its first curve $b_1$. We can now observe that $\gamma$ is 1-greater than $\beta^{-1}$, ie $\gamma'$ is 1-positive, as claimed. This completes the proof of Claim 2.
Finally we turn to the case when \( i \) may not be 1. In this case the first \( i - 1 \) holes are lined up near \(-1\) on the real axis. The same argument as in the case \( i = 1 \), only with the \( i - 1 \)st hole and the line segment \( E_i \) playing the role previously played by \(-1\) and \( E_1 \) respectively, completes the proof of the general case.

The proof of Theorem 4.2 provides an explicit algorithm for converting a braid into its left-consistent canonical form. In the appendix we give a formal version of this algorithm using cutting sequences.

**Remark.** The order on the braid group has the property that inserting a generator \( \sigma_i \) anywhere in a braid word makes the braid larger. A proof of this fact, in the spirit of this paper, is given in [17]. This property is equivalent to the statement that the order extends the *subword order* defined by Elrifai and Morton [8] and an algebraic proof has been given by Laver [11].

5. **Counterexamples.**

We shall call a braid word *\( \sigma \)-consistent* (Dehornoy in [7] calls it *reduced*) if it is \( \sigma \)-positive, \( \sigma \)-negative or trivial. We have seen in the previous chapter that every braid has at least one \( \sigma \)-consistent representative. The aim of this chapter is to disprove some plausible-sounding but over-optimistic conjectures about the ordering and about \( \sigma \)-consistent representatives of braids.
Left invariance on the pure braid group.

Because the pure braid group has an ordering which is simultaneously left and right invariant [15], it would be tempting to think that the geometric ordering is left and right invariant when restricted to the pure braid group. However this is equivalent to saying that a pure positive braid, when conjugated by any pure braid, is again positive and the example in Figure 8 shows this to be false. In $B_3$, the braid group on three strings, we conjugate the pure positive braid $\sigma_1^2\sigma_2^{-2}$ by the pure braid $\sigma_2\sigma_1^2\sigma_2$. The figure shows the equivalence of the resulting braid with the $\sigma$-negative braid $\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$. We are moving first the string segment $\overline{ab}$ and then the segment $\overline{cd}$ ‘over’ the braid ‘to the left of the braid’.

Simultaneously shortest and $\sigma$-consistent representatives.

For any element $b$ of the braid group $B_n$ ($n \geq 2$), there are two ways to represent $b$ by a particularly simple word $w$ in the letters $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$.

(1) $b$ can be represented by a word which is as short as possible. For instance, we shall see later that the word $w_1 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}$ is a shortest possible representative of a braid in $B_4$ (see Figure 9(a)).

(2) $b$ can be represented by a $\sigma$-consistent word. For instance, in the braid word $w_2 = \sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_3\sigma_2$, which represents the same element of $B_4$ as $w_1$, the letter $\sigma_1$ occurs only with positive exponent, see Figure 9(b).
Figure 9. The equivalent braids $\sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1}$ and $\sigma_2^{-1}\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_3\sigma_2$.

**Theorem 5.1.** Every element of $B_n$ for $n = 2, 3$ has a simultaneously shortest and $\sigma$-consistent representative. By contrast, there are braids in $B_n$ for $n \geq 4$ all of whose $\sigma$-consistent representatives have non-minimal length.

**Proof.** The case $n = 2$ is obvious. The case $n = 3$ follows from the fact that in $B_3$ Dehornoy’s handle-reduction algorithm [7] never increases the length of a braid word, and hence turns any shortest representative of a given braid into a simultaneously shortest and $\sigma$-consistent one.

For the case $n \geq 4$ it suffices to prove that the braid $b := \sigma_1\sigma_2\sigma_3^{-1}\sigma_2\sigma_1^{-1} \in B_4$ (Figure 9) has length 5, while every $\sigma$-consistent representative has more than five letters.

To see that every representative has at least five letters we note that the image of $b$ under the natural homomorphism $B_4 \rightarrow S_4$, from the braid group into the symmetric group, is the permutation $(14)$. This permutation cannot be written as a product of less than five adjacent transpositions. The result follows.

We now assume, for a contradiction, that there exists a five-letter representative which is also $\sigma$-consistent. This would be a braid on four strands with the following properties:

(i) its image under the natural map $B_4 \rightarrow S_4$ is $(14)$,
(ii) it has five crossings (i.e., it is a word with five letters),
(iii) if we denote by $c(i, j)$ ($i, j \in \{1, \ldots, 4\}$) the algebraic crossing number of the $i$th and the $j$th string, then the braid must satisfy $c(1, 2) = 1$, $c(1, 3) = 1$, $c(1, 4) = -1$, $c(2, 3) = 0$, $c(2, 4) = -1$, $c(3, 4) = 1$,
(iv) it may contain the letter $\sigma_1$, but not $\sigma_1^{-1}$ (note that there exists a representative of $b$ in which $\sigma_1$ occurs only positively, so there can’t exist a consistently negative one).

There are only three braids satisfying (i)-(iii), pictured in Figure 10, and we observe that none of them satisfies (iv). It follows that no $\sigma$-consistent representative of $b$ with only five crossings exists. 

Minimal number of occurrences of the main generator.
We define the main generator of a braid word to be the generator with lowest index occurring in the word. It is tempting to think that sliding holes along leftmost useful arcs, as in the left consistent canonical form, is the most efficient way of reducing the number of intersections between the curve diagram and the line segment $E_1$. This, however, is wrong:

**Theorem 5.2.** There are braids whose left consistent canonical form does not have the minimal number of occurrences of the main generator among all $\sigma$-consistent representatives.

**Proof.** We shall show that the braid $\Delta^3$, where $\Delta = \sigma_2\sigma_1\sigma_2$, has this property. Note that $\Delta$ is just a half-twist, so $\Delta^2$ generates the centre of $B_3$.

We have $(\sigma_2\sigma_1\sigma_2)^3 = \sigma_2\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2$, so the braid can be represented by a $\sigma$-consistent word in which the main generator $\sigma_1$ occurs only twice. However, as is easy to check with the help of Figure 11, the left consistent canonical form of the braid is the word $(\sigma_2\sigma_1\sigma_2)^3$, which contains the main generator $\sigma_1$ three times. \qed

**Local indicability.**

We are indebted to Stephen P Humphries and Jim Howie for pointing out the following. A group is called locally indicable if every finitely generated subgroup has a nontrivial homomorphism to the integers. It was proved by Burns and Hale [4] that locally indicable groups are right-orderable, but it took almost two decades until G Bergman [1] found an example of a group which is right-orderable but not locally indicable; i.e. the class of locally indicable groups is strictly contained in the class of right-orderable groups. We can now give further examples:
Figure 11. The left consistent canonical form of $\Delta^3$ is $\sigma_2\sigma_1\sigma_2\sigma_2\sigma_2\sigma_1\sigma_2$.

**Theorem 5.3.** The braid group $B_n$ for $n \geq 5$ is right orderable but not locally indicable.

**Proof.** It remains to show that $B_n$ is not locally indicable. The commutator subgroup $B'_n$ of $B_n$ is finitely generated, and for $n \geq 5$ the first and second commutator subgroups coincide: $B'_n = B''_n$ (see [10]). It follows that the abelianization of $B'_n$ is trivial, so $B'_n \subset B_n$ has no nontrivial homomorphism to $\mathbb{Z}$. \qed

### 6. Automatic ordering.

Define a right-invariant ordering to be *automatic* if it can be determined by a finite-state automaton. In this section we shall see that the ordering on the braid group is automatic.

This is proved by comparing the order on the braid group as defined in Section 3 with Mosher’s automatic structure [12, 13]. This comparison gives more. Define a group to be *order automatic* if it is both automatic [9] and right-orderable and such that there is a finite state automaton which detects the order from the automatic normal forms. To be precise, there exists an automatic structure and a finite state automaton, which, given two normal forms for the automatic structure, will decide which represents the greater group element.

**Theorem 6.1.** The braid group $B_n$ is order automatic.

**Remark 6.2.** The algorithm to decide which of two given normal forms is the greater takes linear time in the length of the normal form. Using results from Epstein et al [9] we deduce:

**Corollary 6.3.** There is a quadratic-time algorithm to decide which of two elements of $B_n$ (presented in terms of standard braid generators) is the greater.

Full details of the proof of these results can be found in [16]. Here we shall give a short proof of Theorem 6.1 which yields only a quadratic time
algorithm to order normal forms which is nevertheless sufficient to imply Corollary 6.3.

In [12, 13] Mosher constructs normal forms for elements of mapping class groups by *combing* triangulations (and uses them to prove that mapping class groups are automatic). We shall need to sketch Mosher’s normal form in the special case of the braid group.

We define the base triangulation $B$ of $D_n$ to have vertices at the $n$ missing points and at the four boundary vertices, $±1$ and $±\sqrt{-1}$. The edges of $B$ comprise the four arcs of $\partial D_n$ joining pairs of boundary vertices, $n+1$ edges along the real axis and $2n$ edges joining $±\sqrt{-1}$ to the real vertices not $±1$, see Figure 12. We order and orient the edges as indicated.

![Figure 12. The base triangulation.](image)

An allowable triangulation of $D_n$ is a triangulation with the same vertex set. We identify two allowable triangulations if they differ by a vertex fixing isotopy. A triangulation class is a set of boundary fixing isomorphism classes of allowable triangulations. I.e. two triangulations are in the same class if they are related by the action of an element of the braid group.

We now consider the groupoid $G$ which has for objects the set of triangulation classes of $D_n$ and for morphisms the set of ordered pairs $(T, T')$ of allowable triangulations, where $(T, T')$ is identified with $(h(T), h(T'))$ if $h \in B_n$. The morphism goes from the class of $T$ to the class of $T'$. If $T$ and $T'$ are in the same class, then there is a unique boundary fixing isomorphism from $T'$ to $T$ up to isotopy, i.e. an element of $B_n$. This determines an isomorphism between the vertex group of $G$ and the braid group $B_n$. (Note that for this isomorphism, and for compatibility with Mosher’s conventions, we need to replace the algebraic convention for multiplication in the braid group, described in Section 1, by the opposite functional convention, i.e. $\Phi \Psi := \Phi \circ \Psi$. The functional convention is used throughout this section; the algebraic convention is used in all other sections and in the appendix.)
Combing.

We consider a particular type of morphism in $\mathcal{G}$.

**Definition. Flipping an edge.** An edge $\alpha$ adjacent to two triangles $\delta$ and $\delta'$ is removed (to form a square of which $\alpha$ is a diagonal) and then the square is cut back into two triangles by inserting the opposite diagonal. We call this morphism “flipping $\alpha$” and denote it $f_\alpha$, see Figure 13.

![Figure 13. Flipping an edge.](image)

Every morphism $q = (B,T)$ in $\mathcal{G}$ from the base vertex to another vertex is a product of a canonical sequence of flips. To see this, picture $q$ as given by superimposing $B$ and $T$, and comb $T$ along $B$. To be precise, first reduce $T$ with respect to $B$ and then consider edge 1 of $B$. Suppose that, starting at $-1$, edge one crosses edge $\alpha$ of $T$. Flip $\alpha$. Repeat until there are no more crossings of edge 1 with $T$. (The fact that this process is finite follows from a simple counting argument: one counts the number of intersections of edge 1 with $T$, except with the next edge of $T$ which is to be flipped. For more detail here see [13, pages 321-322].) Now do the same for edge 2 starting at the non-boundary vertex and continue in this way, using the ordering and orientation of edges of $B$ indicated in Figure 12, until $T$ has been converted into a copy of $B$.

The *Mosher normal form* of $q$ is the inverse of the sequence of flips described above. Notice that unlike the general case described in [13], $q$ is completely characterised by the sequence of flips, there is no need to carry the labelling of $T$ along the combing sequence. In particular, there is no relabelling morphism required here. (This is because $\partial D_n$ is fixed throughout.)

**Detecting order from the Mosher normal form.**

To see the connection with order, consider an element $(B,T)$ of the vertex group at the class of $B$. There is an element $g \in B_n$ (a homeomorphism of $D_n$ fixing $\partial D_n$) unique up to isotopy carrying $T$ to $B$. Conversely given $g \in B_n$ the corresponding triangulation pair is $(B, g^{-1}B)$.

---

2Strictly speaking the Mosher normal form is not this flip sequence, which only defines an asynchronous automatic structure, but is derived from it by clumping flips together into blocks called “Dehn twists”, “partial Dehn twists” and “dead ends” (see [13] pages 342 et. seq.). This technicality does not affect any of the results proved here. We prove that order can be detected in linear time from the flip sequence. Since the clumped flip sequence can be unclumped in linear time, this implies that order can be detected in linear time from the strict Mosher normal form.
We observe that if we comb $B$ along $g(B)$ this is combinatorially identical to combing $g^{-1}(B)$ along $B$. We call the sequence of flips defined by this combing the \textit{combing sequence of $g$}. (The reverse of the combing sequence is the Mosher normal form of $g$.)

The curve diagram of $g$ is part of the triangulation $g(B)$ namely the edges numbered $1, 4, 7, \ldots , 3n + 1$. Suppose that $g$ is $i$-positive, then $g$ can be assumed to fix the first $i - 1$ of these edges (i.e. $1, 4, \ldots , 3i - 5$) and then, after reduction, can be assumed to fix the corresponding outlying edges (i.e. $2, 5, \ldots , 3i - 4$ and $3, 6, \ldots , 3i - 3$). But edge $3i - 2$ is carried into the upper half of $D_n$ and must meet edge $3i - 1$ of $B$. Thus the first flip in the combing sequence of $g$ is $f_{3i - 1}$, i.e. flip the edge numbered $3i - 1$. Similarly if $g$ is $i$-negative then the first flip in the combing sequence is $f_{3i}$. We have proved the following:

\begin{algorithm}
(To decide from the Mosher normal form whether a braid element is $i$-positive or negative and provide the correct value of $i$.) \textit{Inspect the combing sequence} (the reverse of the Mosher normal form). \textit{The first flip is either} $f_{3i-1}$ \textit{for some} $i$ \textit{or} $f_{3i}$ \textit{for some} $i$. \textit{In the first case the braid is} $i$-positive \text{and in the second it is} $i$-negative.
\end{algorithm}

This algorithm is visibly executable by a finite-state automaton and linear in the length of the normal form of $g$. Theorem 6.1 and Corollary 6.3 follow from general principles. To decide the relative order of two elements $\alpha$ and $\beta$ we compute the normal form of $\alpha\beta^{-1}$ — this can be done by a finite-state automaton and takes quadratic time, see [9] — and then apply Algorithm 6.4.

\textbf{Final remarks.} (1) We have proved that there is a quadratic time algorithm to decide the relative order of two braid words. In [7] Dehornoy presents an algorithm which does this in practice and is apparently extremely fast — however his formal proof that this algorithm works only provides an exponential bound on time. The algorithm presented here is implementable since the whole Mosher program can be implemented, see [14]. Note that in the appendix we present another algorithm based on cutting sequences.

(2) There is a far stronger connection between the Mosher normal form and the order on $B_n$ than presented here. The relative order of two elements can be detected from their combing sequences by inspecting just the first four differences in the sequences (and this proves Remark 6.2). Full details here are to be found in [16].

\textbf{Appendix A. Cutting sequences.}

In this appendix we define a unique \textit{reduced cutting sequence} for a braid. We give implementable algorithms to read the reduced cutting sequence from the braid word, to decide order from the cutting sequence and to put a braid,
given in terms of standard twist generators, into its left-consistent canonical
form.

**Cutting sequences and curve diagrams.**

A cutting sequence is a finite word $\chi$ in the letters $0, \ldots, n, 0, \ldots, n+1$, ↑ and ↓ such that

(i) $\chi$ starts with 0 and ends with $n+1$,
(ii) each of the letters $0, \ldots, n+1$ occurs precisely once in $\chi$,
(iii) in the word $\chi$ numbers and arrows alternate, with the single possible
exception that strings of the form $i \ i + 1$ or $i + 1 \ i$ ($i = 0, \ldots, n$) may
occur.

Consider now a curve diagram $\Gamma$. It consists of three types of subcurves:
Curves in the upper half plane, curves in the lower half plane, and straight
line segments in the real line. Note that curves in the upper or lower half
plane may be replaced by semicircles since they are determined by their end
points. For convenience we rescale the curve diagram so that it goes from 0
to $n + 1$ and the $n$ holes are the integers $1, 2, \ldots, n$.

Going along $\Gamma$ we can read off a cutting sequence, by reading an ↑ or ↓ for every curve in the upper or lower half plane respectively, an $\underline{i}$ ($i \in \{0, \ldots, n+1\}$) for every intersection with the integer $i$ in the real line (so underlined integers correspond to holes), and an $i$ for every intersection with the real interval $(i, i + 1)$. It is easy to check that a word obtained in this way is indeed a cutting sequence.

For example the curve diagram representing $\sigma_1$ in Figure 2 is coded as $0 \uparrow 2 \downarrow 1 \downarrow 3 \downarrow 4$, whereas $\sigma_1 \sigma_1^{-1}$ is coded $0 \uparrow 1 \downarrow 3 \downarrow 1 \downarrow 3 \uparrow 2 \uparrow 4$.

We define a reduction of a cutting sequence to be a replacement of the
sequence by a shorter one, according to the one of the following rules (where $\uparrow$ denotes $\uparrow$ or $\downarrow$, and $i \in \{0, \ldots, n\}$).

- $i \uparrow i \rightarrow \underline{i}$, $\underline{i + 1} \uparrow \underline{i} \rightarrow i + 1$, $\underline{i} \uparrow \underline{i + 1} \rightarrow i + 1$,
- $\downarrow i \downarrow \rightarrow \downarrow$, $\uparrow i \uparrow \rightarrow \uparrow$,
- $i \uparrow i \rightarrow i$,
- $i \uparrow i + 1 \rightarrow \underline{i} \ i + 1$, $\underline{i + 1} \uparrow \underline{i} \rightarrow \underline{i} + 1 \ \underline{i}$

A cutting sequence is called reduced if it allows no reduction.

**Proposition A.1.** Every braid on $n$ strings has a unique reduced cutting
sequence.

**Proof.** Let $\chi$ be a cutting sequence of a curve diagram $\Gamma$ of the braid. We
observe that a reduced version $\chi'$ of $\chi$ is the same as the cutting sequence
of a curve diagram $\Gamma'$, where $\Gamma'$ is obtained by reducing $\Gamma$ with respect to
the trivial curve diagram $E$. From Proposition 2.3 we deduce that any two
reduced cutting sequences $\chi'$ and $\chi''$ must come from curve diagrams which
are equivalent with respect to $E$. Therefore $\chi'$ and $\chi''$ must agree. \qed
The reduced curve diagram can be reconstructed from the reduced cutting sequence. Thus the cutting sequence classifies the curve diagram, and hence the braid. This is most easily seen by using pen and paper. One reads the cutting sequence, and for every number symbol one encounters, draws one arc in the diagram. If the cutting sequence is reduced, then this involves no choices. Below we shall give an algorithm to do this which is more suitable for computer implementation.

Note that it is easy to construct reduced cutting sequences which do not come from curve diagrams. The pen and paper method can also be used to decide whether a cutting sequence does correspond to a curve diagram. Again we give a more formal algorithm below which will do this.

**Reading the cutting sequence from the braid word.**

We next show how to convert a braid defined in terms of the twist generators $\sigma_i^{\pm 1}$ into a reduced cutting sequence. We do this inductively by defining how $\sigma_i$ and $\sigma_i^{-1}$ act on reduced cutting sequences and then let the whole word act on the trivial sequence $0 \ 1 \ldots n \ n + 1$.

**Algorithm A.2.** Suppose a braid $\beta$ has reduced cutting sequence $\chi$. Then a cutting sequence of $\beta \sigma_i$ is obtained by simultaneously making the following replacements everywhere in the word $\chi$. These rules are to be interpreted as simultaneous, not sequential, replacements.

1. $i \rightarrow i + 1$, $i + 1 \rightarrow i$.
2. $\downarrow (\hat{i}) \rightarrow \downarrow i - 1 \uparrow (i + 1)$, $(\hat{i}) \downarrow \rightarrow (i + 1) \uparrow i - 1 \downarrow$.
3. $i - 1 (\hat{i}) \rightarrow i - 1 \uparrow (i + 1)$, $(\hat{i}) i - 1 \rightarrow (i + 1) \uparrow i - 1$.
4. $\uparrow (\hat{i}) \rightarrow \uparrow (i + 1)$, $(\hat{i}) \uparrow \rightarrow (i + 1) \uparrow$.
5. $\downarrow (i + 1) \rightarrow \downarrow (\hat{i})$, $(i + 1) \downarrow \rightarrow (\hat{i}) \downarrow$.
6. $i + 2 (i + 1) \rightarrow i + 2 \downarrow (\hat{i})$, $(i + 1) \ i + 2 \rightarrow (\hat{i}) \ i + 2$.
7. $\uparrow (i + 1) \rightarrow \uparrow i + 1 \downarrow (\hat{i})$, $(i + 1) \uparrow \rightarrow (\hat{i}) \downarrow i + 1 \uparrow$.
8. $\downarrow i \uparrow \rightarrow \downarrow i - 1 \uparrow i \i + 1 \uparrow$, $(i) \downarrow \rightarrow \uparrow i + 1 \downarrow i \uparrow i - 1 \downarrow$.

**Note.** In rules (ii)-(vii), rule (i) is being applied, and its application is indicated by brackets. Replacements of symbols other than $i, i + 1$ depend on context, e.g. rule (ii) says that if $\downarrow$ is followed by $\hat{i}$, then it is to be replaced by $\downarrow i - 1 \uparrow$, and the $\hat{i}$ is replaced by $i + 1$, by (i). So $\downarrow \hat{i}$ turns into $\downarrow i - 1 \uparrow i + 1$.

The rules for the action of $\sigma_i^{-1}$ are obtained by interchanging the symbols $\uparrow$ and $\downarrow$ everywhere in this list (i.e. replacing up- by down-, and down- by up-arrows). The resulting cutting sequence can then be reduced, to obtain the reduced cutting sequence of the braid $\beta \sigma_i$ or $\beta \sigma_i^{-1}$.

We can now deduce an effective algorithm to decide whether a given braid is positive, trivial, or negative:
Algorithm A.3. (To decide if a given braid is positive, trivial, or negative.) Use Algorithm A.2 to calculate the reduced cutting sequence of the braid. The braid is positive if and only if the first arrow in this sequence is an up-arrow ↑.

Recovering the curve diagram from the cutting sequence.

We now show how to recover a reduced curve diagram from its associated cutting sequence. At the same time this will provide an effective algorithm to decide if a given cutting sequence corresponds to a curve diagram.

To make precise the problem here, we define the real cutting sequence of a curve diagram to be the cutting sequence, with the non-underlined integers replaced by real numbers specifying the precise intersection point of the curve diagram with the real line, up to order preserving bijections. (Taking the integer part of all numbers in the real cutting sequence we retrieve the cutting sequence.) Given the real cutting sequence, we can immediately construct the curve diagram. Moreover it is trivial to check if a real cutting sequence corresponds to an (embedded) curve diagram: One just checks that

1. if \( i \) or \( i + 1 \) occurs in the sequence then no real number in \((i, i + 1)\) occurs,
2. the numbers on each side of two arrows of the same type correspond to nested intervals (so that the corresponding curves do not intersect).

So we need an algorithm to reconstruct the real cutting sequence from the cutting sequence or equivalently to decide for each \( i \) the order in which the corresponding points actually occur in \( \mathbb{R} \).

Algorithm A.4. Suppose the letter \( i \) \((i \in \{0, \ldots, n\})\) appears in two different places, say in the \( r \)th and \( s \)th position, in the cutting sequence. To decide which one represents the smaller number in the interval \((i, i + 1)\) in the real cutting sequence proceed as follows.

Since the cutting sequence is reduced, there are two arrows in opposite direction adjacent to each of the letters \( i \). Starting at the \( r \)th letter we read the sequence either forwards or backwards. We define the up-string at the \( r \)th place to be the word obtained from the cutting sequence by reading forwards or backwards, starting at the \( r \)th letter, up to the next underlined number, with the reading direction specified by the requirement that the first two letters read should be \( i \uparrow \). Similarly, we define the down-string at the \( r \)th place by reading in the opposite direction, such that the resulting word starts with \( i \downarrow \), again up to the next underlined number. We compare the up-string at the \( r \)th with that at the \( s \)th place, and the down-string at the \( r \)th with that at the \( s \)th place. They cannot both agree, for if they did, the curve diagram would have two curves with the same endpoints.

We now manipulate the up- and down strings as follows: Firstly, we increase all non-underlined integers by \( \frac{1}{2} \). Then we remove the underline from all underlined integers. We obtain sequences of the form \( x_0 \downarrow x_1 \downarrow \).
... \uparrow x_{1-1} \uparrow x_1, \text{ where } l \in \mathbb{N}, x_0 = i + \frac{1}{2}, x_1, \ldots, x_{l-1} \in \{\frac{1}{2}, 1\frac{1}{2}, \ldots, n + \frac{1}{2}\}, \text{ and } x_l \in \{0, \ldots, n + 1\}.

From this we can construct a sequence of numbers in \{1, 1\frac{1}{2}, \ldots, n - \frac{1}{2}, n\}, called the \textit{cyclically associated sequence}, as follows. For every string \(x_j \uparrow x_{j+1}\) we write down the unique representative in \(\{\frac{1}{2}, 1, \ldots, n, n + \frac{1}{2}\}\) of \(x_{j+1} - x_j + (n+1)\mathbb{Z} \in \mathbb{R}/(n+1)\mathbb{Z}\); for every string \(x_j \downarrow x_{j+1}\) we write down the unique representative in \(\{\frac{1}{2}, 1, \ldots, n, n + \frac{1}{2}\}\) of \(x_j - x_{j+1} + (n+1)\mathbb{Z} \in \mathbb{R}/(n+1)\mathbb{Z}\). Altogether, this yields a sequence of length \(l\).

We now define an up-string \(u\) to be \textit{cyclically lexicographically larger} than another up-string \(u'\), if the cyclically associated sequence of \(u\) is lexicographically larger than the one of \(u'\). The geometric interpretation is that the curve diagram has two line segments starting in the real interval \((i, i+1)\), going into the upper half plane. The line segment representing the cyclically lexicographically larger up-string is the one turning ‘more to the left’. Since the two line segments must be disjoint (being part of the curve diagram), the starting point of the curve segment yielding the cyclically lexicographically larger up-string must represent a smaller real number in the real cutting sequence. Similarly, we define a cyclic lexicographic ordering on the down-strings; this time, the starting point of a curve segment which gives rise to a cyclically lexicographically larger down-string than another curve segment must represent a \textit{larger} real number in the real cutting sequence. \textit{End of Algorithm A.4.}

To summarise, we have found an algorithm for reconstructing the real cutting sequence from the cutting sequence: Given any two places in the cutting sequence where the letter \(i\) occurs, we compare the up-strings at these places. If they agree, we compare the down-strings instead. In either case we can work out the cyclically associated sequences, and then decide which of the two letters \(i\) represents the smaller number in the interval \((i, i+1)\) in the real cutting sequence.

\textbf{An algorithm to determine order from the cutting sequence.}

Algorithm A.4 also allows us to decide which of two given reduced cutting sequences represents the larger braid. If the two sequences agree on some initial segment, then we remove the underlines from all underlined numbers (except the first letter \(0\)) that lie in this segment. Then we reduce the resulting two sequences. We obtain two new sequences whose initial segments up to the first underlined numbers do not agree. If they differ already on the second letter (after \(0\)), then we know which one is larger. Otherwise, we work out which of them is cyclically lexicographically larger, using Algorithm A.4.

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\(^3\text{Cyclic lexicographic order is used by Birman and Series [3].}\)
The algorithm to determine left-consistent canonical form.

We are finally ready to describe our algorithm to calculate the left-consistent canonical form of a braid. The input is a braid $\beta$ represented as a word $w$ in the twist generators $\sigma_i^\pm 1$. The output is the same braid in left-consistent canonical form of $\beta$, again given as a word in the $\sigma_i^\pm 1$.

The algorithm proceeds by repeating the main step (described below) after each repetition we have a word $W$ and a cyclically reduced cutting sequence $\chi$ which are both modified at the next repetition.

**Start.** We start with $W$ the trivial word, and $\chi$ the reduced cutting sequence of $\beta$ calculated using Algorithm A.2.

**Finish.** If the reduced cutting sequence $\chi$ is $0 \ 1 \ \ldots \ n \ n+1$, then the algorithm stops, and the inverse of the word $W$ is the desired canonical word.

**Main step.** If the reduced cutting sequence starts $0 \ 1 \\ldots \ i \uparrow$, with $i < n+1$, then we hunt for subwords of the following forms

(i) $i \uparrow a_1 \downarrow a_2 \uparrow \ldots \uparrow a_{l-1} \downarrow a_l \uparrow$
(ii) $i \downarrow a_1 \uparrow a_2 \downarrow \ldots \downarrow a_{l-1} \uparrow a_l \downarrow$
(iii) $a_l \downarrow a_{l-1} \downarrow \ldots \downarrow a_2 \uparrow a_1 \uparrow i$
(iv) $a_l \uparrow a_{l-1} \downarrow \ldots \uparrow a_2 \downarrow a_1 \downarrow i$

where the $a_1, \ldots, a_{l-1}$ are not equal to $i$ and not underlined, and $a_l \neq i, i+1$. (If the reduced cutting sequence starts $0 \ 1 \ \ldots \ i \downarrow$, then we hunt for subwords like $i \downarrow a_1 \uparrow \ldots \uparrow a_{l-1} \uparrow a_l$ instead.) We shall call these words *useful subwords*, because they correspond to useful arcs.

We consider the set of all useful subwords, and we want to identify the ‘leftmost one’, i.e. the one whose letter $i$ represents the leftmost point in the interval $(i, i+1)$. One of them starts or ends with a letter $i$, i.e. if one of them is of type (ii) or (iv), then this is it. If not, then we can use Algorithm A.4 to determine the leftmost one. When we have found the leftmost useful subword, we modify it as follows. If it is of type (i) or (ii), then we write it backwards, so that it starts with the letter $a_l$. Irrespectively of the type of the useful subword, we remove the underline from the letter $a_l$. Then we let $c := a_l$, replace all letters $a_k$ ($k \in \{1, \ldots, l\}$) with $a_k \geq c$ by $a_k - 1$ (e.g. $a_l$ turns into $a_l - 1$), and reduce the resulting sequence. By doing this, we obtain a modified sequence $a'_0 \downarrow a'_1 \downarrow \ldots \downarrow a'_{\ell'}$, possibly with the letter $a'_{\ell'} = i$ underlined.

We now multiply $W$ on the right by a word $v_1 \ldots v_{\ell'}$, where $v_k$ is determined by $a'_{\ell'-1}$, $a'_k$, and the arrow in between $a'_{\ell'-1}$ and $a'_k$ as follows:

(i) If the modified leftmost useful subword contains the string $a'_{\ell'-1} \uparrow a'_k$, and $a'_{\ell'-1} < a'_k$, then $v_k = \sigma_{a'_{\ell'-1} + 1} \cdots \sigma_{a'_k}$;
(ii) If the modified leftmost useful subword contains the string \( a'_{k-1} \uparrow a'_{k} \), and \( a'_{k-1} > a'_{k} \), then \( v_{k} = \sigma_{a'_{k-1}}^{-1} \cdots \sigma_{a'_{k}+1}^{-1} \); 

(iii) If the modified leftmost useful subword contains the string \( a'_{k-1} \downarrow a'_{k} \), and \( a'_{k-1} < a'_{k} \), then \( v_{k} = \sigma_{a'_{k-1}+1}^{-1} \cdots \sigma_{a'_{k}}^{-1} \); 

(iv) If the modified leftmost useful subword contains the string \( a'_{k-1} \downarrow a'_{k} \), and \( a'_{k-1} > a'_{k} \), then \( v_{k} = \sigma_{a'_{k-1}} \cdots \sigma_{a'_{k}+1} \).

The word \( v_{1} \ldots v_{l} \) represents the slide of a hole back along the leftmost useful arc. 

Finally, we calculate the new reduced cutting sequence after this slide. This can be done by letting the word \( v_{1} \ldots v_{l} \) act on the reduced cutting sequence, as described above. (An alternative method would be to remove the underline from the letter \( a_{l} \), underline the unique letter \( i \) which belongs to the leftmost useful subword instead, carefully relabel the cutting sequence, using Algorithm A.4, and then reduce the resulting cutting sequence.) \textit{End of main step.}

The proof of Theorem 4.2 implies that the algorithm stops after a finite number of repetitions of the main step.

\textbf{References}


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NUMERICAL SEMIGROUPS GENERATED BY INTERVALS

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We study numerical semigroups generated by intervals and solve the following problems related to such semigroups: the membership problem, give an explicit formula for the Frobenius number, decide whether the semigroup is a complete intersection and/or symmetric, and computation of the cardinality of a (any) minimal presentation of this kind of numerical semigroups.

A numerical semigroup is a finitely generated subsemigroup of the set of nonnegative integers $\mathbb{N}$, such that the group generated by it is the set of all integers $\mathbb{Z}$. In this paper we study the semigroups generated by intervals of nonnegative integers, that is to say, semigroups of the form $S = \langle a, a+1, \ldots, a+x \rangle = \{\sum_{i=0}^{x} n_i(a+i) : n_i \in \mathbb{N}\} \subseteq \mathbb{N}$. Note that if $x \geq a$, then $S = \{a, a+1, \ldots\} = a + \mathbb{N} = \langle a, a+1, \ldots, 2a-1 \rangle$; thus we may assume that $x \leq a-1$. For a semigroup of this kind we solve the following problems:

1) Membership problem. An element $n \in \mathbb{N}$ belongs to $S = \langle a, a+1, \ldots, a+x \rangle$ if and only if $n \mod a \leq \lfloor \frac{n}{a} \rfloor x$, where $\lfloor \frac{n}{a} \rfloor$ is quotient of the integer division of $n$ by $a$, and $n \mod a$ denotes the remainder of this division, $n - \lfloor \frac{n}{a} \rfloor a$.

2) Computation of the Frobenius number of the semigroup. The Frobenius number of a numerical semigroup (also known as the conductor of the semigroup) is the greatest integer not belonging to the given semigroup. The Frobenius number of $S = \langle a, a+1, \ldots, a+x \rangle$ is $\lceil \frac{a-1}{x} \rceil a - 1$, where $\lceil q \rceil$ denotes the least integer greater than or equal to $q \in \mathbb{Q}^+$. stages.

3) Symmetry of the semigroup. A numerical semigroup $T$ with Frobenius number $C$ is symmetric if and only if for each $z \in \mathbb{Z}$ we have that either $z \in S$ or $C - z \in S$. These kinds of semigroups are specially interesting in Ring Theory as Kunz shows in [7]. The semigroup $S = \langle a, a+1, \ldots, a+x \rangle$ is symmetric if and only if $a \equiv 2 \mod x$ (here $a \equiv b \mod c$ denotes the fact $a - b = kc$ for some integer $k$).

4) Cardinality of a minimal presentation of $S$. The semigroup $S = \langle a, a+1, \ldots, a+x \rangle$ is isomorphic to $\mathbb{N}^{x+1} / \sigma$, where $\sigma$ is the kernel congruence of the semigroup morphism

$$\varphi : \mathbb{N}^{x+1} \rightarrow S, \quad \varphi(n_0, \ldots, n_x) = \sum_{i=0}^{x} n_i(a+i),$$

75
that is to say $a \sigma b$ if and only if $\varphi(a) = \varphi(b)$. A minimal presentation of $S$ is a minimal system of generators of the congruence $\sigma$. In this paper we show that the cardinality of a minimal presentation of $S$ is

$$\frac{x(x-1)}{2} + x - ((a-1) \mod x).$$

5) **Complete intersection semigroups.** A numerical semigroup is a complete intersection if the cardinality of a minimal presentation plus one equals the cardinality of a minimal system of generators of the given semigroup (see [6]). We show that the semigroup $S = \langle a, a + 1, \ldots, a + x \rangle$ is a complete intersection if and only if $S = \langle a, a + 1 \rangle$ or $(2k, 2k + 1, 2k + 2)$.

The point of departure to solve these problems is the following lemma.

**Lemma 1.** Let $S = \langle a, a + 1, \ldots, a + x \rangle$ be a numerical semigroup with $1 \leq x < a$. Then, $n \in S$ if and only if $n = qa + i$ with $q \in \mathbb{N}$ and $i \in \{0, \ldots, qx\}$.

**Proof.** If $n \in S$ then there exist $n_0, \ldots, n_x \in \mathbb{N}$ such that

$$n = (\sum_{j=0}^{x} n_j) a + \sum_{j=0}^{x} n_j j.$$  

Thus, $n = (\sum_{j=0}^{x} n_j) a + \sum_{j=1}^{x} n_j j$. Take $q = \sum_{j=0}^{x} n_j$ and $i = \sum_{j=1}^{x} n_j j \leq \sum_{j=0}^{x} n_j x = qx$.

Now, assume that $n = qa + i$, with $0 \leq i \leq qx$. We distinguish two possible cases:

1) If $i = qx$ then $n = q(a + x) \in S$.

2) If $i = kx + r$, with $0 \leq k < q$ and $0 \leq r \leq x - 1$, then $n = qa + kx + r = (q - k - 1)a + k(a + x) + a + r \in S$.

$\square$

From this membership characterization, we can derive the following characterization which is easier to check and it is what we will use later in the paper.

**Corollary 2.** Let $S = \langle a, a + 1, \ldots, a + x \rangle$ be a numerical semigroup with $1 \leq x < a$. Then, $n \in S$ if and only if $(n \mod a) \leq \lfloor \frac{n}{a} \rfloor x$.

**Proof.** If $n \in S$, then using the previous lemma, there exists $q \in \mathbb{N}$ and $0 \leq i \leq qx$ such that $n = qa + i$. Besides, $n = \lfloor \frac{n}{a} \rfloor a + (n \mod a)$. Thus, $n \mod a = qa + i - \lfloor \frac{n}{a} \rfloor a = (q - \lfloor \frac{n}{a} \rfloor) a + i$, and since $q \leq \lfloor \frac{n}{a} \rfloor$, we get that $n \mod a \leq i \leq qx \leq \lfloor \frac{n}{a} \rfloor x$.

Now, assume that $(n \mod a) \leq \lfloor \frac{n}{a} \rfloor x$. Since $n = \lfloor \frac{n}{a} \rfloor a + (n \mod a)$ and $0 \leq n \mod a \leq \lfloor \frac{n}{a} \rfloor x$, applying the previous result, we get that $n \in S$. $\square$

For every $n \in S$, the Apéry set (see [1]) associated to $n$ is defined as

$$S(n) = \{ s \in S : s - n \not\in S \}.$$

Using last result, we can characterize the elements belonging to $S(a)$. This characterization is going to play an important role in the rest of the paper.
Corollary 3. Let \( S = \langle a, a + 1, \ldots, a + x \rangle \) be a numerical semigroup with \( 1 \leq x < a \) and let \( n \geq a \). Then, \( n \in S(a) \) if and only if \((n \mod a) \in \{(\left\lfloor \frac{n}{a} \right\rfloor - 1)x + 1, \ldots, \left\lfloor \frac{n}{a} \right\rfloor x\}\). 

Proof. The element \( n \in S(a) \) if and only if \( n \in S \) and \( n - a \notin S \). By the previous result this occurs if and only if \((n \mod a) \leq \left\lfloor \frac{n}{a} \right\rfloor x \) and \((n - a) \mod a > \left\lfloor \frac{n-a}{a} \right\rfloor x \). But \((n - a) \mod a = n \mod a \) and \( \left\lfloor \frac{n-a}{a} \right\rfloor = \left\lfloor \frac{n}{a} \right\rfloor - 1 \) (note that \( n \geq a \)). Hence, \( n \in S(a) \) if and only if \( n \mod a \in \{(\left\lfloor \frac{n}{a} \right\rfloor - 1)x + 1, \ldots, \left\lfloor \frac{n}{a} \right\rfloor x\}\). □

We can explicitly construct \( S(a) \) as the next corollary shows.

Corollary 4. Let \( S = \langle a, a + 1, \ldots, a + x \rangle \) be a numerical semigroup with \( 1 \leq x < a \). Then, \( S(a) = \{qa + (q - 1)x + r : 1 \leq r \leq x, q \in \mathbb{N} \) and \( 0 \leq (q - 1)x + r < a\}\). 

Proof. Take \( n = qa + (q - 1)x + r \) such that \( 1 \leq r \leq x, q \in \mathbb{N} \) and \( 0 \leq (q - 1)x + r < a\). Then, \( \left\lfloor \frac{n}{a} \right\rfloor = q \) and \( n \mod a = (q - 1)x + r = (\left\lfloor \frac{n}{a} \right\rfloor - 1)x + r \) \( \in \{(\left\lfloor \frac{n}{a} \right\rfloor - 1)x + 1, \ldots, \left\lfloor \frac{n}{a} \right\rfloor x\}\). Using the previous result, we get that \( n \in S(a) \).

Now, take \( n = \left\lfloor \frac{n}{a} \right\rfloor a + (n \mod a) \in S(a) \). Then, by the previous corollary, \( n \mod a \in \{(\left\lfloor \frac{n}{a} \right\rfloor - 1)x + 1, \ldots, \left\lfloor \frac{n}{a} \right\rfloor x\}\) and therefore \( n \mod a = (\left\lfloor \frac{n}{a} \right\rfloor - 1)x + r \), with \( r \in \{1, \ldots, x\} \). Taking \( q = \left\lfloor \frac{n}{a} \right\rfloor \) we are done. □

If \( S \) is a numerical semigroup then the set \( \mathbb{N} \setminus S \) is finite, because the group spanned by \( S \) is \( \mathbb{Z} \). As we have mentioned before, the maximum of this set is called the Frobenius number of the semigroup, which we denote by \( C(S) \). It is well known (see [2]) that \( C(S) = \max(S(a)) - a \). Thus, if we want to compute \( C(S) \), we have to determine the greatest element of \( S(a) \). This is performed in the next result.

Corollary 5. Let \( S = \langle a, a + 1, \ldots, a + x \rangle \) be a numerical semigroup with \( 1 \leq x < a \). Then, \( C(S) = \left\lfloor \frac{a-1}{x} \right\rfloor a - 1 \).

Proof. We must determine the maximum of \( S(a) = \{qa + (q - 1)x + r : 1 \leq r \leq x, q \in \mathbb{N} \) and \( 0 \leq (q - 1)x + r < a\}\). The maximum is reached when \( (q - 1)x + r = a - 1 \). The element \( a - 1 \) is equal to \( \left\lfloor \frac{a-1}{x} \right\rfloor x + ((a - 1) \mod x) \).

Two possibilities arise:

- If \( (a - 1) \mod x \neq 0 \) then take \( (q - 1) = \left\lfloor \frac{a-1}{x} \right\rfloor \) and \( r = (a - 1) \mod x \).

  The greatest element in \( S(a) \) is \( qa + (q - 1)x + r = (\left\lfloor \frac{a-1}{x} \right\rfloor + 1)a + a - 1 \).

- If \( (a - 1) \mod x = 0 \) then, since \( r \) must be in \( \{1, \ldots, x\} \), write \( a - 1 \) as \( a - 1 = (\left\lfloor \frac{a-1}{x} \right\rfloor - 1)x + x \). Take \( q - 1 = \left\lfloor \frac{a-1}{x} \right\rfloor - 1 \) and \( r = x \). In this case, the greatest element in \( S(a) \) is \( qa + (q - 1)x + r = qa + (q - 1)x + x = \left\lfloor \frac{a-1}{x} \right\rfloor a + a - 1 \).

Both cases are represented by \( \left\lfloor \frac{a-1}{x} \right\rfloor a + a - 1 \), which is the greatest element in \( S(a) \). □
Note that this implies that for the numerical semigroups generated by intervals there exists an explicit formula to compute the Frobenius number of the semigroup. It seems that there is no known formula for the general case. Nevertheless, for some specific cases there exists an explicit formula. For instance, in [4], a formula for the Frobenius number of numerical semigroups generated by up to three elements and of symmetric numerical semigroups generated by up to four elements is given (see also [5] for more references).

Another characterization of symmetric numerical semigroups is the following (see [3]). The numerical semigroup $S$ is symmetric if the greatest element, $w$, of $S(a)$ satisfies the condition that for every $s \in S(a)$, the element $w - s$ is in $S$. We use this result to give a characterization of the numerical semigroups generated by intervals that are symmetric.

**Theorem 6.** Let $S = \langle a, a + 1, \ldots, a + x \rangle \neq \mathbb{N}$ be a numerical semigroup with $1 \leq x < a$. Then, $S$ is symmetric if and only if $a \equiv 2 \mod x$.

**Proof.** We can assume that $x \geq 2$, since if $x = 1$ then $S$ is generated by two relatively prime elements and in this case it is well known that $S$ is symmetric (see for instance [5]).

Let $w$ be the greatest element in $S(a)$. We already know that $w = qa + (q - 1)x + r$ with $(q - 1)x + r = a - 1$.

If $S$ is symmetric, since $1 \not\in S$, then $qa + (q - 1)x + r - 1$ cannot be in $S(a)$. Hence, $qa + (q - 1)x + r - 1 \in S \setminus S(a)$, which means that $n = (q - 1)a + (q - 1)x + r - 1 - a \in S$. Since $n \mod a = (q - 1)x + r - 1$, $\lfloor \frac{n}{a} \rfloor = q - 1$ and $n \mod a$ must be less than or equal to $\lfloor \frac{n}{a} \rfloor$, we get that $r - 1$ must be zero. Hence, $a - 1 = (q - 1)x + 1$, which means that $a \equiv 2 \mod x$.

If $a \equiv 2 \mod x$, then $(a - 1) \equiv 1 \mod x$, and therefore $(a - 1) \mod x = 1$, which from the first case in the proof of the last corollary implies that $r = 1$. Consequently, $w = qa + (q - 1)x + 1$. Take $0 \neq m = ka + (k - 1)x + i \in S(a)$, that is to say $0 \neq k \in \mathbb{N}$, $0 \leq (k - 1)x + i < a$ and $i \in \{1, \ldots, x\}$. Then, $w - m = (q - k)a + (q - k - 1)x + (x - i + 1)$. Note that $x - i + 1 \in \{1, \ldots, x\}$, $q - k \in \mathbb{N}$ and $0 \leq (q - k - 1)x + (x - i + 1) = (q - k)x + 1 - i < a$. Hence, $w - m \in S(a) \subset S$ for all $m \in S(a)$ and this means that $S$ is symmetric. 

The rest of the paper is devoted to computing the cardinality of a minimal presentation of the numerical semigroup $S = \langle a, a + 1, \ldots, a + x \rangle$ with $1 \leq x < a$.

It can be shown (see [6] for example) that for a numerical semigroup the cardinality of a set of generators of $\sigma$, the kernel congruence of $\varphi$, is greater than or equal to the number of generators of the semigroup minus one (in our case, this amount is $x$). When this lower bound is reached, the semigroup is a complete intersection. These semigroups are always symmetric.

In [8], the first author gives an algorithm to compute a system of generators, $\rho$, for $\sigma$ with minimal cardinality. From the results given in that paper, it is determined that the concepts of system of generators for $\sigma$ with
minimal cardinality and minimal system (with respect to the inclusion) of
generators of $\sigma$ coincide. Next, we give a sketch of this construction, which
is needed to count the elements in a minimal presentation of $S$.

For every $n \in S$, we define the graph $G_n = (V_n, E_n)$, as

$$V_n = \{a + i \in \{a, \ldots, a + x\} : n - (a + i) \in S\},$$
$$E_n = \{[a + i, a + j] : n - ((a + i) + (a + j)) \in S, i \neq j \in \{0, \ldots, x\}\}.$$

We define $\rho_n$ as:

1) If $G_n$ is not connected and $G_n^1 = (V_n^1, E_n^1), \ldots, G_n^r = (V_n^r, E_n^r)$ are its
connected components, then for every $1 \leq i \leq r$ we select an element
$\alpha_i = (n_{i0}, \ldots, n_{ix}) \in \mathbb{N}^{x+1} \setminus \{0\}$ such that $\varphi(\alpha_i) = n$ and $n_{ki} = 0$ for
all $a + k_i \notin V_n^i$. Define $\rho_n = \{\{\alpha_1, \alpha_2\}, \{\alpha_1, \alpha_3\}, \ldots, (\alpha_1, \alpha_r)\}$.

2) If $G_n$ is connected, we define $\rho_n = \emptyset$.

The set $\rho = \bigcup_{n \in S} \rho_n$ is a system of generators for $\sigma$ with minimal card-
ninality (see [8]). Thus, we must look for the elements $n \in S$ such that $G_n$ is not connected.

**Example 7.** Let $S = \langle 6, 7, 8 \rangle$. The vertices of $G_{14}$ are $\{6, 7, 8\}$ and the only
edge of $G_{14}$ is $[6, 8]$. Thus $G_{14}$ has two connected components: One is the
vertex $7$ and the other is the edge $[6, 8]$. The elements $(0, 2, 0)$ and $(1, 0, 1)$
are in $\mathbb{N}^3$ and verify that $\varphi(0, 2, 0) = \varphi(1, 0, 1) = 14$ (observe that the first
and last coordinates of $(0, 2, 0)$ are zero and that the second coordinate of
$(1, 0, 1)$ is zero). Thus $\rho_{14} = \{((0, 2, 0), (1, 0, 1))\}$, meaning that $2 \times 7 = 6 + 8$
is a relation on $S$.

**Theorem 8.** Let $S = \langle a, a + 1, \ldots, a + x \rangle$ be a numerical semigroup with
$1 \leq x < a$ and $\rho$ as before. Then,

$$\#\rho = \frac{x(x - 1)}{2} + x - ((a - 1) \mod x).$$

**Proof.** From the proof of Lemma 1, it is derived that if $n \in S$ then $n$ can be expressed
in one of the following ways:

- $n = ka + l(a + x)$ for some nonnegative integers $k, l$.
- $n = ka + l(a + x) + (a + i)$ for some nonnegative integers $k, l$ and
  $i \in \{1, \ldots, x - 1\}$.

Hence, if $G_n$ is not connected, then the set $\{a, a + x\} \cap V_n$ is not empty. In
the construction of $\rho$ we take $G_n^1$ to be the connected component containing
$a$, if $a \in V_n$. If $a$ is not a vertex of $G_n$ then we take $G_n^1$ to be the connected
component containing $a + x$ (which must be in $V_n$). We are going to count
the elements in $\rho$ which come from the fact that $a + i, 1 \leq i < x$ is neither
in the connected component containing $a$ nor in the connected component
containing $a + x$ (if $a$ or $a + x$ is not in $V_n$ this is translated to the fact that
$a + i$ is not in $V_n^1$). Next, we will count the elements in $\rho$ which arise when
$a$ and $a + x$ are in different connected components (that is to say, $a + x$ not in $V^1_n$). With this, we count all the elements belonging to $\rho$.

- First, let us assume that $G_n$ is not connected and there is $a + i \in V_n$ such that $a + i$ is not in the connected component(s) containing \{a, a + x\} $\cap V_n$. Hence, $n - (a + i + a) \notin S$ and $n - (a + i + a + x) \notin S$, which implies that $n = w + (a + i)$ with $0 \neq w \in S(a) \cap S(a + x)$. It is easy to check, from the description of $S(a)$ in Corollary 4, that $S(a) \cap S(a + x) = \{0, a + 1, \ldots, a + x - 1\}$. Thus, there exists $j \in \{1, \ldots, x - 1\}$ such that $n = (a + j) + (a + i)$. Note that the reverse is also true: If $n = (a + i) + (a + j)$ with $i, j \in \{1, \ldots, x - 1\}$, then the elements $a$ and $a + i$ are not connected in $G_n$ and the same holds for $a + x$ and $a + i$. This means that every expression of the form $n = (a + i) + (a + j)$ with $1 \leq i, j \leq x - 1$ yields a new element in $\rho_n$ (the element $(a_1, a_2)$, where $G_n$ is the connected component of $G_n$ containing $a + i$). This implies that from these graphs we get as many elements in $\rho$ as pairs $(i, j)$ with $1 \leq i, j \leq x - 1$. This amount is $x(x - 1)/2$.

- Now, let us count the elements in $\rho$ coming from non-connected graphs $G_n$ such that $a$ and $a + x$ are in different connected components. Since $a$ and $a + x$ are in different connected components of $G_n$, the element $n$ can be expressed as $n = w + (a + x)$, where $w \in S(a)$. Note also that $n \notin S(a)$. Thus, we must find the elements $w$ in $S(a)$ such that $w + (a + x) - a = w + x \in S$. By the proof of Corollary 5, depending on $r = (a - 1) \mod x$, the maximum element in $S(a)$ is $qa + (q - 1)x + r$, if $r \neq 0$, or $qa + (q - 1)x + x$, if $r = 0$, where $q = \lceil \frac{a - 1}{x} \rceil$.

1) If $r \neq 0$, then $q = \lceil \frac{a - 1}{x} \rceil + 1 \geq 2$. Let us show that the element $w = ka + (k - 1)x + i \in S(a)$ ($1 \leq i \leq x$ and $0 \leq (k - 1)x + i < a$) satisfies that $w + x$ is not in $S$ when $k \leq q - 2$. Note that $(k - 1)x + i + x \leq (k + 1)x \leq (q - 1)x = \lfloor \frac{a - 1}{x} \rfloor x \leq a - 1 < a$. Thus, $(w + x) \mod a = kx + i$ and $\lfloor \frac{w + x}{a} \rfloor x = kx$. Since $i \geq 1$, by Corollary 2, $w + x \notin S$.

Let us show that $w = (q - 1)a + (q - 2)x + i \in S(a)$ verifies that $w + x \notin S$ when $i \leq r$. Note that $(q - 2)x + i + x = (q - 1)x + i \leq (q - 1)x + r = a - 1$. Thus, $(w + x) \mod a = (q - 1)x + i$ and $\lfloor \frac{w + x}{a} \rfloor x = (q - 1)x$, which by Corollary 2 implies that $w + x \notin S$.

In addition, let us prove that if $w \in \{qa + (q - 1)x + 1, \ldots, qa + (q - 1)x + r\}$, then $a$ and $a + x$ are in the same connected component of $G_{w + (a + x)}$. Take $w = qa + (q - 1)x + i$ in this set. Then, $w + (a + x) = q(a + x) + (a + i)$, which means that $[a + x, a + i] \in E(G_n)$. Besides, $w + (a + x) = (q + 1)a + qx + i = (q + 1)a + ((q - 1)x + r) + x - r + i = (q + 1)a + (a - 1) + x - r + i = qa + (a + i) + (a + (x - r - 1))$ (observe
that $0 \leq x - r - 1 \leq x$). This implies that $[a, a + i] \in E(G_n)$ and consequently $a$ and $a + x$ are connected in $G_n$.

Hence, the elements $w$ we are interested in must be in the set $R = \{(q - 1)a + (q - 2)x + r + 1, \ldots, (q - 1)a + (q - 2)x + x\}$. Let us show that, as a matter of fact, for the elements $n \in S$ of the form $n = w + (a + x)$, with $w$ in the previous set, the graph $G_{w+(a+x)}$ has no path connecting $a$ and $a + x$. In order to show this, it is enough to prove that if $w = (q - 1)a + (q - 2)x + r + i \in R$, $n - ((a + k) + (a + j)) \in S$ and $j \geq i$ then $k$ must be greater or equal than $i$. The element $m = n - (a + k + a + j)$ is equal to $(q - 2)a + (q - 1)x + r + i - k - j$. We distinguish two cases depending on the value of $q - 1$:

- If $q - 1 \geq 2$, then $(q - 1)x + r + i \geq 2x \geq k + j$ and therefore $(q - 1)x + r + i - k - j \geq 0$. Besides, $(q - 1)x + r + i - k - j = (a - 1) + i - k - j \leq a - 1$ since $i - j \leq 0$. Hence, $m \mod a = (q - 1)x + r + i - k - j$ and $\lfloor \frac{m}{a} \rfloor = (q - 2)x$. Using Corollary 2, we get that since $m = n - (a + j + a + k) \in S$, $(q - 1)x + r + i - j - k \leq (q - 2)x$, that is to say, $x + r + i - j - k \leq 0$, and this occurs if and only if $(x - j) + r + i \leq k$ which implies that $k \geq i$, since $x - j + r \geq 0$.

- If $q - 1 = 1$ then $m = x + r + i - k - j \geq 0$, since $m \in S$ and as before $x + r + i - k - j \leq a - 1$. In this case, $m \mod a = x + r + i - k - j$ and $\lfloor \frac{m}{a} \rfloor = 0$. Thus, $m \in S$ implies that $x + r + i - k - j \leq 0$, and this leads to $k \geq i$.

Taking all this into account we have as many new elements in $\rho$ as elements has the set $R$, and this amount is $x - r = x - ((a - 1) \mod x)$.

2) If $r = 0$, then $q = \lfloor \frac{a - 1}{x} \rfloor \geq 1$. Let us show that the element $w = ka + (k - 1)x + i \in S(a)$ ($1 \leq i \leq x$ and $0 \leq (k - 1)x + i < a$) satisfies that $w + x$ is not in $S$ when $k \leq q - 1$. This is due to the fact that $(k - 1)x + i + x = kx + i \leq (q - 1)x + i = a - 1 - x + i < a$ (recall that $i \leq x$), and therefore $(w + x) \mod a = kx + i$ and $\lfloor \frac{w + x}{a} \rfloor = kx$, which, once more, implies that $w + x \not\in S$.

In this case, we get that $w$ must be in the set $R = \{qa + (q - 1)x + 1, \ldots, qa + (q - 1)x + x\}$. In the same way we did for the case $r \neq 0$, it can be shown that all these elements produce graphs such that $a$ and $a + x$ are in different connected components (the proof is the same but for the cases to distinguish, which are $q - 1 \geq 1$ and $q - 1 = 0$). Again, we get $x = x - 0 = x - ((a - 1) \mod x)$ new elements in $\rho$. 
Counting all the elements in $\rho$, we have that
\[
\#\rho = \frac{x(x-1)}{2} + x - ((a-1) \mod x).
\]
\square

With this theorem it is easy to prove the next result.

**Corollary 9.** Let $S = \langle a, a+1, \ldots, a+x \rangle$ be a numerical semigroup with $1 \leq x < a$ and $\rho$ as before. Then, $S$ is a complete intersection if and only if one of the following cases occur:

1) $S = \langle a, a+1 \rangle$ ($x = 1$).

2) $S = \langle 2k, 2k+1, 2k+2 \rangle$ ($x = 2$ and $a \mod 2 = 0$).

**Proof.** The semigroup $S$ is a complete intersection if and only if $\#\rho = x$. Using previous theorem this occurs if and only if $x(x-1)/2 + x - ((a-1) \mod x) = x$, and this happens if and only if $x(x-1)/2 = (a-1) \mod x$. Thus, if $S$ is a complete intersection then $x(x-1)/2 < x$ and this implies that $x < 3$, since $x > 0$. Hence, two cases may occur:

1) $x = 1$. In this case, $S = \langle a, a+1 \rangle$. Note that if $S$ is of this form then $S$ is a complete intersection.

2) $x = 2$. Under this setting then $S = \langle a, a+1, a+2 \rangle$. Besides, if we want $S$ to be a complete intersection then $2(2-1)/2 + 2 - ((a-1) \mod 2) = 2$ and this leads to $a \mod 2 = 0$.

\square

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**References**


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A SELBERG INTEGRAL FORMULA AND APPLICATIONS

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We obtain a 3-fold Selberg integral formula. As a consequence we are able to compute the explicit value of the sharp constant in a trilinear fractional integral inequality due to Beckner.

1. Introduction.

Multilinear fractional integral inequalities have been used in connection with restriction theorems of the Fourier transform and also in obtaining estimates for the $k$-plane and the $x$-ray transform. See for instance [C1], [C2], and [D].

In this article we are interested in a sharp form of a multilinear fractional integral inequality obtained by [B] (Theorem 6).

**Theorem ([B]).** Let $1 < p_1, \ldots, p_k < \infty$, $\sum_{j=1}^k p_j^{-1} > 1$, and $0 \leq \gamma_{ij} = \gamma_{ji} < n$ be real numbers satisfying

$$
\sum_{1 \leq j \leq k, j \neq s} \gamma_{js} = \frac{2n}{p_s} \quad \text{and} \quad \frac{1}{n} \sum_{1 \leq i < j \leq k} \gamma_{ij} + \sum_{j=1}^k \frac{1}{p_j} = k,
$$

where $p_j$ and $p_j'$ are dual exponents. Then

$$
\left| \int_{\mathbb{R}^n} \prod_{j=1}^k f_j(x_j) \prod_{1 \leq i < j \leq k} |x_i - x_j|^{-\gamma_{ij}} dx_1 \ldots dx_k \right| \leq A(\gamma_{ij}, n) \prod_{j=1}^k \|f\|_{p_j}.
$$

Moreover, the best constant $A(\gamma_{ij}, n)$ in (1) is attained for the extremal functions $f_j(x) = C(1 + |x|^2)^{-n/p_j}$ up to a conformal automorphism.

The second condition in (0) is necessary to ensure conformal invariance of the variational inequality (1). It is worth mentioning that the one dimensional form of inequality (1) above when all the exponents are equal was obtained by [C1] without sharp constants (and without the first restriction in (0)).

The value of the best constant in (1) was computed in [B]:

$$
A(\gamma_{ij}, n) = |S^n|^{-k + \frac{2n}{p}} \sum_{1 \leq i < j \leq k} \gamma_{ij} \int_{(S^n)^k} \prod_{1 \leq i < j \leq k} |\xi_i - \xi_j|^{-\gamma_{ij}} d\xi_1 \ldots d\xi_k,
$$
where \(|S^n| = (4\pi)^{n/2}\Gamma(n/2)\Gamma(n)^{-1}\) is the Lebesgue measure of the unit sphere in \(\mathbb{R}^{n+1}\). This formula brings a connection between multilinear fractional integral inequalities and Selberg integrals.

Multiple integrals such as the one in (2) are known as Selberg’s integrals and their exact values are useful in representation theory and in mathematical physics. These integrals have only been computed in special cases, for instance by Selberg himself when \(n = 1\) and \(\gamma_{ij} = \gamma\) (see [Se]), or when \(n = 2\) and \(\gamma_{ij} = 1\) (see [Ca]), but not in general. For a treatment of Selberg integrals, the reader could consult [Me], Section 17.11.

The question we would like to address is the following:

**Question.** Can the constant \(A(\gamma_{ij}, n)\) be computed explicitly?

In this paper we give an answer to this question when \(k = 3\). We are able to compute the three-fold Selberg integral (2) when \(\gamma_{12} + \gamma_{23} + \gamma_{31} = n\) for \(n \geq 1\).

Before we state our first result we would like to discuss the case \(k = 2\). The bilinear version of (1) is the well known Hardy-Littlewood-Sobolev inequality

\[
\left(3\right) \int_{\mathbb{R}^n} |x - y|^{-\gamma} f_1(x) f_2(y) \, dx \, dy \leq E(\gamma, p_1, p_2, n) \|f_1\|_{p_1} \|f_2\|_{p_2}
\]

which holds when \(1/p_1 + 1/p_2 > 1, 1/p_1 + 1/p_2 + d/n = 2\), and \(0 < \gamma < n\). The sharp constant in inequality (3) was derived by [L] when \(p_1 = p_2 = 2n/(2n - \gamma)\) and also when \(p_1 = 2\) or \(p_2 = 2\). When \(p_1 = p_2 = 2n/(2n - \gamma)\), the sharp constant in (3) is

\[
E(\gamma, p_1, p_2, n) = |S^n|^{(\gamma-2n)/n} \int_{(S^n)^2} |\xi - \eta|^{-\gamma} d\xi d\eta,
\]

which can be easily computed since

\[
\left(4\right) \int_{S^n} |\xi - \eta|^{-\gamma} d\xi = 2^{n-\gamma} \pi^{n/2} \frac{\Gamma\left(\frac{n-\gamma}{2}\right)}{\Gamma\left(n - \frac{d}{2}\right)},
\]

for all given \(\eta \in S^n\).

We now turn our attention to the case \(k = 3\). It turns out that in this case we can find a closed form for the constant in (2) when \(1/p_1 + 1/p_2 + 1/p_3 = 2\) or, equivalently, when \(\gamma_{12} + \gamma_{23} + \gamma_{31} = n\). It will be convenient to slightly change our notation in this case. We set \(-\gamma_{12} = d_1 - n, -\gamma_{23} = d_2 - n, \) and \(-\gamma_{31} = d_3 - n\). Now for \(0 < d_1, d_2, d_3 < n\) we denote by

\[
\left(5\right) Q_{d_1, d_2, d_3}[f_1, f_2, f_3] = \int_{\mathbb{R}^n} |x - y|^{d_1-n} |y - z|^{d_2-n} |z - x|^{d_3-n} f_1(x) f_2(y) f_3(z) \, dx \, dy \, dz
\]
the trilinear fractional integral that appears in (1). With this notation, inequality (1) is just
\[
Q_{d_1, d_2, d_3}[f_1, f_2, f_3] \leq C(d_1, d_2, d_3, n)\|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3},
\]
where 0 < d_1, d_2, d_3 < n are real numbers satisfying \(d_1 + d_2 + d_3 > n\), and
\[
p_1 = 2n/(d_1 + d_3), \quad p_2 = 2n/(d_1 + d_2), \quad p_3 = 2n/(d_2 + d_3).
\]
The best constant in the inequality above can be written as
\[
C(d_1, d_2, d_3, n) = |S^n|^{-\frac{d_1+d_2+d_3}{n}} \int_{(S^n)^3} |\xi - \eta|^{d_1-n}|\eta - \zeta|^{d_2-n}|\zeta - \xi|^{d_3-n} d\xi d\eta d\zeta.
\]

We now state our first result:

**Theorem 1.** Let 0 < d_1, d_2, d_3 < n, and d_1 + d_2 + d_3 = 2n. Then, for any distinct \(x, y, z \in \mathbb{R}^n\), the following formula holds
\[
\int_{\mathbb{R}^n} |x - t|^{-d_2}|y - t|^{-d_3}|z - t|^{-d_1} dt = B(d_1, d_2, d_3, n)|x - y|^{d_1-n}|y - z|^{d_2-n}|z - x|^{d_3-n},
\]
where
\[
B(d_1, d_2, d_3, n) = \pi^{n/2} \prod_{j=1}^3 \frac{\Gamma\left(\frac{n-d_j}{2}\right)}{\Gamma\left(d_j\right)}.
\]
Similarly, for any distinct \(\xi, \eta, \zeta \in S^n\) we have
\[
\int_{S^n} |\xi - \tau|^{-d_2}|\eta - \tau|^{-d_1}|\zeta - \tau|^{-d_1} d\tau = B(d_1, d_2, d_3, n)|\xi - \eta|^{d_1-n}|\eta - \zeta|^{d_2-n}|\zeta - \xi|^{d_3-n}.
\]

**Corollary 1.** Let 0 < d_1, d_2, d_3 < n and d_1 + d_2 + d_3 = 2n. Then the following Selberg integral formula holds:
\[
\int_{(S^n)^3} |\xi - \eta|^{d_1-n}|\eta - \zeta|^{d_2-n}|\zeta - \xi|^{d_3-n} d\xi d\eta d\zeta = |S^n|(2\pi)^n \prod_{j=1}^3 \frac{\Gamma\left(d_j\right)}{\Gamma\left(n - d_j\right)},
\]
and thus the exact value of the best constant in (6) when \(d_1 + d_2 + d_3 = 2n\) is
\[
C(d_1, d_2, d_3, n) = (2\pi)^n |S^n|^{-1} \prod_{j=1}^3 \frac{\Gamma\left(d_j\right)}{\Gamma\left(n - d_j\right)}.
\]
We point out that the kernel formula (8) is a trilinear version of the standard beta integral on $\mathbb{R}^n$:

\begin{equation}
\int_{\mathbb{R}^n} |x - t|^{-\alpha_1} |y - t|^{-\alpha_2} dt \\
= \pi^{n/2} \Gamma\left(\frac{n-\alpha_1}{2}\right) \Gamma\left(\frac{n-\alpha_2}{2}\right) \Gamma\left(\frac{\alpha_1 + \alpha_2 - n}{2}\right) \Gamma\left(\frac{\alpha_1}{2}\right) \Gamma\left(\frac{\alpha_2}{2}\right) \Gamma\left(\frac{n - \alpha_1 + \alpha_2}{2}\right) |x - y|^{n-\alpha_1 - \alpha_2},
\end{equation}

which is valid when $0 < \alpha_1, \alpha_2 < n$, $\alpha_1 + \alpha_2 > n$. It is still unclear to us whether or not there is a corresponding $k$-fold analogue of (8) and (9).

2. The proof of Theorem 1.

Clearly both sides of (8) are invariant under translations, dilations, and rotations of $x, y, z$. Therefore, by a translation we can assume that $z = 0$, by a dilation that $|y| = 1$, and by a rotation that $y = e_1 = (1, 0, \ldots, 0)$. Let us denote by $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} f(x) dx$ the Fourier transform of $f$. Recall that

\begin{equation}
(|x|^{d-n})^\wedge(\xi) = \pi^{n/2-d} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{n-d}{2}\right)} |\xi|^{-d} := c(d)|\xi|^{-d}
\end{equation}
in the sense of distributions (see [GS]).

After these reductions, we prove (8) by showing that the Fourier transform of both sides coincide. The function $x \rightarrow |x - e_1|^{d_1-n}|x|^{d_3-n}$ has Fourier transform

\begin{equation}
(|x|^{d_1-n}|x - e_1|^{d_1-n})^\wedge(\xi) \\
= c(d_3)c(d_1)|\xi|^{-d_3} * (|\xi|^{-d_1} e^{-2\pi i \xi \cdot e_1}) \\
= c(d_3)c(d_1) \int_{\mathbb{R}^n} |\xi - \eta|^{-d_3} |\eta|^{-d_1} e^{-2\pi i \eta \cdot e_1} d\eta \\
= c(d_3)c(d_1)|\xi|^{-n-d_3} \int_{\mathbb{R}^n} |\xi' - t|^{-d_1} |t|^{-d_3} e^{-2\pi i |t| \xi' dt},
\end{equation}

where $\xi' = \xi/|\xi|$. Now, for given $\xi$ find a rotation $A_\xi$ so that $A_\xi e_1 = \xi'$. Clearly $|\xi' - t| = |e_1 - A_\xi^{-1} t|$, $|t| = |A_\xi^{-1} t|$, and $t \cdot e_1 = t \cdot A_\xi^{-1} \xi' = A_\xi^{-1} t \cdot A_\xi^{-2} \xi'$. Hence, with $s = A_\xi^{-1} t$ the expression in (13) is equal to

\begin{equation}
c(d_3)c(d_1)|\xi|^{-n-d_3} \int_{\mathbb{R}^n} |e_1 - s|^{-d_3} |s|^{-d_1} e^{-2\pi i s \cdot A_\xi^{-2} \xi'} ds \\
= c(d_3)c(d_1)|\xi|^{-n-d_3} \hat{h}(A_\xi^{-2} \xi),
\end{equation}

where $h(t) := |t - e_1|^{-d_3} |t|^{-d_1}$.

On the other hand, let us denote by $g(x)$ the left hand side of (8) when $z = 0$ and $y = (1, 0, \ldots, 0)$. We have that

\begin{equation}
\hat{g}(\xi) = (\hat{h} \ast |t|^{-d_2})^\wedge(\xi) = c(n - d_2)\hat{h}(\xi)|\xi|^{d_2-n}.
\end{equation}
Using that $d_1 + d_2 + d_3 = 2n$ and that $c(n-d)^{-1} = c(d)$ we deduce that the Fourier transforms of the two sides of (8) are equal if and only if

$$
\hat{h}(\xi) = \hat{h}(A_\xi^{-2}\xi) \quad \text{for almost all } \xi \in \mathbb{R}^n.
$$

We now use the fact that if a function is reflection invariant with respect to a hyperplane then so is its Fourier transform. Modulo rotations and translations it is enough to check this for hyperplanes of the form $x_j = 0$. But the function $h$ is constant along circles orthogonal to $e_1$; in particular $h$ is reflection invariant with respect to the hyperplanes $x_j = 0$, for $j = 2, 3, \ldots, n$, and hence so is $\hat{h}$. But $A_\xi^{-2}\xi$ can be obtained from $\xi$ by finitely many reflections with respect to the above hyperplanes, and this concludes the proof of (8).

To prove (9) we use the stereographic projection $\bar{\pi} : \mathbb{R}^n \rightarrow S^n$. Recall that the Jacobian of $\bar{\pi}$ is

$$
|J_{\bar{\pi}}(t)| = 2^n (1 + |t|^2)^{-n},
$$

and that for any $a, b$ in $\mathbb{R}^n$ we have

$$
|\bar{\pi}(a) - \bar{\pi}(b)| = 2|a - b|(1 + |a|^2)^{-\frac{1}{2}}(1 + |b|^2)^{-\frac{1}{2}}.
$$

Now let $\xi = \bar{\pi}(x)$, $\eta = \bar{\pi}(y)$, $\zeta = \bar{\pi}(z)$, and $\tau = \bar{\pi}(t)$ in the integral on the left hand side of (7). Using formulas (12) and (13) one can obtain (7) by simply rewriting (6) in terms of the coordinates $\xi, \eta, \zeta, \tau$.

**Proof of Corollary 1.** Formula (10) follows by integrating (9) with respect to $\xi, \zeta, \eta$ and using (3). Formula (11) is immediate from (7).

From analyticity considerations it follows that the upper bound for the $d_j$ in Corollary 1 can be extended to $2n$ instead of $n$. Of course this is not the case in Theorem 1 since the integral in (8) may diverge if $d_j \geq n$.

### 3. Application to a sharp Sobolev imbedding.

The purpose if this section is to bring out some connections between multilinear integrals of type (1) with equal exponents and Sobolev imbeddings. These connections enter also in the context of conformal deformations of the metric structure of $S^n$ and spectral theory. Indeed, as shown in [Mo], multilinear fractional integrals arise as explicit computations of zeta functions of natural pseudodifferential operators, in the conformal class of the standard metric on $S^n$.

Let us discuss again the case $k = 2$. Inequality (3) can also be written as

$$
\|I_\alpha(f)\|_q \leq N_{p,\alpha,n}\|f\|_p,
$$


where \( \alpha - n = -\gamma, 1/q = 1/p - \alpha/n, 1 < p, q < \infty \), and \( I_\alpha \) denotes fractional integration given by

\[
I_\alpha(f)(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{n/2} \Gamma\left(\frac{\alpha}{2}\right)} \int_{\mathbb{R}^n} |x - y|^{\alpha - n} f(y) dy.
\]

When \( q' = p \), the sharp constant in (17) is

\[
N_{p,a,n} = 2^{-a} \pi^{-a/2} \frac{\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{n+a}{2}\right)} \frac{\Gamma(n)}{\Gamma(n/2)}^{\alpha/n}
\]

as computed in \([L]\). Inequality (17) expresses the sharp imbedding from \( L^p(\mathbb{R}^n) \hookrightarrow \dot{L}^{q'}_{-\alpha}(\mathbb{R}^n) \), where \( \alpha = n(2/p - 1) > 0, \ 1 < p < 2 \), and

\[
\dot{L}^q_{-\alpha}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{\dot{L}^q_{-\alpha}} = \| I_\alpha(|f|) \|_q < \infty \}
\]

is a homogeneous Sobolev space.

We now consider the case \( k = 3 \). When \( d_1 = d_2 = d_3 = d \) and \( f_1 = f_2 = f_3 = f \), inequality (6) is a special case of a more general sharp inequality derived in \([Mo]\). In the special case \( 0 < d \leq 2 \) it is possible to write \( Q_d[f] \) as a certain path integral, which is an \( L^3 \) norm with an appropriate Wiener measure. This allows us to conclude that the expression

\[
f \rightarrow Q_d[f] = (Q_{d,d,d}[|f|, |f|, |f|])^{1/3}
\]

is a norm when \( 0 < d \leq 2 \). It is quite natural to expect that \( Q_d \) is a norm also when \( 2 < d < n \), although we are not quite certain how to prove this in general. When \( d = 2n/3 \), however, it is an easy consequence of Theorem 1 that \( Q_d \) is a norm since it is the \( L^3 \) norm of a fractional integral.

**Corollary 2.** We have that for all \( f \geq 0 \),

\[
Q_{2n/3}[f] = (2\pi)^{n/3} \| I_{n/3}(f) \|_3 = (2\pi)^{n/3} \| f \|_{\dot{L}^{3}_{-n/3}}.
\]

To prove Corollary 2, take \( d_1 = d_2 = d_3 = 2n/3 \) in (8). Then multiply (8) by \( f(x)f(y)f(z) \) and integrate with respect to \( dx, dy, \) and \( dz \). Apply Fubini’s theorem and use (18) to obtain (20). \( \square \)

Thus, in the special case \( d_1 = d_2 = d_3 = 2n/3 \) inequality (6) is the same as inequality (17), with \( q = 3 = p' \) and \( \alpha = n/3 \). Observe that the constants also coincide since

\[
(\| f \|_{\dot{L}^3_{-n/3}}^2 B(2n/3, 2n/3, 2n/3,n))^{1/3} = C(2n/3, 2n/3, 2n/3,n)
\]

\[
= (2\pi)^{n/3} \| f \|_{\dot{L}^{3}_{-n/3}},
\]

as it should be. This gives a relationship between the sharp imbedding given by (6) and the sharp Sobolev imbedding given in (17) when \( d = 2n/3 \).

It is fairly routine to check that the expression \( Q_d f \) remains unchanged if \( f \) is replaced by \( (f \circ U)|J_U|^{d/n} \), where \( U \) is a conformal transformation of \( \mathbb{R}^n \).
and $J_{U}$ is its Jacobian. This transformation is scaled suitably to preserve $L^{n/d}$. We will denote by $B_{d}$ the space of all measurable functions $f$ on $\mathbb{R}^{n}$ such that $Q_{d}f < \infty$. The main feature of the space $B_{d}$ is the conformal invariance of its ‘norm’ $Q_{d}$. It is reasonable to ask whether $B_{d}$ is related to any $L^{3}$-based homogeneous Sobolev space. By homogeneity it can only be compared to $\hat{L}_{n/3-d}^{3}$. We have the following:

**Theorem 2.** The space $B_{d}$ is contained in $\hat{L}_{n/3-d}^{3}$ when $n > d > 2n/3$ but $B_{d}$ contains $\hat{L}_{n/3-d}^{3}$ when $2n/3 > d > n/3$; furthermore both inclusions are strict. Quantitatively speaking, for any $n > d > 2n/3$ there exists a constant $C = C_{d,n}$ such that for all measurable functions $f$ we have

$$\|I_{d-n/3}(|f|)\|_{3} \leq CQ_{d}[f].$$

For any $2n/3 > d > n/3$ there exists a constant $C = C_{d,n}$ such that for all measurable functions $f$ we have

$$Q_{d}[f] \leq C\|I_{d-n/3}(|f|)\|_{3}.$$

4. The proof of Theorem 2.

Observe that the cube of the left hand side of (21) is equal to

$$C_{d,n} \int_{\mathbb{R}^{3n}} K_{d}(x,y,z)f(x)f(y)f(z)dx dy dz,$$

where

$$K_{d}(x,y,z) = \int_{\mathbb{R}^{n}} |x - t|^{d-\frac{4n}{3}} |y - t|^{d-\frac{4n}{3}} |z - t|^{d-\frac{4n}{3}} dt.$$ 

If we establish that for $2n/3 < d < 2n$ we have

$$K_{d}(x,y,z) \leq C_{d,n}|x - y|^{d-n}|y - z|^{d-n}|z - x|^{d-n},$$

then (21) will follow immediately. Similarly, if we prove that for $n/3 < d < 2n/3$ we have

$$|x - y|^{d-n}|y - z|^{d-n}|z - x|^{d-n} \leq C_{d,n}K_{d}(x,y,z),$$

then (22) will follow as well. Now a simple dilation implies that (23) and (24) are valid for $|x|, |y|, |z| \leq 1$, then they are valid for $|x|, |y|, |z| \leq R$ with the same constant for all $R > 0$. Letting $R \to \infty$ we conclude that (23) and (24) are valid for all $R > 0$. Therefore, it suffices to prove (23) and (24) for $|x|, |y|, |z| \leq 1$.

Given any three points $x, y, z$ in $\mathbb{R}^{n}$, let $M(x,y,z) = \max(|x - y|, |y - z|, |z - x|)$ be their maximum and $m(x,y,z) = \min(|x - y|, |y - z|, |z - x|)$ be their minimum. Let us also call $\mu(x,y,z)$ the number in the middle. Then we have that $\mu(x,y,z) \geq \frac{1}{2} M(x,y,z)$. The following lemma gives us asymptotic estimates for $K_{d}(x,y,z).$
Lemma. Let $|x|, |y|, |z| \leq 1$. Then for $5n/6 < d < n$ we have
\begin{equation}
K_d(x, y, z) \sim M(x, y, z)^{3(d-n)}.
\end{equation}
For $d = 5n/6$ we have
\begin{equation}
K_d(x, y, z) \sim M(x, y, z)^{-\frac{n}{2}} \log \frac{M(x, y, z)}{m(x, y, z)},
\end{equation}
and for $n/3 < d < 5n/6$ we have
\begin{equation}
K_d(x, y, z) \sim m(x, y, z)^{-n+2(d-\frac{n}{3})} \frac{M(x, y, z)}{d-\frac{4n}{3}}.
\end{equation}

Now (23) and (24) are easy consequences of this lemma and of the observation that $\mu(x, y, z)$ is always comparable to $M(x, y, z)$.

Let us now give sketch the proof of the lemma above. Since the problem is translation invariant, it suffices to study the asymptotic behavior of the integral below as $|\alpha|, |\beta| \to 0$
\begin{equation}
\int_{\mathbb{R}^n} |\alpha - t|^{-n+\lambda} |\beta - t|^{-n+\lambda} |t|^{-n+\lambda} dt,
\end{equation}
where we set $\lambda = d - n/3$ and $\alpha = z - x$ and $\beta = z - y$. Since both $|\alpha|, |\beta| \leq 2$ the problem is local and we consider the following five cases:
Case 1. $|\alpha| \to 0$, $|\beta| \sim |\beta - \alpha| \sim 1$.
Case 2. $|\alpha - \beta| \to 0$, $|\beta| \sim |\alpha| \sim 1$.
Case 3. $|\beta - \alpha| \ll |\alpha| \sim |\beta| \to 0$.
Case 4. $|\beta - \alpha| \sim |\alpha| \sim |\beta| \to 0$.
Case 5. $|\alpha| \ll |\beta| \sim |\beta - \alpha| \to 0$.

It is easy to see that in Case 1, the integral (28) behaves like a constant when $\lambda > n/2$, blows up like $|\alpha|^{-n+2\lambda}$ when $\lambda < n/2$ and also blows up like $\log |\alpha|^{-1}$ when $\lambda = n/2$.

Case 2 is similar to Case 1 where the roles of $|\alpha|$ and $|\beta - \alpha|$ are interchanged.

In Case 3 the situation is slightly different. The integral (28) behaves asymptotically like $|\alpha|^{-n+\lambda} |\alpha - \beta|^{-n+2\lambda}$ when $\lambda < n/2$, as $|\alpha|^{-2n+3\lambda}$ when $\lambda > n/2$, and as $|\alpha|^{-2n+3\lambda} \log(|\alpha| |\alpha - \beta|^{-1})$ when $\lambda = n/2$.

In Case 4, the integral (28) behaves asymptotically like $|\alpha - \beta|^{-2n+3\lambda}$.

Finally, Case 5 follows from Case 1. In this case one has asymptotic behavior $|\alpha|^{-n+2\lambda} |\beta|^{-n+\lambda}$ when $\lambda < n/2$, $|\beta|^{-2n+3\lambda}$ when $\lambda > n/2$, and $|\beta|^{-2n+3\lambda} \log(|\beta|/|\alpha|)$ when $\lambda = n/2$.

The derivation of the asymptotics of (28) in each case involves different splitting of the integral (28) and use of formula (12). The details are rather tedious and are omitted.
The exceptional case $\lambda = n/2$ corresponds to $d = 5n/6$ and only in this case a logarithmic term appears.

We now indicate how the behavior of $K(x, y, z)$ follows from the asymptotic behavior of the integral (28). First take $5n/6 < d < n$, equivalently $n/2 < \lambda < 2n/3$. Recalling that $|\alpha| = |z - x|$ and $|\beta| = |z - y|$, we observe that the asymptotics in the five cases above, (i.e. $C$ in Cases 1 and 2, $|x - z|^{-2n+3\lambda}$ in Case 3, $|x - y|^{-2n+3\lambda}$ in Case 4, and $|y - z|^{-2n+3\lambda}$ in Case 5) is a restatement of (25). Likewise, the statements in the five cases above can be summarized in (26) when $d = 5n/6$, and in (27) when $n/3 < d < 5n/6$.

Using again the asymptotics for $K_d$, one can construct examples to show that the converse inequalities to (21) and (22) are false. The details are omitted. This concludes the proof of Theorem 2.

5. An application to fractional integrals.

Formula (8) can be used to give an alternative proof of inequality (6) in the particular case $d_1 + d_2 + d_3 = 2n$. Observe that in this case $1/p_1 + 1/p_2 + 1/p_3 = 2$ while $1/p'_1 + 1/p'_2 + 1/p'_3 = 1$. Use (8) to rewrite $Q_{d_1, d_2, d_3}(f_1, f_2, f_3)$ as

$$
\frac{1}{B(d_1, d_2, d_3, n)} \int_{\mathbb{R}^n} (f_1 * |\cdot|^{-d_2})(t) (f_2 * |\cdot|^{-d_3})(t) (f_3 * |\cdot|^{-d_1})(t) \, dt.
$$

Apply Hölder’s inequality to estimate (29) by

$$
\frac{1}{B(d_1, d_2, d_3, n)} \|f_1 * |\cdot|^{-d_2}\|_{p'_1} \|f_2 * |\cdot|^{-d_3}\|_{p'_2} \|f_3 * |\cdot|^{-d_1}\|_{p'_3}.
$$

(17), (18), and (19) now imply that (30) is bounded by

$$
\frac{1}{B(d_1, d_2, d_3, n)} \prod_{j=1}^{3} \pi^{\frac{j}{2}} \frac{\Gamma(n-d_j)}{\Gamma(n)} \Gamma(n/2) \Gamma(n)^{-1} \frac{\Gamma(n/2)}{\Gamma(n)} \frac{\Gamma(n/2)}{\Gamma(n)} \frac{\Gamma(n/2)}{\Gamma(n)} \|f_j\|_{p_j}
$$

which is nothing else than $C(d_1, d_2, d_3, n) \prod_{j=1}^{3} \|f_j\|_{p_j}$.

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ASCENT AND DESCENT FOR FINITE SEQUENCES OF COMMUTING ENDOMORPHISMS

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Homological techniques involving the Koszul complex are used to define and explore two invariants, ascent and descent, for a finite sequence of commuting endomorphism of a module. It is shown in particular that, as in the case of a single endomorphism, if ascent and descent are both finite then they are equal, and that this finiteness condition is equivalent to a certain strong Fitting type property.

1. Introduction.

Let $A$ be an algebra over a commutative ring $R$ and let $M$ be a left $A$-module. If $a : M \to M$ is an $A$-endomorphism then $\ker a^i \subseteq \ker a^{i+1}$ and $\operatorname{im} a^i \supseteq \operatorname{im} a^{i+1}$ for every $i \geq 0$. The ascent of $a$ is the least positive integer $r$ for which $\ker a^r = \ker a^{r+1}$ and the descent is the least positive integer $s$ for which $\operatorname{im} a^s = \operatorname{im} a^{s+1}$, if such integers exist and $\infty$ if they don’t. If both the ascent $r$ and the descent $s$ of $a$ are finite then $r = s$ and $M = \ker a^r \oplus \operatorname{im} a^r$. This is Fitting’s Lemma. It holds in particular for every $a \in \operatorname{End}_A(M)$ if the $A$-module $M$ is both Artinian and Noetherian. More generally, we may say that $a$ has the Fitting property if $M = K \oplus I$ and $a I = I$, where $K = \bigcup_r \ker a^r$ and $I = \cap_r a^r M$.

In this paper we consider $n$-tuples $a = (a_1, a_2, \ldots, a_n)$ of commuting endomorphisms of the $A$-module $M$. The ascent and the descent of $a$ are defined and investigated by means of homological techniques. The role of the single endomorphism $a$ in the classical case will be taken by the Koszul complex $K(M, \partial_a)$ of the $n$-tuple $a$. Composition by the endomorphism $a$ is replaced by the procedure of forming the diagonal complex of the double complex obtained via tensoring by the Koszul complex $K(A, \partial_a)$, where $A$ is a commutative subalgebra of $\operatorname{End}_A(M)$ containing the $n$-tuple $a$. This leads to a natural extension of the concepts of ascent, descent and of Fitting’s Lemma to finite sequences of commuting endomorphisms in Sections 2 to 5.

The whole approach works for any module $M$ over a commutative ring $A$ and any finite sequence of elements of $A$, without specific reference to $R$ and to an $A$-module structure on $M$. However, in our context the Fitting decomposition is of course $A$-invariant. In general, we say that the $n$-tuple $a$ has the Fitting property if $M = K \oplus I$ and $a I = I$, where
\[ K = \bigcup_r \text{Hom}_A(A/a^r, \mathcal{M}), \mathcal{I} = \cap_a a^r \mathcal{M} \text{ and } a \text{ is the ideal in } A \text{ generated by the } n\text{-tuple } a. \] The main result of Section 5 is that \( a \) has finite ascent and finite descent if and only if \( a \) has the Fitting property and \( a \) acts nilpotently on \( K \).

In case \( R \) is a field, we show in Section 7 and 8 that the concepts of ascent and descent are actually functions defined on finite dimensional subspaces of commuting elements of \( \text{End}_A(\mathcal{M}) \). More precisely, if the commuting \( n\)-tuple \( a \) and the commuting \( m\)-tuple \( b \) span the same subspace in \( \text{End}_A(\mathcal{M}) \) then they yield the same ascent, the same descent and the same Fitting decomposition of \( \mathcal{M} \). Localization techniques are then employed to see that these concepts are functions defined on finitely generated commutative subalgebras of \( \text{End}_A(\mathcal{M}) \), i.e. if the finite sequences \( a \) and \( b \) generate the same ideal in the commutative subalgebra \( A \) of \( \text{End}_A(\mathcal{M}) \) then they yield the same ascent and the same descent, even in the absence of the Fitting property.

Apart from their independent interest, the results presented in this paper have also been motivated by some open problems in functional analysis and operator theory. It is folk knowledge in this theory that the spectral behaviour of a single compact operator in the vicinity of a non-zero spectral point can be studied using an analog of Fitting’s Lemma [TAE, p. 271ff]. It was J.S. Taylor [T] who introduced the Koszul complex into the discussion of the joint spectrum of a commuting \( n\)-tuple of bounded operators. The question of how to study the spectral behaviour of such an \( n\)-tuple is of considerable importance, in particular in applications like multiparameter spectral theory [R] and the theory of elementary operators [C]. In [BT] the Taylor spectrum was applied to study the spectral properties of a commuting \( n\)-tuple of compact operators by combining the ideas of Fitting’s lemma and the Koszul complex. Our approach provides a Fitting type decomposition for any point in the Taylor spectrum. To establish notation and for the convenience of the reader some of the well-known results concerning the homology of Koszul complexes [S] are reviewed in Section 2.

2. Preliminaries on Koszul Complexes.

2.1. Throughout this article let \( A \) be a fixed associative algebra with unit over the commutative ring \( R \), and let \( \mathcal{M} \) be a fixed left \( A\)-module. Furthermore, let \( A \) be a commutative subalgebra of \( \text{End}_A(\mathcal{M}) \). For any \( n\)-tuple \( a = (a_1, a_2, \ldots, a_n) \) of elements of \( A \) we may construct the Koszul complexes \( K(A, a) \) and \( K(\mathcal{M}, a) \) as follows. If \( e = (e_1, e_2, \ldots, e_n) \) is a basis of the free \( A\)-module \( A^n \) then the differential graded \( A\)-algebra \( K(A, a) \) consists of the exterior algebra \( \Lambda(A^n) \) together with the differential of degree one

\[ \partial_a : \Lambda(A^n) \to \Lambda(A^n) \]
defined by \( \partial_a(x) = \sum_{i=1}^{n} a_i e_i \wedge x \). The Koszul cochain complex of \( \mathcal{M} \) is then the differential graded \( A \)-module \( K(\mathcal{M}, a) = \mathcal{M} \otimes_A K(A,a) \). It is often useful to use the recursive definition \( K(A,a) = K(A,a') \otimes_A K(A,a_n) \) of the Koszul complex, where \( a' = (a_1,a_2,\ldots,a_{n-1}) \) and where \( K(A,a_n) \) is just the complex \( a_n : A \to A \).

**Proposition 2.2.** Let \( X \) be any cochain complex of \( A \)-modules and let \( a \) be an element of \( A \). Then there is a short exact sequence in cohomology

\[
0 \to H^1(H^{p-1}(X) \otimes_A K(A,a)) \to H^p(X \otimes_A K(A,a)) \to H^0(H^p(X) \otimes_A K(A,a)) \to 0
\]

for each integer \( p \geq 0 \).

**Proof.** Considering \( K(A,a) \) as the complex \( a : K^0(A,a) \to K^1(A,a) \) then we get the short exact sequence of cochain complexes

\[
0 \to (X \otimes A K^1(A,a))^{p-1} \to (X \otimes A K(A,a))^p \to (X \otimes A K^0(A,a))^p \to 0
\]

and the associated long exact sequence in cohomology

\[
H^{p-1}(X) \otimes_A K^0(A,a)^{1 \otimes A a} K^{p-1}(X) \otimes_A K^1(A,a) \to H^p(X \otimes_A K(A,a)) \to H^0(H^p(X) \otimes_A K(A,a)).
\]

The assertion is now established, by taking the kernel of \( 1 \otimes A a \) on the right \( H^0(H^p(X) \otimes_A K(A,a)) = \ker(1 \otimes A a) = \ker(H^{p-1}(X) \overset{a}{\to} H^p(X)) \) and of course its cokernel on the left \( H^1(H^{p-1}(X) \otimes_A K(A,a)) = \coker(1 \otimes A a) = H^{p-1}(X)/aH^{p-1}(X) \). \( \square \)

**Corollary 2.3.** Let \( X \) be any cochain complex of \( A \)-modules and let \( a \) be an element of \( A \). If \( a \) is invertible then \( H^*(X \otimes_A K(A,a)) = 0 \) and if \( a = 0 \) then \( H^p(X \otimes_A K(A,a)) = H^{p-1}(X) \oplus H^0(X) \).

**Corollary 2.4.** If \( \mathcal{M} \neq 0 \) is Artinian and Noetherian as an \( A \)-module and if \( \mathcal{a} \subset \text{rad } A \) then the \( H^p(K(\mathcal{M}, a)) = 0 \) for \( 0 \leq p \leq n \).

**Proof.** Let \( \mathcal{a} = (a_1,a_2,\ldots,a_n) \) be contained in the Jacobson radical \( \text{rad } A \). The result will be established by induction on \( n \). For \( n = 1 \) the exact sequence

\[
0 \to H^0(\mathcal{M}, a) \to \mathcal{M} \overset{a}{\to} \mathcal{M} \to H^1(\mathcal{M}, a) \to 0
\]

of Artinian and Noetherian \( A \)-modules describes the situation. If \( H^0(\mathcal{M}, a) = 0 \) then the map \( a : \mathcal{M} \to \mathcal{M} \) is injective. The descending chain of submodules

\[
\mathcal{M} \supseteq \text{im } a \supseteq \text{im } a^2 \supseteq \ldots \supseteq \text{im } a^j \supseteq \ldots
\]

must become stationary after finitely many steps since \( \mathcal{M} \) is Artinian. But, \( \text{im } a^j = \text{im } a^{j+1} = a(\text{im } a^j) \) means that \( a^j(\text{im } a) = a^j(\mathcal{M}) \) so that \( a\mathcal{M} = \text{im } a = \mathcal{M} \), since \( a^j \) is injective. But if \( H^1(\mathcal{M}, a) = \mathcal{M}/a\mathcal{M} = 0 \) then
\( M = 0 \) by Nakayama’s Lemma, hence a contradiction. Thus, the case \( n = 1 \) is established. Now assume that \( n > 1 \), \( a' = (a_1, a_2, \ldots, a_{n-1}) \). The induction hypothesis is that the \( A \)-modules \( H^q(M, a') \neq 0 \) for all \( q \) satisfying \( 0 \leq q \leq n - 1 \) and that they are both Artinian and Noetherian.

If \( H^p(M, a) = 0 \) then by the exact sequences of Proposition 2.2

\[
0 \to H^1(H^{p-1}(X) \otimes_A K(A, a)) \to H^p(X \otimes_A K(A, a)) \to H^0(H^p(X) \otimes_A K(A, a)) \to 0
\]

with \( X = K(M, a') \) and \( a = a_n \), we see that \( H^{p-1}(X)/a_n H^{p-1}(X) = 0 \) and \( \ker(a_n : H^p(X) \to H^p(X)) = 0 \). For \( p > 0 \), Nakayama’s Lemma implies that \( H^{p-1}(M, a') = 0 \), which contradicts the induction hypothesis unless \( p > n \).

If \( p = 0 \), then \( H^0(M, a) = \ker(a_n : H^0(M, a') \to H^0(M, a')) = 0 \) so that \( a_n : H^0(M, a') \to H^0(M, a') \) is an injective endomorphism of the Artinian \( A \)-module \( H^0(M, a') \), hence bijective. Again Nakayama’s Lemma implies that \( H^0(M, a') = 0 \) in contradiction to the induction hypothesis. \( \square \)

**Corollary 2.5.** For any \( A \)-module \( M \) the commuting \( n \)-tuple \( a \) is in the annihilator of \( H^r(K(M, a)) \).

**Proof.** This follows easily by induction on \( n \) using the exact sequence of Proposition 2.2 with \( X = K(M, a') \) and \( a = a_n \). \( \Box \)

2.6. If \( X \) is a cochain complex of \( A \)-modules then \( X \otimes_A K(A, a) \) becomes a bicomplex with differentials \( \partial \otimes_A 1 \) and \( 1 \otimes_A \partial_a \). The diagonal complex \( D = D(X \otimes_A K(A, a)) \) with \( D_i = X_i \otimes_A K_1 \) has differential \( \partial \otimes_A \partial_a \). Let \( K^{(0)} = A = D^{(0)} \) and for \( r \geq 1 \) define inductively the iterated diagonal complex \( D^{(r)} = D(D^{(r-1)} \otimes_A K(A, a)) \), which is a differential graded \( A \)-algebra with \( D_i^{(r)} = \bigotimes_A^r K_i(A, a) \) and differential \( \partial^{(r)} = \bigotimes_A^r \partial_a = \partial^1 \partial^2 \cdots \partial^r \), where \( \partial^j : \bigotimes_A^j K_j(A, a) \to \bigotimes_A^{j+1} K_{j+1}(A, a) \) is acting like \( \partial_a \) on the \( j \)-th and like the identity on the remaining tensor factors. Finally we introduce the differential graded \( D^{(r)} A \)-module \( M^{(r)} = M \otimes A D^{(r)} \). Since there is no danger of confusion, we shall use the same notation \( \partial^i, \partial^{(r)} \) for the corresponding differentials on both \( D^{(r)} \) and \( M^{(r)} \).

From the degree-wise and coordinate-wise point of view we may introduce an additional upper index on the basis elements indicating the factor in the tensor power \( D_i^{(r)} = \bigotimes_A^r K_i(A, a) \). A basis of \( D_p^{(r)} \) is given by \{ \( e_{i_1, i_2}^{1} \otimes_A e_{i_2}^{2} \otimes_A \cdots \otimes_A e_{i_p}^{p} \) \}, where \( e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \) for the multi-index \( I = (i_1, i_2, \ldots, i_p) \) of type \( p \) with \( 1 \leq i_1 < i_2 < \cdots < i_p \leq n \). Notice that \( M_i^{(r)} = M \otimes_A D_i^{(r)} \) and that \( \partial_i^{(r)} : M_i^{(r)} \to M_{i+1}^{(r)} \) is the composite \( \partial_i^{(r)} = \partial_i^{1} \partial_i^{2} \cdots \partial_i^{r} \), where \( \partial_i^{j} x = (\sum_{k=1}^{n} a_k e_{i_k}^{j}) \wedge x \) for every \( x \in M_i^{(r)} \). We can also define \( M_i^{(r)} \) and the differentials \( \partial_i^{(r)} \) inductively using the commutative diagram in Figure 1. Observe that the main diagonal of this commutative diagram represents the
left most vertical of the diagram on the next step, i.e. with \( r \) replaced by \( r + 1 \). Every row and every column of the diagram represents a cochain complex. Also, the left most and the right most columns are actually isomorphic. Since \( D_j \) is a free \( A \)-module, the cohomology of the \( j \)-th column is

\[
H^*(M^{(r)} \otimes_A D_j) \cong H^*(M^{(r)}) \otimes_A D_j.
\]

The \( i \)-th row is of course isomorphic to the Koszul complex

\[
K(M_i^{(r)}, a) \cong K(M, a) \otimes_A D_i^{(r)}
\]

of the \( A \)-module \( M_i^{(r)} \cong M \otimes_A D_i^{(r)} \) and, since \( D_i^{(r)} \) is a free \( A \)-module, its cohomology is

\[
H^*(K(M_i^{(r)} \otimes_A D, a)) \cong H^*(K(M, a)) \otimes A D_i^{(r)}.
\]

In particular, if the \( A \)-module \( M \) is both Artinian and Noetherian and if \( a \subset \text{rad} \ A \), then it follows from Corollary 2.4 that \( H^p(K(M_i^{(r)} \otimes A D)) \neq 0 \) for \( 0 \leq p \leq n \).

\[
\begin{array}{cccc}
\mathcal{M}_0^{(r)} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_0^{(r)} \otimes_A D_1 & \ldots & \mathcal{M}_0^{(r)} \otimes_A D_{n-1} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_0^{(r)} \otimes_A D_n \\
\downarrow \partial_0^{(r)} & & \downarrow \partial_0^{(r)} & & \downarrow \partial_0^{(r)} & & \downarrow \partial_0^{(r)} \\
\mathcal{M}_1^{(r)} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_1^{(r)} \otimes_A D_1 & \ldots & \mathcal{M}_1^{(r)} \otimes_A D_{n-1} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_1^{(r)} \otimes_A D_n \\
\downarrow \partial_1^{(r)} & & \downarrow \partial_1^{(r)} & & \downarrow \partial_1^{(r)} & & \downarrow \partial_1^{(r)} \\
\mathcal{M}_2^{(r)} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_2^{(r)} \otimes_A D_1 & \ldots & \mathcal{M}_2^{(r)} \otimes_A D_{n-1} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_2^{(r)} \otimes_A D_n \\
\vdots & & \vdots & & \vdots & & \vdots \\
\mathcal{M}_{n-1}^{(r)} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_{n-1}^{(r)} \otimes_A D_1 & \ldots & \mathcal{M}_{n-1}^{(r)} \otimes_A D_{n-1} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_{n-1}^{(r)} \otimes_A D_n \\
\downarrow \partial_{n-1}^{(r)} & & \downarrow \partial_{n-1}^{(r)} & & \downarrow \partial_{n-1}^{(r)} & & \downarrow \partial_{n-1}^{(r)} \\
\mathcal{M}_n^{(r)} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_n^{(r)} \otimes_A D_1 & \ldots & \mathcal{M}_n^{(r)} \otimes_A D_{n-1} & \xrightarrow{\partial_0^{r+1}} & \mathcal{M}_n^{(r)} \otimes_A D_n
\end{array}
\]

**Figure 1.**

The composite of any of either a horizontal or a vertical by a consecutive diagonal map is zero and so is the composite of any diagonal by a consecutive horizontal or vertical map. Therefore, any path which changes
from horizontal to vertical direction or vice versa only through a diagonal
is a cochain complex. Of particular interest are “Yoneda composites” of
columns by rows and of rows by columns.


At each point $\mathcal{M}_{i}^{(r)} \otimes_{A} D_{j}$ of the double complex in Fig. 1, i.e. at that point in the commutative diagram

\begin{align*}
\mathcal{M}_{i-1}^{(r)} \otimes_{A} D_{j-1} & \xrightarrow{\partial_{i-1}^{(r)}} \mathcal{M}_{i-1}^{(r)} \otimes_{A} D_{j} \xrightarrow{\partial_{i-1}^{(r)+1}} \mathcal{M}_{i-1}^{(r)} \otimes_{A} D_{j+1} \\
\mathcal{M}_{i}^{(r)} \otimes_{A} D_{j-1} & \xrightarrow{\partial_{i}^{(r)}} \mathcal{M}_{i}^{(r)} \otimes_{A} D_{j} \xrightarrow{\partial_{i}^{(r)+1}} \mathcal{M}_{i}^{(r)} \otimes_{A} D_{j+1} \\
\mathcal{M}_{i+1}^{(r)} \otimes_{A} D_{j-1} & \xrightarrow{\partial_{i+1}^{(r)-1}} \mathcal{M}_{i+1}^{(r)} \otimes_{A} D_{j} \xrightarrow{\partial_{i+1}^{(r)+1}} \mathcal{M}_{i+1}^{(r)} \otimes_{A} D_{j+1}
\end{align*}

we may consider the vertical, the vertical-diagonal and the diagonal-vertical
cohomologies, denoted by $H_{ij}^{(r)}$, $L_{ij}^{(r)}$ and $R_{ij}^{(r)}$, respectively. It follows in particular that

$H_{00}^{(r)} = \ker \partial_{0}^{(r)} \cong \operatorname{Hom}(A/a^{r}, \mathcal{M})$, $H_{nn}^{(r)} = \operatorname{im} \partial_{n-1}^{(r)} \cong M/a^{r}M$,

$L_{00}^{(r)} = H_{00}^{(r)+1}$ and $R_{nn}^{(r)} = H_{nn}^{(r)+1}$ for all $r \geq 1$, where $a$ is the ideal in $A$
generated by the $n$-tuple $a$.

**Proposition 3.1.** Let $a$ be an $n$-tuple of endomorphisms contained in the
commutative subalgebra $A$ of $\operatorname{End}_{A}(\mathcal{M})$.

(a) Suppose $0 \leq j < n$. If $L_{ij}^{(r)} = H_{ij}^{(r)}$ for some $l$, where $0 \leq l < n$, then

$L_{kj}^{(r)} = H_{kj}^{(r)}$ for all $k$, where $0 \leq k \leq l$.

(b) Suppose $0 < j \leq n$. If $R_{ij}^{(r)} = H_{ij}^{(r)}$ for some $l$, where $0 < l \leq n$, then

$R_{kj}^{(r)} = H_{kj}^{(r)}$ for all $k$, where $l \leq k \leq n$.

**Proof.** a) Observe that, due to the fact that the vertical and the vertical-diagonal cohomologies at a point are defined by the same incoming homomorphism, we only have to show that the outgoing maps have equal kernels. However, since the diagonal map $\partial_{j}^{(r)+1} \partial_{k}^{(r)}$ is a composite by the vertical map $\partial_{k}^{(r)}$, it suffices to prove that $\ker(\partial_{j}^{(r)+1} \partial_{k}^{(r)}) \subseteq \ker(\partial_{k}^{(r)})$. In order to see this
pick an element \(x \in \mathcal{M}_k^{(r)} \otimes_A D_j\) from the kernel of the outgoing diagonal map and write

\[
x = \sum_{I_1, I_2, \ldots, I_{r+1}} v_{I_1 I_2 \ldots I_r I_{r+1}} e_{I_1} \otimes e_{I_2} \otimes \cdots \otimes e_{I_r} \otimes e_{I_{r+1}}.
\]

Here every \(I_i\) for \(1 \leq i \leq r\) is a multi-index of type \((j_1, j_2, \ldots, j_k)\), where \(1 \leq j_1 < j_2 < \cdots < j_k \leq n\), and the corresponding \(e_{I_i} = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_k}\). Moreover, \(e_{I_{r+1}}\) is a multi-index of type \((h_1, h_2, \ldots, h_j)\) with \(1 \leq h_1 < h_2 < \cdots < h_j \leq n\) and \(e_{I_{r+1}} = e_{h_1} \wedge e_{h_2} \wedge \cdots \wedge e_{h_j}\). This implies that

\[
\partial_k^{(r)} x = \sum_{I_1, I_2, \ldots, I_{r+1}} a_{p_1} a_{p_2} \cdots a_{p_r} v_{I_1 I_2 \ldots I_r I_{r+1}}
\]

\[
(e_{p_1} \wedge e_{I_1}) \otimes (e_{p_2} \wedge e_{I_2}) \otimes \cdots \otimes (e_{p_r} \wedge e_{I_r}) \otimes e_{I_{r+1}}.
\]

By assumption we have

\[
0 = \partial_j^{(r+1)} \partial_k^{(r)} x = \sum_{I_1, I_2, \ldots, I_{r+1}} a_{p_1} a_{p_2} \cdots a_{p_r} a_{p_{r+1}} v_{I_1 I_2 \ldots I_r I_{r+1}}
\]

\[
(e_{p_1} \wedge e_{I_1}) \otimes (e_{p_2} \wedge e_{I_2}) \otimes \cdots \otimes (e_{p_r} \wedge e_{I_r}) \otimes (e_{p_{r+1}} \wedge e_{I_{r+1}}).
\]

For \(i = 1, 2, \ldots, r\) choose any multi-index \(J_i\) of type \((j_1, j_2, \ldots, j_{i-1})\), where \(1 \leq j_1 < j_2 < \cdots < j_{i-1} \leq n\), and define \(e_{J_i} = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{i-1}}\). Letting \(e = e_{J_1} \otimes e_{J_2} \otimes \cdots \otimes e_{J_r}\), we see that \(x \wedge e\) belongs to \(\mathcal{M}_l^{(r)} \otimes_A D_j\). Now \(\partial_j^{(r+1)} \partial_i^{(r)} (x \wedge e) = (\partial_j^{(r+1)} \partial_i^{(r)} x) \wedge e\), since the diagram

\[
\begin{array}{ccc}
\mathcal{M}_k^{(r)} \otimes_A D_j & \xrightarrow{\partial_k^{(r)}} & \mathcal{M}_{k+1}^{(r)} \otimes_A D_j & \xrightarrow{\partial_j^{(r+1)}} & \mathcal{M}_{k+1}^{(r)} \otimes_A D_{j+1} \\
\wedge e \downarrow & & \wedge e \downarrow & & \wedge e \downarrow \\
\mathcal{M}_{l+1}^{(r)} \otimes_A D_j & \xrightarrow{\partial_i^{(r)}} & \mathcal{M}_{l+1}^{(r)} \otimes_A D_j & \xrightarrow{\partial_l^{(r+1)}} & \mathcal{M}_{l+1}^{(r)} \otimes_A D_{j+1}
\end{array}
\]

is commutative. Thus, \(x \in \ker(\partial_j^{(r+1)} \partial_i^{(r)} x)\) implies that \(x \wedge e \in \ker(\partial_j^{(r+1)} \partial_i^{(r)} \wedge e)\) and by the assumptions in part a) also that \(x \wedge e \in \ker(\partial_i^{(r)} x)\) for every \(e\) as described above. But then \((\partial_k^{(r)} x) \wedge e = \partial_i^{(r)} (x \wedge e) = 0\) for every such \(e\). Clearly there are sufficiently many maps \(\wedge e\) of that kind to conclude that \(\partial_k^{(r)} x = 0\).

b) Similar arguments apply to the second part of the Proposition. First observe that at a given point both the vertical and the diagonal-vertical cohomologies are defined by the same outgoing homomorphism. It therefore suffices to prove that the ingoing maps have equal images. However, since the diagonal map \(\partial_k^{(r)} \partial_i^{(r)} x\) is a composite by the vertical map \(\partial_k^{(r)}\), it remains to show that \(\text{im}(\partial_k^{(r)} x) \subseteq \text{im}(\partial_k^{(r)} \partial_i^{(r)} x)\). To this effect pick \(x \in \mathcal{M}_k^{(r)} \otimes_A D_j\)
and write
\[ x = \sum_{I_1, I_2, \ldots, I_{r+1}} u_{I_1 I_2 \ldots I_r I_{r+1}} e_{I_1} \otimes e_{I_2} \otimes \ldots \otimes e_{I_r} \otimes e_{I_{r+1}}. \]

Here every \( I_i \) for \( 1 \leq i \leq r \) is a multi-index of type \((j_1, j_2, \ldots, j_{k-1})\), where \( 1 \leq j_1 < j_2 < \ldots < j_{k-1} \leq n \), and the corresponding \( e_{I_i} = e_{j_1}^{i_1} \wedge e_{j_2}^{i_2} \wedge \ldots \wedge e_{j_{k-1}}^{i_{k-1}} \). Moreover, \( I_{r+1} \) is a multi-index of type \((h_1, h_2, \ldots, h_j)\) with \( 1 \leq h_1 < h_2 < \ldots < h_j \leq n \) and \( e_{I_{r+1}} = e_{h_1} \wedge e_{h_2} \wedge \ldots \wedge e_{h_j} \). This implies that

\[ u = \partial^{(r)}_{k-1} x = \sum_{I_1, I_2, \ldots, I_{r+1}} \sum_{p_1, p_2, \ldots, p_r} a_{p_1} a_{p_2} \ldots a_{p_r} u_{I_1 I_2 \ldots I_r I_{r+1}} \]

\[ (e_{p_1} \wedge e_{I_1}) \otimes (e_{p_2} \wedge e_{I_2}) \otimes \ldots \otimes (e_{p_r} \wedge e_{I_r}) \otimes e_{I_{r+1}}. \]

For \( i = 1, 2, \ldots, r \) choose any multi-indices \( J_i \) of type \((j_1, j_2, \ldots, j_{l-1})\), where \( 1 \leq j_1 < j_2 < \ldots < j_{l-1} \leq n \), in such a way that \( J_i \) is a subindex of \( I_i \), i.e. such that every index that belongs to \( J_i \) also belongs to \( I_i \). Next, let \( e_{J_i} = e_{j_1}^{i_1} \wedge e_{j_2}^{i_2} \wedge \ldots \wedge e_{j_{l-1}}^{i_{l-1}} \). Then \( e = e_{J_1} \otimes e_{J_2} \otimes \ldots \otimes e_{J_r} \) defines a homomorphism \( \wedge e : M_{s}^{(r)} \otimes A D_j \rightarrow M_{s+k-l}^{(r)} \otimes A D_j \) for each \( s \) so that the diagram

\[
\begin{array}{ccc}
M_{l-1}^{(r)} \otimes A D_j & \xrightarrow{\partial^{(r)}_{l-1}} & M_{l-1}^{(r)} \otimes A D_j \\
\wedge e \downarrow & & \wedge e \downarrow \\
M_{l-1}^{(r)} \otimes A D_j & \xrightarrow{\partial^{(r)}_{l-1}} & M_{l-1}^{(r)} \otimes A D_j \\
\end{array}
\]

is commutative. According to the construction of \( e \) there must be an element \( y \in M_{l-1}^{(r)} \otimes A D_j \) such that \( x = y \wedge e \). By the assumption of the Proposition that \( R_{ij}^{(r)} = H_{ij}^{(r)} \) we have \( \partial^{(r)}_{l-1} y \in \text{im}(\partial^{(r)}_{l-1} \partial^{(r)}_{l-1}) \), and there exists a \( z \in M_{l-1}^{(r)} \otimes A D_j \) such that \( \partial^{(r)}_{l-1} \partial^{(r)}_{l-1} z = \partial^{(r)}_{l-1} y \). By the commutativity of the diagram we see that \( \partial^{(r)}_{l-1} \partial^{(r)}_{l-1} (z \wedge e) = (\partial^{(r)}_{l-1} \partial^{(r)}_{l-1} z) \wedge e = (\partial^{(r)}_{l-1} y) \wedge e = \partial^{(r)}_{k-1} (y \wedge e) = \partial^{(r)}_{k-1} x \), proving the assertion. \( \square \)

**Corollary 3.2.** Let \( \alpha \) be an \( n \)-tuple of endomorphisms contained in the commutative subalgebra \( A \) of \text{End}_A(\mathcal{M}) \).

(a) If \( L_{0}^{(r)} = H_{0}^{(r)} \) for some \( l, 0 \leq l < n \), then \( L_{k}^{(r)} = H_{k}^{(r)} \) for all \( k, 0 \leq k \leq l \).

(b) If \( R_{0}^{(r)} = H_{0}^{(r)} \) for some \( l, 0 < l \leq n \), then \( R_{k}^{(r)} = H_{k}^{(r)} \) for all \( k, l \leq k \leq n \).

3.3. Example 5.5 will show that the opposite of this Corollary is not true in general. However, we will also show that in some cases, and in particular in
the case of an $A$-module $M$ which is both Artinian and Noetherian, equality of the vertical and the vertical-diagonal cohomology modules at $M_0^{(r)}$, i.e. at the top left corner of Fig. 1, implies their equality at $M_{n-1}^{(r)}$. Thus, this condition imposed on the cohomology modules at some point $M_k^{(r)}$, where $0 \leq k < n$, implies that the same condition holds for all $k$. Similar remarks apply to the case of the cohomology modules on the right-most vertical of Fig. 1.

4. Ascent and Descent.

In the following we shall be mainly interested in the cohomology modules at the left most and the right most verticals of the double complexes $M^{(r)} \otimes_A D$, that is in the situation of Corollary 3.2. Throughout the modules $M_n^{(r)}$ and $M_0^{(r)}$ will be identified with $M$ and $a$ is the ideal in $A$ generated by $a$.

**Proposition 4.1.** If $L_{n-1,0}^{(r)} = H_{n-1,0}^{(r)}$ for some index $r \geq 1$, then

(a) $\ker(\partial_0^{r+l} \cdots \partial_0^{r+1} \partial_{n-1}^{(r)}) = \ker(\partial_{n-1}^{(r)})$ for all $l \geq 1$;

(b) $\ker \partial_0^{(r)} \cap \im \partial_{n-1}^{(r)} = 0$, i.e. $\Hom(A/a^r, M) \cap a^r M = 0$;

(c) $L_{n-1,0}^{(s)} = H_{n-1,0}^{(s)}$ for all indices $s \geq r$.

**Proof.** (a) The case $l = 1$ is just our assumption. Assume the claim true for a certain index $l - 1$ and observe that $\ker(\partial_0^{r+l} \cdots \partial_0^{r+1} \partial_{n-1}^{(r)}) \supseteq \ker \partial_{n-1}^{(r)}$, so that we only have to show that in fact the opposite inclusion holds. To this end pick $x \in M_{n-1}^{(r)}$ such that $\partial_0^{r+l} \cdots \partial_0^{r+1} \partial_{n-1}^{(r)} x = 0$ and write

$$y = \partial_0^{r+l-1} \cdots \partial_0^{r+1} x = \sum_{i_1, i_2, \ldots, i_{n-1}} x_{i_1 i_2 \ldots i_{n-1}} e_{i_1}^{r+1} \otimes e_{i_2}^{r+2} \otimes \cdots \otimes e_{i_{n-1}}^{r+l-1}.$$

Clearly, $x_{i_1 i_2 \ldots i_{n-1}} \in M_{n-1}^{(r)}$, and since $\partial_0^{r+l}$ commutes with $\partial_i^{(r)}$ for $i \neq j$ we see that $\partial_0^{r+l} \partial_{n-1}^{(r)} y = 0$. This forces $\partial_{n-1}^{(r)} y = 0$ by the assumption in the statement of our proposition. Since the maps $\partial_0^{(r)}$ commute with $\partial_{n-1}^{(r)}$, we conclude that $\partial_0^{r+l-1} \cdots \partial_0^{r+1} \partial_{n-1}^{(r)} x = 0$ and the assertion follows by the induction hypothesis.

(b) The homomorphism $\partial_{n-1}^{(r)}$ is a composite of maps of the type $\partial_i^{(r)}$ for $1 \leq i \leq r$. After identifying $M_0^{(r)}$ with $M_{n-1}^{(r)}$ we may write $\partial_0^{(r)} = \partial_0^{2r} \partial_0^{2r-1} \cdots \partial_0^{r+1}$. If $x \in \ker \partial_0^{(r)} \cap \im \partial_{n-1}^{(r)}$ then by assumption $x = \partial_{n-1}^{(r)} y$ for some $y \in M_{n-1}^{(r)}$ and $\partial_0^{2r} \partial_0^{2r-1} \cdots \partial_0^{r+1} x = \partial_0^{(r)} x = 0$. Consequently $\partial_0^{2r} \partial_0^{2r-1} \cdots \partial_0^{r+1} \partial_{n-1}^{(r)} y = 0$ and it follows by (a) that $x = \partial_{n-1}^{(r)} y = 0$. 
(c) Choose any \( x \in \ker(\partial_n^{r+1} \partial_{n-1}^{r+2}) \) and observe that it suffices to show that \( x \in \ker(\partial_{n-1}^{r+1}) \). Recall that \( \partial_n^{r+1} = \partial_{n-1}^{r+1} \partial_n^{r+1} \) and that the two homomorphisms commute. We see that \( y = \partial_{n-1}^{r+1} x \in \ker(\partial_n^{r+1} \partial_{n-1}^{r+2}) \) and hence by assumption \( y \in \ker(\partial_{n-1}^{r+1}) \). This finally shows that \( 0 = \partial_n^{r+1} \partial_{n-1}^{r+1} x = \partial_{n-1}^{r+1} x \).

Proposition 4.2. If \( R_{l_{n_1}}^{(r)} = H_{l_{n_1}}^{(r)} \) for some index \( r \geq 1 \), then

(a) \( \mathrm{im}(\partial_{n-1}^{r+1} \partial_{n-1}^{r+2}) = \mathrm{im}(\partial_n^{r+1}) \) for all \( l \geq 1 \);

(b) \( \ker(\partial_n^{(r)}) + \ker(\partial_{n-1}^{(r)}) = \mathcal{M} \), i.e. \( \text{Hom}_A(A/\mathfrak{a}, \mathcal{M}) + a^r \mathcal{M} = \mathcal{M} \);

(c) \( R_{l_{n_1}}^{(r)} = H_{l_{n_1}}^{(r)} \) for all indices \( s \geq r \).

Proof. (a) The case \( l = 1 \) is identical with our assumption. Suppose that the claim is true for an index \( l - 1 \) and observe that \( \ker(\partial_{n-1}^{r+1}) \), so that it suffices to show that the opposite inclusion holds. To this end pick \( x \in \mathcal{M}_{l_{n_1}}^{(r)} \odot_A D_n \) such that \( x \in \ker(\partial_n^{(r)}) \) and assume by the induction hypothesis that \( x = (\partial_{n-1}^{r+1} \partial_{n-1}^{r+2}) \partial_{l_{n_1}}^{(r)} y \) for some \( y \in \mathcal{M}_{l_{n_1}}^{(r)} \odot_A D_n \). Then

\[
y = \sum_{i_1, i_2, \ldots, i_l} y_{i_1} i_2 r+2 \odot_A e_{i_1} r+3 \odot_A \cdots \odot_A e_{i_l}^{r+1},
\]

where the exterior products \( e_i^j = \bigwedge_{j \neq i} e_i^j \) form the canonical basis elements of \( D_n \) and where \( y_{i_1} i_2 \ldots i_l \in \mathcal{M}_{l_{n_1}}^{(r)} \odot \mathcal{M}_{l_{n_1}}^{(r)} \odot_A D_n \). Clearly, \( \partial_n^{(r)} y_{i_1} i_2 \ldots i_l \in \mathcal{M}_{l_{n_1}}^{(r)} \odot_A D_n \), so that by the assumption in our proposition there is an element \( z_{i_1} i_2 \ldots i_l \in \mathcal{M}_{l_{n_1}}^{(r)} \odot_A D_n \) such that \( \partial_n^{(r)} y_{i_1} i_2 \ldots i_l = \partial_{n-1}^{r+1} \partial_n^{(r)} z_{i_1} i_2 \ldots i_l \). Since the maps \( \partial_{n-1}^{r+1} \) and \( \partial_{n-1}^{r+1} \) commute for \( i \neq j \), we conclude for

\[
z = \sum_{i_1, i_2, \ldots, i_l} z_{i_1} i_2 \ldots i_l e_{i_1} r+2 \odot_A e_{i_1} r+3 \odot_A \cdots \odot_A e_{i_l}^{r+1}
\]

that \( (\partial_{n-1}^{r+1} \partial_{n-1}^{r+2}) \partial_n^{(r)} z = x \), which proves the assertion.

(b) Choose any \( x \in \mathcal{M} \). The homomorphism \( \partial_n^{(r)} \) is a composite of maps of the type \( \partial_n^{r+1} \) for \( 1 \leq i \leq r \). After identifying \( \mathcal{M}_{l_{n_1}}^{(r)} \) with \( \mathcal{M}_{l_n}^{(r)} \) we may write \( \partial_{n-1}^{r+1} = \partial_{n-1}^{r+1} \partial_{n-1}^{r+1} \). By Corollary 3.2. (b) the element \( \partial_n^{(r)} x \) belongs to \( \ker(\partial_{n-1}^{r+1} \partial_n^{(r)}) \), yielding the existence of an element \( y \) for which \( \partial_{n-1}^{r+1} \partial_{r}^{(r)} y = \partial_{r}^{(r)} x \) and hence \( \partial_n^{(r)} (x - \partial_n^{(r)} y) = 0 \).

(c) Pick any \( x \in \ker(\partial_n^{(r+1)}) \) and observe that it obviously suffices to show that \( x \in \ker(\partial_n^{(r+1)} \partial_{n-1}^{r+2}) \). Recall first that \( \partial_n^{(r+1)} \partial_{n}^{(r+1)} = \partial_{n-1}^{r+1} \partial_{n}^{(r)} \) and that the two maps commute. If \( y \) is such that \( x = \partial_n^{(r+1)} \partial_n^{(r)} y \) then \( \partial_n^{(r+1)} y \in \ker(\partial_n^{(r+1)}) \).
im(∂^{r+2}_n \partial_0^{(r)}) by the induction hypothesis and by the assumption in the Proposition. Thus we conclude that \( x = \partial_0^{(r)} \partial_0^{r+1} y \in \text{im}(\partial^{r+2}_n \partial_0^{(r)}) \). \( \square \)

4.3. If there exists an index \( r \) for which at the point \( M^{(r)}_{n-1} \otimes_A D_0 \) the left vertical cohomology \( H^{(r)}_{n-1,0} \) is equal to the vertical-diagonal cohomology \( L^{(r)}_{n-1,0} \), then by Proposition 4.1 the same holds for all larger indices. The smallest index \( r \) with this property will be called the \textit{ascent} of the \( n \)-tuple \( a \). If such an index does not exist, we shall say that the ascent of the \( n \)-tuple is infinite.

Similarly, if there is an index \( s \) for which at the point \( M^{(s)}_1 \otimes_A D_n \) the right vertical cohomology \( H^{(s)}_1 \) is equal to the diagonal-vertical cohomology \( R^{(s)}_1 \), then by Proposition 4.2 the same holds for all larger indices. The smallest index \( s \) with this property will be called the \textit{descent} of the \( n \)-tuple \( a \). If such an index does not exist, we shall say that the descent of the \( n \)-tuple is infinite.

More generally, for any pair \((i, j)\) satisfying \( 0 \leq i, j \leq n \), let \( d^0_{ij} \) be the smallest among the indices \( r \) for which \( L^{(r)}_{ij} = H^{(r)}_{ij} \) and \( d^1_{ij} \) the smallest among the indices \( r \) for which \( R^{(r)}_{ij} = H^{(r)}_{ij} \). We let any of these indices be infinite whenever there is no \( r \) satisfying the respective condition. According to this notation the ascent of the \( n \)-tuple \( a \) is \( d^0_{0,n-1} \) and the descent is \( d^1_{1n} \).

5. Relations between ascent, descent and Fitting’s Lemma.

In this section we will show that both ascent and descent of a commuting \( n \)-tuple \( a \) are finite if and only if \( a \) has the Fitting property and \( a \) acts nilpotently on \( K \), and that ascent and descent are equal in this case. Let us recall from Section 4, that the notation for the ascent and the descent are \( d^0 = d^0_{0,n-1} \) and \( d^1 = d^1_{1n} \), respectively. It is also useful to remember that \( H^{(r)}_{00} \cong \text{Hom}_A(A/a^r, M) \), \( H^{(r)}_{nn} \cong M/a^r M \), \( L^{(r)}_{00} = H^{(r+1)}_{00} \) and \( R^{(r)}_{nn} = H^{(r+1)}_{nn} \).

Proposition 5.1. Let \( a \) be an \( n \)-tuple of endomorphisms contained in the commutative subalgebra \( A \) of \( \text{End}_A(M) \).

(a) If \( L^{(r)}_{00} = H^{(r)}_{00} \) for some index \( r \geq 1 \), then \( L^{(s)}_{00} = H^{(s)}_{00} \) for all \( s \geq r \).
(b) If \( R^{(r)}_{nn} = H^{(r)}_{nn} \) for some index \( r \geq 1 \), then \( R^{(s)}_{nn} = H^{(s)}_{nn} \) for all \( s \geq r \).
(c) If \( L^{(r)}_{00} = H^{(r)}_{00} \) for a fixed index \( r \geq 1 \) and \( R^{(s)}_{1n} = H^{(s)}_{1n} \) for some index \( s \geq 1 \), then this holds for some \( s \leq r \).
(d) If \( R^{(r)}_{nn} = H^{(r)}_{nn} \) for a fixed index \( r \geq 1 \) and \( L^{(s)}_{n-1,0} = H^{(s)}_{n-1,0} \) for some index \( s \geq 1 \), then this holds for some \( s \leq r \).

Proof. For (a) and (b) use similar arguments as in the proofs of Propositions 4.1(a) and 4.2(a).
(c) Let $s$ be the smallest index for which $R_{1n}^{(s)} = \mathcal{H}_{1n}^{(s)}$ and suppose that the assertion is false. Then, there must be an $r < s$ with $L_{00}^{(r)} = \mathcal{H}_{00}^{(r)}$. By (a) we may assume with no loss of generality that $r = s - 1$. We will show that in this case we must also have $R_{1n}^{(s-1)} = \mathcal{H}_{1n}^{(s-1)}$, in contradiction to the minimality of $s$, thus proving the assertion. Notice that we always have $\text{im}(\partial_{n-1}^{s} \partial_{1}^{(s-1)}) \subseteq \text{im}(\partial_{0}^{(s-1)})$ at the point $\mathcal{M}_{1}^{(s)} \otimes A D_{n}$, so that only the opposite inclusion has to be established. For this purpose pick any $x \in \text{im}(\partial_{0}^{(s-1)})$. Then $x = \partial_{0}^{(s-1)} y$ for some $y \in \mathcal{M}_{0}^{(s-1)} \otimes A D_{n}$ and it follows that $\partial_{0}^{s} x \in \text{im}(\partial_{0}^{(s)}) = \text{im}(\partial_{0}^{s}) \partial_{n-1}^{s+1}$ by the second assumption in (c). This yields the existence of an element $z \in \mathcal{M}_{0}^{(s)} \otimes A D_{n-1}$ such that $\partial_{0}^{s} \partial_{n-1}^{s+1} z = \partial_{0}^{s} \partial_{n-1}^{s+1} z$, and hence $y - \partial_{n-1}^{s+1} z \in \ker(\partial_{0}^{(s)})$. Thus $\mathcal{H}_{00}^{(s)} = L_{00}^{(s-1)} = \mathcal{H}_{00}^{(s-1)}$, where the first equality holds by definition and the second by assumption. Therefore, $x = \partial_{0}^{s} y = \partial_{0}^{s} \partial_{n-1}^{s+1} z$ which belongs to $\text{im}(\partial_{0}^{(s-1)} \partial_{n-1}^{s+1})$, establishing the inclusion $\text{im}(\partial_{0}^{(s-1)}) \subseteq \text{im}(\partial_{0}^{(s-1)} \partial_{n-1}^{s+1})$ and hence the required contradiction.

(d) Let $s$ be the smallest index for which $L_{n-1,0}^{(s)} = \mathcal{H}_{n-1,0}^{(s)}$ and suppose that the assertion is false. Then, there must be an $r < s$ such that $R_{n}^{(r)} = \mathcal{H}_{n}^{(r)}$. By (b) we may assume with no loss of generality that $r = s - 1$. We shall show that in this case we must also have $L_{n-1,0}^{(s-1)} = \mathcal{H}_{n-1,0}^{(s-1)}$, contradicting the minimality of $s$ and therefore proving our assertion. Recall that we always have $\ker(\partial_{n-1}^{(s-1)}) \subseteq \ker(\partial_{n-1}^{(s-1)} \partial_{0}^{(s)})$, so that only the opposite inclusion has to be established. Pick any $x \in \ker(\partial_{n-1}^{(s-1)} \partial_{0}^{(s)})$, then $y = \partial_{n-1}^{(s-1)} x \in \ker(\partial_{0}^{(s)}) \cap \text{im}(\partial_{n-1}^{(s-1)})$. Since $\mathcal{H}_{n}^{(s)} = R_{n}^{(s-1)} = \mathcal{H}_{n}^{(s-1)}$, where the first equality holds by definition and the second by assumption, it follows that $y = \partial_{n-1}^{(s)} z$ for some $z \in \mathcal{M}_{n-1}^{(s)}$ and hence $0 = \partial_{0}^{(s)} y = \partial_{0}^{(s)} \partial_{n-1}^{(s)} z$. This implies, by the assumption $L_{n-1,0}^{(s)} = \mathcal{H}_{n-1,0}^{(s)}$, that $y = \partial_{n-1}^{(s)} z = 0$ and hence $x \in \ker(\partial_{n-1}^{(s-1)})$, establishing the required contradiction.

Theorem 5.2. The ascent $d^{0}$ and the descent $d^{1}$ of a commuting $n$-tuple $\mathbf{a}$ are both finite if and only if $\mathcal{M} = \mathcal{K} \oplus \mathcal{I}$ as an $A$-module, where the ideal $\mathbf{a}$ generated by the $n$-tuple $\mathbf{a}$ acts nilpotently on $\mathcal{K}$,

$$\text{Hom}_{A}(A/\mathbf{a}, \mathcal{I}) = 0 \quad \text{and} \quad \mathbf{a}\mathcal{I} = \mathcal{I}.$$  

If these equivalent conditions are satisfied, then

(a) $d^{0} = r = d^{1}$.

(b) $d_{j-1,0}^{0} = r = d_{j}^{1}$ for $1 \leq r \leq n$.

(c) $\mathcal{K} = \ker(\partial_{0}^{(r)}) \cong \text{Hom}_{A}(A/\mathbf{a}^{r}, \mathcal{M})$ and $\mathcal{I} = \text{im}(\partial_{n-1}^{(r)}) = \mathbf{a}^{r}\mathcal{M}$. 

(d) The restrictions of the $a_i$ to $\ker \partial_0^{(r)}$ are nilpotent. Moreover,

$$\text{Hom}_A(A/a^{(k)}I, I) = 0$$

for every $k \geq 1$, where $a^{(k)}$ is the ideal in $A$ generated by the $n$-tuple $a^{(k)} = (a_1^k, a_2^k, \ldots, a_n^k)$.

(e) $r$ is an invariant of the ideal $a$, i.e. independent of the particular finite generating set $a$.

Proof. Recall that the ascent is $d^0 = d_{n-1,0}$ and the descent $d^1 = d_{1n}$. Suppose that they are both finite. By Propositions 4.1(c) and 5.1(c) we have that $d_{n-1,0}^0 \geq d_{n-2,0}^0 \geq \ldots \geq d_{0,0}^0 \geq d_{1n}^0$, proving in particular that the ascent is no smaller than the descent. On the other hand, Propositions 4.2(c) and 5.1(d) yield $d_{1n}^1 \geq d_{2n}^1 \geq \ldots \geq d_{nn}^1 \geq d_{n-1,0}^0$ and, in particular, that the descent is no smaller than the ascent. Thus all the indices and, in particular the ascent and the descent, are equal as required in (b) and in (a). Assertion (c) is an easy consequence of Propositions 4.1(b) and 4.2(b).

That the decomposition is invariant under the actions of $A$ and of $A$ is due to the fact that the differentials are compatible with both actions. Now, (d) follows directly from (c).

Conversely, suppose that $M = K \oplus I$, where the restriction of the $a_i$ to $K$ are nilpotent and the restrictions $a'_i$ of the $a_i$ to $I$ satisfy

$$\bigcap_{1=1}^n \ker a'_i = 0 \quad \text{and} \quad \sum_{i=1}^n \text{im} a'_i = I.$$

Then for each $r \geq 1$ there is obviously an isomorphism of complexes

$$M^{(r)} \cong K^{(r)} \oplus I^{(r)},$$

and so it suffices to prove the assertion separately for $K$ and for $I$.

For $N$ equal to $K$ or $I$, as the case may be, consider first the diagram

$$\begin{array}{ccc}
N_n^{(r)} & \xrightarrow{\partial_n^{(r)}} & N_n^{(r+1)} \\
| \downarrow{\partial_n^{(r)}} & & \downarrow{\partial_{n+1}^{(r)}} \\
N_{n-1}^{(r)} & \xrightarrow{\partial_{n-1}^{(r)}} & N_{n-1}^{(r)} \otimes_A D_1.
\end{array}$$

Let $E_i = e_1 \wedge \ldots \wedge e_i \wedge \ldots e_n$. Then for $N = K$, we see that

$$\partial_{n-1}^{(r)} \left( \sum_{i_1, i_2, \ldots, i_r} x_{i_1i_2\ldots i_r} E_{i_1} \otimes E_{i_2} \otimes \ldots \otimes E_{i_r} \right) = \sum_{i_1, i_2, \ldots, i_r} a_{i_1} a_{i_2} \ldots a_{i_r} x_{i_1i_2\ldots i_r} = 0$$

if $r$ is large enough, since each coefficient $x_{i_1i_2\ldots i_r}$ is in $K$. It suffices to choose $r \geq \sum_i r_i$, where $a_i^r K = 0$. Thus, there is an index $r_0$, such that $\partial_{n-1}^{(r)} = 0$.
for all \( r \geq r_0 \), and \((a, \mathcal{K})\) has finite ascent. On the other hand, when \( \mathcal{N} = \mathcal{I} \) then

\[
\partial^{r+1}_n (y) = \sum_j a'_j y \otimes e_j = 0
\]

if and only if \( y = 0 \), since \( y \in \mathcal{I} \) and \( \cap_j \ker a'_j = 0 \). Hence, \( \partial^{r+1}_n \) is injective for every \( r \) and \((a', \mathcal{I})\) has finite ascent. Now consider the diagram

\[
\begin{array}{c}
\mathcal{N}^{(r)}_0 \otimes_A D_{n-1} \xy (600,0)*+!R{\partial^{r+1}_n} \ar 0;0 \end{array} \rightarrow \mathcal{N}^{(r)} \otimes_A D_n \\
\downarrow \partial_0^{(r)} \\
\mathcal{N}^{(r)}_1 \otimes_A D_n.
\end{array}
\]

Then, setting \( \mathcal{N} = \mathcal{K} \), we see that

\[
\partial_0^{(r)} (x) = \sum_{j_1, j_2, \ldots, j_n} a_{j_1} a_{j_2} \ldots a_{j_n} x e_{j_1} \otimes e_{j_2} \otimes \ldots \otimes e_{j_n} = 0
\]

if \( r \) is large enough, since \( x \in \mathcal{K} \). Again, it suffices to choose \( r \geq \sum_i r_i \), where \( a_i' \mathcal{K} = 0 \). Thus, there is an index \( s_0 \) such that \( \partial_0^{(r)} = 0 \) for all \( r \geq s_0 \), so that \((a, \mathcal{K})\) has finite descent. If \( \mathcal{N} = \mathcal{I} \) then

\[
\partial^{r+1}_{n-1} \left( \sum_i x_i e_1 \wedge \ldots \wedge \hat{e}_i \wedge \ldots \wedge e_n \right) = \sum_i (-1)^{i-1} a'_i x_i.
\]

Thus, \( \partial^{r+1}_{n-1} \) is surjective for every \( r \), since \( \mathcal{I} = \sum_i \text{im} a'_i \), so that \((a', \mathcal{I})\) has finite descent. The ascent and the descent of \( a \) are therefore both finite, hence equal. Moreover, it follows directly from our conditions, which are independent of the particular finite generating set of the ideal \( a \), that this number \( r \) is the least positive integer for which one and hence all the conditions \( a' \mathcal{K} = 0 \), \( \mathcal{I} = a' \mathcal{M} \) and \( \mathcal{K} = \text{Hom}_A(A/a', \mathcal{M}) \) are satisfied, so that (e) holds.

5.3. The conditions \( \text{Hom}_A(A/a, \mathcal{I}) = 0 \) and \( \mathcal{I} = a\mathcal{I} \) say that the \( n \)-tuple \( a \) acts ‘jointly bijectively’ on \( \mathcal{I} \). Observe that a slight modification of the proof of Theorem 5.2 shows that ascent and descent are either both finite or both infinite whenever \( a \) has the Fitting property, i.e. whenever \( \mathcal{M} = \mathcal{K} \oplus \mathcal{I} \) and \( \mathcal{I} = a\mathcal{I} \), where \( \mathcal{K} = \bigcup_r \text{Hom}_a(A/a', \mathcal{M}) \) and \( \mathcal{I} = \cap_r a'^r \mathcal{M} \). As a more general version of assertion (e) we shall see in Section 8 that, even in the absence of the Fitting property, both ascent and descent are invariants of the ideal \( a \).

We say that the commuting \( n \)-tuple satisfies condition (F) if the \( A \)-module \( \mathcal{M} \) decomposes into a direct sum of \( A \)-submodules

\[
\mathcal{M} = \mathcal{K} \oplus \mathcal{I}_1 \oplus \mathcal{I}_2 \oplus \ldots \oplus \mathcal{I}_p,
\]
Let $V$ be a $\mathcal{A}$-module, then every commuting $n$-tuple of endomorphisms satisfies this condition. Indeed, in this case the ascent and the descent of the endomorphism $a_1$ must be finite by Fitting’s Lemma [P, B]. By Theorem 5.2, applied to the 1-tuple $a_1$ and the fixed commutative algebra $A$, the $\mathcal{A}$-module $\mathcal{M}$ decomposes into $\mathcal{M} = \mathcal{K}_1 \oplus \mathcal{I}_1$, where $\mathcal{K}_1 = \ker a_1^r$ and $\mathcal{I}_1 = \text{im} a_1^r$ for some sufficiently large exponent $r$. By commutativity, $\mathcal{K}_1$ and $\mathcal{I}_1$ are $\mathcal{A}$-invariant. If the restrictions of all the $a_i$ to $\mathcal{K}_1$ are nilpotent, then we are done. If not, pick one whose restriction to $\mathcal{K}_1$ is not nilpotent, say $a_2$, and decompose $\mathcal{K}_1$ into a direct sum of $\mathcal{A}$-submodules $\mathcal{K}_2 \oplus \mathcal{I}_2$ as above. Obviously, the inductive procedure must terminate after finitely many steps and we get the desired decomposition. These considerations actually show slightly more. In order that condition (F) is satisfied it suffices that each member of the $n$-tuple $a$ has finite ascent and finite descent.

**Corollary 5.4.** If each member of the $n$-tuple $a$ has finite ascent and finite descent, in particular, when the $\mathcal{A}$-module $\mathcal{M}$ is both Artinian and Noetherian, then both ascent and descent of the $n$-tuple are finite and the conclusions of Theorem 5.2 are valid.

**Proof.** The “Fitting” decomposition of $\mathcal{M}$, given in condition (F) and guaranteed by our assumptions, decomposes $\mathcal{M}^{(r)} = [\mathcal{M}] \otimes A D^{(r)}_{n-1}$ into

$$\mathcal{M}^{(r)}_{n-1} = \left(\mathcal{K} \otimes A D^{(r)}_{n-1}\right) \oplus \left(\mathcal{I}_1 \otimes A D^{(r)}_{n-1}\right) \oplus \cdots \oplus \left(\mathcal{I}_p \otimes A D^{(r)}_{n-1}\right).$$

An element $x \in \ker(\partial^{(r+1)} - \partial^{(r)}_{n-1})$ decomposes accordingly into $x = u_0 + u_1 + \cdots + u_p$. Since the differentials $\partial^{(r)}_{n-1}$ and $\partial^{(r+1)}_{n-1}$ preserve the decomposition it follows that

$$0 = \partial^{(r+1)}_{n-1} x = \partial^{(r+1)}_{n-1} u_0 + \partial^{(r+1)}_{n-1} u_1 + \cdots + \partial^{(r+1)}_{n-1} u_p.$$

For $0 \leq i \leq p$ the term $\partial^{(r)}_{n-1} u_i$ must therefore be in the kernel of each $a_j$ restricted to $\mathcal{K}$ and $\mathcal{I}_i$, respectively. This forces $\partial^{(r)}_{n-1} u_i$ to be zero for $1 \leq i \leq p$, since then at least one of these restrictions is invertible. Hence $\partial^{(r)}_{n-1} x = \partial^{(r)}_{n-1} u_0$, and by the nilpotency of the restriction of each $a_j$ to $\mathcal{K}$ we see that $\partial^{(r)}_{n-1} x = \partial^{(r)}_{n-1} u_0 = 0$ for $r$ sufficiently large. Similarly, if $y = \partial^{(r)}_{n-1} x \in \text{im} \partial^{(r)}_{n-1}$, decompose $x$ according to (F) into $x = u_0 + u_1 + \cdots + u_p$. For sufficiently large $r$ we may assume $u_0 = 0$ with no loss of generality. But then $x \in \text{im} \partial^{(r+1)}_{n-1}$, since the restriction $\partial_{n-1} : \oplus_{i=1}^p \mathcal{I}_i \otimes A D_{n-1} \to \oplus_{i=1}^p \mathcal{I}_i \otimes A D_n$ is surjective. $\square$

**Examples 5.5.** a) The converse of Corollary 5.4 is false. Here is an example. Let $V = \mathcal{F}(\mathcal{N}, F)$ be the vector space of all sequences in the field $F$. Consider
the pair of commuting linear operators $a = (a, b)$ on $V$ defined by

$$(af)(n) = \begin{cases} 0, & \text{for } n = 1 \\ f(n), & \text{for } n > 1 \end{cases} \quad \text{and} \quad (bf)(n) = \begin{cases} 0, & \text{for } n = 1 \\ f(n + 1), & \text{for } n > 1 \end{cases}.$$  

Since $a$ is idempotent, $\ker a^n = \ker a = \{f | f(n) = 0 \text{ for } n > 1\}$ and $\im a^n = \im a = \{f | f(1) = 0\}$, we see that the ascent and the descent of $a$ are both equal to 1. On the other hand, $\ker b^n = \{f | f(n) = 0 \text{ for } n \geq r + 2\}$ and $\im b^n = \im b = \{f | f(1) = 0\}$, so that $b$ has infinite ascent but descent 1.

b) There is a pair of commuting endomorphisms on an infinite dimensional vector space $M$ for which the ascent $d^0 = d^0_{n-1,0}$ is infinite and the index $d^0_{00}$ is finite.

**Construction.** Let $F$ be an arbitrary field and let $M$ be the vector space of pairs $(\alpha, p)$, where $\alpha \in F$ and $p = (p_i(x))$ is a sequence of polynomials with coefficients in $F$. If we define $a : M \rightarrow M$ by

$$a(\alpha, p_1(x), p_2(x), \ldots) = (0, xp_1(x), xp_2(x), \ldots)$$

and $b : M \rightarrow M$ by

$$b(\alpha, p_1(x), p_2(x), \ldots) = (p_1(0), p_2(x), p_3(x), \ldots)$$

then it is easy to verify that $a$ and $b$ are commuting endomorphisms of $M$. Moreover, we see that $a^n(\alpha, p_1(x), p_2(x), \ldots) = (0, x^n p_1(x), x^n p_2(x), \ldots)$ and hence $\ker a^n = \{(\alpha, 0)\}$ for all $n \geq 1$. Since $\ker a \subset \ker b$, we must necessarily have $\cap_{i+j=n} \ker(a^i b^j) = \ker a$ for all $n \geq 1$, showing that $d^0_{00} = 1$.

In order to see that the ascent of this pair is infinite, choose for $r \geq 1$ an element $u = (\alpha, p) \in M$ such that $p_r(0) \neq 0$ and $p_k(x) = 0$ for $k > r$. It then follows that $0 \neq b^r u = (p_r(0), 0, 0, \ldots) \in \ker a$. Let $\hat{u} = \hat{u}_1 \otimes \hat{e}_1^r \otimes \cdots \otimes \hat{e}_1^r \in M_{(r)}^{(r)}$. Then $b^r u = \partial_{n-1}^{(r)} \hat{u}$ is a non-zero vector from $\ker \partial_{n-1}^{(r+1)} = \ker a$. Hence, $\ker \partial_{n-1}^{(r)} \neq \ker(\partial_{n-1}^{(r)} \partial_{n-1}^{(r)})$ for every $r \geq 1$.

### 6. Some remarks and comparisons.

**6.1.** If $a$ is the ideal generated by $a = (a_1, a_2, \ldots, a_n)$ and $a^{(s)}$ the ideal generated by $a^{s} = (a_1^s, a_2^s, \ldots, a_n^s)$ in $A$ then clearly $a^{(ns)} \subseteq a^{ns} \subseteq a^{s}$, $\operatorname{lim}_j \operatorname{Hom}_A(A/a^j, M) = \operatorname{lim}_j \operatorname{Hom}_A(A/a^{(j)}, M)$ and $\cap_j a^j M = \cap_j a^{(j)} M$. The local cohomology with coefficients in the $A$-module $M$ of the ideal $a$ is defined by

$$H^n_a(M) = \operatorname{lim}_s \operatorname{Ext}_A^s(A/a^s, M).$$

An equivalent description in terms of Koszul complexes can be given as follows. For each $s \geq 1$ the maps

$$\chi : K^*(M, a^s) \rightarrow K^*(M, a^{s+1}) \quad \text{and} \quad \eta : K^*(M, a^{s+1}) \rightarrow K^*(M, a^s),$$

and
defined by $\chi(me_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}) = a_{i_1}a_{i_2} \ldots a_{i_l}me_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}$ and by $\eta(me_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}) = a_{i_{l+1}}a_{i_{l+2}} \ldots a_{i_n}me_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}$, respectively, are cochain maps. It is known [BH, Theorem 3.5.6] that

$$H^*_a(M) = H^*(\lim_{\to} K^*(M, a^*)) = \lim_{\to} H^*(K^*(M, a^*))$$

and in particular,

$$H^0_a(M) = \lim_{\to} \hom_A(A/a^j, M) = \lim_{\to} \hom_A(A/a^{(j)}, M),$$

where the direct limit is taken along the maps $\chi$.

We may also take the inverse limit along the maps $\eta$ and get a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \cap_j a^jM & \longrightarrow & M & \longrightarrow \lim_j H^n(K^*(M, a^{(j)})) \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & a^{(s)}M & \longrightarrow & M & \longrightarrow H^n(K(M, a^{(s)})) & \longrightarrow 0
\end{array}
$$

for each $s \geq 1$.

6.2. With the notation already used earlier we also have

$$\mathcal{K} = \lim_{\to} \hom_A(A/a^j, M) = \lim_{\to} \hom_A(A/a^{(j)}, M),$$

$$\mathcal{I} = \cap_j a^jM = \cap_j a^{(j)}M$$

and we get the obvious commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{K} & \longrightarrow & M & \longrightarrow M/\mathcal{K} & \longrightarrow 0 \\
\alpha \downarrow & & \| & & \uparrow \beta & & . \\
0 & \longleftarrow & M/\mathcal{I} & \longleftarrow & M & \longleftarrow \mathcal{I} & \longleftarrow 0
\end{array}
$$

It is easy to see that:

1. $\mathcal{K} \cap \mathcal{I} = 0$ iff $\alpha$ is injective iff $\beta$ is injective;
2. $\mathcal{K} + \mathcal{I} = M$ iff $\alpha$ is surjective iff $\beta$ is surjective;
3. $\mathcal{K} \oplus \mathcal{I} = M$ iff $\alpha$ is bijective iff $\beta$ is bijective.

The ascending chain of submodules

$$\hom_A(A/a, M) \subseteq \ldots \subseteq \hom_A(A/a^j, M) \subseteq \hom_A(A/a^{j+1}, M) \subseteq \ldots$$

of $M$ becomes stationary if and only if the ascending chain

$$\hom_A(A/a, M) \subseteq \ldots \subseteq \hom_A(A/a^{(j)}, M) \subseteq \hom_A(A/a^{(j+1)}, M) \subseteq \ldots$$

becomes stationary. It follows from Proposition 4.1 that this happens at stage $s$ when the ascent of $a$ is finite and equal to $s$; in that case

$$\hom_A(A/a^j, M) = \hom_A(A/a^{(j)}, M)$$
and \( \text{Hom}_A(A/a^s, \mathcal{M}) \cap a^j \mathcal{M} = 0 \) for all \( j \geq s \). The descending chain of submodules

\[
\mathcal{M} \supseteq a \mathcal{M} \supseteq a^2 \mathcal{M} \supseteq a^3 \mathcal{M} \supseteq \ldots
\]

becomes stationary if and only if the descending chain

\[
\mathcal{M} \supseteq a \mathcal{M} \supseteq a^j \mathcal{M} \supseteq a^{j+1} \mathcal{M} \supseteq \ldots
\]

becomes stationary. By Proposition 4.2 this happens at stage \( r \) when the descent of \( a \) is finite and equal to \( r \); in that case

\[
a^j \mathcal{M} = a^{(j)} \mathcal{M}
\]

and \( \text{Hom}_A(A/a^s, \mathcal{M}) + a^j \mathcal{M} = \mathcal{M} \) for all \( j \geq r \). In each case the two chains may not terminate at the same stage as the following example shows. If \( a_1 \) and \( a_2 \) are the linear operators defined by

\[
a_1(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0) \quad \text{and} \quad a_2(x_1, x_2, x_3, x_4) = (x_2, 0, x_4, 0)
\]
on the vector space \( \mathcal{M} = \mathbb{R}^4 \), then \( a_1^2 = 0 = a_2^2 \) but \( a_1 a_2(x_1, x_2, x_3, x_4) = (x_4, 0, 0, 0) \), so that in each case the second chain terminates at stage two and the first at stage three.

If both the ascent and the descent of \( a \) are finite then by Theorem 5.2 they are equal, \( \mathcal{K} = \text{Hom}_A(A/a^s, \mathcal{M}) = \text{Hom}_A(A/a^{(s)}, \mathcal{M}) \), \( I = a^s \mathcal{M} = a^{(s)} \mathcal{M} \) and we have the Fitting decomposition \( \text{Hom}_A(A/a^s, \mathcal{M}) \oplus a^s \mathcal{M} = \mathcal{M} \).

However, Example 5.5. b) shows that the stationarity of the above chains of submodules of \( \mathcal{M} \) alone will not guarantee a Fitting type decomposition of \( \mathcal{M} \). Therefore stationarity of these chains alone will not suffice to get a useful definition for finite ascent and finite descent.

### 7. Comparison maps.

#### 7.1. Let \( a = (a_1, a_2, \ldots, a_n) \) be an \( n \)-tuple of commuting endomorphisms of the \( \mathcal{A} \)-module \( \mathcal{M} \) and let \( b = (b_1, b_2, \ldots, b_m) \) be an \( m \)-tuple of commuting endomorphisms of the \( \mathcal{A} \)-module \( \mathcal{N} \). A map of differential graded modules

\[
u : K(\mathcal{M}, \partial_a) \rightarrow K(\mathcal{N}, \partial_b)
\]

must satisfy the equation \( \partial_b u = u \partial_a \), i.e. the diagram

\[
\begin{array}{ccc}
K(\mathcal{M}, \partial_a) & \xrightarrow{\partial} & K(\mathcal{M}, \partial_a) \\
u & & \downarrow u \\
K(\mathcal{N}, \partial_b) & \xrightarrow{\partial} & K(\mathcal{N}, \partial_b)
\end{array}
\]

commutes. In particular, if \( u_0 : \mathcal{M} \rightarrow \mathcal{N} \) is an \( \mathcal{A} \)-module map and \( U = [u_{ij}] \) is an \( m \times n \)-matrix of \( \mathcal{A} \)-endomorphisms of \( \mathcal{N} \) such that entries from different
rows commute, then a map of graded modules \( u = (u_0, U) : K(M, \mathbf{a}) \to K(N, \mathbf{b}) \) can be defined by

\[
u(ve_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_p}) = \sum_{i_1, i_2, \ldots, i_p} u_{i_1} u_{i_2} \ldots u_{i_p} u_0(v) f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_p} = \sum_{I \in J(p)} \det([u_{i_k, j_l}]_{k,l=1}^p) u_0(v) f_{i_1} \wedge f_{i_2} \wedge \ldots \wedge f_{i_p},
\]

where \( J(p) = \{ I = (i_1, i_2, \ldots, i_p) | 1 \leq i_1 < i_2 < \ldots < i_p \leq n \} \). Using multi-indices this can be written as

\[u(ve_J) = \sum_{I \in J(p)} u_{IJ} u_0(v) f_I,\]

where \( u_{IJ} = \det([u_{i_k, j_l}]_{k,l=1}^p) \). The following result is essentially [R, Lemma 2.1], where the condition that \( u_{ij} \) and \( b_k \) commute for \( i \neq k \) should be added.

**Lemma 7.2.** If the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\partial} & K_1(M, \mathbf{a}) \\
\downarrow u_0 & & \downarrow u_1 \\
N & \xrightarrow{\partial} & K_1(N, \mathbf{b})
\end{array}
\]

commutes, and if \( u_{ij} \) commutes with \( u_{kl} \) and with \( b_k \) for \( i \neq k \), then

\[u = (u_0, U) : K(M, \partial_\mathbf{a}) \to K(N, \partial_\mathbf{b})\]

is a cochain map. If, in addition, \( u_0 \) is an isomorphism and the matrix \( [u_{ij}] \) is invertible then \( u \) is an isomorphism of complexes.

**Proof.** The commutativity of the diagram says that

\[
\sum_i b_i u_0(v) f_i = \partial u_0(v) = u_1 \partial(v) = \sum_{i,j} u_{ij} u_0(a_j v) f_i
\]

for every \( v \in M \) and hence \( b_i u_0 = \sum_{i,j} u_{ij} u_0 a_j \) for \( 1 \leq i \leq m \). But then

\[
u(ve_{j_1} \wedge \ldots \wedge e_{j_p}) = \sum_{i_0, i, j_0} u_{i_0, j_0} u_{i_1, j_1} \ldots u_{i_p, j_p} u_0(a_{j_0} v) f_{i_0} \wedge f_{i_1} \wedge \ldots \wedge f_{i_p} = \sum_{i_0, I} b_{i_0} u_{i_1, j_1} \ldots u_{i_p, j_p} u_0(v) f_{i_0} \wedge f_{i_1} \wedge \ldots \wedge f_{i_p, j_p} = \partial u(ve_J),
\]

is a cochain map. If, in addition, \( u_0 \) is an isomorphism and the matrix \( [u_{ij}] \) is invertible then \( u \) is an isomorphism of complexes.
if $u_{ij}$ and $b_k$ commute for $i \neq k$, so that the diagram
\[
\begin{array}{ccc}
K_p(M, a) & \xrightarrow{\partial} & K_{p+1}(M, a) \\
\downarrow u_p & & \downarrow u_{p+1} \\
K_p(N, b) & \xrightarrow{\partial} & K_{p+1}(N, b)
\end{array}
\]
commutes for every $p \geq 0$. In terms of multi-indices this reads
\[
u\partial(ve_J) = \sum_{j} u_{ij}u_1ju_0(a_jv)f_i \wedge f_l = \sum_{i,l} b_iu_{IJ}u_0(ve_J).
\]
If, in addition, $u_0$ and $[u_{ij}]$ are invertible then a direct calculation shows that
\[
(u_0^{-1}, [u_0^{-1}u_{ij}u_0]^{-1}) = u^{-1} : K(N, b) \to K(M, a)
\]
is the inverse of $u$. □

It is the following special case of the above situation that is of particular importance for us.

**Proposition 7.3.** Let $A$ and $B$ be commutative $R$-subalgebras of $\text{End}_A(M)$ containing the $n$-tuples $a$ and $b$, respectively, such that $bu_0 = Uu_0a$ for some $n \times n$ matrix $U = [u_{ij}]$ of $A$-endomorphisms and some $A$-endomorphism $u_0$ of $M$. If $u_{ij}$ commutes with $u_{kl}$ and with $b_k$ for $k \neq i$, then $U$ induces a cochain map
\[
u = (u_0, U) : K(M, \partial_a) \to K(M, \partial_b).
\]
Moreover, if $U$ is invertible and $u_0$ is an automorphism then $\nu$ is an isomorphism.

**7.4.** As the first of the following example shows, the conditions of Proposition 7.3 are not sufficient to guarantee that $a$ and $b$ have the same ascent or the same descent. In case $M = R^2$ consider the matrices
\[
a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
\]
in $M_2(R)$. Then $b = ua$, $a^2 = 0$, $b^2 = b$ and $c = uau^{-1}$. Thus, with $(u_0, U) = (id, u)$ the conditions of Proposition 7.3 are met, but the ascent and the descent of $a$ are both equal to 2, while those of $b$ are both equal to 1. On the other hand, the conditions are met again if we replace $b$ by $c = uau^{-1}$ and take $(u_0, U) = (u, id)$. As expected, $a$ and $c$ have of course the same ascent and the same descent.

This, and considerations of how $(u_0, U)$ could induce a map of complexes from $\mathcal{M}^{(r)}(a)$ to $\mathcal{M}^{(r)}(b)$ for $r \geq 2$, suggest that what is needed to guarantee
equal ascent and equal descent, aside from \( u_0 \) being invertible, is that the entries of \( U \) commute pairwise and that they commute with the entries of \( b \). It is obvious that ascent and descent do not depend on the particular subalgebra \( A \) of \( \text{End}_A(\mathcal{M}) \) containing \( a \). If \( u_0 \in \text{Aut}_A(\mathcal{M}) \) then \( bu_0 = Uu_0a \) is equivalent to \( b = Uu_0a_0^{-1}U \). If, in addition, \( U : \mathcal{M}^n \to \mathcal{M}^n \) is invertible then the entries of \( u_0a_0^{-1} \) commute with those of \( U \) and those of \( b \). It therefore remains to compare ascent or descent of \( a, u_0a_0^{-1} \) and \( b = Uu_0a_0^{-1} \).

**Corollary 7.5.** Let \( A \) and \( B \) be commutative \( R \)-subalgebras of \( \text{End}_A(\mathcal{M}) \) containing the \( n \)-tuples \( a \) and \( b \), respectively, such that \( bu_0 = Uu_0a \) for some \( U \in \text{GL}_n(B) \) and some \( u_0 \in \text{Aut}_A(\mathcal{M}) \). Then \( (u_0, U) \) induces a cochain map \( u^{(r)} : \mathcal{M}^{(r)}(a) \to \mathcal{M}^{(r)}(b) \) for each \( r \geq 1 \). Moreover, if \( U \in \text{GL}_n(B) \) then \( u^{(r)} \) is an isomorphism for \( r \geq 1 \) so that \( a \) and \( b \) have the same ascent and the same descent.

**Proof.** Since \( U \in \text{GL}_n(B) \), the conditions of Proposition 7.3 are obviously satisfied, and the assertion for \( r = 1 \) follows. Moreover, since the entries of \( U \) commute and they also commute with the entries of \( b \) it follows that

\[
b_i u(ve_j) = \sum_j b_i u_{IJ} u_0 v f_I = \sum_I u_{IJ} b_i u_0 v f_I
\]

which means that \( b_i u = \sum_j u_{ij} u_0 a_j \).

Now proceed by induction on \( r \). Suppose that \( u^{(r)} : \mathcal{M}^{(r)}(a) \to \mathcal{M}^{(r)}(b) \) has already been defined such that \( \partial_b u^{(r)} = u^{(r)} \partial_a \) and that \( b_i u^{(r)} = \sum_j u_{ij} u^{(r)} a_j \). Then define \( u^{(r+1)} : \mathcal{M}^{(r+1)}(a) \to \mathcal{M}^{(r+1)}(b) \) by \( u^{(r+1)} (x \otimes e_l) = \sum_j u_{Jl} u^{(r)} x \otimes f_j \). The commutativity conditions and the induction hypothesis then imply that

\[
\partial_b^{(r+1)} u^{(r+1)} (x \otimes e_l) = \sum_{j,l} b_j \partial_b^{(r)} u_{Jl} u^{(r)} x \otimes (f_j \wedge f_J)
\]

\[
= \sum_{j,l} b_j u_{Jl} u^{(r)} \partial_a^{(r)} x \otimes (f_j \wedge f_J)
\]

\[
= \sum_{i,j,l} u_{ij} u_{Jl} u^{(r)} a_j \partial_a^{(r)} x \otimes (f_j \wedge f_J)
\]

\[
= u^{(r+1)} \left( \sum_i a_i \partial_a^{(r)} x \otimes (e_i \wedge e_I) \right)
\]
and also
\[ u^{(r+1)} \partial_a^{(r+1)} (x \otimes e_I) = \sum_j u_{IJ} b_1 u^{(r)} x \otimes f_J \]
\[ = \sum_{J,i} u_{IJ} u_{ij} u^{(r)} (a_j x) \otimes f_J = \sum_j u_{ij} u^{(r+1)} (a_j x \otimes e_I). \]

This means that \( \partial_b^{(r+1)} u^{(r+1)} = u^{(r+1)} \partial_a^{(r+1)} \) and \( b_1 u^{(r+1)} = \sum_j u_{ij} u^{(r+1)} a_j. \)

If \( u_0 \in \text{Aut}_A(\mathcal{M}) \) and \( U \in \text{GL}_n(B) \) then one shows directly that the inverse of \( u^{(r)} \) is induced as above by \( (u_0^{-1}, [u_0^{-1} u_{ij} u_0]^{-1}) \) for each \( r \geq 1 \). There is another, more conceptual way to show that \( u^{(r)} \) is an isomorphism. It is easy to see that \( (u_0, id) \) induces an isomorphism \( \mathcal{M}^{(r)}(\partial_a) \cong \mathcal{M}^{(r)}(\partial_b) \).

The fact that \( (id, U) \) induces an isomorphism \( \mathcal{M}^{(r)}(\partial_{u_0 u_0^{-1}}) \cong \mathcal{M}^{(r)}(\partial_b) \) uses the assumption that the entries of \( u_0 a u_0^{-1} \) commute with those of \( U \) and those of \( b \) so that we may assume that all these entries are contained in \( B \). The rest is then a consequence of the following general observation. If \( \phi : X \to X' \) and \( \psi : Y \to Y' \) are maps of differential graded \( B \)-modules then \( \phi \otimes_B \psi : X \otimes_B Y \to X' \otimes_B Y' \) is a map of bicomplexes and \( D(\phi \otimes_B \psi) : D(X \otimes_B Y) \to (X' \otimes_B Y') \) is a homomorphism of the associated diagonal complexes. Moreover, if \( \phi \) and \( \psi \) are isomorphisms then so are \( \phi \otimes_B \psi \) and \( D(\phi \otimes_B \psi) \).

8. Invariance Properties.

8.1. In case \( u_0 = id \), the assumption that \( b = U a \) with \( U \in \text{GL}_n(A) \) in 7.5 clearly implies that \( a \) and \( b \) generate the same ideal in \( A \). The question now arises whether an \( n \)-tuple \( a \) and an \( m \)-tuple \( b \), which generate the same ideal in \( A \), also have the same ascent and the same descent. This is indeed the case, as we shall demonstrate using localization arguments together with the following basic result.

**Theorem.** Let \( A \) be a commutative \( R \)-subalgebra of \( \text{End}_A(\mathcal{M}) \) containing the \( n \)-tuple \( a \) and the \( m \)-tuple \( b \). Suppose \( a \) and \( b \) generate the same ideal in \( A \) and that \( b = U a' \) for some \( a' \subseteq a \) and some \( U \in \text{GL}_m(A) \). Then \( K(\mathcal{M}, \partial_a) \cong K(\mathcal{M}, \partial_b) \otimes K(R^{n-m}, 0) \); hence \( H(K(\mathcal{M}, \partial_a)) \cong H(K(\mathcal{M}, \partial_b)) \otimes \Delta(R^{n-m}) \). Moreover, \( a \) and \( b \) have the same ascent and the same descent.

**Proof.** In view of Corollary 7.5 we may assume that \( b = (a_1, a_2, \ldots, a_m) \), and that \( a \) and \( b \) generate the same ideal in \( A \). Thus, if the \( n \)-tuple \( c = b \cup 0 \) is obtained from \( b \) by adjoining \( n - m \) zeros, then there is a system of equations \( \sum_{j=1}^i u_{ij}(a_j) a_j = c_i \) with \( u_{ii} = 1 \) for \( i = 1, 2, \ldots, n \). The matrix
$U = [u_{ij}]$ is in $\text{GL}_n(A)$, since $\det U = 1$. So, by Corollary 7.5 we have
\[
K(M, \partial_a) \cong K(M, \partial_c) \cong K(M, \partial_b) \otimes K(R^{n-m}, 0)
\]
as complexes and thus $H(K(M, \partial_a)) \cong H(K(M, \partial_b)) \otimes \Lambda(R^{n-m})$ as graded modules. Moreover, since the differential of $D$ is the right hand parts as direct sums, the diagrams commute for every $i$. Moreover, let $D$ be obtained from $b$ by adjoining a single zero. For this purpose let $D(b) = K(A, \partial_b)$ and $E = K(R, 0)$, so that $D(c) = K(A, \partial_c) = D(b) \otimes E$ and moreover $K(M, \partial_b) = M \otimes_A D(b)$, $K(M, \partial_c) = M \otimes_A D(c) \cong M \otimes_A D(b) \otimes E$.

By the results of Section 4, we only have to analyse the diagram
\[
\begin{array}{ccc}
\mathcal{M}_m^{(r)}(c) & \xrightarrow{\partial_{m+1}^{(r)}(c)} & \mathcal{M}_{m+1}^{(r)}(c) \\
\downarrow & & \downarrow \\
\mathcal{M}_{m+1}^{(r)}(c) & \to & \mathcal{M}_{m+1}^{r+1}(c) \otimes_A D_1(c)
\end{array}
\]
to solve the ascent problem. First notice that $D(c) = D(b) \otimes E$ means in particular that
\[
D_i(c) = (D_i(b) \otimes E_0) \oplus (D_{i-1}(b) \otimes E_1) \cong D_i(b) \oplus D_{i-1}(b)
\]
for $0 \leq i \leq m + 1$, where of course $D_j(b) = 0$ for $j < 0$ and for $j > m$. Moreover, since the differential of $E$ is zero, the diagram
\[
\begin{array}{ccc}
D_i(c) & \xrightarrow{=} & (D_i(b) \otimes E_0) \oplus (D_{i-1}(b) \otimes E_1) \\
\partial_i(c) & & \downarrow \\
D_{i+1}(c) & \xrightarrow{=} & (D_{i+1}(b) \otimes E_0) \oplus (D_{i}(b) \otimes E_1)
\end{array}
\]
commutes for every $i$, $0 \leq i \leq m$. More generally, since $D_i^{(r)}(c) \cong \otimes_A^{r}(D_i(b) \oplus D_{i-1}(b))$, the $r$-fold tensor product, we see that $\partial_i^{(r)}(c) = \otimes_A^{r}(\partial_i(b) \oplus \partial_{i-1}(b))$. In particular for $i = m$ and for $i = 0$, after expressing the right hand parts as direct sums, the diagrams
\[
\begin{array}{ccc}
D_m^{(r)}(c) & \xrightarrow{=} & Z \oplus D_{m-1}^{(r)}(b) \\
\partial_m^{(r)}(c) & & \downarrow (0, \partial_{m-1}^{(r)}(b)) \\
D_{m+1}^{(r)}(c) & \xrightarrow{=} & D_{m}^{(r)}(b)
\end{array}
\]
\[
\begin{array}{ccc}
D_0^{(r)}(c) & \xrightarrow{=} & D_0^{(r)}(b) \\
\partial_0^{(r)}(c) & & \downarrow (\partial_{0}^{(r)}(b), 0) \\
D_1^{(r)}(c) & \xrightarrow{=} & D_1^{(r)}(b) \oplus U
\end{array}
\]
commute. It follows that the differential $\partial^{(r)}_m(c)$ is related to $\partial^{(r)}_{m-1}(b)$ by an isomorphism of the form

$$
\begin{array}{ccc}
\mathcal{M}_m^{(r)}(c) & \overset{\cong}{\longrightarrow} & \mathcal{M} \otimes_A D_m^{(r)}(c) \\
\partial^{(r)}_m(c) \downarrow & & \downarrow \cong \\
\mathcal{M}_{m+1}^{(r)}(c) & \overset{\cong}{\longrightarrow} & \mathcal{M} \otimes_A D_{m+1}^{(r)}(c) \\
& & \downarrow \cong \\
& & \mathcal{M}_m^{(r)}(b),
\end{array}
$$

while the relation between the differentials $\partial^{r+1}_0(c)$ and $\partial^{r+1}_0(b)$ is given by

$$
\begin{array}{ccc}
\mathcal{M}_{m+1}^{(r)}(c) & \overset{\partial^{r+1}_m(c)}{\longrightarrow} & \mathcal{M}_{m+1}^{(r)}(c) \otimes_A D_1(c) \\
\cong \downarrow & & \cong \downarrow \\
\mathcal{M} \otimes_A D_{m+1}^{(r)}(c) & & \mathcal{M} \otimes_A D_{m+1}^{(r)}(c) \otimes_A D_1(c) \\
& & \cong \downarrow \\
& & (\mathcal{M}_m^{(r)}(b) \otimes_A D_1(b)) \oplus (\mathcal{M}_m^{(r)}(b) \otimes_A D_0(b)).
\end{array}
$$

With the proper identifications we therefore get the matrix equation

$$
\partial^{r+1}_0(c)\partial^{(r)}_m(c) = \begin{pmatrix}
\partial^{r+1}_0(b) \\
0 & \partial^{(r)}_{m-1}(b)
\end{pmatrix} = \begin{pmatrix}
0 & \partial^{r+1}_0(b)\partial^{(r)}_{m-1}(b)
\end{pmatrix}.
$$

This shows that $\ker \partial^{(r)}_m(c) = (\mathcal{M} \otimes_A Z) \oplus \ker \partial^{(r)}_m(b)$ and $\ker(\partial^{r+1}_0(c)\partial^{(r)}_m(c)) = (\mathcal{M} \otimes_A Z) \oplus \ker(\partial^{r+1}_0(b)\partial^{(r)}_m(b))$, and that the inclusion of the former into the latter is componentwise. Hence, $\ker \partial^{(r)}_m(c) = \ker(\partial^{r+1}_0(c)\partial^{(r)}_m(c))$ if and only if $\ker \partial^{(r)}_m(b) = \ker(\partial^{r+1}_0(b)\partial^{(r)}_m(b))$, which implies the invariance of ascent.

To solve the descent problem we have, again by the results of Section 4, to analyse the diagram

$$
\begin{array}{ccc}
\mathcal{M}_0^{(r)}(c) \otimes_A D_m(c) & \overset{\partial^{r+1}_m(c)}{\longrightarrow} & \mathcal{M}_0^{(r)}(c) \otimes_A D_{m+1}(c) \\
\downarrow \partial^{(r)}_0(c) & & \downarrow \partial^{(r)}_0(c) \\
\mathcal{M}_1^{(r)}(c) \otimes_A D_{m+1}(c).
\end{array}
$$
Now, $\partial^{r+1}_m(c)$ is related to $\partial^{r+1}_{m-1}(b)$ by the isomorphism
\[
\begin{align*}
\mathcal{M}^{(r)}_0(c) \otimes_A D_m(c) & \xrightarrow{\partial^{r+1}_m(c)} \mathcal{M}^{(r)}_0(c) \otimes_A D_{m+1}(c) \\
\cong & \\
\mathcal{M} \otimes_A D^{(r)}_0(c) \otimes_A D_m(c) & \xrightarrow{\partial^{r+1}_0(c)} \mathcal{M} \otimes_A D^{(r)}_1(c) \otimes_A D_{m+1}(c) \\
\cong & \\
(\mathcal{M}^{(r)}_0(b) \otimes_A D_{m-1}(b)) \oplus V & \xrightarrow{(\partial^{r+1}_{m-1}(b), 0)} (\mathcal{M}^{(r)}_0(b) \otimes_A D_m(b)) \oplus W
\end{align*}
\]
with $V = \mathcal{M}^{(r)}(b) \otimes_A D_m(b)$, while $\partial^{(r)}_0(c)$ and $\partial^{(r)}_0(b)$ are related by
\[
\begin{align*}
\mathcal{M}^{(r)}_0(c) \otimes_A D_{m+1}(c) & \xrightarrow{\partial^{(r)}_0(c)} \mathcal{M}^{(r)}_1(c) \otimes_A D_{m+1}(c) \\
\cong & \\
\mathcal{M} \otimes_A D^{(r)}_0(c) \otimes_A D_{m+1}(c) & \xrightarrow{\partial^{(r)}_0(c)} \mathcal{M} \otimes_A D^{(r)}_1(c) \otimes_A D_{m+1}(c) \\
\cong & \\
\mathcal{M}^{(r)}_0(b) \otimes_A D_m(b) & \xrightarrow{(\partial^{(r)}_0(b), 0)} (\mathcal{M}^{(r)}_1(b) \otimes_A D_m(b)) \oplus W
\end{align*}
\]
if $W = \mathcal{M} \otimes_A U \otimes_A D_m(b)$. Considering the isomorphisms as identifications we have
\[
\partial^{(r)}_0(c) \partial^{r+1}_m(c) = \begin{pmatrix} \partial^{(r)}_0(b) \\ 0 \end{pmatrix} \begin{pmatrix} \partial^{r+1}_{m-1}(b) \\ 0 \end{pmatrix} = \begin{pmatrix} \partial^{(r)}_0(b) \partial^{r+1}_{m-1}(b) \\ 0 \end{pmatrix}.
\]
Thus, $\text{im} \partial^{(r)}_0(c) = \text{im} \partial^{(r)}_0(b)$ and $\text{im}(\partial^{(r)}_0(c) \partial^{r+1}_m(c)) = \text{im}(\partial^{(r)}_0(b) \partial^{r+1}_{m-1}(b))$. In particular, we may conclude now that $\text{im}(\partial^{(r)}_0(c) \partial^{r+1}_m(c)) = \text{im} \partial^{(r)}_0(c)$ if and only if $\text{im}(\partial^{(r)}_0(b) \partial^{r+1}_{m-1}(b)) = \text{im} \partial^{(r)}_0(b)$, establishing the invariance of descent.

\[\square\]

**Corollary 8.2.** Let $R$ be a field and let $L$ be a finite dimensional subspace of commuting elements in $\text{End}_A(M)$. Any two finite spanning subsets of $L$ have the same ascent and the same descent.

**Proof.** If $a$ and $b$ are two finite spanning subsets of $L$ then they generate the same commutative subalgebra $A$ and the same ideal in it. Moreover, they each contain a basis $a'$ and $b'$ of $L$, so that $b' = Ua'$ for some $U \in \text{GL}_n(R)$, where $n = \text{dim}_R L$. The assertion now follows from Theorem 8.1. \[\square\]

**8.3.** Additional conditions on the commuting $n$-tuple $a$ or the commutative subalgebra $A$ of $\text{End}_A(M)$ it generates may give tighter invariance results. Such a situation occurs when $A$ is a commutative local algebra; this happens for example when $R$ is a field and $a$ consists of nilpotent endomorphisms.
Theorem. Suppose $A$ is a commutative local subalgebra of $\text{End}_A(M)$ with maximal ideal $M$ and residue class field $F$ and let $I$ be a finitely generated ideal of $A$. Then any two finite generating subsets of $I$ have the same ascent and the same descent.

Proof. The $n$-tuple $a = (a_1, a_2, \ldots, a_n) \subseteq I$ forms a generating subset for the ideal $I$ of the commutative local algebra $A$ if and only if the residue classes $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ span $I/MI$ as a vector space over the field $F = A/M$, i.e., if and only if $\bar{a}$ contains an $F$-basis of $I/MI$. This follows from Nakayama's Lemma [A, S]. Thus, if this is the case, then $a$ contains a subset $a'$ which generates the ideal $I$ and such that $a'$ is an $F$-basis of $I/MI$. By Theorem 8.1 we see that $a$ and $b = a'$ have the same ascent and the same descent. It remains to show that the same is true for $m$-tuples $a$ and $b$ for which $\bar{a}$ and $b$ are $F$-bases of $I/MI$. But in this case $b = Ua$ for some matrix $U \in M_m(A)$ with $\bar{U} \in \text{GL}_m(F)$. Then $\det U = \det \bar{U} \neq 0$, so that $\det U$ is invertible and $U \in \text{GL}_m(A)$. Again by Theorem 8.1 we conclude that $a$ and $b$ have the same ascent and the same descent. □

8.4. We are now in a position to prove a general invariance result for any finitely generated ideal in a commutative subalgebra $A$ of $\text{End}_A(M)$.

Theorem. Let $A$ be a commutative $R$-subalgebra of $\text{End}_A(M)$ and let $J$ be a finitely generated ideal of $A$. Then any two finite generating subsets of $J$ have the same ascent and the same descent.

Proof. Suppose that $J$ can be generated by the $n$-tuple $a = (a_1, a_2, \ldots, a_n)$ as an ideal of $A$. For any prime ideal $P$ of $A$ let $A_P$ and $M_P = A_P \otimes_A M$ denote the localizations of $A$ and $M$ at $P$. The ideal $I = JP_P$ of the local ring $A_P$ is then generated by $a$ as well. We shall see that $d^j(a, M) = \sup_P \{d^j(a, M_P)\}$ for $j = 0, 1$. Since the localization functor $(\ )_P : \text{Mod}_A \rightarrow \text{Mod}_{A_P}$ is exact [A, Proposition 3.9], [S], we see that $K^r(M, \partial_a)_P \cong K^r(M_P, \partial_{\tilde{a}})$, $H^r(M, \partial_a)_P \cong H^r(M_P, \partial_{\tilde{a}})$, $L^r(M, \partial_a)_P \cong L^r(M_P, \partial_{\tilde{a}})$ and $R^r(M, \partial_a)_P \cong R^r(M_P, \partial_{\tilde{a}})$ for all $r \geq 1$. Thus, by Propositions 4.1 and 4.2, we see that $d^j(a, M) \geq d^j(a, M_P)$ for every prime ideal $P$ of $A$. But, by [A, Proposition 3.9], [S], a homomorphism of $A$-modules $f : X \rightarrow Y$ is an isomorphism if and only if for each prime ideal $P$ of $A$ the induced homomorphism of $A_P$-modules $f_P : X_P \rightarrow Y_P$ is an isomorphism. By the definition of ascent and descent in 4.3 we conclude that $d^j(a, M) = d^j(a, M_P)$.
for some prime ideal $P$ of $A$. If $b$ is another generating subset of the ideal $J$ then invoke Theorem 8.3 to see that

$$d^j(a, M) = \sup_P \{d^j(a, M_P)\} = \sup_P \{d^j(b, M_P)\} = d^j(b, M)$$

which proves our assertion. □

References


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TOPOLOGY VERSUS CHERN NUMBERS
FOR COMPLEX 3-FOLDS

Claude LeBrun

We show by example that the Chern numbers $c_1^3$ and $c_1c_2$ of a complex 3-fold are not determined by the topology of the underlying smooth compact 6-manifold. In fact, we observe that infinitely many different values of a Chern number can be achieved by (integrable) complex structures on a fixed 6-manifold.

1. Introduction.

Suppose that $X$ is a smooth compact oriented 6-manifold. Recall that an almost-complex structure on $X$ means an endomorphism $J : TX \to TX$ of the tangent bundle of $X$ with $J^2 = -1$ which determines the given orientation of $X$. Such a structure makes $TX$ into a complex vector bundle, so that one can speak of the Chern classes $c_j \in H^{2j}(X, \mathbb{Z})$ of $(X, J)$, and therefore of the Chern numbers

$$c_1^3 = \int_X c_1^3$$

$$c_1c_2 = \int_X c_1c_2$$

$$c_3 = \int_X c_3$$

of the almost-complex manifold $(X, J)$. The only obstruction [10] to the existence of an almost-complex structure $J$ on $X$ is that $X$ be spin$^c$. This happens precisely when the second Stiefel-Whitney class $w_2(X) \in H^2(X, \mathbb{Z}_2)$ can be written as the mod-2 reduction of an element of $H^2(X, \mathbb{Z})$, in which case each preimage of $w_2$ in $H^2(X, \mathbb{Z})$ can be realized as $c_1$ for some almost-complex structure $J$. It follows that the Chern numbers $c_1^3$ and $c_1c_2$ of the almost-complex $(X, J)$ are certainly not topological invariants of the 6-manifold $X$. For example, if $X = \mathbb{CP}_3$, every integer of the form $8j$ can be realized as $c_1c_2$, and every integer of the form $8j^3$ can be realized as $c_1^3$ for some almost-complex structure $J$ on $\mathbb{CP}_3$. On the other hand, $c_3$ is the Euler class of $TX$, so that $c_3 = \chi(X)$ is actually a homotopy invariant of $X$. 

123
In this note, we will observe that the above situation persists even if one demands that the almost-complex structures under consideration be integrable. Recall that an almost-complex structure $J$ on $X$ is called a complex structure if it is integrable, in the sense of being locally isomorphic to the standard, constant-coefficient structure on $\mathbb{R}^6 = \mathbb{C}^3$. The question of whether the Chern numbers $c_3$ and $c_1c_2$ of a complex 3-fold might actually be topological invariants of the underlying 6-manifold was raised, for example, in an interesting survey article by Okonek and van de Ven [7, p. 317].

Our principal results are as follows:

**Theorem A.** There is a compact simply connected 6-manifold $X$ which admits a sequence $J_m$, $m \in \mathbb{Z}^+$, of (integrable) complex structures with

$$c_1c_2(X, J_m) = 48m.$$  

Indeed, there are infinitely many homeotypes of $X$ with this property.

**Theorem B.** Let $(m, n)$ be any pair of integers. Then for any integer $\tilde{n} \ll n$, there is a complex projective 3-fold $(X, J)$ with Chern numbers

$$c_1c_2 = 24m, \quad c_3 = 8n,$$

which admits a second complex structure $\tilde{J}$ with

$$c_1c_2 = 48m, \quad c_3 = 8\tilde{n}.$$

2. Infinitely Many Complex Structures.

The fact that the Chern classes of a complex 3-fold are not determined by the topology of the underlying 6-manifold was observed long ago by Calabi [2]. While his examples all have vanishing Chern numbers, they nonetheless contain the seeds of a natural class of examples which lead to Theorem A:

**Theorem 1.** For each positive integer $m$, the 6-manifold $X = K3 \times S^2$ admits a complex structure $J_m$ with

$$c_1c_2(X, J_m) = 48m$$

and $c_3(X, J_m) = 0$.

**Proof.** Let $M$ denote the underlying oriented 4-manifold of the $K3$ surface, and let $g$ be any hyper-Kähler metric on $M$; such metrics exist by Yau’s solution of the Calabi conjecture [11]. Let $Z$ be the twistor space [1, 8] of $(M, g)$, and let $\pi : Z \to \mathbb{CP}_1$ be the holomorphic projection induced by the hyper-Kähler structure. Differentiably, $\pi$ is the trivial fiber bundle with fiber $M$, so that $Z$ may be thought of as $X = M \times S^2$ equipped with a complex structure.

Now let $f_m : \mathbb{CP}_1 \to \mathbb{CP}_1$ be a holomorphic map of degree $m - 1$; for example, we may take $f_m([u, v]) = [u^{m-1}, v^{m-1}]$, where $m$ is any positive
integer. We may then define a holomorphic family $f_m^*\varpi$ of K3’s over $\mathbb{CP}_1$ by pulling back the family $\varpi$ via $f_m$:

$$
f_m^*Z \longrightarrow Z \quad f_m^*\varpi \downarrow \varpi \downarrow \mathbb{CP}_1 \quad f_m \mathbb{CP}_1.
$$

In other words, $f_m^*Z$ is the inverse image, via $1 \times \varpi$, of the graph of $f_m$. Since $\varpi$ is differentiably trivial, so is $f_m^*\varpi$, and $f_m^*Z$ may therefore be viewed as $X = K3 \times S^2$ equipped with a complex structure $J_m$.

Now if $\pi : Z \to M$ is the (non-holomorphic) twistor projection, an explicit diffeomorphism $Z \to X$ is given by $\tau \times \varpi$, and $f_m^*Z$ is similarly trivialized by $f_m^*\varpi \times f_m^*\pi$. Let $L \subset Tf_m^*Z$ be the kernel of the derivative of the pulled-back twistor projection $f_m^*\pi$. Then $L$ is $J_m$ invariant, despite the fact that $\pi$ is not holomorphic, and so may be viewed as a complex line-bundle. Moreover, $L$ may be identified with the pull-back of the (holomorphic) tangent bundle of $\mathbb{CP}_1$ via $f_m^*\varpi$, so that $c_1^2(L) = 0$, and hence $p_1(L) = 0$. If, on the other hand, we use $H$ to denote the kernel of the derivative of $f_m^*\varpi$, then the underlying real bundle of $H$ is $(f_m^*\pi)^*TM$, and so $p_1(H) = f_m^*[p_1(M)] = -48F$, where, by Poincaré duality, $F$ is represented by a fiber $S^2$ of $f_m^*\pi$. It follows that $p_1(f_m^*Z) = p_1(L) + p_1(H) = -48F$. On the other hand, any K3 has trivial canonical line bundle, and the fibers of $\pi$ are $\mathbb{CP}_1$’s with normal bundle $O(1) \oplus O(1)$, so $c_1(H) = c_1(L)$ when $m = 2$. For general $m$, it follows that $c_1(H) = (m - 1)c_1(L)$, and hence that

$$
c_1 = mc_1(L).
$$

We therefore have $c_1^2 = m^2c_1^2(L) = 0$, so that

$$
c_1c_2 = \frac{c_1^3 - c_1p_1}{2} = 24 \int_F mc_1(L) = 48m,
$$

and $c_3^1 = 0$. □

When $m = 1$, the above complex structure is simply an arbitrary product complex structure on $K3 \times \mathbb{CP}_1$, and so is of Kähler type; indeed, we may even arrange for it to be projective-algebraic if we like. On the other hand, the $m = 2$ complex structure is that of a twistor space, and so is never of Kähler type by Hitchin’s classification of Kählerian twistor spaces [4]. For large values of $m$, one can prove something even stronger: $J_m$ isn’t even homotopic to a complex structure of Kähler type. This is because the Todd genus

$$
1 - h^1(O) + h^2(O) - h^3(O) = \chi(O) = \frac{c_1c_2}{24} = 2m,
$$
so that $h^2(\mathcal{O})$ will eventually exceed $b_2(X)$, in violation of the Hodge decomposition. In the next section, we will see that this phenomenon is actually quite typical.

In order to show that there is more than one 6-manifold for which infinitely many different values of a Chern number are achieved by (integrable) complex structures, we may now invoke the standard process of blowing up. The following facts about blow-up 3-folds are left as exercises for the reader.

**Proposition 2.** Let $(X, J)$ be any compact complex 3-fold, and let $(\hat{X}, \hat{J})$ be obtained from $(X, J)$ by blowing up a point. Then $\hat{X}$ is diffeomorphic to the connected sum $X \# \mathbb{CP}_3$, and if $X$ is spin, so is $\hat{X}$. Moreover, the Chern numbers of the blow-up are related to those of the original 3-fold by

\[
\begin{align*}
    c_1^3(\hat{X}, \hat{J}) &= c_1^3(X, J) + 8 \\
    c_1c_2(\hat{X}, \hat{J}) &= c_1c_2(X, J) \\
    c_3(\hat{X}, \hat{J}) &= c_3(X, J) + 2. 
\end{align*}
\]

Iterated blow-ups of the previous examples thus prove the following precise form of Theorem A:

**Corollary 3.** For each integer $n \geq 0$, the 6-dimensional spin manifold

\[ X = (K3 \times S^2) \# n\mathbb{CP}_3 \]

admits a sequence $J_m$ of complex structures with

\[
\begin{align*}
    c_1c_2(X, J_m) &= 48m \\
    c_3^3(X, J_m) &= 8n. 
\end{align*}
\]

3. Kähler Type.

We saw in Theorem 1 that it is possible to find 6-manifolds with sequences of complex structures for which a Chern number takes on infinitely many different values. On the other hand, if one requires that the complex structures in question be of Kähler type, one arrives at essentially the opposite conclusion. This is illustrated by our next result.

**Theorem 4.** Let $X$ be the underlying compact oriented 6-manifold of any Kählerian 3-fold. Then there exist infinitely many homotopy classes of almost-complex structures on $X$ which cannot be represented by complex structures of Kähler type.

**Proof.** By assumption, there is a Kähler class $[\omega] \in H^2(X, \mathbb{R})$ with $[\omega]^3 \neq 0$. Now $H^2(X, \mathbb{Q}) \subset H^2(X, \mathbb{R})$ is dense, and the cup form is continuous, so approximating $[\omega]$ with rational classes will produce classes $\alpha_0 \in H^2(X, \mathbb{Q})$ with $\alpha_0^3 > 0$. Multiplying by a suitable positive integer $k$ to clear denominators, we may thus obtain a class $k\alpha_0$ which is the image of an integer
class $\alpha \in H^2(X, \mathbb{Z})$ in rational cohomology. Now let $\beta \in H^2(X, \mathbb{Z})$ be the first Chern class of the given complex structure on $X$. Then $2n\alpha + \beta$ is an integer lift of $w_2$, and so [10] can be realized as $c_1$ for some homotopy class $[J_n]$ of almost-complex structures. Now if $J_n \in [J_n]$ is integrable, the Todd genus of $(X, J_n)$ is

$$\tau = \frac{c_1 \cdot c_2}{24} = \frac{(2n\alpha + \beta)^3 - (2n\alpha + \beta) \cdot p_1}{48},$$

which is cubic in $n$, with the coefficient of $n^3$ non-vanishing. It therefore follows that there is an integer $n_0$ such that $|\sum_k (-1)^k h_0,k| > \sum_j b_j(X)$ whenever $|n| > n_0$. For $n$ in this range, the Hodge theorem must therefore fail, and so an integrable $J_n$ could not possibly be of Kähler type.

The reader should note that we have only used two mild consequences of the Kähler condition: the degeneration of the Fröhlicher spectral sequence, and the non-triviality of the cup form on $H^2$. The same argument would thus apply if one instead wished to consider, say, complex structures of Moishezon type.

### 4. Independence of Chern Numbers.

So far, we have seen that the Chern number $c_1c_2$ of a complex 3-fold is not an invariant of the underlying 6-manifold. We will now see see that the same is true of $c_3^2$.

To this end, let $N$ be any smooth, compact oriented 4-manifold. By [9], the connected sum $M = N \# k\mathbb{CP}^2$ admits anti-self-dual metrics $g$ provided that $k$ is sufficiently large. The twistor space of such an anti-self-dual metric is a complex 3-fold $(Z, J_2)$, the underlying 6-manifold $Z$ of which is formally the fiber-wise projectivization $\mathbb{P}(S_+)$ of the bundle of positive spinors on $M$. This description may seem a bit paradoxical, insofar as we are interested in choices of $M$ which definitely are not spin, but it may be made quite concrete by choosing a spin$^c$ structure on $M$. This then gives rise to a well defined “twisted spinor” bundle $V_+$ which formally satisfies

$$V_+ = S_+ \otimes L^{1/2}$$

for a line bundle $L$ with $c_1(L) \cong w_2(M) \mod 2$. This done, we then have a canonical identification of $Z$ with the total space of the $\mathbb{CP}^1$-bundle $\mathbb{P}(V_+)$.

The naturally defined complex structure $J_2$ then makes each fiber of the projection $\pi : Z \to M$ into a holomorphically embedded $\mathbb{CP}^1$ with normal bundle $O(1) \oplus O(1)$. For more details, see [1, 4, 8].

Now let us now specialize to the case in which $N$ is a complex surface, and notice that $M = N \# k\mathbb{CP}^2$ may then be thought of as an iterated blow-up of $N$, and so, in particular, carries a complex structure. This complex structure induces a spin$^c$ structure on $M$ such that, for any metric $g$, the associated
twisted spin bundle $V_+$ is smoothly bundle-isomorphic to the holomorphic vector $O \oplus K^{-1}$, where $K$ is the canonical line bundle of $M$. Indeed, for a Hermitian metric on $M$, there is even a canonical isomorphism $V_+ \cong O \oplus K^{-1}$; and, up to abstract the bundle equivalence, the twisted spinor bundle $V_+$ is metric-independent once a spin$^c$ structure is specified. In this way, the twistor space $Z$ of an anti-self-dual metric $g$ on $M$ is diffeomorphic to the complex manifold $\mathbb{P}(O \oplus K^{-1})$, and so carries a second complex structure $J_1$. Notice that we do not need to assume any compatibility between the metric $g$ and the complex structure of $M$. Also notice that $(Z, J_1)$ is projective algebraic (respectively, Kählerian) if $M$ is.

Let us now calculate the Chern numbers of $(Z, J_1)$. To do this, first notice that $Z = \mathbb{P}(O \oplus K^{-1})$ carries two canonical hypersurfaces, $\Sigma$ and $\bar{\Sigma}$, corresponding to the factors of the direct sum $O \oplus K^{-1}$. These are both copies of the complex surface $M$, but their normal bundles are respectively $K^{-1}$ and $K$. Moreover, the divisor $\Sigma + \bar{\Sigma}$ precisely represents the vertical line bundle $L$ of $Z$. We thus have

$$c_1(Z, J_1) = \Sigma + \bar{\Sigma} + \pi^* c_1(M) = 2\Sigma.$$ 

Hence

$$c_1^3(Z, J_1) = (2\Sigma)^3 = 8(2\chi + 3\tau)$$

and

$$c_1c_2(Z, J_1) = 2(c_1^2 + c_2)(M) = 6(\chi + \tau),$$

where $\chi$ and $\tau$ are the Euler characteristic and signature of $M$, respectively.

On the other hand, there is a fiber-wise antipodal map which acts anti-holomorphically on $(Z, J_2)$, so $c_1(J_2)$ is Poincaré dual to an element of $H_4(Z)$ which is invariant under this antipodal map. We also know that the integral of $c_1(J_2)$ on a fiber is 4. It follows that

$$c_1(Z, J_2) = 2\Sigma + 2\bar{\Sigma}.$$ 

Since the restrictions of $J_1$ and $J_2$ to a tubular neighborhood of $\Sigma$ are also homotopic, we therefore deduce the formulae

$$c_1^3(Z, J_2) = 16(2\chi + 3\tau)$$

and

$$c_1c_2(Z, J_2) = 12(\chi + \tau)$$

derived (with opposite orientation conventions) by Hitchin [4] in greater generality. For us, the point is that the invariants $c_1^3$ and $c_1c_2$ of $(Z, J_2)$ are precisely double the corresponding Chern numbers of $(Z, J_1)$.

We are now in a position to prove a more precise version of Theorem B.
Theorem 5. Let \((m,n)\) be any pair of integers. Then for any integer \(\tilde{n} \ll n\), there is a spin, complex projective 3-fold \((X,J)\) with Chern numbers
\[
\begin{align*}
c_1c_2 &= 24m, \ c_1^3 = 8n, \\
c_1c_2 &= 48m, \ c_1^3 = 8\tilde{n}.
\end{align*}
\]
which admits a second complex structure \(\tilde{J}\) with
\[
\begin{align*}
c_1c_2 &= 48m, \ c_1^3 = 8\tilde{n}.
\end{align*}
\]
If \(m > 0\), moreover, we may even arrange for \(X\) to be simply connected.

Proof. For each integer \(m\), we begin by choosing a complex algebraic surface \(N_m\) with Todd genus \((\chi + \tau)/4 = m\). For example, if \(m \leq 1\), let us take \(N_m = \mathbb{C} \times \mathbb{C}P_1\), where \(\mathbb{C}\) is a Riemann surface of genus \(1 - m\). On the other hand, if \(m > 1\), we may let \(N_m\) be the minimal resolution of \((E \times C)/\mathbb{Z}_2\), where \(E\) is an elliptic curve, \(C\) is a hyperelliptic curve of genus \(m - 1\), and \(\mathbb{Z}_2\) acts simultaneously on both factors by the Weierstrass involution. Notice that our choice of \(N_m\) is simply connected when \(m > 0\), and that, incidentally, this is the best one can do in principal.

Now, for each \(m\), let \(k_0(m)\) be chosen so that \(N_m \# k\mathbb{C}P_2\) admits anti-self-dual metrics for each \(k \geq k_0(m)\). By Taubes’ theorem [9], such an integer \(k_0(m)\) exists. Moreover, with the above choices, we may even take \(k_0(m) = 0\) for \(m < 0\), \(k_0(0) = 6\), \(k_0(1) = 13\), and \(k_0(2) = 3\) [5, 6, 3].

Now let \((m,n)\) be any pair of integers, and let \(\tilde{n}\) be any integer such that
\[
\tilde{n} \leq \min(n - k_0(m) + c_2^2(N_m), 2n).
\]
We may then define integers \(k \geq k_0(m)\) and \(\ell \geq 0\) by
\[
\begin{align*}
k &= n - \tilde{n} + c_2^2(N_m), \\
\ell &= 2n - \tilde{n}.
\end{align*}
\]

Let \(Z(k,m)\) be the twistor space of an anti-self-dual metric on \(M = N_m \# k\mathbb{C}P_2\), and let
\[
X(k,\ell,m) = Z(k,m) \# \ell\mathbb{C}P_3.
\]
Notice that \(X\) is a spin manifold. Moreover, it comes equipped with two different complex structures.

First, because \(M = N_m \# k\mathbb{C}P_2\) is a projective algebraic surface, \(Z\) carries a projective algebraic complex structure \(J_1\), and \(X\) may then be identified with the blow-up of \((Z, J_1)\) at \(\ell\) points. Let us denote this complex structure on \(X\) by \(J\). Using Proposition 2 and the above computations, we therefore have
\[
\begin{align*}
c_1^3(X, J) &= 8(2\chi + 3\tau)(M) + 8\ell = 8n \\
c_1c_2(X, J) &= 6(\chi + \tau)(M) = 24m.
\end{align*}
\]
On the other hand, each \(Z\) also admits its twistor complex structure \(J_2\), and we may instead choose to think of \(X\) as the blow-up of this twistor space.
at ℓ points. Let us denote the corresponding complex structure on X by ˜J. Thus

\[ c_3(X, \tilde{J}) = 16(2\chi + 3\tau)(M) + 8\ell = 8\tilde{n}, \]
\[ c_1c_2(X, \tilde{J}) = 12(\chi + \tau)(M) = 48m, \]

as claimed. □

Since the Todd genus of any complex 3-fold is given by \( c_1c_2/24 \), this result realizes all possible values of \( c_1c_2 \). The divisibility of \( c_3^3 \) by 8 is also necessary for X to be spin, so the result is also essentially optimal in this respect.

On the other hand, we have chosen to ignore \( c_3 \), which is determined by \((m, n, \tilde{n})\) in these examples. The abundance of rational curves also forces all our 3-folds all have Kodaira dimension \(-\infty\). And finally, most of our manifolds are in no sense minimal. It would obviously be of great interest to produce new examples which overcome these limitations.

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COHOMOLOGY OF COMPLETE INTERSECTIONS IN TORIC VARIETIES

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We explicitly describe cohomology of complete intersections in compact simplicial toric varieties.

In this paper we will study intersections of hypersurfaces in compact simplicial toric varieties \( \mathbf{P}_\Sigma \). The main purpose is to relate naturally the Hodge structure of a complete intersection \( X_{f_1} \cap \ldots \cap X_{f_s} \) in \( \mathbf{P}_\Sigma \) to a graded ring. Originally this idea appears in [Gr], [St], [Dol], [PS]. The case of a hypersurface in a toric variety has been treated in [BC]. Also the Hodge structure of complete intersections in a projective space was described in [Te], [Ko], [L], [Di], [Na]. The common approach was to reduce studying of the Hodge structure on a complete intersection to studying of the Hodge structure on a hypersurface in a higher dimensional projective variety. This is the idea of a “Cayley trick”. About a Cayley trick in the toric context see [GKZ], [DK], [BB]. A special case of a complete intersection (when it is empty) in a complete simplicial toric variety was elaborated in [CCD]. The basic references on toric varieties are [F1], [O], [Da], [C].

The paper is organized as follows:

Section 1 establishes notation and studies cohomology of subvarieties in a complete simplicial toric variety. In Section 2 we describe a Cayley trick for toric varieties. In Section 3 we prove the main result where we relate the Hodge components \( H^{d-s-p,p}(X_{f_1} \cap \ldots \cap X_{f_s}) \) in the middle cohomology group to homogeneous components of a graded ring. Section 4 treats a special case of complete intersections: a nondegenerate intersection.

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1. Quasi-smooth intersections.

We first fix some notation. Let \( M \) be a lattice of rank \( d \), \( N = \text{Hom}(M, \mathbb{Z}) \) the dual lattice; \( M_\mathbb{R} \) (resp. \( N_\mathbb{R} \)) denotes the \( \mathbb{R} \)-scalar extension of \( M \) (resp. of \( N \)). Let \( \Sigma \) be a rational simplicial complete \( d \)-dimensional fan in \( N_\mathbb{R} \) [BC], \( \mathbf{P}_\Sigma \) a complete simplicial toric variety associated with this fan.

Such a toric variety can be described as a geometric quotient [C]. Let \( S(\Sigma) = \mathbb{C}[x_1, \ldots, x_n] \) be the polynomial ring over \( \mathbb{C} \) with variables \( x_1, \ldots, x_n \).
corresponding to the integral generators $e_1, \ldots, e_n$ of the 1-dimensional cones of $\Sigma$. For $\sigma \in \Sigma$ let $\hat{x}_\sigma = \prod_{e_i \in \sigma} x_i$, and let $B(\Sigma) = \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle \subset S$ be the ideal generated by the $\hat{x}_\sigma$'s. This ideal gives the variety $Z(\Sigma) = V(B(\Sigma)) \subset \mathbb{A}^{n}$. The toric variety $\mathbb{P} = \mathbb{P}_\Sigma$ will be a geometric quotient of $U(\Sigma) := \mathbb{A}^{n} \setminus Z(\Sigma)$ by the group $\mathbb{D} := \text{Hom}_{\mathbb{Z}}(A_{d-1}(\mathbb{P}), \mathbb{C}^*)$, where $A_{d-1}(\mathbb{P})$ is the Chow group of Weil divisors modulo rational equivalence.

Each variable $x_i$ in the coordinate ring $S(\Sigma)$ corresponds to a torus-invariant irreducible divisor $D_i$ of $\mathbb{P}$. As in [C], we grade $S = S(\Sigma)$ by assigning to a monomial $\prod_{i=1}^{n} x_i^{a_i}$ its degree $[\sum_{i=1}^{n} a_i D_i] \in A_{d-1}(\mathbb{P})$. A polynomial $f$ in the graded piece $S_\alpha$ corresponding to $\alpha \in A_{d-1}(\mathbb{P})$ is said to be $\mathbb{D}$-homogeneous of degree $\alpha$.

Let $f_1, \ldots, f_s$ be $\mathbb{D}$-homogeneous polynomials. They define a zero set $V(f_1, \ldots, f_s) \subset \mathbb{A}^n$, moreover $V(f_1, \ldots, f_s) \cap U(\Sigma)$ is stable under the action of $\mathbb{D}$ and hence descends to a closed subset $X \subset \mathbb{P}$, because $\mathbb{P}$ is a geometric quotient.

**Definition 1.1.** We say that $X$ is a *quasi-smooth intersection* if $V(f_1, \ldots, f_s) \cap U(\Sigma)$ is either empty or a smooth subvariety of codimension $s$ in $U(\Sigma)$.

**Remark 1.2.** This notion generalizes a nonsingular complete intersection in a projective space. Notice that since the $(n-d)$-dimensional group $\mathbb{D}$ has only zero dimensional stabilizers [BC], $X$ is of pure dimension $d - s$ or empty.

We can now relate this notion to a $V$-submanifold (see Definition 3.2 in [BC]).

**Proposition 1.3.** If $X \subset \mathbb{P}$ is a closed subset of codimension $s$ defined by $\mathbb{D}$-homogeneous polynomials $f_1, \ldots, f_s$, then $X$ is a quasi-smooth intersection if and only if $X$ is a $V$-submanifold of $\mathbb{P}$.

The proof of this is very similar to the proof of the Proposition 3.5 in [BC].

The next result is a Lefschetz-type theorem.

**Proposition 1.4.** Let $X \subset \mathbb{P}$ be a closed subset, defined by $\mathbb{D}$-homogeneous polynomials $f_1, \ldots, f_s$, in a complete simplicial toric variety $\mathbb{P}$. If $f_1, \ldots, f_s \in B(\Sigma)$, then the natural map $i^* : H^i(\mathbb{P}) \to H^i(X)$ is an isomorphism for $i < d - s$ and an injection for $i = d - s$. In particular, this is valid if $X$ is an intersection of ample hypersurfaces.

**Proof.** We can present $X = X_{f_1} \cap \ldots \cap X_{f_s}$, where $X_{f_i} \subset \mathbb{P}$ is a hypersurface defined by $f_i$. As it was shown in the proof of the Proposition 10.8 [BC], if $f \in B(\Sigma)$ then $\mathbb{P} \setminus X_f = (\mathbb{A}^{n} \setminus V(f))/\mathbb{D}(\Sigma)$ is affine, hence $H^i(\mathbb{P} \setminus X_f) = 0$ for $i > d$. We will prove by induction on $s$ that $H^i(\mathbb{P} \setminus (X_{f_1} \cap \ldots \cap X_{f_s})) = 0$ for $i > d + s - 1$. Consider the Mayer-Vietoris sequence

$$
\cdots \to H^i(U \cap V) \to H^{i+1}(U \cup V) \to H^{i+1}(U) \oplus H^{i+1}(V) \to H^{i+1}(U \cap V) \to \cdots
$$
with \( U = P \setminus (X_{f_1} \cap \cdots \cap X_{f_{s-1}}) \), \( V = P \setminus X_{f_s} \). Notice that \( U \cup V = P \setminus (X_{f_1} \cap \cdots \cap X_{f_s}) \) and \( U \cap V = \cup_{i=1}^{s-1} P \setminus (X_{f_i} \cup X_{f_s}) = P \setminus (X_{f_1} \cap \cdots \cap X_{f_{s-1}}) \). So, using the induction and the above sequence, we obtain that \( H^i(P \setminus X) = 0 \) for \( i > d + s - 1 \). As a consequence of this, \( X \) is nonempty unless \( s > d \) because the dimension \( h^{2d}(P) = 1 \). Since \( P \setminus X \) is a \( V \)-manifold, Poincaré duality implies that \( H^i_c(P \setminus X) = 0 \) for \( i \leq d - s \). Now the desired result follows from the long exact sequence of the cohomology with compact supports \((X \text{ and } P \text{ are compact})\):

\[
\cdots \to H^i_c(P \setminus X) \to H^i_c(P) \to H^i_c(X) \to H^{i+1}_c(P \setminus X) \to H^{i+1}_c(P) \to \cdots.
\]

If \( X \) is an intersection of ample hypersurfaces defined by \( f_1, \ldots, f_s \), then Lemma 9.15 [BC] gives us that \( f_1, \ldots, f_s \) belong to \( B(\Sigma) \). \( \square \)

**Corollary 1.5.** A quasi-smooth intersection \( X = X_{f_1} \cap \cdots \cap X_{f_s} \), defined by \( f_1, \ldots, f_s \in B(\Sigma) \), has pure dimension \( d - s \).

Since the dimension of \( H^0(X, C) \) is the number of connected components of \( X \), we obtain another important result.

**Corollary 1.6.** An intersection \( X_{f_1} \cap \cdots \cap X_{f_s} \), defined by \( f_1, \ldots, f_s \in B(\Sigma) \), in a complete simplicial toric variety \( P_{\Sigma} \) is connected provided \( s < \dim P_{\Sigma} \).

**Remark 1.7.** If the polynomials \( f_1, \ldots, f_s \) have ample degrees, then this corollary follows from a more general statement in [FL1] (see also [FL2] and [FH] for connectedness theorems).

### 2. “Cayley trick”

We will explore a Cayley trick to reduce studying of the cohomology of quasi-smooth intersections to results already known for hypersurfaces.

Let \( L_1, \ldots, L_s \) be line bundles on a complete \( d \)-dimensional toric variety \( P = P_{\Sigma} \), and let \( \pi : P(E) \to P \) be the projective space bundle associated to the vector bundle \( E = L_1 \oplus \cdots \oplus L_s \). Then the \( \mathbb{P}^{s-1} \)-bundle \( P(E) \) is a toric variety. The fan corresponding to it can be described as follows [O, p. 58]. Suppose that support functions \( h_1, \ldots, h_s \) give rise to the isomorphism classes of line bundles \([L_1], \ldots, [L_s] \in \text{Pic}(P)\), respectively. Introduce a \( \mathbb{Z} \)-module \( N' \) with a \( \mathbb{Z} \)-basis \( \{n_2, \ldots, n_s\} \) and let \( \tilde{N} := N \oplus N' \) and \( n_1 := -n_2 - \cdots - n_s \). Denote by \( \tilde{\sigma} \) the image of each \( \sigma \in \Sigma \) under the \( \mathbb{R} \)-linear map \( N_{\mathbb{R}} \to \tilde{N}_{\mathbb{R}} \) which sends \( y \in N_{\mathbb{R}} \) to \( y - \sum_{j=1}^s h_j(y)n_j \). On the other hand, let \( \sigma'_1 \) be the cone in \( N_{\mathbb{R}}' \) generated by \( n_1, n_1, n_1 + 1, \ldots, n_s \) and let \( \Sigma' \) be the fan in \( N_{\mathbb{R}}' \) consisting of the faces of \( \sigma'_1, \ldots, \sigma'_s \). Then \( P(E) \) corresponds to the fan \( \Sigma := \{ \tilde{\sigma} + \sigma' : \sigma \in \Sigma, \sigma' \in \Sigma' \} \). From this description it is easy to see that if \( \Sigma \) is a complete simplicial fan then \( P(L_1 \oplus \cdots \oplus L_s) \) is a complete simplicial toric variety. We see that the integral generators of
the 1-dimensional cones in $\tilde{\Sigma}$ are given by

$$\tilde{e}_i = e_i - \sum_{1 \leq j \leq s} h_j(e_i)n_j, \quad i = 1, \ldots, n,$$

$$\tilde{n}_1 = -n_2 - \cdots - n_s,$$

$$\tilde{n}_j = n_j, \quad j = 2, \ldots, s,$$

where $e_1, \ldots, e_n$ are the integral generators of the 1-dimensional cones in $\Sigma$.

The homogeneous coordinate ring of $P(E)$ is the polynomial ring

$$R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_s],$$

where $x_i$ corresponds to $\tilde{e}_i$ and $y_j$ corresponds to $\tilde{n}_j$. This ring has a grading by the Chow group $A_{d+s-2}(P(E))$. Since $P$ is a normal variety, there is an embedding of the Picard group Pic($P$) $\hookrightarrow A_{d-1}(P)$. We want to show that if some polynomials $f_j \in S(\Sigma) = \mathbb{C}[x_1, \ldots, x_n]$ have the property $\deg(f_j) = [L_j] \in \text{Pic}(P)$, then the polynomials $y_j f_j$ all have the same degree in $R$.

This will allow us to consider a hypersurface defined by the homogeneous polynomial $F = \sum_{j=1}^s y_j f_j$.

**Lemma 2.1.** Let $f_1, \ldots, f_s \in S(\Sigma)$ be D-homogeneous polynomials, such that $\deg(f_j) = [L_j]$ for some line bundles $L_1, \ldots, L_s$. Then $F = \sum_{j=1}^s y_j f_j$ is homogeneous in $R$ and its degree is the isomorphism class $[O_{P(E)}(1)]$ of the canonical line bundle on $P(E) = P(L_1 + \cdots + L_s)$.

**Proof.** To prove that $F$ is a homogeneous polynomial we will repeat the arguments in the proof of Lemma 3.5 in [CCD]. Let $D_1, \ldots, D_n$ be the torus-invariant divisors on $P = P_{\Sigma}$ corresponding to the 1-dimensional cones of the fan $\Sigma$. Then the pullback $\pi^*D_i$ is the torus-invariant divisor of $P(E)$ corresponding to the cone generated by $\tilde{e}_i$. Also denote by $D'_j$ the torus-invariant divisor corresponding to $\tilde{n}_j$. Let $\tilde{M} = M \oplus M'$ be the lattice dual to $\tilde{N} = N \oplus N'$ with $M' = \text{Hom}(N', \mathbb{Z})$ having $\{n_2^*, \ldots, n_s^*\}$ as a basis dual to $\{n_2, \ldots, n_s\}$. The divisor corresponding to the character $\chi^{n_j^*}$ is

$$\text{div}(\chi^{n_j^*}) = \sum_{i=1}^n \langle n_j^*, \tilde{e}_i \rangle \pi^*D_i + \sum_{k=1}^s \langle n_j^*, \tilde{n}_k \rangle D'_k$$

$$= \sum_{i=1}^n (h_1(e_i) - h_j(e_i)) \pi^*D_i - D'_1 + D'_j.$$ 

Therefore, $[D'_j] + [\pi^*L_j]$ all have the same degree in the Chow group $A_{d+s-2}(P(E))$, and, consequently, $F$ is a homogeneous polynomial.

Now consider the following exact sequence [M]:

$$0 \rightarrow O_{P(E)} \rightarrow \pi^*E^* \otimes O_{P(E)}(1) \rightarrow T_{P(E)} \rightarrow \pi^*T_P \rightarrow 0,$$
where $T_X$ denotes the tangent bundle, $E^*$ is the dual bundle. From here we can compute the Chern class
\[
c_1(T_{P(E)}) = c_1(\pi^*T_P) + c_1(\pi^*E^* \otimes O_{P(E)}(1)) = \pi^*c_1(T_P) - \pi^*c_1(E) + s \cdot c_1(O_{P(E)}(1)).
\]
Hence, $s \cdot c_1(O_{P(E)}(1)) = \pi^*c_1(L_1) + \cdots + \pi^*c_1(L_s) + c_1(T_{P(E)}) - \pi^*c_1(T_P)$.

On the other hand, from the generalized Euler exact sequence [BC, §12] we get
\[
0 \to O_P^{n-d} \to \oplus_{i=1}^{n} O_P(D_i) \to T_P \to 0.
\]
This implies that $c_1(T_P) = [D_1] + \cdots + [D_n]$. Similarly we have $c_1(T_{P(E)}) = [\pi^*D_1] + \cdots + [\pi^*D_n] + [D'_1] + \cdots + [D'_s]$. Under the identification $\text{Pic}(P(E)) \hookrightarrow A_{d+s-2}(P(E))$ the first Chern class of a line bundle on $P(E)$ is exactly its isomorphism class in the Picard group $\text{Pic}(P(E))$. Therefore
\[
s \cdot [O_{P(E)}(1)] = [\pi^*L_1] + \cdots + [\pi^*L_s] + [D'_1] + \cdots + [D'_s] = s \cdot ([\pi^*L_2] + [D'_2]).
\]

It can be easily checked that $D'_2$ is a Cartier divisor on $P(E)$. Hence all classes $[O_{P(E)}(1)]$, $[\pi^*L_2]$ and $[D'_2]$ lie in the Picard group $\text{Pic}(P(E))$. But this group is free abelian, because $P(E)$ is complete. So the above equality is divisible by $s$: $[O_{P(E)}(1)] = [\pi^*L_2] + [D'_2] = \deg(F)$.

From now on we assume that $P = P_\Sigma$ is a complete simplicial toric variety and that $\deg(f_j) \in \text{Pic}(P)$, $j = 1, \ldots, s$. Denote by $Y$ the hypersurface in $P(E)$ defined by $F = \sum_{j=1}^{s} y_j f_j$.

**Lemma 2.2.** $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection iff the hypersurface $Y$ is quasi-smooth.

**Proof.** $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection means that whenever $x \in V(f_1, \ldots, f_s) \setminus Z(\Sigma)$, the rank$(\frac{\partial F}{\partial x_i}(x))_{i,j} = s$. And $Y$ is quasi-smooth iff $z = (x, y) \in V(F) \setminus Z(\Sigma)$ implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j = 1, \ldots, s$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \ldots, n$, is nonzero.

So let $(x, y) \in V(F) \setminus Z(\Sigma)$, then there is a cone $\Sigma'$ with $\sigma \in \Sigma$, $\sigma' \in \Sigma'$, such that $\prod_{i \not\in \sigma} x_i \prod_{i \in \sigma} y_j \neq 0$ where $x_i, y_j$ are the coordinates of $(x, y)$. If $f_1(x) = \cdots = f_s(x) = 0$, then $x \in V(f_1, \ldots, f_s) \setminus Z(\Sigma)$ because $\prod_{i \not\in \sigma} x_i \neq 0$. And if $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \ldots, n$, is nonzero.

Conversely, suppose $Y$ is quasi-smooth. Pick any $x \in V(f_1, \ldots, f_s) \setminus Z(\Sigma)$, then $(x, y) \in V(F) \setminus Z(\Sigma)$ for each $y = (y_1, \ldots, y_s) \neq 0$. Therefore $\sum_{j=1}^{s} y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some $i$, which means the rank$(\frac{\partial f_i}{\partial x_i}(x))_{i,j}$ is maximal. \qed

Since a quasi-smooth intersection is a compact $V$-manifold (Proposition 1.3), the cohomology on it has a pure Hodge structure. Using Proposition 1.4 and the Poincaré duality, we can compute the cohomology of a quasi-smooth intersection except for the cohomology in the middle dimension $d - s$. So we introduce the following definition.

**Definition 3.1.** The variable cohomology group $H_{\text{var}}^{d-s}(X)$ is $\text{coker}(H^{d-s}(P) \xrightarrow{i^*} H^{d-s}(X))$.

The variable cohomology group also has a pure Hodge structure.

**Proposition 3.2.** Let $X = X_{f_1} \cap \ldots \cap X_{f_s}$ be a quasi-smooth intersection of ample hypersurfaces. Then there is an exact sequence of mixed Hodge structures

$$0 \to H^{d-s-1}(P) \xrightarrow{[X]} H^{d+s-1}(P) \to H^{d+s-1}(P \setminus X) \to H_{\text{var}}^{d-s}(X) \to 0,$$

where $[X] \in H^{2s}(P)$ is the cohomology class of $X$.

**Proof.** Consider the Gysin exact sequence:

$$\cdots \to H^{i-2s}(X) \xrightarrow{i^*} H^i(P) \to H^i(P \setminus X) \to H^{i-2s+1}(X) \xrightarrow{i^*} H^{i+1}(P) \to \cdots.$$ 

Since $i^*$ is Poincaré dual to the Gysin map $i_!$, it follows that $H_{\text{var}}^{d-s}(X)$ is isomorphic to the kernel of $i_!: H^{d-s}(X) \to H^{d+s}(P)$. So we get an exact sequence

$$H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(P) \to H^{d+s-1}(P \setminus X) \to H_{\text{var}}^{d-s}(X) \to 0.$$

Now we use a commutative diagram

$$\begin{array}{ccc}
H^{d-s-1}(X) & \xrightarrow{i_!} & H^{d+s-1}(P) \\
\text{by} & \uparrow & \text{bijection} \\
H^{d-s-1}(P) & \xrightarrow{i^*} & \end{array}$$

By Proposition 1.4 $i^*$ is an isomorphism in this diagram, so it suffices to prove that the Gysin map $i_!$ is injective in the above diagram.

**Lemma 3.3.** If $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a quasi-smooth intersection of ample hypersurfaces, then the Gysin map $H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(P)$ is injective.

**Proof.** Since the odd dimensional cohomology of a complete simplicial toric variety vanishes [F1, pp. 92-94] and $i^*: H^{d-s-1}(P) \to H^{d-s-1}(X)$ is an isomorphism by Proposition 1.4, it follows that $H^{d-s-1}(X) = H^{d+s-1}(P) = H^{d+s-1}(P \setminus X) = 0$ when $d + s - 2$ is odd. So the Gysin exact sequence (1) it is enough to show that $H^{d+s-2}(P \setminus X) = 0$ when $d + s - 2$ is odd. To prove this we use the Cayley trick again. Let $Y$ be the hypersurface defined by
Consider the commutative diagram
\[
\begin{array}{ccc}
H_{d+s-3}(Y) & \xrightarrow{j_!} & H^{d+s-1}(P(E)) \\
\uparrow j^* & & \\
H^{d+s-3}(P(E)) & \rightarrow & H^{d+s-1}(P(E))
\end{array}
\]
where \([Y] \in H^2(P(E))\) is the cohomology class of \(Y\). The canonical line bundle \(O_{P(E)}(1)\) is ample [H, III, §1], whence by Lemma 2.1, \(Y\) is ample. So by Proposition 10.8 [BC] \(j^*: H^{d+s-3}(P(E)) \rightarrow H^{d+s-3}(Y)\) is an isomorphism and by Hard Lefschetz \(\cup[Y]: H^{d+s-3}(P(E)) \rightarrow H^{d+s-1}(P(E))\) is injective. Thus, from the above diagram the lemma follows. \(\Box\)

**Definition 3.4.** For a nonzero polynomial \(F \in R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_s]\) the Jacobian ring \(R(F)\) denotes the quotient of \(R\) by the ideal generated by the partial derivatives \(\frac{\partial F}{\partial y_j}, j = 1, \ldots, s, \frac{\partial F}{\partial x_i}, i = 1, \ldots, n.\)

**Remark 3.5.** If \(F = y_1f_1 + \cdots + y_sf_s\) is as in Lemma 2.1 with \(f_j \in S_{\alpha_j}\), then \(R(F)\) carries a natural grading by the Chow group \(A_{d+s-2}(P(E))\). Moreover, there are canonical isomorphisms \(A_{d+s-2}(P(E)) \cong A_{d-1}(P) \oplus A_d(P) \cong A_{d-1}(P) \oplus \mathbb{Z} ([F2]).\) With respect to this bigrading of the Chow group \(A_{d+s-2}(P(E))\) we have that \(\deg(F) = (0, 1), \deg(f_j) = (\alpha_j, 0), \deg(y_j) = (-\alpha_j, 1),\) which is very similar to the case when \(P\) is a projective space.

We now can state the main result.

**Theorem 3.6.** Let \(P\) be a \(d\)-dimensional complete simplicial toric variety, and let \(X \subset P\) be a quasi-smooth intersection of ample hypersurfaces defined by \(f_j \in S_{\alpha_j}, j = 1, \ldots, s.\) If \(F = y_1f_1 + \cdots + y_sf_s\), then for \(p \neq \frac{d+s-1}{2}\), we have a canonical isomorphism

\[
R(F)_{(d+s-p)\beta-\beta_0} \cong H_{\text{var}}^{p-s,d-p}(X)
\]

where \(\beta_0 = \deg(x_1 \cdots x_n, y_1 \cdots y_s), \beta = \deg(F) = \deg(f_j) + \deg(y_j).\) In the case \(p = \frac{d+s-1}{2}\) there is an exact sequence

\[
0 \rightarrow H^{d-s-1}(P) \xrightarrow{\cup[X]} H^{d+s-1}(P) \rightarrow R(F)_{\frac{d+s+1}{2}\beta-\beta_0} \rightarrow H_{\text{var}}^{\frac{d-s+1}{2}, \frac{d+s+1}{2}}(X) \rightarrow 0.
\]
Proof. Since $H^i(P)$ vanishes for $i$ odd and has a pure Hodge structure of type $(p, p)$ for $i$ even, from Proposition 3.2 we get $Gr_P H^{d+s-1}(P \setminus X) \cong H^{d+s-1}_\var(H^{d+s-1}(P \setminus X)$ if $p \neq \frac{d+s-1}{2}$, and in case $p = \frac{d+s-1}{2}$ the following sequence

$$0 \to H^{d-s-1}(P) \underleftarrow{\cup[X]} H^{d+s-1}(P) \to Gr_P H^{d+s-1}(P \setminus X) \to H^{d+s-1}_\var(H^{d+s-1}(P \setminus X) \to 0$$

is exact.

Now use the isomorphism of mixed Hodge structures $H^i(P \setminus X) \cong H^i(P(E) \setminus Y)$ and by the Theorem 10.6 [BC] the desired result follows. □


In this section we consider a special case of quasi-smooth intersections.

**Definition 4.1.** A closed subset $X = X_{f_1} \cap \ldots \cap X_{f_s}$, defined by $D$-homogeneous polynomials $f_1, \ldots, f_s$, is called a nondegenerate intersection if $X_{f_1} \cap \ldots \cap X_{f_k} \cap T_\tau$ is a smooth subvariety of codimension $k$ in $T_\tau$ for any \{j_1, \ldots, j_k\} \subset \{1, \ldots, s\}$ and $\tau \in \Sigma$. (Here $T_\tau$ denotes the torus in $P_\Sigma$ associated with a cone $\tau \in \Sigma$.)

We will show how to define a nondegenerate intersection in terms of the polynomials $f_1, \ldots, f_s$. For $\sigma \in \Sigma$, let $U_\sigma = \{x \in A^n : x_\sigma \neq 0\}$. We know that $P_\Sigma$ has an affine toric open cover by $A_\sigma = U_\sigma / D(\Sigma)$, $\sigma \in \Sigma$ [BC]. Also $T_\tau = (U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma) / D(\Sigma)$. Notice that $U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma = \{x \in A^n : \tilde{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$ is a torus. So each $T_\tau$ is a quotient of a torus by a $D$-subgroup, because $D$ is diagonalizable [BC].

**Lemma 4.2.** Let $T = (C^*)^n / G$ be the quotient of a torus by a $D$-subgroup $G$. Suppose that $X \subset (C^*)^n$ is an invariant subvariety with respect to the action of $G$. Then the geometric quotient $X/G$ is smooth iff $X$ is smooth.

**Proof.** By the structure theorem of a $D$-group [Hu, §16.2] we can assume that $(C^*)^n = G_0 \times (C^*)^k$, where $G_0 \cong (C^*)^{n-k}$ is the identity component of $G$, and $G = G_0 \times H$ for some finite subgroup $H$ in $(C^*)^k$. Now it suffices to show the Lemma if $G$ is a torus or a finite group. If $G = G_0$ then $X = (C^*)^{n-k} \times p(X)$, where by $p(X)$ we mean the projection of $X$ onto $(C^*)^k$. Notice that $p(X) \cong X/G$, hence $X$ is smooth iff $X/G$ is smooth. In the case $G = H$ is a finite group it can be easily checked that $X \to X/G$ is an unramified cover [Sh, p. 346]. So $X$ and $X/G$ are smooth simultaneously. □

From this Lemma it follows that $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection iff $V(f_1, \ldots, f_k) \cap V_\tau$ is a smooth subvariety of codimension $k$ in the torus $V_\tau = \{x \in A^n : \tilde{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$. 
As in Section 2 we can consider the hypersurface $Y \subset \mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$.

**Lemma 4.3.** $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection iff $Y$ is a nondegenerate hypersurface.

**Proof.** As shown above, $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection if the rank $(\frac{\partial f_j}{\partial x_i}(x))_{i \in \{1, \ldots, \tilde{r}\}} = k$ for all $x \in \mathbf{V}(f_{j_1}, \ldots, f_{j_k}) \cap V_\tau$, $\tau \in \Sigma$ and $\{j_1, \ldots, j_k\} \subset \{1, \ldots, s\}$. Similarly $Y$ is nondegenerate iff $z = (x, y) \in \mathbf{V}(F) \cap V_{\tilde{r}+\tau'}$, $\tilde{r} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$ (recall the definition of $\mathbf{P}(E)$ associated with $\tilde{\Sigma}$ in the Section 2) implies that one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = f_j(x)$, $j \in \{j : \tilde{n}_j \notin \tau'\}$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tau\}$, is nonzero.

Let $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{r}+\tau'}$, where $\tilde{r} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$. Then $\prod_{\tilde{e}_i \notin \tilde{r}} x_i \prod_{\tilde{n}_j \notin \tilde{r}'} y_j \neq 0$ and $x_i = 0$ if $\tilde{e}_i \notin \tilde{r}$, $y_j = 0$ if $\tilde{n}_j \notin \tilde{r}'$. If $f_j(x) = 0$ for all $j \in \{j : \tilde{n}_j \notin \tilde{r}'\}$, then $x \in \mathbf{V}(f_{j_1}, \ldots, f_{j_k}) \cap V_\tau$ where $\{j_1, \ldots, j_k\} = \{j : \tilde{n}_j \notin \tilde{r}'\}$. So if $X = X_{f_1} \cap \ldots \cap X_{f_s}$ is a nondegenerate intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tilde{r}\}$, is nonzero.

Conversely, suppose $Y$ is nondegenerate. Take any $x \in \mathbf{V}(f_{j_1}, \ldots, f_{j_k}) \cap V_\tau$ with $\tau \in \Sigma$, $\{j_1, \ldots, j_k\} \subset \{1, \ldots, s\}$. Then $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{r}+\tau'}$ for each $y \in V_{\tau'} = \{y \in \mathbf{A}^s : y_j \neq 0 \text{ if } \tilde{n}_j \notin \tilde{r}', y_j = 0 \text{ if } \tilde{n}_j \in \tilde{r}'\}$ where $\tau'$ is the cone generated by the complement of $\{\tilde{n}_j, \ldots, \tilde{n}_s\}$ in the set $\{\tilde{n}_1, \ldots, \tilde{n}_s\}$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some $i$, which means the rank $(\frac{\partial f_j}{\partial x_i}(x))_{i \in \{1, \ldots, \tilde{r}\}} = s$.

Since a nondegenerate hypersurface is quasi-smooth [BC], Lemma 2.2 shows that a nondegenerate intersection is quasi-smooth.

**Definition 4.4 ([BC]).** Given a polynomial $f \in S = \mathbf{C}[x_1, \ldots, x_n]$, we get the ideal quotient $J_1(f) = (x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n}) : x_1 \ldots x_n$ (see [CLO, p. 193]) and the ring $R_1(f) = S/J_1(f)$.

**Remark 4.5.** If $F = \sum_{j=1}^s y_j f_j \in R$ is as in Lemma 2.1, then $R_1(F) = R/J_1(F)$ has a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$.

**Theorem 4.6.** Let $X = X_{f_1} \cap \ldots \cap X_{f_s}$ be a nondegenerate intersection of ample hypersurfaces given by $f_j \in S_{\alpha_j}$, $j = 1, \ldots, s$. If $F = \sum_{j=1}^s y_j f_j \in R$, then there is a canonical isomorphism

$$H^{p-s-d-p}_{\operatorname{var}}(X) = R_1(F)_{(d+s-p)\beta - \beta_0},$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s)$, $\beta = \deg(F)$. 

Proof. First we will show that there is an isomorphism of Hodge structures $H^{d-s}_{\text{var}}(X)(1-s) \cong H^{d+s-2}_{\text{var}}(Y)$. Let $\varphi : Y \to P$ be the composition of the inclusion $j : Y \hookrightarrow P(E)$ and the projection $\pi : P(E) \to P$. As in [Te], consider the following morphism of the Leray spectral sequences

$$E_{2}^{p,q} = H^{p}(P, R^{\varphi}h_{*}C) \Rightarrow H^{p+q}(P(E));$$

$$E_{2}^{p,q} = H^{p}(P, R^{\varphi}h_{*}C) \Rightarrow H^{p+q}(Y).$$

Since

$$\varphi^{-1}(X) = \begin{cases} \mathbb{P}^{s-1} & \text{if } x \in X, \\ \mathbb{P}^{s-2} & \text{if } x \notin X, \end{cases}$$

we have that (see [Go, p. 202], [De])

$$R^{q}\varphi_{*}C = \begin{cases} C_{P}(\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q < 2s - 2, \\ C_{X}(1-s) & \text{if } q = 2s - 2, \\ 0 & \text{otherwise.} \end{cases}$$

Also we have

$$R^{q}\pi_{*}C = \begin{cases} C_{P}(\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q < 2s - 2, \\ 0 & \text{otherwise.} \end{cases}$$

The first spectral sequence degenerates at $E_{2}$, because for $p$ or $q$ odd $E_{r}^{p,q}$ vanishes. The second spectral sequence also degenerates at $E_{2}$:

$$h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(P) = \sum_{p+q=l} \dim' E_{2}^{p,q} \geq \sum_{p+q=l} \dim' E_{\infty}^{p,q} = h^{l}(Y).$$

To show the degeneracy of $\dim' E_{2}^{p,q}$ it suffices to show that the above inequality is an equality. From Proposition 10.8 [BC] and Proposition 3.2 we get

$$h^{d+s-2}(Y) = h^{d+s-2}(P(E)) + h^{d+s-1}(P(E) \setminus Y)$$

$$- h^{d+s-1}(P(E)) + h^{d+s-3}(P(E)),$$

$$h^{d-s}(X) = h^{d-s}(P) + h^{d+s-1}(P \setminus Y) - h^{d+s-1}(P) + h^{d-s-1}(P).$$

Hence, using the spectral sequence $E_{2}^{p,q}$, we can easily compute the Hodge numbers of $P(E)$ and check that $h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(P) = h^{l}(Y)$ for $l = d+s-2$. Using Proposition 1.4, we can similarly show the above equality for $l \neq d+s-2$ as well. So the spectral sequence $\dim' E_{2}^{p,q}$ degenerates at $E_{2}$. Since $E_{2}^{d+s-2-q,q} = \dim' E_{2}^{d+s-2-q,q}$ for $q \neq 2s - 2$ and, by Proposition 1.4, $E_{2}^{d-s,2s-2} \hookrightarrow E_{2}^{d-s,2s-2}$, we get an isomorphism of Hodge structures (for details see [Te]):

$$H^{d+s-2}_{\text{var}}(Y) \cong E_{2}^{d-s,2s-2} / E_{2}^{d-s,2s-2} \cong H^{d-s}_{\text{var}}(X)(1-s).$$

Now we only need to apply Theorem 11.8 [BC] to finish the proof. \qed
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CONNECTED SUMS OF SELF-DUAL MANIFOLDS AND EQUIVARIANT RELATIVE SMOOTHINGS

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In this paper we address a problem in differential geometry using tools from algebraic geometry and the theory of singular complex spaces. We obtain examples of compact four dimensional self-dual conformal manifolds with torus symmetry and positive scalar curvature from twistor spaces with divisors and we study the local moduli of such geometries.

1. Introduction.

Recall that a conformal 4-manifold is called self-dual if its Weyl curvature, considered as a bundle valued 2-form, is in the +1 eigenspace of the Hodge star-operator \([1]\). Due to Schoen’s proof \([19]\) of the Yamabe conjecture it is known that within any conformal class on a compact manifold is a metric whose scalar curvature is constant and the sign of this constant is a conformal invariant. The main objective of this paper is to ensure that the scalar curvature is positive for the self-dual structures on the connected sums \(n\mathbb{CP}^2\) of complex projective planes found in \([13]\). The metrics admit a torus \(T^2\) of orientation preserving conformal isometries.

For self-dual metrics the total space \(Z\) of the bundle of anti-self-dual 2-forms of unit length is a complex 3-manifold. This complex manifold is the twistor space \([1]\) and it gives an alternative description of self-duality. Indeed, Donaldson and Friedman \([2]\) used a desingularisation of a singular model of the desired twistor space to prove existence of self-dual structures on \(n\mathbb{CP}^2\). The self-dual metric on \(\mathbb{CP}^2\) is the Fubini-Study metric and the full moduli on \(2\mathbb{CP}^2\) had previously been obtained \([15]\) via a different twistor construction. In \([13]\) we adapted the theory of Donaldson and Friedman to obtain equivariant connected sums of compact self-dual manifolds.

If the symmetry group is at least three-dimensional it is known \([16]\) that the conformal metric is of non-negative type. In contrast Kim \([7]\) obtained \(S^1\)-symmetric examples of negative scalar curvature while LeBrun \([11]\) gave examples on \(n\mathbb{CP}^2\) of positive scalar curvature and with symmetry group \(S^1\). These metrics were obtained via an ansatz involving monopoles on hyperbolic 3-space. Similarly, Joyce \([6]\) obtained \(T^2\)-symmetric metrics on \(n\mathbb{CP}^2\) of positive type using hyperbolic monopoles in two dimensions. These constructions give relatively easy access to knowledge about scalar curvature

145
while such insight is absent from our equivariant smoothings [13]. However, it is known [3] that if the complex manifold \( Z \) carries effective divisors, then the corresponding metric is of non-negative type and if furthermore the intersection form of the 4-manifold is positive definite the scalar curvature is in fact positive [10].

Kim and Pontecorvo [8] extended the work of Donaldson and Friedman obtaining a way of constructing scalar-flat Kähler surfaces based on relative complex deformations of singular twistor spaces with divisors of degree 1. We combine the equivariant and the relative smoothing programme to obtain a local moduli space of dimension \( n - 1 \) of \( T^2 \)-symmetric self-dual structures on \( n\mathbb{CP}^2 \) such that the associated twistor spaces all have smooth degree 1 and 2 divisors. In particular, all these self-dual metrics have positive scalar curvature.

We also use our equivariant relative smoothing approach to obtain anti-self-dual Hermitian metrics with non-semi-free circle action on the blow up \((S^1 \times S^3)\# n\mathbb{CP}^2\), for \( n \geq 3 \), of the Hopf surface in more than two points. This should be compared with LeBrun’s examples on the same spaces [12] but with semi-free circle symmetry.

2. The Geometrical Construction.

Our study of self-duality on a 4-manifold \( M \) takes place on the associated twistor space \( Z \). As a smooth manifold \( Z \) is the total space of the sphere bundle \( S(\Lambda^2_\mathbb{C}) \to M \) of the bundle of anti-self-dual 2-forms. Equivalently, \( Z \) is the total space of the bundle \( P(V^-) \to M \) of projectivised half spinors. The twistor space is a complex 3-manifold and for any point \( x \in M \) the twistor line in \( Z \), \( \pi^{-1}(x) = L_x \), is a rational curve with normal bundle \( N \) the sum \( \mathcal{O}(1) \oplus \mathcal{O}(1) \) of degree 1 line bundles. Invariants \( N = H^- \otimes V_x^+ \) where \( H^- \to P(V_x^-) \) is the hyperplane bundle. The antipodal map on the fibers of \( \pi : Z \to M \) induces a real structure on \( Z \) [1]. The anti-canonical bundle \( K^{-1} = \Lambda^{3,0}TZ \) of \( Z \) has a square root \( K^{-\frac{1}{2}} \) which, when restricted onto a twistor line, coincides with the degree 2 holomorphic line bundle \( \mathcal{O}(2) \). The zero set in the twistor space of a real holomorphic section of \( K^{-\frac{1}{2}} \) is called a degree 2 divisor. Such a divisor \( S \) may either be irreducible, in which case it is smooth [14], or it may decompose into a conjugate pair \( D, D \) of smooth degree 1 divisors meeting in exactly one twistor line [17].

Now, assume we have two self-dual manifolds \( M_1 \) and \( M_2 \) with associated twistor spaces \( Z_1 \) and \( Z_2 \). Furthermore, assume \( Z_1 \) contains an irreducible degree 2 divisor \( S \) and let \( L_1 = L_{x_1} \) be a twistor line above \( x_1 \in M_1 \) meeting \( S \) transversely in a conjugate pair of points \( q \) and \( \bar{q} \). Also, let \( D_2, \bar{D}_2 \) be a conjugate pair of divisors in \( Z_2 \) meeting in the twistor line \( L_2 = L_{x_2} \) above \( x_2 \in M_2 \). Then, following Donaldson and Friedman [2], we blow up the twistor spaces \( Z_i \) along the twistor lines \( L_i, i = 1, 2 \). The proper transform
$\tilde{S}$ of $S$ in the blow-up $\tilde{Z}_1$ of $Z_1$ meets the exceptional divisor $Q_1$ in two rational curves $C_1$ and $\tilde{C}_1$. The normal bundles $N^S_{\tilde{C}_1}, N^S_C$ relative to $\tilde{S}$ are both equal to $\mathcal{O}(-1)$. In the blow-up $\tilde{Z}_2$ of $Z_2$ the proper transforms $D_2, \tilde{D}_2$, meet the exceptional divisor $Q_2$ in a conjugate pair of rational curves $C_2, \tilde{C}_2$ with normal bundles $N^D_{C_2} = \mathcal{O}(1)$ and $N^D_{\tilde{C}_2} = \mathcal{O}(1)$.

From an orientation reversing isometry $\phi : T_{x_1}M_1 \to T_{x_2}M_2$ we obtain a holomorphic isomorphism

$$\phi = \phi_+ \times \phi_- : Q_1 = P(V^+_x) \times P(V^-_x) \to Q_2 = P(V^+_x) \times P(V^-_x)$$

and we construct the singular complex space $Z = Z_1 \cup_{\phi} \tilde{Z}_2$ with normal crossing singularity along $Q = Q_1 = \phi^{-1}(Q_2)$. In $Z$ we have the singular divisor $S$ with normal crossing singularities along $C = C_1 = \phi^{-1}(C_2)$ and $\tilde{C} = \tilde{C}_1 = \phi^{-1}(\tilde{C}_2)$.

Inspired by Kim and Pontecorvo [8] we now proceed to study smoothings of the pair $(\mathcal{Z}, \mathcal{S})$. The aim is to show that in some cases the connected sum has a twistor space with a degree 2 divisor. Let us introduce the necessary notation: If $Y_1, \ldots, Y_n$ are smooth submanifolds of a compact complex manifold $X$ we consider the sheaf $\Theta_{XY_1, \ldots, Y_n}$ of holomorphic vector fields on $X$ which are tangent to $Y_i$ along $Y_i$, $i = 1, \ldots, n$. It is well known [9] that infinitesimal relative deformations are given by the first cohomology of this sheaf and that obstructions lie in the second cohomology. For a compact singular complex space $\mathcal{X}$ the deformation theory is described in terms of global extension groups $T^\mathcal{X}_\tau = \text{Ext}^1(\Omega_{\mathcal{X}}, \mathcal{O}_\mathcal{X})$ where $\Omega_{\mathcal{X}}$ is the sheaf of Kähler differentials and $\mathcal{O}_\mathcal{X}$ is the structure sheaf. These groups are computed from the sheaves $\tau^\mathcal{X}_\tau = \mathcal{E}xt^1(\Omega_{\mathcal{X}}, \mathcal{O}_\mathcal{X})$ using the local to global spectral sequence $E^{p,q}_2 = H^p(\tau^\mathcal{X}_q) \Rightarrow T^{p+q}_{\mathcal{X}}$ [2], [4]. Here $T^0_{\mathcal{X}}$ is the Lie algebra of the group of automorphisms of $\mathcal{X}$, the first order deformations lie in $T^1_{\mathcal{X}}$ and obstructions are in $T^2_{\mathcal{X}}$. For normal crossings $\tau^\mathcal{X}_\tau = 0$ and the local to global spectral sequence is given as

$$0 \to H^1(\tau^0_{\mathcal{X}}) \to T^1_{\mathcal{X}} \to H^0(\tau^0_{\mathcal{X}}) \to H^2(\tau^0_{\mathcal{X}}) \to T^2_{\mathcal{X}} \to H^1(\tau^1_{\mathcal{X}}).$$

In the situations we are going to investigate, we have $H^1(\tau^0_{\mathcal{X}}) = 0$. Furthermore, we impose conditions implying the vanishing of $H^2(\tau^0_{\mathcal{X}})$ so that $T^2_{\mathcal{X}} = 0$ and the deformations are unobstructed and parametrised by $T^1_{\mathcal{X}}$. The subspace $H^1(\tau^0_{\mathcal{X}})$ corresponds to deformations for which the singularities remain locally a product. If the image of an element in the projection $T^1_{\mathcal{X}} \to H^0(\tau^1_{\mathcal{X}})$ does not vanish the corresponding deformed space is smooth. Furthermore, for $\mathcal{X}$ equal to the singular twistor space $\mathcal{Z}$ the smoothing results in a twistor space with real structure [2].

The singular theory above will be applied to $\mathcal{X} = \mathcal{Z}$ and $\mathcal{X} = \mathcal{S}$. To study deformations of the singular pair $f : \mathcal{S} \hookrightarrow \mathcal{Z}$ we employ the theory of Ran [18]: Let $T^{\mathcal{Z}|\mathcal{S}}$ denote the extension groups $\text{Ext}^1(\Omega_{\mathcal{Z}|\mathcal{S}}, \mathcal{O}_\mathcal{S})$ [8] with the local
to global spectral sequence $E_{2}^{p,q} = H^{p}(\mathcal{Ext}^{q}(\Omega_{\mathcal{Z}}|\mathcal{S}), \mathcal{O}_{\mathcal{S}})$ where $E_{2}^{p,q} \Rightarrow T_{2}^{p+q}$. This gives the sequence

$$(2.2) \quad 0 \to H^{1}(\tau_{\mathcal{Z}|\mathcal{S}}) \to T_{1}^{1}|\mathcal{S} \to H^{0}(\tau_{\mathcal{Z}|\mathcal{S}}) \to H^{2}(\tau_{\mathcal{Z}|\mathcal{S}}).$$

Furthermore, the groups $T_{i}^{i}|\mathcal{S}$ fit into a sequence

$$(2.3) \quad \cdots \to T_{1}^{i} \to T_{1}^{i} \oplus T_{1}^{i} \to T_{2}^{i}|\mathcal{S} \to T_{2}^{i} \oplus T_{2}^{i} \to \cdots$$

where $T_{i}^{i}$ is a natural derived functor and $T_{1}^{i}$ gives infinitesimal deformations of the singular pair. Also, $T_{0}^{i}$ is the Lie algebra of the symmetries of the pair $(\mathcal{Z}, \mathcal{S})$ and $T_{2}^{i}$ is the obstruction space. For pairs $(\mathcal{X}, \mathcal{Y})$ of complex spaces we also have the sheaf $\tau_{\mathcal{X}|\mathcal{Y}}^{i}$ of derivations of $\mathcal{X}$ preserving the ideal sheaf of $\mathcal{Y}$. This sheaf of relative derivations coincide with $\Theta_{\mathcal{X}|\mathcal{Y}}$ for pairs of smooth manifolds. From inclusions and restrictions we obtain the exact sequence

$$(2.4) \quad 0 \to \tau_{\mathcal{X}|\mathcal{Y}}^{0} \to \tau_{\mathcal{Y}}^{0} \oplus \tau_{\mathcal{X}}^{0} \to \tau_{\mathcal{X}|\mathcal{Y}}^{0} \to 0.$$
Q \subseteq \mathcal{Z} \text{ and is the sheaf of sections of } N\tilde{Z}_1 \otimes \phi^* \tilde{N}\tilde{Z}_2 = \mathcal{O}_Q \ [2]. \text{ Likewise, a choice of trivialization } \tau^1_2 \cong \mathcal{O}_Q \text{ induces trivializations } \tau^1_3 \cong \mathcal{O}_C \text{ and } \tau^1_{Z|S} \cong \mathcal{O}_{Q|S} \cong \mathcal{O}_C. \text{ Then } H^0(\tau^1_S) = H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_C), \text{ } H^0(\tau^1_1) = H^0(\mathcal{O}_Q), \text{ } H^0(\tau^1_{Z|S}) = H^0(\mathcal{O}_C) \oplus H^0(\mathcal{O}_C) \text{ and } \beta((a,b),c) = (a-c, b-c) \text{ which certainly is surjective. The kernel of } \beta \text{ is } \langle c,c,c \rangle \cong \mathbb{C}. \text{ Now we shall make the following assumptions which are satisfied in examples to be considered later:}

**Assumption 2.1.** In the case where the degree 2 divisor is irreducible we assume the vanishing of the following cohomology groups: \( H^2(Z_i, \Theta_{Z_i}) \), \( i = 1, 2; \) \( H^2(S, \Theta_S) \); \( H^2(D_2, \Theta_{D_2}) \); \( H^2(D_1, \Theta_{D_1}) \); \( H^2(Z_1, \Theta_{Z_1}) \) and \( H^2(Z_2, \Theta_{Z_2D_2D_1}) \).

We shall show how these assumptions lead to the vanishing of \( H^2(\tau^0_S) \), \( H^2(\tau^0_1) \) and \( H^2(\tau^0_{Z|S}) \) and therefore from (2.1) we have the vanishing of the obstruction spaces \( T^2_\mathcal{Z} \) and \( T^2_S \). We also see that \( \Delta \) is equal to the kernel of \( \beta \). Now a diagram chasing gives the surjectivity of the morphism \( \gamma \) so \( T^1 = 0 \). Therefore, any element \( \omega \) in the complement of the hyperplane \( H^1(\tau^0_S) \) of \( T^1_f \) gives smoothings of \( \mathcal{Z} \) and \( \mathcal{S} \) as \( \omega \) maps to non-zero elements in \( H^0(\tau^1_S) \) and \( H^0(\tau^1_1) \). Thus the twistor space of the connected sum \( M_1 \# M_2 \) has a degree 2 divisor. Later we return to a calculation of the dimension of the space \( T^1 \) of infinitesimal deformations.

We shall also consider the situation where the degree 2 divisor \( S \) decomposes into a conjugate pair \( D_1, \bar{D}_1 \) of degree 1 divisors meeting in a twistor line \( L \). We still blow up \( Z_1 \) along a twistor line \( L_1 \) intersecting \( D_1 + \bar{D}_1 \) transversely in a conjugate pair of points. Also the proper transforms \( \tilde{D}_1, \bar{D}_1 \) of \( D_1 + \bar{D}_1 \) meet \( Q_1 \) in \( C_1 \) and \( \bar{C}_1 \). As before we construct the singular twistor space \( \mathcal{Z} \) with the singular divisors \( \mathcal{D}, \bar{\mathcal{D}} \) having normal crossing singularities along \( C \) and \( \bar{C} \) respectively. The divisors \( \mathcal{D} \) and \( \bar{\mathcal{D}} \) meet in \( L \). In this situation \( \mathcal{S} \) is the singular space \( \mathcal{D} \cup_L \bar{\mathcal{D}} \) with normal crossing singularities along \( L, C \) and \( \bar{C} \). We make the

**Assumption 2.2.** In the case where the degree 2 divisor is reducible we assume the vanishing of the following cohomology groups: \( H^2(Z_i, \Theta_{Z_i}) \), \( H^2(D_i, \Theta_{D_i}) \), \( H^2(D_i, \Theta_{D_i}) \), \( i = 1, 2; \) \( H^2(Z_1, \Theta_{Z_1}) \) and \( H^2(Z_2, \Theta_{Z_2D_2D_1}) \).

Furthermore, we get a diagram similar to (2.7) and as in the case where \( S \) is irreducible the smoothing is unobstructed. However, we do not control the singularities of \( \mathcal{S} \) along \( L \). Therefore, a smoothing of \( \mathcal{Z} \) may produce an irreducible smooth degree 2 divisor. We shall discuss this problem at the end of Section 5.
3. Symmetries.

In this section we assume $M_1$ and $M_2$ have a torus group $T^2$ of orientation preserving conformal transformations with fixed points $x_1$ and $x_2$ respectively.

Furthermore, the orientation reversing isometry $\phi : T_{x_1}M_1 \to T_{x_2}M_2$, used to make the connected sum, is assumed to be an equivariant map. In this situation we proved in [13] that if the smoothing is unobstructed then also the equivariant smoothing is unobstructed. We refer to that paper for details. Now we want to impose the symmetries onto the relative smoothing described in Section 2.

Recall [13] that we may choose metrics in $M_1$ and $M_2$ for which the torus acts as isometries. Therefore, the isotropy representation $\iota$ at the fixed points map into $SO(4)$. Also, the symmetries lift to the twistor space to give real holomorphic automorphisms. If $x_1$ is a fixed point in $M_1$, and $L_1$ is the twistor line with normal bundle $N$, then the well known isomorphisms from twistor theory

$$ (T_{x_1} M_1)^C = H^0(L_1, N) = V_{x_1}^+ \otimes V_{x_1}^- $$

are equivariant. This means the isotropy representation on the complexified tangent space coincide with the induced holomorphic action on the sections of the normal bundle and with the tensor product $\iota_+ \otimes \iota_-$. Here $\iota_{\pm}$ are defined modulo $Z_2$ from the isotropy representation $\iota$ and the projections onto the factors of $SU(2) \times SU(2)$.

Since the twistor line $L_1$ intersects the divisor $S$ transversely at two distinct points, we assume that $S$ is reducible to $D_1 + D_1$ when we consider the twistor geometry in a neighbourhood of $L_1$. Now suppose the divisors $D_i, \bar{D}_i$ in $Z_i, i = 1, 2$ are invariant. Then the points $q = P(V_{x_1}^-) \cap D_1$ and $\bar{q} = P(V_{x_1}^-) \cap \bar{D}_1$ are fixed by the action on $Z_1$. Therefore if $v \in V_{x_1}^-$ represents $q$ we have $V_{x_1}^- = \text{span}(v) \oplus \text{span}(\bar{v})$ as a decomposition into $T^2$-invariant subspaces. Furthermore, from the definition of the complex structure on $Z_1$ the horizontal space $T_q D_1$ is equal to $V_{x_1}^+ \otimes \text{span}(v)^*$ with the action $\iota_+ \otimes \iota_-^*$.

We have $C_1 = P(T_q D_1)$ and the normal bundle $N_{C_1}^{\bar{D}_1}$ is the universal subbundle of $T_q D_1$ over $C_1$.

On the other side we have $L_2 = P(V_{x_2}^-)$ and the normal bundles $N_{C_2}^{\bar{D}_2} = N_{L_2}^{\bar{D}_2}, N_{\bar{C}_2}^{\bar{D}_2} = N_{L_2}^{\bar{D}_2}$. As $L_2$ is equal to the transversal intersection $D_2 \cap \bar{D}_2$ we have $N_{L_2}^{Z_2} = N_{L_2}^{D_2} \oplus N_{L_2}^{\bar{D}_2}$ and this is a decomposition into invariant subbundles. From twistor theory we have $N_{L_2}^{Z_2} = V_{x_2}^+ \otimes H^-$. Also, the torus action on $V_{x_2}^+$ decomposes, $V_{x_2}^+ = E \oplus \bar{E}$, and the only decomposition of $V_{x_2}^+ \otimes H^-$ into invariant subspaces is $(E \otimes H^-) \oplus (\bar{E} \otimes H^-)$. We may therefore assume that $N_{L_2}^{D_2} = E \otimes H^-$ and $N_{L_2}^{\bar{D}_2} = \bar{E} \otimes H^-$. 
The $T^2$-equivariant isomorphism $\phi_- : V_{x_1}^- \to V_{x_2}^+$ induced by $\phi : T_{x_1}M_1 \to T_{x_2}M_2$ can be assumed to satisfy $\phi_-(\text{span}(v)) = E$ and $\phi_-(\text{span}(\bar{v})) = \bar{E}$. Now we have $D = \hat{D}_1 \cup_{\phi_+} \hat{D}_2$ where

$$\phi_+ : C_1 = P(T_qD_1) = P(V_{x_1}^+) \to P(V_{x_2}^-) = C_2$$

and similarly with $\check{D}$. Recall the fact [13] that $H^0(\tau_2^1) = H^0(\tau_2^1) \cong \mathbb{C}$ which is true because $\tau_2^1 = N_{\hat{D}_1}^{\hat{D}_2} \otimes \phi^*(\mathcal{N}_{\hat{D}_2}) = \mathcal{O}_Q$ and the action is trivial. Also, $H^0(\tau_2^1|S) = H^0(\tau_2^1|S) \cong \mathbb{C}^2$ because $\tau_2^1$ is supported by $Q$ and $Q$ intersects $S$ transversely along $C$ and $C$.

Likewise we shall need the following:

**Lemma 3.1.** The sheaf $\tau_D^1$ is supported on the curve $C$ and is trivial. All the sections are $T^2$-invariant, i.e. $H^0(\tau_D^1) = H^0(\tau_D^1) = \mathbb{C}$. Similarly $H^0(\tau_D^1) = H^0(\tau_D^1) = \mathbb{C}^2$ when the degree 2 divisor $S$ is irreducible.

*Proof.* Since the singularity is a normal crossing we have $\tau_D^1 = N_{C_1}^D \otimes \phi^*N_{C_2}^D$ [2]. The curve $C_1$ is the exceptional divisor of the blowing up of $q$ in $D_1$ so $N_{C_1}^D = \mathcal{O}(-1)$. Also, $N_{C_2}^D$ is isomorphic to $N_{L_2}^D$ so $N_{C_2}^D = \mathcal{O}(1)$. Thus $\tau_D^1 = \mathcal{O}_C$. Furthermore, the torus action is trivial on $\tau_D^1$: the action on $N_{C_1}^D$ is $\iota_+ \otimes \iota_-$ as this normal bundle is the universal subbundle of $T_qD_1 = V_{x_1}^+ \otimes (\text{span}(v))^*$. On the other hand $\phi^*N_{C_2}^D = \phi^*N_{L_2}^D = \phi^*(E \otimes H^-) = (\phi^*E) \otimes (\phi^*H^-) = \text{span}(v) \otimes H^+ = ((\text{span}(v))^* \otimes U^+)^* = (N_{C_1}^D)^*$. The torus action on $N_{L_2}^D = E \otimes H^-$ is $\iota_+ \otimes \iota_-$. Therefore the action on $\phi^*N_{L_2}^D$ is $\iota_- \otimes \iota_+$ and we get the trivial action on $\tau_D^1 = N_{C_1}^D \otimes \phi^*N_{C_2}^D$ from the tensor product $\iota_+ \otimes \iota_- \otimes \iota_- \otimes \iota_+$. \hfill \Box

**4. Vanishing Theorems and the Unobstructed Case.**

We consider an irreducible degree 2 divisor $S$ and address the problem of proving the vanishing of $H^2(\tau_0^1)$, $H^2(\tau_0^1)$, and $H^2(\tau_0^1|S)$. This will lead to a theorem on the unobstructed $T^2$-equivariant relative smoothing.

First some technical lemmata:

**Lemma 4.1.** Let $Y \subseteq X$ be a smooth hypersurface in a complex manifold $X$. Then for any sufficiently small open set $U \subseteq X$ we have $H^i(U, \Theta_X) = 0$ and $H^i(U, \Theta_{XY}) = 0$, $i \geq 1$.

*Proof.* We have $H^i(U, \Theta_X) = 0$, $i \geq 1$, as $\Theta_X$ is locally free and $H^i(U, \mathcal{O}) = 0$ by the $\beta$-Poincaré Lemma. Similarly $H^i(U \cap Y, \mathcal{O}_Y(N_X^Y)) = 0$, $i \geq 1$.

From the sequence

$$0 \to \Theta_{XY} \to \Theta_X \to \mathcal{O}_Y(N_X^Y) \to 0$$
we get $H^i(U, \Theta_{XY}) = 0$, $i \geq 2$. We may assume $U$ is a coordinate patch where the vanishing $x_1 = 0$ of the first coordinate defines $Y$. Therefore

$$0 \to H^0(U, \Theta_{XY}) \to H^0(U, \Theta_X) \to H^0(U \cap Y, \mathcal{O}_Y(N_{Y/Z}^2)) \to 0$$

is exact and it follows that $H^1(U, \Theta_{XY}) = 0$. \hfill \Box

**Lemma 4.2.** Let $b_i : \tilde{Z}_i \to Z_i$ be the blow-up of $Z_i$ along the real twistor line $L_i$, $i = 1, 2$. Then for all $j \geq 0$: $H^j(\tilde{Z}_i, \Theta_{\tilde{Z}_i}) = H^j(Z_i, \Theta_{Z_i L_i})$.

**Proof.** We drop the subscript “i”. The direct image sheaf $b_* \Theta_{\tilde{Z}}$ is isomorphic to $\Theta_{ZL}$ via the differential of $b$

$$db : H^0(b^{-1}(U), \Theta_{\tilde{Z}}) \to H^0(U, \Theta_{ZL})$$

for $U \subseteq Z$ open. Certainly $db$ maps into $H^0(U, \Theta_{ZL})$ if we can prove that any vector field on $b^{-1}(U)$ is tangential to $Q$ along $Q$. But this follows from the vanishing of $H^0(Q, [Q]) = H^0(\mathbb{CP}^1 \times \mathbb{CP}^1, \mathcal{O}(1, -1))$ and from the sequence

$$0 \to TQ \to T\tilde{Z}|_Q \to [Q]|_Q \to 0.$$

Conversely, let $L$ be given in local coordinates $(z_1, z_2, z_3)$ by $z_2 = z_3 = 0$. Then, a vector field $X = a_1 \frac{\partial}{\partial z_1} + (a_2 z_2 + a_3 z_3) \frac{\partial}{\partial z_2} + (a_4 z_2 + a_5 z_3) \frac{\partial}{\partial z_3}$ in $H^0(U, \Theta_{ZL})$ can be lifted to $\tilde{X} = a_1 \frac{\partial}{\partial z_1} + (a_2 z_2 + a_3 z_3) \frac{\partial}{\partial \ell_2} + (a_4 + a_5 \ell) \frac{\partial}{\partial \ell_3}$ in $H^0(b^{-1}(U), \Theta_{\tilde{Z}})$ where $\ell = \frac{1}{z_2}$ is a coordinate on $b^{-1}(U) = \{(z_1, z_2, z_3), [\ell_2, \ell_3] \in U \times \mathbb{CP}^1 | \ell_2 z_3 = \ell_3 z_2 \}$. The lemma now follows from the Leray spectral sequence [4] if we can prove that the sheaf $R^ib_* (\Theta_{\tilde{Z}})$ vanishes for all $i \geq 1$ where $R^ib_* (\Theta_{\tilde{Z}})(U) = H^i(b^{-1}(U), \Theta_{\tilde{Z}})$. Thus, consider the Leray cover $V_1 \cup V_2$ of $b^{-1}(U)$ given by coordinates $(z_1, z_2, \ell)$ and $(z_1, z_3, \frac{1}{\ell})$ respectively. Then $H^i(b^{-1}(U), \Theta_{\tilde{Z}}) = 0$, $i \geq 2$, and $H^1(b^{-1}(U), \Theta_{\tilde{Z}}) = H^1(V_1 \cap V_2, \Theta_{\tilde{Z}}) = H^1(\mathbb{CP}^3 \setminus \mathbb{CP}^2, \Theta_{\tilde{Z}}) = 0$ as $H^1(\mathbb{CP}^3 \setminus \mathbb{CP}^2, \mathcal{O}) = 0$ by the $\partial$-Poincaré lemma. \hfill \Box

Now, let us study the cohomology groups $H^j(\tau_{\tilde{Z}S}^0)$. Let $Z', S'$ be the normalizations of $Z, S$ respectively and let also $Q'$ denote the disjoint union of $Q_1$ and $Q_2$. Then we have the following sequence on $Z$

$$0 \to \tau_{\tilde{Z}S}^0 \to q_* \Theta_{Z'S'Q'} \to i_* \Theta_{QCC} \to 0 \tag{4.1}$$

where $q : Z' \to Z$ is the identification map and $i : Q \hookrightarrow Z$ is the inclusion.

**Lemma 4.3.** For all $j \geq 0$ we have:

$$H^j(Z, q_* \Theta_{Z'S'Q'}) = H^j(Z', \Theta_{Z'S'Q'}) = H^j(\tilde{Z}_1, \Theta_{Z_1 S_0 Q_1}) \oplus H^j(\tilde{Z}_2, \Theta_{Z_2 \tilde{D}_2 \tilde{D}_2 Q_2}).$$
Proof. This follows from the Leray Spectral sequence if we can prove the vanishing of $R^i \Theta Z_s Q$, $i \geq 1$, or equivalently we need the vanishing of $H^i(U_1, \Theta Z_s Q_1)$ and $H^i(U_2, \Theta Z_s Q_2)$, $i \geq 1$ for small open sets $U_k$, $k = 1, 2$. On $U_1$, where $U_1 \cap S \cap Q$ is non-empty, we have
\begin{equation}
0 \to \Theta Z_1 S \to \Theta Z S \to \mathcal{O}_S(N_{Z_1}) \to 0
\end{equation}
which is exact by the transversality of $Q_1$ and $S$: Choose coordinates $(x_1, x_2, x_3)$ such that $x_1 = 0$ defines $\tilde{S}$ and $x_2 = 0$ defines $Q_1$. Then on $U_1 \pi = r \circ dx_2 : \Theta Z S \to \mathcal{O}_Q(N_{Z_1})$ where $r$ is the restriction onto $Q_1$ and the exactness is now easily seen. Then, from Lemma 4.1 and the sequence (4.2) we get $H^i(U_1, \Theta Z_s Q_1) = 0$, $i \geq 2$ and as (4.2) is true on the level of presheaves we have also the vanishing for $i = 1$. The vanishing of $H^i(U_2, \Theta Z_s Q_2)$, $i \geq 1$, may be obtained as above by choosing $U_2$ so small that it meets only $D_2$ (or $\hat{D}_2$).

Lemma 4.4. For all $j \geq 0$, $H^j(\tilde{Z}_1, \Theta Z_1 S Q_1) = H^j(Z_1, \Theta Z_1 S L_1)$.

Proof. As in the proof of Lemma 4.2 we have $b_1 \Theta Z_1 S Q_1 = \Theta Z_1 S L_1$. The result follows from the Leray spectral sequence once we have established the vanishing of $R^i b_1 \Theta Z_1 S Q_1$ for $i \geq 1$. We consider the sequences (4.2) and
\begin{equation}
0 \to \Theta Z_1 S \to \Theta Z_1 S \to \mathcal{O}_S(N_{Z_1}) \to 0.
\end{equation}
Assume $\tilde{q} \notin U \subseteq Z_1$ and that the bundles are trivial on $U$. Then on $b_1^{-1}(U)$ we have $\mathcal{O}_{\tilde{S}}(N_{Z_1}) = [\tilde{S}]_{\tilde{S}} = b_1[S]_{\tilde{S}} \otimes Q_1^{-1} = b_1[S]_{\tilde{S}} \otimes C_1^{-1}$. Thus on $b_1^{-1}(U) \cap \tilde{S}$
\begin{equation}
0 \to \mathcal{O}_S(N_{Z_1}) \to b_1[S] \to \mathcal{O}_{C_1}(b_1[S]) \to 0
\end{equation}
is exact. Also, $H^j(b_1^{-1}(U) \cap C_1, b_1[S]) = H^j(q[S]) = 0$ for $j \geq 1$, $H^0(q[S]) = [S]^* = \mathbb{C}$ and $H^j(b_1^{-1}(U) \cap \tilde{S}, b_1[S]) = H^j(U \cap S, [S]) = 0$, for $j \geq 1$, by the $\bar{\partial}$-Poincaré Lemma. Thus, $H^j(b_1^{-1}(U) \cap \tilde{S}, \mathcal{O}_{\tilde{S}}(N_{Z_1})) = 0$, $j \geq 1$, if we can prove surjectivity of
\[ H^0(b_1^{-1}(U) \cap \tilde{S}, b_1[S]) \to H^0(b_1^{-1}(U) \cap C_1, b_1[S]). \]
This corresponds via $b_1$ to the map $H^0(U \cap S, [S]) \to H^0(q, [S])$ which is onto by evaluation. From the proof of Lemma 4.2 we also have $H^j(b_1^{-1}(U), \Theta Z_1) = 0$, $j \geq 1$. Thus, from (4.3), $H^j(b_1^{-1}(U), \Theta Z_1 S) = 0$, $j \geq 1$, once we have proved that
\[ H^0(b_1^{-1}(U), \Theta Z_1) \to H^0(b_1^{-1}(U) \cap \tilde{S}, \mathcal{O}_S(N_{Z_1})). \]
is onto. However, it is not hard to describe the vector fields on \( b_1^{-1}(U) \) in local coordinates on patches \( V_1, V_2 \) as in the proof of Lemma 4.2 and to see that the map is surjective.

Now, as \( H^j(b_1^{-1}(U), \mathcal{O}_{Q_1}(N_{Q_1}^Z)) = 0, j \geq 1 \), we get from (4.2) that \( H^j(b_1^{-1}(U), \Theta_{Z_1\mathcal{S}Q_1}) = 0, j \geq 2 \). The map

\[
H^0(b_1^{-1}(U), \Theta_{Z_1\mathcal{S}}) \rightarrow H^0(b_1^{-1}(U), \mathcal{O}_{Q_1}(N_{Q_1}^Z))
\]

is onto which again is seen using local coordinates on \( b_1^{-1}(U) \). Thus we have \( R^ib_*(\Theta_{Z_1\mathcal{S}Q_1}) = 0, i \geq 1 \) and the lemma is proved. \( \square \)

Similarly we have:

**Lemma 4.5.** For all \( j \geq 0 \), \( H^j(\tilde{Z}_2, \Theta_{Z_2D_2\tilde{D}_2Q_2}) = H^j(Z_2, \Theta_{Z_2D_2\tilde{D}_2}). \)

**Proof.** Again \( b_2\Theta_{Z_2D_2\tilde{D}_2Q_2} = \Theta_{Z_2D_2\tilde{D}_2} \) via the differential \( db_2 \). Furthermore the sheaf \( R^ib_2(\Theta_{Z_2D_2\tilde{D}_2Q_2}) \) vanishes for \( i \geq 1 \): First we prove the vanishing of \( H^j(b_2^{-1}(U_2), \Theta_{Z_2D_2\tilde{D}_2}) \), \( j \geq 1 \), \( U_2 \subseteq Z_2 \). Use coordinates \((z_1, z_2, \ell), (z_1, z_3, \ell)\) on a cover \( V_1 \cup V_2 \) of \( b_2^{-1}(U_2) \) such that \( D \) is given by \( z_2 = 0 \), \( \tilde{D} \) by \( z_3 = 0 \) and \( L_2 \) by \( z_2 = z_3 = 0 \). Lemma 4.1 gives that this is a Leray cover of \( b_2^{-1}(U_2) \) because \( \Theta_{Z_2D_2\tilde{D}_2}(V_2) = \Theta_{Z_2\tilde{D}_2}(V_2) \) and \( \tilde{D}_2 \) in \( V_2 \) is a smooth hypersurface. Also, \( \Theta_{Z_2\tilde{D}_2}(V_1 \cap V_2) = \Theta_{\tilde{Z}_2}(\mathbb{C}^3 \setminus \mathbb{C}^2) \). Therefore \( H^j(b_2^{-1}(U_2), \Theta_{Z_2\tilde{D}_2}) \) vanishes, \( j \geq 1 \). Then consider the sequence

\[
0 \rightarrow \Theta_{Z_2\tilde{D}_2\tilde{D}_2Q_2} \rightarrow \Theta_{Z_2\tilde{D}_2} \rightarrow \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2}) \rightarrow 0.
\]

The vanishing of \( H^j(b_2^{-1}(U_2) \cap Q_2, \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2})) \), \( j \geq 1 \) is seen using the Leray cover \( V_1, V_2 \). Also, using coordinates on \( V_1, V_2 \) we can show that vector fields may be lifted so we have surjectivity of

\[
H^0(b_2^{-1}(U_2), \Theta_{Z_2\tilde{D}_2\tilde{D}_2}) \rightarrow H^0(b_2^{-1}(U_2), \mathcal{O}_{Q_2}(N_{Q_2}^{\tilde{Z}_2})).
\]

Then the short exact sequence gives \( H^j(b_2^{-1}(U_2), \Theta_{Z_2\tilde{D}_2\tilde{D}_2Q_2}) = 0, j \geq 1 \), and we have proved the lemma. \( \square \)

To make use of the sequence (4.1) we prove the following lemma.

**Lemma 4.6.** The cohomology groups \( H^j(Q, \Theta_{QCC}) \) vanishes for \( j \geq 1 \) and \( H^0(Q, \Theta_{QCC}) = su(V^+) \oplus u(1) \).

**Proof.** \( H^j(Q, \Theta_Q) = 0, j \geq 1 \) and \( H^0(Q, \Theta_Q) = H^0(P(V_x^+), \mathcal{O}(2)) \oplus H^0(P(V_x^-), \mathcal{O}(2)) = S^2(V_x^+) + S^2(V_x^-) = su(V_x^+) \oplus su(V_x^-) \). From the sequence

\[
0 \rightarrow \Theta_{QCC} \rightarrow \Theta_Q \rightarrow \mathcal{O}_{CC}(N_{CC}^Q) \rightarrow 0
\]

(4.4)
and the vanishing of $H^j(C, \mathcal{O}_C(N^Q_C))$, $j \geq 1$ we get $H^2(Q, \Theta_{QCC}) = 0$. Now, $H^0(Q, \Theta_Q)$ is generated by global $SU(2) \times SU(2)$ holomorphic transformations. Elements in $H^0(Q, \Theta_{QCC})$ are the vector fields generated by actions leaving $C$ and $\tilde{C}$ invariant, i.e. by rotations on $L_1$ leaving $q$ and $\tilde{q}$ fixed. Thus $H^0(Q, Q_{CC})$ is generated by $SU(V^+_x) \times U(1)$ where $U(1) \leq SU(V^-_)$. Also, the map

$$H^0(Q, \Theta_Q) \rightarrow H^0(C, \mathcal{O}_C(N^Q_C)) \otimes H^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(N^Q_{\tilde{C}}))$$

is surjective so $H^1(Q, \Theta_{QCC}) = 0$. □

We are now able to prove the vanishing of the relative obstruction group.

**Proposition 4.7.** Assume $H^2(Z_1, \Theta_{Z_1S}) = 0$ and $H^2(Z_2, \Theta_{Z_2D_2\tilde{D}_2}) = 0$. Then the obstruction group $H^2(\tau^0_{ZS})$ vanishes.

**Proof.** The long exact sequence associated to (4.1) together with Lemmae 4.3, 4.4, 4.5 and 4.6 gives

$$H^2(Z, \tau^0_{ZS}) = H^2(Z_1, \Theta_{Z_1S L_1}) \oplus H^2(Z_2, \Theta_{Z_2D_2\tilde{D}_2}).$$

The second summand vanishes due to the assumption. Consider the exact sequence

$$0 \rightarrow \Theta_{Z_1S L_1} \rightarrow \Theta_{Z_1S} \rightarrow \mathcal{O}_{L_1}(N^Z_{L_1}) \rightarrow 0.$$ (4.5)

We have $H^1(L_1, \mathcal{O}_{L_1}(N^Z_{L_1})) = 0$ and also $H^2(Z_1, \Theta_{Z_1S}) = 0$ so the first summand also vanishes. □

As explained in Section 2 we also need the following vanishing results.

**Proposition 4.8.** Assume $H^2(Z_i, \Theta_{Z_i}) = 0$, $i = 1, 2$, $H^2(S, \Theta_S) = 0$ and $H^2(D_2, \Theta_{D_2}) = 0$. Then the obstruction groups $H^2(\tau^0_Z)$ and $H^2(\tau^0_S)$ vanishes.

**Proof.** Consider the normalization in the following sequence

$$0 \rightarrow \tau^0_Z \rightarrow q_* \Theta_{Z'Q'} \rightarrow i_* \Theta_Q \rightarrow 0.$$ (4.6)

Then, from Lemma 4.1 and the Leray spectral sequence, we get

$$H^2(Z, q_* \Theta_{Z'Q'}) = H^2(\tilde{Z}_1, \Theta_{\tilde{Z}_iQ_i}) \oplus H^2(\tilde{Z}_2, \Theta_{\tilde{Z}_2Q_2}).$$

From the sequence

$$0 \rightarrow \Theta_{\tilde{Z}_iQ_i} \rightarrow \Theta_{\tilde{Z}_i} \rightarrow \mathcal{O}_{Q_i}(N^Z_{Q_i}) \rightarrow 0$$

and the vanishing of $H^j(Q_i, \mathcal{O}_{Q_i}(N^Z_{Q_i}))$, $j \geq 1$, we have $H^2(\tilde{Z}_i, \Theta_{\tilde{Z}_iQ_i}) = H^2(\tilde{Z}_i, \Theta_{\tilde{Z}_i})$ which is equal to $H^2(Z_i, \Theta_{Z_iL_i})$ by Lemma 4.2. Since $H^j(L_i, \mathcal{O}_{L_i}(N^Z_{L_i})) = 0$, $j \geq 1$, we have $H^2(Z_i, \Theta_{Z_iL_i}) = H^2(Z_i, \Theta_{Z_i})$, $i = 1, 2$, and since $H^j(Q_i, \Theta_{Q_i}) = 0$, $i = 1, 2$, $j \geq 1$, the long exact sequence
associated to (4.6) gives \( H^2(\mathcal{Z}_2^0) = H^2(Z_1, \Theta_{Z_1}) \oplus H^2(Z_2, \Theta_{Z_2}) \) which vanishes by assumptions. This was just a repetition of the result of Donaldson and Friedman [2].

Next we concentrate on the divisor. Again we have a normalization sequence

\[(4.7) \quad 0 \rightarrow \tau^0_S \rightarrow q_* \Theta_{S'}C' \rightarrow i_* \Theta_{C'} \rightarrow 0.\]

Lemma 4.1 together with the Leray spectral sequence give

\[H^2(S, q_* \Theta_{S'}C') = H^2(\tilde{S}, \Theta_{\tilde{S}C_1}) \oplus H^2(\tilde{D}_2, \Theta_{\tilde{D}_2C_2}) \oplus H^2(\tilde{D}_2, \Theta_{\tilde{D}_2C_2}).\]

Then, as \( H^j(C_1, \mathcal{O}_{C_1}(N_{\tilde{S}_1}^\tilde{S})) = H^j(\mathbb{P}^1, \mathcal{O}(−1)) = 0, j \geq 0 \), the sequence

\[0 \rightarrow \Theta_{\tilde{S}_C_1C_1} \rightarrow \Theta_S \rightarrow \mathcal{O}_{C_1}(N_{\tilde{S}_1}^\tilde{S}) \oplus \mathcal{O}_{C_1}(N_{\tilde{S}_1}^\tilde{S}) \rightarrow 0\]

gives \( H^j(\tilde{S}, \Theta_{\tilde{S}_C_1C_1}) = H^j(S, \Theta_{\tilde{S}}), j \geq 0 \). Essentially by repeating the arguments in Lemma 4.2 we get \( H^j(\tilde{S}, \Theta_{\tilde{S}}) = H^j(S, \Theta_{Sqq}), j \geq 0 \). For dimensional reasons we have \( H^j(q\tilde{q}, \mathcal{O}_{qq}(N_{\tilde{qq}}^\tilde{S})) = 0, j \geq 1 \), so \( H^2(S, \Theta_{Sqq}) = H^2(\tilde{S}, \Theta_{\tilde{S}}) \). For \( \tilde{D}_2 \) we have \( N_{\tilde{D}_2} \cong \mathcal{O}(1) \) so the sequence

\[0 \rightarrow \Theta_{\tilde{D}_2C_2} \rightarrow \Theta_{\tilde{D}_2} \rightarrow \mathcal{O}_{C_2}(N_{\tilde{D}_2}^\tilde{S}) \rightarrow 0\]

gives \( H^2(\tilde{D}_2, \Theta_{\tilde{D}_2C_2}) = H^2(\tilde{D}_2, \Theta_{\tilde{D}_2}). \) Then as \( \Theta_{C_2} = \mathcal{O}(2) \) the long exact sequence associated to (4.7) gives

\[H^2(\tau_S^0) = H^2(S, \Theta_S) \oplus H^2(D_2, \Theta_{D_2}) \oplus H^2(\tilde{D}_2, \Theta_{\tilde{D}_2})\]

which vanishes by assumptions.

This ends our proof of the fact that under the general Assumption 2.1, the twistor space of the connected sum \( M_1 \# M_2 \) has an irreducible degree 2 divisor.

Finally we want to summarize and at the same time bring the symmetries back into considerations.

**Theorem 4.9.** Let \( M_1, M_2 \) be compact self-dual conformal 4-manifolds with torus symmetry and fixed points. Assume the isotropy representations at the fixed points are intertwined via an orientation reversing isometry. If the Assumption 2.1 is satisfied for \( T^2 \)-invariant divisors \( S, D_2 \) and \( \tilde{D}_2 \), then there is a complex equivariant smoothing of the singular twistor space \( Z \) and the singular divisor \( S \) into a twistor space with torus action and an invariant irreducible degree 2 divisor.

**Proof.** Due to the equivariance of the various sheaf morphisms we get an analogue of diagram (2.7) where the morphisms are between the \( G \)-invariant part of the groups. Consider the vanishing results in Propositions 4.7 and 4.8. The spectral sequence (2.1) now gives the vanishing of the obstruction spaces \( T_Z^2 \) and \( T_S^2 \). Then, chasing the diagram (2.7) gives surjectivity of the
Remark. We can also prove $H^2(\tau^0_{Z|S}) = 0$ if we assume that each component $M_i$, $i = 1, 2$, has positive scalar curvature. As this vanishing result is not used in this paper, we only give a short outline of the proof: Start with the normalization sequence

$$(4.8) \quad 0 \rightarrow \tau^0_{Z|S} \rightarrow q_*\Theta_{Z'|Q'|S} \rightarrow i_*\Theta_{Q'|CC} \rightarrow 0.$$ 

Then, as $(q_*\Theta_{Z'|Q'})|_S = q_*(\Theta_{Z'|Q'|S})$, we get

$$H^j\left(S, q_*\Theta_{Z'|Q'}|_S \right)$$

$$= H^j\left(\tilde{S}, \Theta_{\tilde{Z}_iQ_i|\tilde{S}} \right) \oplus H^j\left(\tilde{D}_2, \Theta_{\tilde{Z}_2Q_2|\tilde{D}_2} \right) \oplus H^j\left(\tilde{D}_2, \Theta_{\tilde{Z}_2Q_2|\tilde{D}_2} \right), \quad j \geq 0.$$ 

We study each component in this sum. From the sequence

$$(4.9) \quad 0 \rightarrow \Theta_{\tilde{Z}_iQ_i|\tilde{S}} \rightarrow \Theta_{\tilde{Z}_i|\tilde{S}} \rightarrow \mathcal{O}_{C_1C_1} \left(N_{\tilde{Z}_i} \right) \rightarrow 0,$$

and the vanishing $H^j(\tilde{S}, \mathcal{O}_C, C_1, C_1 \left(N_{\tilde{Z}_i} \right)) = 0, j \geq 0$, we get $H^j(\tilde{S}, \Theta_{\tilde{Z}_iQ_i|\tilde{S}}) = H^j(\tilde{S}, \Theta_{\tilde{Z}_i|\tilde{S}}), j \geq 0$. Consider therefore the sequence

$$(4.10) \quad 0 \rightarrow \Theta_{\tilde{S}} \rightarrow \Theta_{\tilde{Z}_i|\tilde{S}} \rightarrow \mathcal{O}_{\tilde{S}} \left(N_{\tilde{Z}_i} \right) \rightarrow 0.$$ 

As $N_{\tilde{Z}_i} = b_1[S]|_\tilde{S} \otimes [C_1]^{-1} \otimes [\tilde{C}_1]^{-1}$ and $\mathcal{O}_{C_1C_1} \left(b_1[S] \right) = \mathcal{O}_{C_1C_1}$ we have

$$(4.11) \quad 0 \rightarrow N_{\tilde{Z}_i} \rightarrow b_1[S]|_\tilde{S} \rightarrow \mathcal{O}_{C_1C_1} \rightarrow 0.$$ 

Also, $H^j(\tilde{S}, b_1[S]) = H^j(S, [S]) = H^j(S, K^{-\frac{1}{2}}), j \geq 0$. Then, due to the positive scalar curvature and the vanishing results of Hitchin [5], the sequence

$$0 \rightarrow \mathcal{O}_{Z_1} \rightarrow K^{-\frac{1}{2}} \rightarrow \mathcal{O}_S(K^{-\frac{1}{2}}) \rightarrow 0$$

gives $H^2(\tilde{S}, b_1[S]) = 0$. Then (4.11) gives $H^2(\tilde{S}, \mathcal{O}_S(N_{\tilde{Z}_i})) = 0$. From the sequence (4.10) and from $H^2(\tilde{S}, \Theta_{\tilde{S}}) = H^2(S, \Theta_{Sqq}) = H^2(S, \Theta_S) = 0$ we get $H^2(\tilde{S}, \Theta_{\tilde{Z}_i|\tilde{S}}) = 0$. Thus $H^2(\tilde{S}, \Theta_{\tilde{Z}_iQ_i|\tilde{S}}) = 0$. To prove the vanishing of $H^2(\tilde{D}_2, \Theta_{\tilde{Z}_2Q_2|\tilde{D}_2})$ we proceed as above with sequences similar to (4.9), (4.10), (4.11) and the use of vanishing results due to the positive scalar curvature of $M_2$. Thus we have $H^2(S, q_*\Theta_{Z'|Q'|S}) = 0$ and (4.8) then gives $H^2(S, \tau^0_{Z|S}) = 0$.
5. A Local Moduli Space of Self-Dual Metrics on $n\mathbb{CP}^2$.

We shall continue our study [13] of the local moduli of $T^2$-symmetric self-dual structures on the connected sums of $n$ copies of the complex projective plane. As in [13] it is assumed that the only orbits are tori on which $T^2$ acts freely, circles stabilized by some $S^1$-subgroup inside $T^2$ and isolated fixed points. We construct $n\mathbb{CP}^2$ from $\mathbb{CP}^2$ by attaching planes step by step and at the same time keeping divisors and symmetries. However, let us take the general approach a little further in the case of an irreducible degree 2 divisor.

From the equivariant version of (2.7) we get the sequence

$$0 \rightarrow H^1(\tau^0_{\mathbb{Z}S})_{T^2} \rightarrow (T^1_f)_{T^2} \rightarrow \Delta_{T^2} \rightarrow 0. \tag{5.1}$$

Similarly (4.1) and Lemmas 4.3, 4.4, 4.5 and 4.6 give the sequence

$$0 \rightarrow H^0(\tau^0_{\mathbb{Z}S})_{T^2} \rightarrow H^0(Z_1, \Theta_{Z_1SL_1})_{T^2} \oplus H^0(Z_2, \Theta_{Z_2D_2D_2})_{T^2} \rightarrow (su(V^+) \oplus u(1))_{T^2} \rightarrow H^1(\tau^0_{\mathbb{Z}S})_{T^2} \rightarrow H^1(Z_1, \Theta_{Z_1SL_1})_{T^2} \oplus H^1(Z_2, \Theta_{Z_2D_2D_2})_{T^2} \rightarrow 0. \tag{5.2}$$

Furthermore, from (4.5) we have the sequence

$$0 \rightarrow H^0(Z_1, \Theta_{Z_1SL_1})_{T^2} \rightarrow H^0(Z_1, \Theta_{Z_1S})_{T^2} \rightarrow H^0(L_1, \mathcal{O}(N^2))_{T^2} \rightarrow H^1(Z_1, \Theta_{Z_1SL_1})_{T^2} \rightarrow H^1(Z_1, \Theta_{Z_1S})_{T^2} \rightarrow 0. \tag{5.3}$$

Thus, putting (5.1), (5.2) and (5.3) together, we get a formula for the dimensions: Let $\chi_1(\mathcal{F}_X)_{T^2} = h^0(X, \mathcal{F}_X)_{T^2} - h^1(X, \mathcal{F}_X)_{T^2}$ for a sheaf $\mathcal{F}_X$ on a complex manifold $X$.

**Proposition 5.1.** Let $(T^1_f)_{T^2}$ denote the real vector space of $T^2$-invariant tangent vectors at $x_1$ and let $C(T^2)$ denote the Lie algebra of the centralizer of $T^2$ in $SO(3) \times SO(3)$. Then

$$\dim(T^1_f)_{T^2} = 1 + h^1(\tau^0_{\mathbb{Z}S})_{T^2} = 1 + h^0(\tau^0_{\mathbb{Z}S})_{T^2} + \dim_{\mathbb{R}}(T^1_f M_1)_{T^2} + \dim_{\mathbb{R}} C(T^2) - \chi_1(\Theta_{Z_1S})_{T^2} - \chi_1(\Theta_{Z_2D_2D_2})_{T^2}.$$

**Proof.** We have used the Kodaira equivalence (3.1) between sections of the normal bundle of the line $L_1$ in the twistor space and tangent vectors at the corresponding point $x_1$ in $M_1$. The symmetry group $T^2$ sits in $SO(3) \times SO(3)$ via the composition of the isotropy representation $i : T^2 \rightarrow SO(4)$ and the representation of $SO(4)$ on $\Lambda^2_+ \oplus \Lambda^2_\pm$. \hfill $\square$

Now we need a series of lemmata which will ensure that we have the general Assumption 2.1 satisfied and will make it possible to find the number $\dim(T^1_f)_{T^2}$ in the case of irreducible degree 2 divisors in the connected sums of complex projective planes.
Lemma 5.2. Let $Z_2$ be the twistor space of $\mathbb{CP}^2$. Then $H^2(Z_2, \Theta_{Z_2}) = 0$, $H^2(D_2, \Theta_{D_2}) = 0$, $H^2(D_2, \Theta_{\tilde{D}_2}) = 0$, $H^0(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) \cong (u(1) \oplus su(2))_C \subseteq su(3)_C$ and $H^1(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = H^2(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = 0$.

Proof. The space $H^0(Z_2, \Theta_{Z_2})$ is generated by the complexifications $SU(3)_C$ of the lifts of the isometries on $\mathbb{CP}^2$ to the flag

\[
Z_2 = \left\{ ([v_0, v_1, v_2], [\ell_0, \ell_1, \ell_2]) \mid \sum v_i \ell_i = 0 \right\}.
\]

Then $H^0(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2})$ consists of the vector fields generated by the group that leaves $D_2 \cup \tilde{D}_2$ invariant. For $D_2$ given by $v_0 = 0$ and $\tilde{D}_2$ by $\ell_0 = 0$ it is clear that $H^0(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2})$ must contain and therefore be equal to the maximal subalgebra $(u(1) \oplus su(2))_C$ of $su(3)_C$. Since $Z_2$ is a flag manifold and $D_2$ is the blow-up of $\mathbb{CP}^2$ at one point, $H^2(Z_2, \Theta_{Z_2})$ and $H^2(D_2, \Theta_{D_2})$ vanish.

Lemma 4.5 and the vanishing of $H^j(Q_2, \mathcal{O}_{Q_2}(N_{Z_2}^{\tilde{2}}))$, $j \geq 0$, give $H^j(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = H^j(Z_2, \Theta_{Z_2 \tilde{2} D_2 \tilde{2} Q_{\tilde{2}}}) = H^j(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}).$ Consider the sequence

\[
0 \to \Theta_{\tilde{Z}_2 D_2 \tilde{D}_2} \to \Theta_{Z_2 \tilde{2} D_2 \tilde{2} Q_{\tilde{2}}} \to 0.
\]

Here $H^j(Z_2, \mathcal{O}_{D_2 \tilde{D}_2} \Theta_{Z_2 D_2 \tilde{D}_2})$ is equal to the sum of $H^j(D_2, [D_2] \otimes [L_{D_2}]^{-1})$ and the conjugate part. This follows because $N_{\tilde{Z}_2 D_2} = b_2^2[D_2] \otimes [C_2]^{-1}$ and $b_2$ gives isomorphisms $\tilde{D}_2 \cong D_2, C_2 \cong L_2$. Identify $D_2$ to the blow-up of $\mathbb{CP}^2$ at one point with hyperplane class $H$ and exceptional divisor $E$, then $D_2|D_2 = H - E$ and $D_2|D_2 = H - E$. It follows that $H^j(D_2, [D_2] \otimes [L_2]^{-1})$ vanishes for $j \geq 0$ as $H^0(D_2, -E) = 0$, $H^0(D_2, -3H + 2E) = 0$ and $\chi(-E) = 0$ by the Riemann-Roch formula. The sequence (5.5) now shows that $H^j(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = H^j(Z_2, \Theta_{Z_2})$ which coincide with $H^j(Z_2, \Theta_{Z_2 L_2})$ as shown in Lemma 4.2. Tracing these identities we now have $H^j(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = H^j(Z_2, \Theta_{Z_2 L_2})$, $j \geq 0$. As $H^j(L_2, \mathcal{O}_{L_2}(N_{Z_2}^{\tilde{2}})) = 0$, $j \geq 1$, the sequence

\[
0 \to \Theta_{Z_2 L_2} \to \Theta_{Z_2} \to N_{L_2}^{\tilde{2}} \to 0
\]

gives $H^2(Z_2, \Theta_{Z_2 L_2}) = H^2(Z_2, \Theta_{Z_2})$ which vanishes for the flag. Thus, $H^2(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = 0$ as claimed. Also, as $D_2 \cap D_2 = L_2$, $H^0(Z_2, \Theta_{Z_2 L_2})$ contains $H^0(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = \left( u(1) \oplus su(2) \right)_C$, which is maximal, so $H^0(Z_2, \Theta_{Z_2 L_2})$ is 4-dimensional. Since $H^0(L_2, \mathcal{O}_{L_2}(N_{Z_2}^{\tilde{2}}))$ is 4-dimensional so (5.6) gives $H^1(Z_2, \Theta_{Z_2 L_2}) = H^1(Z_2, \Theta_{Z_2})$ which vanishes for the flag. Thus we get $H^1(Z_2, \Theta_{Z_2 D_2 \tilde{D}_2}) = 0$ and the lemma has been proved. □

Consider the twistor space (5.4) of $\mathbb{CP}^2$ with the action of $T^2$ given by

\[
([v_0 e^{i\Phi_1}, v_1 e^{i\Phi_2}, v_2 e^{-i(\Phi_1 + \Phi_2)}], [\ell_0 e^{-i\Phi_1}, \ell_1 e^{-i\Phi_2}, \ell_2 e^{i(\Phi_1 + \Phi_2)}]).
\]
where $\Phi_i = p_i\theta + q_i\psi, i = 1, 2$, $(\theta, \psi) \in T^2$ and the integers $p_i, q_i$ satisfy $p_1q_2 - p_2q_1 = \pm 1$. Then the irreducible degree 2 divisor $S$ given by $\sum \lambda_i v_i \ell_i = 0$, where $\lambda_i \neq \lambda_j, i \neq j$, is $T^2$-invariant. Indeed $T^2$ is exactly the symmetry group of $S$. The twistor fibration onto $\mathbb{CP}^2$ is $[z_0, z_1, z_2] = [v_1 \ell_2 - v_2 \ell_1, v_2 \ell_0 - v_0 \ell_2, v_0 \ell_1 - v_1 \ell_0]$, and it induces the action $[z_0 e^{-i\Phi_1}, z_1 e^{-i\Phi_2}, z_2 e^{i(\Phi_1 + \Phi_2)}]$ on $\mathbb{CP}^2$. Take $x = [0, 0, 1] \in \mathbb{CP}^2$. Then the twistor line $L_x$ is given by $v_2 = \ell_2 = 0$ with two fixed points $q = ([0, 1, 0], [1, 0, 0])$ and $\bar{q} = ([1, 0, 0], [0, 1, 0])$. A generic $T^2$-invariant degree 2 divisor is transversal to $L_x$ and contains $q$ and $\bar{q}$.

We get a reducible invariant divisor with $D_1$ given by $v_0 = 0$ and $D_1$ given by $\ell_0 = 0$. We have $L_x \cap D_1 = q$ and $L_x \cap D_1 = \bar{q}$. Finally take $D_2$ given by $v_2 = 0$ and $D_2$ corresponding to $\ell_2 = 0$. Then $L_x = D_2 \cap D_2$. Thus we have the following building blocks.

**Lemma 5.3.** The twistor space $Z$ of $\mathbb{CP}^2$ contains $T^2$-invariant divisors $S$, $D_i, D_i, i = 1, 2$ as in Section 2 with arbitrary isotropy representation at fixed points and satisfying $H^0(Z, \Theta_{ZS})_{T^2} = \mathbb{C}^2$, $H^2(S, \Theta_S) = 0$, $H^2(D_1, \Theta_{D_1}) = 0$, $H^2(D_2, \Theta_{D_2}) = 0$ and $H^2(Z, \Theta_{ZD_1D_2}) = 0$.

**Proof.** The vanishing of $H^2(S, \Theta_S)$ follows because $H^2(Z, \Theta_Z) = 0$ and because $S$ is contained in $Z$ with positive normal bundle. $H^0(Z, \Theta_{ZS}_{T^2})$ contains at least the algebra generated by $T^2$ as $S$ is invariant and for the generic $S$ described above where $\lambda_i \neq \lambda_j, i \neq j$, the symmetry group can at most be two dimensional. With Lemma 5.2 and the discussion above in mind, the lemma is proved.

Note that in order to fulfill all the vanishing conditions stated in Assumptions 2.1 and 2.2 we still need to prove $H^2(Z, \Theta_{ZS}) = 0$ for the flag manifold $Z$. This could be done, mutatis mutandis, as for $\Theta_{ZDD}$ in Lemma 5.2. However, as we focus on the $T^2$-equivariant situation we only prove the vanishing of the $T^2$-invariant part of this cohomology group. Indeed it follows from the next more general lemma.

**Lemma 5.4.** Let $Z_1$ be the twistor space of $n\mathbb{CP}^2$ and let $S \subseteq Z_1$ be an irreducible degree 2 divisor with canonical bundle $K_S$. Then $H^0(S, \mathcal{O}(K_S^{-1}))_{T^2} \cong \mathbb{C}$, $H^1(S, \mathcal{O}(K_S^{-1}))_{T^2} = 0$ and $H^2(Z_1, \Theta_{S})_{T^2} = 0$.

**Proof.** It is known [14] that $S$ is the blow-up $S_n$ of a real quadric $S_0$ $n$ times in a pair of conjugate points. Let $b_k : S_k \to S_{k-1}, 1 \leq k \leq n$, be the blow-down map from the blow-up of $S_0$ $k$ times to the blow-up $k - 1$ times.

The torus action has only isolated fixed points on $S_n$ so the action on $S_k, 1 \leq k \leq n$, induced by blowing down, also only has isolated fixed points. Therefore, the points $q_{k-1}, q_{k-1}$ of blowing-up from $S_{k-1}$ to $S_k$ is at the intersection of invariant divisors. For topological reasons the action on $S_0$ has four isolated fixed points.
Let $K_k = K_{S_k}$ and let $E_k, \tilde{E}_k$ be the exceptional divisor of the blowing-up $b_k$. Then $K_{k}^{-1} = b_k^* K_{(k-1)}^{-1} \otimes E_k^{-1} \otimes \tilde{E}_k^{-1}$ and

$$0 \to \mathcal{O}(K_{k}^{-1}) \to \mathcal{O}(b_k^* K_{(k-1)}^{-1}) \to \mathcal{O}_{E_k} \cup \tilde{E}_k (b_k^* K_{(k-1)}^{-1}) \to 0$$

is exact and induces

$$(5.7) \quad 0 \to H^0(S_k, \mathcal{O}(K_{k}^{-1}))_{T^2} \to H^0 \left(S_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}) \right)_{T^2}$$

$$\to H^0 \left(E_k \cup \tilde{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}) \right)_{T^2}$$

$$\to H^1(S_k, \mathcal{O}(K_{k}^{-1}))_{T^2} \to H^1 \left(S_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}) \right)_{T^2} \to 0.$$ 

Now, $H^0(E_k \cup \tilde{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}))$ is isomorphic to $K_{(k-1), q_{k-1}}^{-1} \oplus K_{(k-1), q_{k-1}}^{-1}$. Furthermore, the $T^2$-action on $H^0(E_k \cup \tilde{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}))$ is given as non-trivial rotations on each factor: Let $q_{k-1} \in S_{k-1}$ and let $A, B$ be two invariant divisors such that $q_{k-1} \in A \cap B$ and such that in local coordinates $(z_1, z_2)$, $A$ is given by $z_2 = 0$ and $B$ is given as $z_1 = 0$. As the $T^2$-action does not have 2-dimensional fixed point set but only isolated fixed points, we may assume the action on $T_{q_{k-1}} S_{q_{k-1}} = T_{q_{k-1}} \mathbb{P}^1 A \oplus T_{q_{k-1}} \mathbb{P}^1 B$ is given as diagonal $(e^{i\Phi_1}, e^{i\Phi_2})$. On $K_{(k-1), q_{k-1}}^{-1} = \Lambda^2 T_{q_{k-1}} S_{q_{k-1}}$ the action is $e^{i(\Phi_1 + \Phi_2)}$. This action will not be trivial due to the assumption $p_1 q_2 - p_2 q_1 = \pm 1$. On the blow-up the action is $(e^{i\Phi_1}, e^{i(\Phi_2 - \Phi_1)})$ near the intersection of the exceptional divisor and the proper transform $\tilde{A}$ and near $\tilde{B}$ it is $(e^{i\Phi_2}, e^{i(\Phi_1 - \Phi_2)})$.

Therefore the weights must be linear independent at all the points $q_{k-1}$.

Thus $H^0(E_k \cup \tilde{E}_k, \mathcal{O}(b_k^* K_{(k-1)}^{-1}))_{T^2}$ vanishes. By an inductive argument we get from $(5.7)$ that $H^j(S, \mathcal{O}(S^{-1}))_{T^2} = H^j(S_0, \mathcal{O}(K_0^{-1}))_{T^2}$, $j = 0, 1, 2$.

As $S_0$ is a quadric surface we have $H^2(S_0, \mathcal{O}(K_0^{-1})) = 0$. Choose coordinates $([s_0, s_1], [t_0, t_1])$ on $S_0 = \mathbb{C}P^1 \times \mathbb{C}P^1$ such that the action on $S_0$ is $([s_0, s_1 e^{i\Phi_1}], [t_0, t_1 e^{i\Phi_2}])$. The nine monomials $(s_0^2 t_0^2, \ldots, s_1^2 t_1^2)$ of order four give a basis of $H^0(S_0, \mathcal{O}(K_0^{-1})) = H^0(\mathbb{C}P^1 \times \mathbb{C}P^1, \mathcal{O}(2, 2))$. The weights of the induced $T^2$-action vanishes on $s_0^2 t_0^2$ and is of the form $n\Phi_1 + m\Phi_2$ on the eight other monomials. We claim that $s_0^2 t_0^2$ is the only invariant section: Note that if the weights $(\Phi_1, \Phi_2)$ of the isotropy representation at $([1, 0], [1, 0]) \in S_0$ are linearly dependent then the weights of the isotropy representations at the other three fixed points on $S_0$ are also linearly dependent. Since the weights at the fixed points of the $T^2$-action on the blow-up are linear combinations of $\Phi_1$ and $\Phi_2$ the weights on $S$ will be linearly dependent too. It follows, by restricting the twistor projection onto $S$, that on $n\mathbb{C}P^2$ there is a fixed point where $p_1 q_2 - p_2 q_1 = 0$ which is a contradiction. Therefore, $H^0(S, \mathcal{O}(S^{-1}))_{T^2} \cong \mathbb{C}$. Finally, as $[S] = K_S^{-1}$, the sequence

$$(5.8) \quad 0 \to \Theta_{Z, S} \to \Theta_{Z_1} \to \mathcal{O}_S(K_S^{-1}) \to 0,$$
gives $H^2(Z_1, \Theta_{Z_1S})_{T^2} = 0$. □

We are now able to find the number $\dim(T^1_f)_{T^2}$ in a concrete example.

**Proposition 5.5.** It is possible to construct a $T^2$-symmetric twistor space of $n\mathbb{CP}^2$ with invariant irreducible degree 2 divisors by equivariant relative smoothings using $\mathbb{CP}^2$ as building blocks and proceeding step by step. Furthermore, in the final step we have $\dim(T^1_f)_{T^2} = h^1(Z, \Theta_{ZS})_{T^2} = n$, where $Z$ is the smooth twistor space of $n\mathbb{CP}^2$.

**Proof.** The assumptions which give unobstructed equivariant relative smoothings of two planes are satisfied. This follows from Lemmae 5.2, 5.3 and 5.4. Using upper semi-continuity of the dimension of cohomology we may indeed proceed step by step to get $T^2$-symmetric twistor spaces of $n\mathbb{CP}^2$ with degree 2 irreducible divisors.

With the assumption, $p_1q_2 - p_2q_1 = \pm 1$, on the $T^2$-action we have $\dim(R(T^1_f)_{M^1})_{T^2} = 0$ and $\dim(R(C(T^2)) = 2$. Also, from Lemma 5.2, we get $\chi_1(\Theta_{Z_2D_2})_{T^2} = \dim(u(1) \oplus su(2))_{T^2} = 2$. Lemma 5.3 and upper semi-continuity gives $h^0(Z_1, \Theta_{Z_1S})_{T^2} = h^0(\tau_{ZS})_{T^2} = 2$ in each step. From the sequence (5.8) and Lemma 5.4 we get

$$0 \to H^0(S, O(K_{Z_1}^{-1}))_{T^2} \to H^1(Z_1, \Theta_{Z_1S})_{T^2} \to H^1(Z_1, \Theta_{Z_1})_{T^2} \to 0 (5.9)$$

and $h^0(S, O(K_{Z_1}^{-1}))_{T^2} = 1$. In [13] we got $h^1(Z_1, \Theta_{Z_1})_{T^2} = n - 2$ if $Z_1$ corresponds to $(n - 1)\mathbb{CP}^2$, so $h^1(Z_1, \Theta_{Z_1S})_{T^2} = n - 1$. Now, putting all these data into the formula in Proposition 5.1, we get $\dim(T^1_f)_{T^2} = n$ which by upper semi-continuity coincide with $h^1(Z, \Theta_{ZS})_{T^2}$ for the smooth twistor space $Z$ of $n\mathbb{CP}$ with smooth irreducible degree 2 divisor $S$. □

Thus, we may summarize and formulate the following result concerning the local moduli space of self-dual structures on $n\mathbb{CP}^2$ constructed in [13].

**Theorem 5.6.** Consider the local moduli space of $T^2$-symmetric self-dual structures on $n\mathbb{CP}^2$ obtained by equivariant smoothings. The self-dual structures are all associated to $T^2$-symmetric twistor spaces $Z$ with invariant irreducible degree 2 divisors. In particular, the scalar curvature of each Yamabe metric is positive. For $n \geq 3$, the $T^2$-action can be chosen such that all $S^1$ subgroups are non-semi-free. The dimension of the local moduli is equal to $h^1(Z, \Theta_{Z})_{T^2} = n - 1$.

**Proof.** Note that we have ensured that all twistor spaces have symmetry group of dimension 2 so a local moduli space is well-defined.

To prove this theorem, we need to prove that the deformation is target stable [18]. By taking the second components of the maps from the first
column to the second column in diagram (2.7), we have

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^1(\tau^0_{ZS})_{T^2} & \longrightarrow & (T^1_3)_{T^2} & \longrightarrow & \Delta_{T^2} & \longrightarrow & 0 \\
\downarrow \varepsilon & & \downarrow \delta & & \downarrow \hat{\alpha} & & \\
0 & \longrightarrow & H^1(\tau^0_{Z})_{T^2} & \longrightarrow & (T^1_3)_{T^2} & \longrightarrow & H^0(\tau^0_{Z})_{T^2} & \longrightarrow & 0
\end{array}
\]

The target stability is equivalent to \( \delta \) being surjective. Since \( \hat{\alpha} \) is the identity map, \( \delta \) is surjective if \( \varepsilon \) is surjective.

By Lemma 5.3 and upper semi-continuity, the exact sequence (5.2) gives the isomorphism

\[H^1(\tau^0_{ZS})_{T^2} \cong H^1(\tau^0_{Z})_{T^2}\]

Also (5.3) gives the isomorphism

\[H^1(\tau^0_{ZS})_{T^2} \cong H^1(Z_1, \Theta_{Z_1, S_{L_1}})_{T^2}\]

Since the two compositions

\[H^1(\tau^0_{ZS})_{T^2} \cong H^1(Z_1, \Theta_{Z_1, L_1})_{T^2} \quad \text{and} \quad H^1(\tau^0_{ZS})_{T^2} \cong H^1(Z_1, \Theta_{Z_1})_{T^2}\]

are induced by normalizations and restrictions, the inclusions induce a commutative diagram

\[
\begin{array}{ccc}
H^1(\tau^0_{ZS})_{T^2} & \overset{\varepsilon}{\longrightarrow} & H^1(\tau^0_{Z})_{T^2} \\
\downarrow \cong & & \downarrow \cong \\
H^1(Z_1, \Theta_{Z_1, S_{L_1}})_{T^2} & \overset{\lambda}{\longrightarrow} & H^1(Z_1, \Theta_{Z_1})_{T^2} \\
\end{array}
\]

By (5.9), \( \lambda \) is surjective. Therefore \( \varepsilon \) is surjective.

Thus, the local moduli space obtained in [13] does indeed correspond to twistor spaces carrying divisors as claimed. Therefore the corresponding Yamabe metrics are of non-negative type [3]. In fact the scalar curvature must be positive because the intersection form for \( n\mathbb{CP}^2 \) is positive definite [10]. The fact that we may assume the \( S^1 \) subgroups are all non-semi-free was proved in [13]. \( \square \)

Turning to the problem of finding degree 1 divisors we first prove:

**Lemma 5.7.** If \( L \) is the twistor line over a fixed point of a \( T^2 \)-symmetric \( n\mathbb{CP}^2 \) and if \( S \) is an invariant degree 2 divisor containing \( L \), then \( S \) is reducible.
Proof. Assume on the contrary that $S$ is irreducible. Then $S$ is the blow-up of a quadric surface with $L$ a smooth fiber \cite{14}. In particular it does not pass through any points of blowing-up and this implies that $L$ is not invariant: On the quadric the only $T^2$-invariant curves are the two conjugate pair of generator lines passing through the four fixed points which are the only points of blowing-up as $S$ has $T^2$-symmetry.

\begin{proposition}
Given a $T^2$-symmetric twistor space over $n\mathbb{CP}^2$ with an invariant irreducible divisor $S$. Let $L$ be a twistor line above an isolated fixed point. Then there exists a conjugate pair $D, \bar{D}$ of invariant degree 1 divisors intersecting along $L$.
\end{proposition}

Proof. Let $V_1, V_2$ be a pair of independent holomorphic vector fields generated by the torus action on the twistor space $Z$. Since $S$ is invariant the restrictions of $V_1$ and $V_2$ to $S$ are tangential to $S$. Therefore we have an invariant section $\hat{s} = V_1 \wedge V_2|_S \in H^0(S, K_S^{-1})$. We claim that this section is nontrivial. From Lemma 5.7 $L$ intersects $S$ transversely at $q, \bar{q}$. Since the $T^2$-action only has isolated fixed points, there exist complex coordinates $(z_1, z_2)$ centered at $q$ such that the $T^2$-action is given as $(e^{in\theta} z_1, e^{im\psi} z_2)$ with $n$ and $m$ both non-zero. Therefore $\hat{s} = nm z_1 z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ near $q$.

Since $H^0(S, K_S^{-1})$ is non-trivial and $h^1(Z, \mathcal{O}) = h^1(n\mathbb{CP}^2, \mathbb{R}) = 0$ \cite{5}, the exact sequence

$$0 \to \mathcal{O} \to K^{-\frac{1}{2}} \to \mathcal{O}_S(K_S^{-1}) \to 0$$

implies that the section $\hat{s}$ lifts to an invariant section of $K^{-\frac{1}{2}}$.

Let $s \in H^0(Z, K^{-\frac{1}{2}})$ be the section such that $s^{-1}(0) = S$. Then the subspace $\mathcal{V} = \text{span}\{s, \hat{s}\} \in H^0(Z, K^{-\frac{1}{2}})$ is $T^2$-invariant. Every element of this system contains $q$ and $\bar{q}$ and we may choose an element $S' \in |\mathcal{V}|$ containing $p \in L$ where $p \neq q$ and $p \neq \bar{q}$. As $S'$ is of degree 2 it also contains $L$. We may assume $S'$ is real. The divisor $S'$ is invariant, otherwise as $L$ is invariant, $L$ would be contained in the base locus of the system $|\mathcal{V}|$ but $S$ does not contain $L$. Then by Lemma 5.7, $S'$ must be reducible.

\begin{corollary}
The twistor spaces described in Theorem 5.6 contain $T^2$-invariant reducible degree 2 divisors.
\end{corollary}

Remark. Given a $T^2$-symmetric twistor space over $n\mathbb{CP}^2$ with an invariant reducible degree 2 divisor as above, we construct $T^2$-symmetric singular twistor spaces with degree 2 divisors over $(n + 1)\mathbb{CP}^2$ as outlined at the end of Section 2. As in Section 4, mutatis mutandis, we can prove that the obstructions to the equivariant relative smoothing vanish provided Assumption 2.2 is satisfied. The cohomology group $H^2(Z_i, \Theta_{Z_i})$ vanishes by upper semi-continuity. The groups $H^2(D_i, \Theta_{D_i}), H^2(D_i, \Theta_{D_i}), i = 1, 2; H^2(Z_1, \Theta_{Z_1} S)_{T^2}$ and $H^2(Z_2, \Theta_{Z_2 D_1 D_2})_{T^2}$ is proved to vanish as outlined at the end of the
paper. However, as we do not control the singularity of the degree 2 divisors in the smoothing process, we cannot conclude a priori that there are reducible degree 2 divisors after the smoothing. Furthermore, the notion of local moduli becomes dubious when the degree 2 divisor is reducible. We do not pursue this issue. However, the existence of reducible degree 2 divisors is secured by Corollary 5.9. This has applications as shown in the next section.

6. Anti-Self-Dual Hermitian Surfaces.

In [13] we constructed self-dual metrics on \((S^1 \times S^3)^\#n\mathbb{CP}^2, n \geq 3\), such that the symmetry group is \(S^1\) and the action is non-semi-free. We begin with a \(\mathbb{CP}^2\) with \(S^1\)-action \([z_0 e^{i\theta}, z_1, z_2]\). We label the fixed points as \(P_0 = [0, 1, 0], P_1 = [1, 0, 0]\) and \(A_1 = [0, 0, 1]\). Then we attach a \(\mathbb{CP}^2\) with \(S^1\)-action \([z_0 e^{-i\theta}, z_1, z_2]\) and fixed points \(A_2 = [0, 0, 1], P_2 = [1, 0, 0], P_\infty = [0, 1, 0]\). We attach \(A_1\) to \(A_2\) via an orientation reversing isometry and of course using the smoothing of the twistor space. Note that the isotropy representation of \(P_0\) is \((e^{i\theta}x, y)\) while near \(P_\infty\) it is \((e^{-i\theta}x, y)\). Therefore, we may consider the possibility of making a self sum by identifying \(P_0\) and \(P_\infty\). Note that at this stage the \(S^1\)-action is semi-free but if we equivariantly attach a \(\mathbb{CP}^2\) to \(P_2\) we get a \(S^1\)-symmetric \(3\mathbb{CP}^2\) with non-semi-free action and we can still make a self-sum at \(P_0\) and \(P_\infty\). Indeed we can go on attaching more \(\mathbb{CP}^2\)-blocks away from \(P_0\) and \(P_\infty\) and then make a self-sum at \(P_0\) and \(P_\infty\) to obtain self-dual structures on \((S^1 \times S^3)^\#n\mathbb{CP}^2, n \geq 3\), with non-semi-free \(S^1\)-symmetry.

To get a complex structure on \((S^1 \times S^3)^\#n\mathbb{CP}^2\) compatible with the conformal metric but with opposite orientation we bring in the relative smoothing: Due to Corollary 5.9, we may assume that on \(n\mathbb{CP}^2\) we have a non-semi-free \(S^1\)-symmetric twistor space with a reducible divisor \(D + \bar{D}\) such that \(D \cap \bar{D} = L_{P_0}\) and such that the twistor line above \(P_\infty\) intersect \(D\) and \(\bar{D}\) transversely. Then make an equivariant self-sum at \(P_0, P_\infty\) relative to the divisor \(D + \bar{D}\). Following the notation from Section 2 we have curves \(C_1, \bar{C}_1\) in \(D, \bar{D}\) above \(P_\infty\). Also, there are curves \(C_2, \bar{C}_2\) above \(P_0\) which are both mapped onto \(L_{P_0}\) by the blowing-down.

**Theorem 6.1.** There exist anti-self-dual conformal Hermitian metrics on \((S^1 \times S^3)^\#n\mathbb{CP}^2, n \geq 3\), such that the symmetry group is \(S^1\) and the action is non-semi-free.

**Proof.** We claim the isotropy data ensures that \(C_1\) in \(D\) is identified to \(C_2\) in \(\bar{D}\) and not to \(\bar{C}_2\) in \(\bar{D}\). On \(2\mathbb{CP}^2\) we have the four fixed points \(P_0, P_1, P_2, P_\infty\).
We draw the spheres between points as lines and get a diagram:

<table>
<thead>
<tr>
<th></th>
<th>$P_0$</th>
<th>1</th>
<th>$P_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$P_2$</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

(6.1)

Here the numbers represent the weights of the $S^1$-action along the spheres near the fixed points. These numbers are easily obtained from the information above about the $S^1$ action. The twistor space of $\mathbb{CP}^2$ was described in (5.4). We see that the twistor line $L_{P_0}$ is given by $v_1 = 0 = \ell_1$ and the divisor corresponds to $v_1 = 0$ and its conjugate corresponds to $\ell_1 = 0$. On $L_{P_0}$ the $S^1$-action has the two fixed points $q_0 = ([0,0,1],[1,0,0])$ and $\bar{q}_0 = ([1,0,0],[0,0,1])$. On the divisor the isotropy data is given by

<table>
<thead>
<tr>
<th></th>
<th>$q_0$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(6.2)

Now, the divisor $D$ in the twistor space of $2\mathbb{CP}^2$ is the blow up of $\mathbb{CP}^2$ twice [17]. It has the following configuration

(6.3)
with $\pi(q_i) = p_i$, where $\pi$ is the restriction of the twistor projection

$$\pi : D \to \mathbb{C}P^2.$$ 

Note that $\pi$ maps the whole line $L_{P_0}$ to $P_0$ and otherwise is an orientation reversing equivariant diffeomorphism. Therefore, by comparing with the data (6.1) on $2\mathbb{C}P^2$, we obtain the full isotropy data on $D$:

From the real structure $\sigma : D \to \bar{D}$ we can obtain the isotropy data on $\bar{D}$ near $q_0, \bar{q}_0, q_1, \bar{q}_2, q_{\infty}$. Then, as in the proof of Lemma 5.4 we get the data
on the blow-up of $q_{\infty}$: on $\tilde{D}$ we get

\begin{equation}
\begin{array}{c}
a \\
0 \\
1 \\
\alpha \\
0 \\
1 \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
C_2 \\
-1 \\
b \\
-1 \\
1 \\
C_1 \\
-1 \\
\beta \\
1 \\
\end{array}
\end{equation}

(6.5)

and on $\tilde{D}$ we get

\begin{equation}
\begin{array}{c}
\tilde{a} \\
0 \\
-1 \\
\tilde{\alpha} \\
0 \\
-1 \\
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
\tilde{C}_2 \\
1 \\
\tilde{b} \\
1 \\
-1 \\
\tilde{C}_1 \\
-1 \\
\tilde{\beta} \\
1 \\
\end{array}
\end{equation}

(6.6)

Here the points $a, \tilde{b}$ are mapped to $\tilde{q}_0$ by the blow-down and $\tilde{a}, b$ are mapped to $q_0$. The points $a, \tilde{a}$ corresponds to $q_{\infty}, \tilde{q}_{\infty}$ while $\beta, \tilde{\beta}$ correspond to $q_2, \tilde{q}_2$. For the sake of completeness we may compute the isotropy data at $a, b, \alpha, \beta, \tilde{a}, \tilde{b}, \tilde{\alpha}, \tilde{\beta}$ also in directions transversal to the surfaces $\tilde{D}, \tilde{\tilde{D}}$: At
each point let \((z, v, w)\) denote coordinates with \((v, w)\) coordinates along the exceptional divisor, the quadric surface, and with \((z, v)\) coordinates along the blown up divisor. Then with respect to such coordinates the weights of the isotropy is \(a(0, 1, 1), b(-1, -1, 1), \alpha(0, 1, 1), \beta(1, -1, 1)\) and \(\bar{a}(0, -1, -1), \bar{b}(1, 1, -1), \alpha(0, -1, -1), \beta(-1, 1, -1)\). The identification map \((z, v, w) \mapsto (\bar{z}, v, w)\) gives the orientation change on the divisor and it is now clear that \(a\) is identified to \(\alpha\), and \(b\) to \(\beta\).

This shows that we need to attach \(C_2\) to \(C_1\) and \(\bar{C}_2\) to \(\bar{C}_1\). For \(n\mathbb{CP}^2\), in a neighbourhood of the invariant sphere joining the fixed points \(P_0\) and \(P_{\infty}\), the isotropy data is identical to the \(2\mathbb{CP}^2\) case. Therefore, the arguments above can be applied to prove the claim also for \(n\mathbb{CP}^2\).

It follows that \(\bar{D}\) and \(\tilde{D}\) are glued to themselves respectively. Therefore, the singular divisor \(S\) in the singular twistor space \(Z\) is a disjoint union of two degree 1 divisors \(D\) and \(\tilde{D}\). In the next paragraph, we prove that the obstructions to smoothing the pair \((Z, S)\) vanish. The resulting smooth twistor space carries a conjugate pair of disjoint degree 1 divisors intersecting all twistor lines transversely. This pair corresponds to complex structures \(\pm I\) on \((S^1 \times S^1) \# n\mathbb{CP}^2\) compatible with the metric but inducing the opposite orientation.

To prove that the obstructions to the \(S^1\)-equivariant smoothing of the pair \((Z, S)\) vanish, we proceed as in Section 4. It suffices to prove the vanishing of \(H^2(Z, \Theta_{Z\tilde{D}D})_{S^1}\) and \(H^2(D, \Theta_D)_{S^1}\).

By Serre duality, \(h^2(D, \Theta_D) = h^0(D, K_D \otimes \Omega^1)\). Since \(D\) is \(T^2\)-invariant, \((V_1 \wedge V_2)|_D\) is a non-trivial section of \(K_D^{-1}\), where \(V_1\) and \(V_2\) is a pair of independent vector fields generated by the torus action on the twistor space. Therefore, if \(h^0(D, K_D \otimes \Omega^1)\) was not equal to zero, there would have been non-trivial holomorphic 1-forms on \(D\). Since \(D\) is a rational surface, it follows that \(H^2(D, \Theta_D)\) vanishes.

To deal with \(H^2(Z, \Theta_{Z\tilde{D}D})_{S^1}\), we consider the following exact sequences

\[
0 \to \Theta_{Z\tilde{D}D} \to \Theta_{ZD} \to \mathcal{O}_D(N_{\tilde{D}}^Z) \to 0, 
\]

and

\[
0 \to \Theta_{ZD} \to \Theta_Z \to \mathcal{O}_D(N_{\tilde{D}}^Z) \to 0. 
\]

By upper semi-continuity, \(H^2(Z, \Theta_Z)\) vanishes. With the next lemma, we conclude that \(H^2(Z, \Theta_{Z\tilde{D}D})_{S^1}\) vanishes.

**Lemma 6.2.** Let \(Z\) be the twistor space of \(n\mathbb{CP}^2\) with \(T^2\) symmetry and let \(D \subseteq Z\) be an invariant degree 1 divisor. Then \(H^1(D, \mathcal{O}(N_{\tilde{D}}^Z))_{S^1} = 0\) and \(H^1(D, \mathcal{O}_D(N_{\tilde{D}}^Z)) = 0, j \geq 1\).

**Proof.** Recall [17] that \(D\) is the blow-up of \(\mathbb{CP}^2\) \(n\) times. For \(1 \leq k \leq n\) consider the sequence \(b_k : D_k \to D_{k-1}\) of blowing down to \(D_0 = \mathbb{CP}^2\). Let \(p_{k-1} \in D_{k-1}\) be the point which is blown up and let \(E_k\) be the exceptional
divisor. On $D = D_n$, $[D_n]$ denotes the line bundle of $H - \sum_{i=1}^n E_i$ and on $D_k$, $[D_k]$ is the line bundle of $H - \sum_{i=1}^k E_i$ [17]. Then $[D_k] = b_k^*[D_{k-1}] \otimes E_k$ and from

$$0 \to [D_k] \to b_k^*[D_{k-1}] \to \mathcal{O}_{E_k}(b_k^*[D_{k-1}]) \to 0$$

we get

$$(6.7) \quad 0 \to H^0(D_k, [D_k]) \mathbb{S}_1 \to H^0(D_{k-1}, [D_{k-1}]) \mathbb{S}_1 \to ([D_{k-1}]_{p_{k-1}}) \mathbb{S}_1$$

$$\to H^1(D_k, [D_k]) \mathbb{S}_1 \to H^1(D_{k-1}, [D_{k-1}]) \mathbb{S}_1 \to 0$$

and $H^2(D_k, [D_k]) = H^2(D_{k-1}, [D_{k-1}])$. Thus by induction we have

$$H^2(D, \mathcal{O}(N^2_D)) = H^2(D_n, [D_n]) = H^2(D_0, [D_0]) = H^2(\mathbb{CP}^2, [H]) = 0.$$

Now we prove $([D_k]_{p_k}) \mathbb{S}_1 = 0$, $0 \leq k \leq n - 1$. Let $g_k$ be the blow-down from $D_k$ to $\mathbb{CP}^2$. Then on $D_k$ we have [17]

$$K^{-1}_{D_k} = [\bar{D}_k]]_{D_k} \otimes [D_k]|_{D_k} = (g_k^*H)^2 \otimes [D_k]|_{D_k}$$

so $(K^{-1}_{D_k})_{p_k} = (g_k^*H)^2_{p_k} \otimes [D_k]_{p_k}$. In general, we may assume that the action on $\mathbb{CP}^2$ is $[z_0, z_1e^{i\phi_1}, z_2e^{i\phi_2}]$ in homogeneous coordinates. Then for $p = g_k(p_k)$, $(g_k^*H)^3_{p_k} = H^3_p \cong \Lambda^2 T_p\mathbb{CP}^2$ so the weight of $(g_k^*H)^2_{p_k}$ is $2(\Phi_1 + \Phi_2)$. Since $p_k$ is on the intersection of invariant curves $T_{p_k}D_k$ has weight $(n_1\Phi_1 + n_2\Phi_2, m_1\Phi_1 + m_2\Phi_2)$. Then $(K^{-1}_{D_k})_{p_k}$ has weight $m\Phi_1 + n\Phi_2$ for some integers $m, n$. From diagram (6.4), after the curves joining $q_1$ to $q_2$, and $q_2$ to $q_\infty$ are blown down, we see that $\Phi_1 = 0$ and $\Phi_2 = 1$. Therefore, the representation on $[D_k]_{p_k}$ is non-trivial. This gives $([D_k]_{p_k}) \mathbb{S}_1 = 0$, $0 \leq k \leq n - 1$, and then by induction (6.7) gives $H^1(D, [D]) \mathbb{S}_1 = 0$.  

Theorem 6.1 should be compared with LeBrun's examples [12] of $S^1$-symmetric anti-self-dual Hermitian metrics on blow-up of Hopf surfaces. Note that the $S^1$-action in his examples is semi-free.

References


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SOME SUMMATIONS OF $q$-SERIES BY TELESCOPING

M.V. SUBBARAO AND A. VERMA

Summation formulae for $q$-series with independent bases are obtained and used to derive transformation and expansion of $q$-series involving independent bases.

1. Introduction.

The sum of the first $(n + 1)$-terms of the non-terminating very-well-poised $6\phi_5[q][1]$

\[
\sum_{k=0}^{n} \frac{(1 - aq^{2k})}{(1 - a)} \frac{(a, b, c; \frac{aq}{bc}; q)_k}{(q, \frac{aq}{b}, \frac{aq}{c}, bcq; q)_k} q^k = \frac{(aq, bq, cq, \frac{aq}{bc}; q)_n}{(\frac{aq}{b}, \frac{aq}{c}, bcq, q; q)_n}
\]

follows from Jackson's $q$-analogue of Whipple's summation formula for a terminating very-well-poised balanced $8\phi_7[q]$ (in [4, 2.6.2] setting $e = aq^{n+1}$). A bibasic analogue of (1.1) was obtained by Gasper [3],

\[
\sum_{k=0}^{n} \frac{(1 - aq^kp^k)(1 - bp^kq^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k(c, \frac{aq}{bc}; q)_k}{(q, \frac{aq}{b}, \frac{aq}{c}, bcq; p)_k} q^k
\]

\[
= \frac{(cq, \frac{aq}{bc}; q)_n}{(q, \frac{aq}{b}, \frac{aq}{c}; q)_n} \frac{(c, \frac{ad^2}{bc}; q)_n}{(ap, bp; p)_n}, \quad n = 0, 1, 2, \ldots
\]

and he used it for obtaining quadratic and cubic summation and transformations formulae for $q$-hypergeometric series. A little later Gasper and Rahman [5] obtained a bilateral extension of Gasper's bibasic summation formula (1.2):

If $m, n$ are non-negative integers, then

\[
\sum_{k=-m}^{n} \frac{(1 - adp^kq^k)}{(1 - ad)} \frac{(1 - bp^kq^{-k})}{(1 - b)} \frac{(a, b; p)_k}{(dq, \frac{adq}{b}; q)_k} \frac{(c, \frac{ad^2}{bc}; q)_k}{(\frac{adp}{c}, \frac{bcp}{d}; p)_k} q^k
\]

\[
= \frac{(1 - a)(1 - b)(1 - c)(1 - \frac{ad^2}{bc})}{d(1 - ad)(1 - b)(1 - \frac{c}{d})(1 - \frac{ad}{bc})}
\]

\[
\times \left\{ \frac{(ap, bp; p)_n}{(dq, \frac{adq}{b}; q)_n} \frac{(cq, \frac{ad^2}{bc}; q)_n}{(\frac{adp}{c}, \frac{bcp}{d}; q)_n} - \frac{(\frac{c}{d}; \frac{b}{ad}; \frac{p}{q})_{m+1}(\frac{1}{a}; \frac{b}{ad}; \frac{q}{m+1})}{(\frac{1}{d}; \frac{bc}{d}; \frac{q}{m+1})} \right\}.
\]
Jain and Verma [6] used transformations of \(q\)-hypergeometric series to obtain a summation formula involving three independent bases:

\[
\sum_{k=-m}^{n} \frac{(\beta;p)_k(c; q)_k(y; p)_k((\frac{\beta y}{d^2}; \frac{p^2}{q})_k)((1 - \frac{\beta y}{d^2}p^k P^k)(1 - \frac{y}{d} p^k q^{-k})(1 - \frac{\beta}{d} p^k q^{-k}))q^k}{(d q; q)_k((\frac{d y}{d^2}; \frac{p^2}{q})_k)((\frac{\beta y}{d^2}; \frac{p^2}{q})_k)_k}
\]

\[
= \frac{(1 - \beta)(1 - c)(1 - y)(1 - \frac{\beta y}{d^2} p^k q^{-k})}{(c - d) \left\{ \left( \frac{\beta y}{d^2} p^k q^{-k} \right)_m + 1 \left( \frac{d y}{d^2} p^k q^{-k} \right)_m + 1 \right\} \left( \frac{\beta y}{d^2} p^k q^{-k} \right)_n \left( \frac{d y}{d^2} p^k q^{-k} \right)_n}
\]

which for \(P = q\) reduces to the Gasper-Rahman’s summation formula (1.3).

The proof of (1.4) could be given by considering

\[
\beta_k = \frac{(\beta;p)_k(c; q)_k(y P; P)_k((\frac{\beta y}{d^2}; \frac{p^2}{q})_k)_k}{(d q; q)_k((\frac{\beta y}{d^2}; \frac{p^2}{q})_k)_k}
\]

and observing that

\[
\Delta \beta_k = \beta_k \left( \frac{\beta y}{d^2} p^k q^{-k} \right) \left( \frac{d y}{d^2} p^k q^{-k} \right) \left( \frac{\beta y}{d^2} p^k q^{-k} \right) \left( \frac{d y}{d^2} p^k q^{-k} \right)
\]

and summing for \(k\) from \(-m\) to \(n\) (\(m, n\) are non-negative integers) and using the usual convention:

\[
\prod_{k=m}^{n} A_k = \begin{cases} A_m A_{m+1} \cdots A_n & m \leq n \\ 1 & m = n \ - 1 \\ (A_{n+1} A_{n+2} \cdots A_{n-1})^{-1} & m \geq n \ - 2. \end{cases}
\]

Chu [2] obtained a generalization of Gasper-Rahman’s formula (after renaming suitably the sequences so as to remove redundant sequences)

\[
\prod_{j=0}^{k-1} [(1 - a_j)(1 - \frac{a_j}{d})(1 - c b_j)(1 - \alpha^2 d b_j)]
\]

\[
\sum_{k=-m}^{n} \frac{(1 - a_k a b_k)(b_k - \frac{a_k}{d})}{(1 - a_0 b_0)(b_0 - \frac{a_0}{d})} \prod_{j=0}^{k-1} [(1 - a b_j)(1 - a d b_j)(1 - \alpha a b_j)(1 - \frac{a^2}{c} b_j)]
\]

\[
\sum_{k=-m}^{n} \frac{1}{(1 - a_0)(1 - \frac{a_0}{d})(1 - b_0 c)(1 - \frac{a^2}{c} b_0)} \prod_{j=0}^{k} [(1 - a b_j)(1 - a d b_j)(1 - \alpha a b_j)(1 - \frac{a^2}{c} b_j)]
\]

\[
\sum_{k=-m}^{n} \frac{1}{\alpha(1 - a_0 b_0)(1 - \frac{a_0}{d})(1 - b_0 c)(1 - \frac{a^2}{c} b_0)} \prod_{j=0}^{k} [(1 - a b_j)(1 - a d b_j)(1 - \alpha a b_j)(1 - \frac{a^2}{c} b_j)]
\]
where \( \langle a_j \rangle \) and \( \langle b_j \rangle \) are arbitrary sequences such that none of the terms in the denominators vanish. This reduces to the Gasper-Rahman summation formula on setting \( a_k = aq^k \), \( b_k = q^k \) and replacing \( \alpha \) and \( d \) by \( d \) and \( a/b \), respectively.

In this paper we obtain in §2 a generalization of Chu’s summation formula (1.7) involving four arbitrary sequences, which on specialization yields an extension of (1.1) to a summation formula with four independent bases \( p, q, P \) and \( Q \) and incorporating (1.7) as a special case. An expansion of the series \( \sum_{n=0}^{\infty} A_n B_n (wx)^n \) into a series involving three independent bases is developed. A transformation of a series involving eight independent sequences is also developed. The note is concluded by obtaining in §3 some summation formulas which are different from the known ones by telescoping of series including \( q \)-Paff-Saalshütz’s summation formula for a terminating balanced \( 3\phi_2 \).

§2.

We begin this section by proving the summation formula:

If \( \langle u_k \rangle, \langle v_k \rangle, \langle w_k \rangle \) and \( \langle z_k \rangle \) are arbitrary sequences such that none of the terms in the denominators vanish and \( M, N \) are non-negative integers then

\[
\prod_{j=1}^{n} \left[ \frac{(1 - a_j)(1 - \frac{a_j}{d})(1 - b_j)(1 - \frac{a_j^2d}{c}b_j)}{(1 - \alpha b_j)(1 - \alpha b_j d)(1 - \frac{\alpha a_j}{c})(1 - \frac{\alpha b_j}{d}a_j)} \right] = \\
\prod_{j=-m}^{0} \left[ \frac{(1 - \alpha b_j)(1 - \alpha db_j)(1 - \frac{\alpha a_j}{c})(1 - \frac{\alpha b_j}{d}a_j)}{(1 - a_j)(1 - \frac{a_j}{d})(1 - cb_j)(1 - \frac{a_j^2d}{c}b_j)} \right]
\]

where \( \langle a_j \rangle \) and \( \langle b_j \rangle \) are arbitrary sequences such that none of the terms in the denominators vanish. This reduces to the Gasper-Rahman summation formula on setting \( a_k = aq^k \), \( b_k = q^k \) and replacing \( \alpha \) and \( d \) by \( d \) and \( a/b \), respectively.
\[ - \prod_{j=-m}^{0} \left[ \frac{(1 - u_j^m w_j z_j)(1 - u_j v_j z_j)(1 - u_j v_j w_j)}{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)} \right] \]

**Proof.** Let

\[ \tau_k = \prod_{j=1}^{k} \left[ \frac{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)}{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)} \right] \]

Then by straightforward calculations, we get

\[ \Delta \tau_k = \prod_{j=1}^{k-1} \left[ (1 - u_j^2)(1 - v_j^2)(1 - w_j^2) \right] \]

where \( \Delta \tau_k = \tau_k - \tau_{k-1} \). Now summing with respect to \( k \) from \(-m\) to \( n\), and using the fact that \( \sum_{k=-m}^{n} \tau_k = \tau_n - \tau_{-m-1} \) and keeping in mind (1.6), we get (2.1) on simplification.

By setting \( u_j = \sqrt{a} \ p^j, v_j = \sqrt{c} \ q^j, w_j = \sqrt{b} \ P^j, z_j = d \sqrt{Q} j \) in (2.1), we get a summation formula involving four independent bases:

\[ \sum_{k=-m}^{n} \frac{(1 - adp^k q^k P^k Q^k)(c - dP^k Q^k)(1 - bp^k Q^k)(1 - \frac{ad}{bc} \ p^k Q^k)}{(1 - ad)(c - d)(1 - \frac{b}{a})(1 - \frac{ad}{bc})} \]

\[ \times \frac{(a; p^2)_{k} (c; q^2)_{k} (b; P^2)_{k} (d; Q^2)_{k}}{(d; PQ; \ p^2 Q^2)_{k} \ (d; \ p^2 Q^2; \ P^2)_{k} \ (d; \ v^2 P^2; \ P^2)_{k}} \]

\[ = \frac{(1 - a)(1 - b)(1 - c)(1 - \frac{ad^2}{bc})}{(1 - ad)(c - d)(1 - \frac{b}{a})(1 - \frac{ad}{bc})} \]

\[ \times \left\{ \frac{(ap^2; p^2)_{n} (cq^2; q^2)_{n} (bp^2; P^2)_{n} (dP^2 Q^2; Q^2)_{n}}{(dPQ; \ p^2 Q^2)_{n} (dPQ; \ p^2 Q^2)_{n} (dPQ; \ v^2 P^2)_{n} (dPQ; \ v^2 P^2)_{n}} \right\} \]

Summation formula (2.2), on setting \( Q = \frac{p^2}{q} \) and replacing \( p^2, q^2, P^2, a, b, d \) by \( p, q, P, \beta, y, \frac{m}{2} \), respectively, reduces to the summation formula (1.4), which in turn incorporates (1.3) and (1.2) as special cases.
It may be pointed out that (2.2) reduces to Chu’s summation formula (1.7) on setting $u_j = \sqrt{a_j}$, $v_j = \sqrt{a_j}/\sqrt{d}$, $w_j = \sqrt{d/b_j}$, $z_j = \alpha \sqrt{d/c} \sqrt{b_j}$.

Setting $m = 0$ in (2.2), replacing $z_i$ by $\alpha z_i$ and setting $\alpha = \frac{x_0}{v_0 w_0 w_0}$, we get that

\begin{equation}
\sum_{j=0}^{n} \frac{(v_0 w_0 z_0 - u_j v_j w_j z_j)(v_0 w_0 z_0 - w_j z_j)(v_0 w_0 z_0 - v_j z_j)(v_0 w_0 z_0 - u_j v_j w_j z_j)}{(1 - u_0^2)(1 - v_0^2)(1 - w_0^2)} \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} - z_i^2 \right)
\end{equation}

Next, using (2.3), the following transformation involving eight arbitrary sequences is obtained:

\begin{equation}
\sum_{k=0}^{n} \prod_{i=1}^{n-k} \left[ (1 - U_i^2)(1 - V_i^2)(1 - W_i^2) \left( \frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i^2 \right) \left( \frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i^2 \right) \left( \frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i^2 \right) \right] \times \frac{(v_0 w_0 z_0 - u_k v_k w_k z_k)(v_0 w_0 z_0 - w_k z_k)(v_0 w_0 z_0 - v_k z_k)(v_0 w_0 z_0 - u_k v_k w_k z_k)}{(1 - u_k^2)(1 - v_k^2)(1 - w_k^2)} \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right) \left( \frac{v_0^2 w_0^2 Z_0^2}{u_0^2} \right)
\end{equation}
\[
\times \frac{U_i V_i W_i}{Z_i} \\
\times \prod_{i=0}^{j-1} \left[ (1 - U_i^2)(1 - V_i^2)(1 - W_i^2) \left( \frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i \right) \times \left( \frac{v_0 w_0 z_0}{u_0} - \frac{u_{n-i} w_{n-i} z_{n-i}}{u_{n-i}} \right) \left( \frac{v_0 w_0 z_0}{u_0} - \frac{u_{n-i} w_{n-i} z_{n-i}}{v_{n-i}} \right) \right]
\times \prod_{i=0}^{j-1} \left[ \left( \frac{v_0 w_0 z_0}{u_0} - \frac{u_{n-i} w_{n-i} z_{n-i}}{u_{n-i}} \right) \left( \frac{u_0}{v_0 w_0 z_0} - \frac{u_{n-i} w_{n-i} z_{n-i}}{v_{n-i}} \right) \right] \\
\prod_{i=0}^{j} \left[ 1 - u_i^2 w_i^2 (1 - u_i^2)(1 - w_i^2) \left( \frac{v_0^2 w_0^2 z_0^2}{w_0^2} - z_i^2 \right) \right].
\]

Transformation (2.4) can be proved by expanding the first product on the left hand side by using (2.3) (with \( n \) replaced by \( n - k \)), interchanging the order of summations and evaluating the inner sum by using (2.3) once again and simplifying to get the right hand side of (2.4).

Transformation (2.4), on replacing \( u_i, v_i, w_i, z_i, U_i, V_i, W_i, Z_i \) by \( \sqrt{a} p^i, \sqrt{c} q^i, \sqrt{b} P^i, Q^i, \sqrt{A} q^i, \sqrt{C} q^i, \sqrt{B} P^i, \tilde{Q}^i \), respectively, reduces on some simplification to the following transformation of \( q \)-series involving eight independent bases:

\[
(2.5) \sum_{k=0}^{n} \left( \frac{1 - a p^k q^k P^k Q^k}{p^k} \frac{(1 - b q^k P^k Q^k)}{q^k} \right) \frac{(1 - a_{bc})}{(1 - a_{bc})} \\
\times \left( \frac{a p^k q^k P^k Q^k}{p^k} \right) \left( \frac{a_{bc} p^k q^k P^k Q^k}{q^k} \right) \left( \frac{a_{bc} P^k q^k P^k Q^k}{P^k} \right) \left( \frac{a_{bc} P^k q^k P^k Q^k}{Q^k} \right) \\
\times \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \\
\times \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \left( \frac{\tilde{p}_{n} q_{n} \tilde{P}_{n}}{\tilde{Q}_{n}} \right) \\
= \left( \frac{a p^2}{p^2} \right)_n \left( \frac{c q^2}{q^2} \right)_n \left( \frac{b P^2}{P^2} \right)_n \left( \frac{a Q^2}{Q^2} \right)_n \\
\left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{a Q^2}{Q^2} \right)_n \\
\left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{q^2 P^2}{p^2} \right)_n \left( \frac{a Q^2}{Q^2} \right)_n.
\[
\sum_{j=0}^{n} \frac{(1 - A p^j q^j P^j \bar{Q}^j)(q^j \bar{Q}^j p^j - C q^j q^j C)(1 - \frac{b p^j \bar{Q}^j}{q^j q^j})}{(1 - a)(1 - q^j)(1 - b)(1 - \frac{a}{b} q^j)}
\]

Transformation (2.5) is a generalization of Gasper’s [4, Ex. 3.21] quadratic transformation (from which Gasper deduces a transformation of a half-poised \(_{10}\phi_9\) into another half-poised \(_{10}\phi_9\) [4, Ex. 3.24]) to which it reduces on setting \(Q = q\), \(P = p\), \(\bar{P} = \bar{p}\), \(Q = \bar{q}\) and then replacing \(p^2\), \(q^2\), \(\bar{p}\) and \(\bar{q}^2\) by \(p\), \(q\), \(P\) and \(Q\), respectively.

Next, we obtain an expansion of \(\sum_{r=0}^{\infty} A_r B_r (xw)^r / (q;q)_r\) terms of \(q\)-series having three independent bases. In the summation formula (2.2), setting \(m = 0\), \(c = q^{-2n}\), where \(n\) is a non-negative integer and letting \(d \to 1\), we get

\[
\delta_{m,0} = \sum_{j=0}^{m} \frac{(1 - a p^j q^j P^j \bar{Q}^j)(q^j \bar{Q}^j p^j - q^j q^j C)(1 - \frac{b p^j \bar{Q}^j}{q^j q^j})}{(1 - a)(1 - q^j)(1 - b)(1 - \frac{a}{b} q^j)}
\]

Replacing \(a\) and \(b\) by \(a p^r q^r P^r Q^r\) and \(b p^r q^{-r} P^r Q^{-r}\) respectively, where \(r\) is a non-negative integer, setting \(Q = \frac{p^2}{q}\) and then replacing \(p^2\), \(q^2\), \(P^2\), \(n\) by \(p\), \(q\), \(P\), \(m\), respectively, we get

\[
\delta_{m,0} = \sum_{j=0}^{m} \frac{(1 - a p^j q^j P^j \bar{Q}^j)(q^j \bar{Q}^j p^j - q^j q^j C)(1 - \frac{b p^j \bar{Q}^j}{q^j q^j})}{(1 - a)(1 - q^j)(1 - b)(1 - \frac{a}{b} q^j)}
\]
But, we know that

\[ B_r x^r = \sum_{m=0}^{\infty} \frac{(1 - \frac{a}{b} p^r q^m q^{2r+2m+P-r-m}) (\frac{a}{b} p^r q^{2r+P-r} \left( \frac{P}{P} \right) m apq^r; p_r (bpq^{-r} \left( \frac{P}{P} \right) x \sum_{m=0}^{\infty} \delta m, 0) \{ q^{-r} B_{r+m} C_{r,m} x^{r+m} \delta m, 0}, \]

where \( \langle B_r \rangle \) and \( \langle C_{r,m} \rangle \) are arbitrary sequences of complex numbers such that \( C_{r,0} = 1 \) for \( r = 0, 1, 2, \ldots \). Substituting for \( \delta m, 0 \) from (2.6), interchanging the order of summation and setting \( j = n - r \) and \( m = n + k - r \), we get

\[ B_r x^r = \sum_{k=0}^{\infty} \sum_{n=r}^{\infty} (-1)^n \left( 1 - ap^n q^n \right) \left( 1 - bp^n q^{-n} \right) (bpq^{-r} \left( \frac{P}{P} \right) x \sum_{m=0}^{\infty} \delta m, 0) \{ q^{-r} B_{r+m} C_{r,m} x^{r+m} \delta m, 0}, \]

Multiplying both sides by \( \frac{A_r w^r}{(q; q)_r} \) and summing from \( r = 0 \) to \( \infty \) and interchanging the order of summation on the right hand side, we get

\[ \sum_{r=0}^{\infty} A_r B_r \left( \frac{w^r}{(q; q)_r} \right) = \sum_{n=0}^{\infty} \left( 1 - ap^n q^n \right) \left( 1 - bp^n q^{-n} \right) \left( \frac{q^n}{(q; q)_n} \right) \times \sum_{k=0}^{\infty} \frac{(1 - \frac{a}{b} p^{n+k} q^{2n+2k+P-n-k}) x^k B_{n+k}}{(q; q)_k (apq^{n+k}; p)_n (bpq^{-n-k} \left( \frac{P}{P} \right) x \sum_{m=0}^{\infty} \delta m, 0) \{ q^{-r} B_{r+m} C_{r,m} x^{r+m} \delta m, 0}, \]

which is a generalization of Gasper’s bibasic expansion formula [4, (3.7.6)] to which it reduces on setting \( P = p \). It may be noted that on setting \( P = p \) the
terms in \{\cdots\} of the above expression combine to yield \((\frac{q}{q^2})^{n+r+1}; q\)_n as in the bibasic expansion formula of Gasper [4, (3.7.6)].

§3.

All the summation formulae proved so far are for one generalization of (1.1), a very-well-poised \(q\)-series. We next derive a summation formula which gives the sum of a balanced series.

Let \(\langle x_i \rangle, \langle y_i \rangle\) and \(\langle z_i \rangle\) be arbitrary sequences and \(a\) an indeterminate so that none of the terms in the denominators vanish and \(m, n\) are non-negative integers. Then

\[
\sum_{k=-m}^{n} \frac{(1 - y_k/a_{z_k})(1 - x_k/z_k)z_k}{(1 - y_0/a_{z_0})(1 - x_0/z_0)} \prod_{i=0}^{k-1} [(1 - x_i)(1 - y_i)] \prod_{i=1}^{k} [(1 - a z_i)(1 - x_i y_i)]
\]

\[
= \frac{(1 - x_0)(1 - y_0)}{a(1 - y_0)(1 - x_0)} \prod_{i=1}^{n} \left[ \frac{(1 - x_i)(1 - y_i)}{(1 - a z_i)(1 - x_i y_i)} \right] - \prod_{i=-m}^{0} \left[ \frac{(1 - a z_i)(1 - x_i y_i)}{(1 - x_i)(1 - y_i)} \right].
\]

(3.1)

For proving (3.1) we consider

\[
\tau_k = \prod_{i=1}^{k} \left[ \frac{(1 - x_i)(1 - y_i)}{(1 - a z_i)(1 - x_i y_i)} \right].
\]

Then by straight forward calculations we find that

\[
\Delta \tau_k = a z_k \left( 1 - \frac{y_k}{a_{z_k}} \right) \left( 1 - \frac{x_k}{a_{z_k}} \right) \prod_{i=1}^{k-1} [(1 - x_i)(1 - y_i)] \prod_{i=1}^{k} [(1 - a z_i)(1 - x_i y_i)],
\]

which on summing over \(k\) from \(-m\) to \(n\), gives (3.1) after using (1.6).

In view of this it is natural to look for a telescoping proof of the \(q\)-Paff-Salschütz summation formula [4, (1.7.2)]

\[
S_n \equiv \phi_2 \left[ a, b, q^{-n}; q, q \right] \frac{(c/a, c/b; q)_n}{(c, c/a b q^{1-n}; q)_n}.
\]

(3.2)

To this end define for non-negative integers \(n\) and \(r\)

\[
F(n, r) = \frac{(a, b, q^{-n}; q)_n q^r}{(q, c, a/b q^{1-n}; q)_r}.
\]
and \[ G(n, r) = \frac{(a, b; q)_r (q^{-n}; q)_r}{(q, c, \frac{ab}{c}, q^{1-n}; q)_r}. \]

Notice that \( S_0 = 1, G(n, n + 1) = G(0, -1) = 0 \). By straightforward calculations we can verify that

\[ \left(1 - \frac{c}{a} q^n\right) \left(1 - \frac{c}{b} q^n\right) F(n, r) - \left(1 - c q^n\right) \left(1 - \frac{c q^n}{ab}\right) F(n + 1, r) \]

\[ - \frac{c}{ab} q^n [G(n, r) - G(n, r - 1)] = 0. \]

Summing over \( r \) from 0 to \( n + 1 \) and using \( G(n, n + 1) = 0 = G(0, -1) \), we get

\[ \left(1 - \frac{c}{a} q^n\right) \left(1 - \frac{c}{a} q^n\right) S_n = \left(1 - c q^n\right) \left(1 - \frac{c}{ab}\right) S_{n+1} \]

which yields

\[ S_n = \frac{(\frac{c}{a}, \frac{c}{a}; q)_n}{(c, \frac{c}{ab}; q)_n} S_0. \]

By using \( S_0 = 1 \), we get (3.2).

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SOME SUMMATIONS OF $q$-SERIES BY TELESCOPING

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DEHORNIOY’S ORDERING OF THE BRAID GROUPS EXTENDS THE SUBWORD ORDERING

Bert Wiest

We give a simple proof of the result of Laver and Burckel that inserting a conjugate of a positive standard generator of the braid group anywhere in a given braid yields a braid which is larger in the sense of Dehornoy.

P. Dehornoy has defined a right invariant total ordering of the braid group $B_n$ for all $n \in \mathbb{N}$ (see [3], [4]), and this was reinterpreted in [6] in more geometrical terms. Here we are using this interpretation to give a quick proof of the following:

Theorem 1. For any braids $\alpha, \beta_1, \beta_2 \in B_n$ and any $i \in \{1, \ldots, n - 1\}$ we have $\beta_1(\alpha \sigma_i \alpha^{-1}) \beta_2 > \beta_1 \beta_2$.

Since the ordering is right-invariant this is equivalent to the following, with $\gamma = \beta_1 \alpha$:

Theorem 1'. For any braid $\gamma \in B_n$ and any $i \in \{1, \ldots, n - 1\}$ we have $\gamma \sigma_i > \gamma$.

It follows that Dehornoy’s ordering extends the (partial) subword ordering defined in [5]. Theorem 1 was first proved by Laver [8] and Burckel [2] using very different methods. As explained in [8], it can be combined with a theorem of Higman [7] to prove that the restriction of the ordering to the positive braid monoid $B_n^+$ is a well-ordering.

We briefly recall the definition of the ordering of $B_n$ given in [6]. Let $D^2$ be the unit disk in $\mathbb{C}$, and let $D_n$ be equal to $D^2$ with $n$ holes in the real line, labelled 1 to $n$. The holes divide the real interval $[-1, 1]$ into $n + 1$ line segments which we label 1 to $n + 1$. Now any braid $\gamma$ determines a way of sliding the holes about in $D^2$. Extending this movement to an isotopy of $D^2$ which is fixed on $\partial D^2$, we obtain, at the end of the isotopy, a self homeomorphism of $D_n$; this self homeomorphism maps the $n + 1$ line segments to $n + 1$ disjoint simple curves, again numbered 1 to $n + 1$, and the image of the whole interval $[-1, 1]$ under the self homeomorphism is called a curve diagram.

If $\Gamma$ is a curve-diagram in $D_n$ of some braid $\gamma$, and $\Delta$ is another curve diagram of some braid $\delta$, then we can reduce $\Gamma$ and $\Delta$ with respect to each other, i.e. we can ‘pull the diagrams tight’. Then the braid $\gamma$ is called $j$-larger than $\delta$ if the curves number 1, $\ldots$, $j - 1$ of $\Gamma$ and $\Delta$ coincide and...
an initial segment of the $j$th curve of $\Gamma$ lies in the upper component of $D_n - \Delta$. It is proved in [6] that this is equivalent to the braid $\gamma\delta^{-1}$ being representable by a word $w_1\sigma_jw_2\ldots w_{l-1}\sigma_jw_l$, where $w_1, \ldots, w_l$ are words in the letters $\sigma_{j+1}^{\pm 1}, \ldots, \sigma_n^{\pm 1}$. If $\gamma$ is $j$-larger than $\delta$ for some $j$, then we say $\gamma$ is larger than $\delta$. If the curves number $1, \ldots, j - 1$ of $\Gamma$ and $\Delta$ coincide (with no further restrictions on the subsequent curves), then we say $\gamma$ and $\delta$ are $j - 1$-parallel ($j \in \{1, \ldots, n\}$); equivalently, $\gamma\delta^{-1}$ can be represented by a word not containing the letters $\sigma_1^{\pm 1}, \ldots, \sigma_j^{\pm 1}$.

The proof of Theorem 1' is in two steps. We shall deduce Theorem 1' from the following, seemingly weaker, result:

**Proposition 2.** For any braid $\gamma \in B_n$ and any $i \in \{1, \ldots, n - 1\}$ the braid $\gamma\sigma_i$ is not 1-smaller than the braid $\gamma$; equivalently, $\gamma\sigma_i$ is either 1-larger than, or 1-parallel to $\gamma$.

**Proof of Proposition 2.** We consider the intersections of a curve diagram $\Gamma$ with the vertical lines indicated in Figure 1, which divide $D_n$ into $n + 1$ regions, labelled 1 to $n + 1$. We say $\Gamma$ is $v$-reduced if there are no disks in $D_n$ bounded by precisely one segment of some curve of $\Gamma$ and one segment of vertical line. By a sequence of isotopies across such disks we can turn $\Gamma$ into a $v$-reduced diagram.

![Figure 1. Vertical lines dividing $D_n$ into a number of regions.](image)

We equip all curves of $\Gamma$ consistently with an orientation such that, when stuck together, they form an oriented curve in $D^2$ starting at $-1$, through all holes of $D_n$, ending at 1. For every curve of $\Gamma$ we consider the segments in
which it intersects the augmented \(i\)-region. If the whole curve is contained in the \(i+1\)st region itself, connecting the \(i\)th and the \(i+1\)st hole, then right multiplication by \(\sigma_i\) has simply the effect of reversing the orientation of the curve (see Figure 2(a)). If a segment of curve can, by a \(v\)-equivalence, be made disjoint from the straight line from the \(i\)th to the \(i+1\)st hole, then it is unaffected by the right multiplication by \(\sigma_i\) (Figure 2(b)).

\[
\sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i.
\]

\textbf{Figure 2.} Unaffected by performing \(\sigma_i\).

If a segment of curve cannot be made disjoint from the straight line from the \(i\)th to the \(i+1\)st hole, then it is affected by the right multiplication by \(\sigma_i\). We shall find that there are ten \(Z\)-families of possibilities for what such a segment of curve can look like.

If both ends of the segment of curve lie in the \(i-1\)st vertical line (i.e. the left boundary of the augmented \(i+1\)-region), then it is easy to check that in a neighbourhood of the \(i+1\)st region it looks like one of the curves in Figure 3, and under right multiplication by \(\sigma_i\) the diagram changes as indicated.

\[
\sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i \cdot \sigma_i.
\]

\textbf{Figure 3.} Two \(Z\)-families of possibilities ((1) and (2)).

Note that all curves in Figure 3 can have two different orientations so Figure 3 represents two different \(Z\)-families of curve segments. Also note that we can assume that the right multiplication by \(\sigma_i\) leaves the curve diagram fixed in a neighbourhood of the boundary of the augmented \(i+1\)-region, and Figure 3 shows its effect only ‘well inside’ this region.

If both ends of the segment of curve lie in the \(i+2\)nd vertical line (i.e. the right boundary of the augmented \(i+1\)-region), then we have another two \(Z\)-families of possibilities ((3) and (4)). Figure 3, turned by 180°, illustrates these two families.

In the case that the two ends of the segment of curve lie on opposite sides of the augmented \(i+1\)-region (i.e. one in the \(i-1\)st and one in the \(i+2\)nd vertical line), we obtain two more \(Z\)-families of possibilities ((5) and (6)), which differ only in the orientation of the curve segment (Figure 4). Again, it is easy to see that in this case these are the only possibilities.
Finally, for the case that one of the ends of a curve segment lies in the boundary of the augmented $i$-region and the other end is in a hole of $D_n$, we have four more $Z$-families: The one in Figure 5 (Family (7)), the one represented by Figure 5 turned by $180^\circ$ (Family (8)), and the same two families with the orientation of the curve segments reversed (Families (9) and (10)).

Figure 4. Two $Z$-families of possibilities ((5) and (6)).

This completes our construction of a $v$-reduced curve diagram for $\gamma\sigma_i$ from the diagram $\Gamma$.

Our next aim is to compare the $v$-reduced curve diagrams of $\gamma$ and $\gamma\sigma_i$, in order to decide which one is larger. The proof of the following lemma is similar to the proof in [6] that three curve diagrams can be reduced with respect to each other.

**Lemma 3.** If $\Gamma$ and $\Delta$ are two $v$-reduced curve diagrams, then $\Delta$ can be reduced with respect to $\Gamma$ by an isotopy which is a $v$-equivalence.

Given a $v$-reduced curve diagram $\Gamma$, we use the above recipe to determine a $v$-reduced curve diagram $\Delta$ of $\gamma\sigma_i$. Then we reduce $\Delta$ with respect to $\Gamma$ to obtain a curve diagram $\Delta'$, and Lemma 3 tells us that this can be done by an isotopy of $\Delta$ which is a $v$-equivalence; that means, the intersections of $\Delta$ with the vertical lines can slide up and down without crossing the holes of $D_n$, and no intersections are created or cancelled in the course of this isotopy.

There are now two possibilities:

1. $\gamma$ and $\gamma\sigma_i$ are 1-parallel, i.e. the first curves $\Gamma_1$ and $\Delta'_1$ of $\Gamma$ and $\Delta'$ coincide.
2. $\Gamma_1$ and $\Delta'_1$ do not coincide.

In case (1) we are done. In case (2) we walk along the curve $\Gamma_1$, starting at $-1$, reading off a finite sequence of numbers according to Figure 6. The first number lies in $\{1, 2, 3\}$, the subsequent ones in $\{1, \ldots, 4\}$. The sequence ends with a 2 or a 3. We do the same for $\Delta'_1$, and obtain a different sequence.
of numbers (for if the sequences were the same, then the curves $\Gamma_1$ and $\Delta'_1$ would coincide). Since the reduction of $\Delta$ with respect to $\Gamma$ was a v-equivalence, the number sequences obtained by reading along $\Delta_1$ and $\Delta'_1$ agree.

![Figure 6. Reading a sequence of numbers off the curves $\Gamma_1$ and $\Delta'_1$.](image)

Now comes the key step of the whole proof. By carefully checking through all ten $\mathbb{Z}$-families described above it can be seen that the number sequence associated with $\Delta_1$ (obtained from $\Gamma_1$ by applying $\sigma_i$) is always lexicographically larger than the number sequence of $\Gamma_1$; i.e. the first $k - 1$ terms of the two sequences agree ($k \in \mathbb{N}$), but the $k$th number read off $\Delta_1$ is larger than the $k$th number read off $\Gamma_1$. For instance, from the two leftmost diagrams in Figure 4 one reads $\ldots 114314 \ldots$ and $\ldots 13 \ldots$ respectively (independently of orientation), and the second sequence is lexicographically larger. Similarly, from the diagrams in Figure 5 one reads $\ldots 1142, \ldots 12, \ldots 2, \ldots 33, \ldots 3413$, which is lexicographically increasing. The other cases are similar and left to the reader.

Consider now the subarcs of the curves $\Gamma_1$ and $\Delta'_1$ consisting of the first $k$ curve segments as in Figure 6. Since $\Gamma$ and $\Delta'_1$ are reduced with respect to each other, these arcs do not intersect. It follows that an initial segment of $\Delta'_1$ lies in the upper component of $D_n - \Gamma$, i.e. that $\gamma \sigma_i \gamma^{-1}$ is $1$-larger than $\gamma$. This completes the proof of Proposition 2.

**Proof of Theorem 1'.** To see that Proposition 2 implies Theorem 1' we assume, for a contradiction, that there exists a braid $\gamma$ and an $i \in \{1, \ldots, n\}$ such that $\gamma \sigma_i \gamma^{-1}$ is $j$-negative with $j > 1$. Then $\gamma \sigma_i \gamma^{-1}$ can be represented by a word not containing the letters $\sigma_1^{\pm 1}, \ldots, \sigma_{j-1}^{\pm 1}$, and only negative powers of $\sigma_j$.

We consider the natural epimorphism $\pi: B_n \to S_n$, from the braid group to the symmetric group. We have $\pi(\sigma_i) = (i, i+1)$, $\pi(\gamma^{-1}) = (\pi(\gamma))^{-1}$, and $\pi(\gamma \sigma_i \gamma^{-1})(k) = k$ for $k = 1, \ldots, j - 1$. It follows that $\pi(\gamma)(k) \notin \{i, i+1\}$ for $k = 1, \ldots, j - 1$. Therefore the braid $\bar{\sigma}$ on $n - j + 1$ strings which is obtained from the braid $\sigma_i$ by removing the strings number $\pi(\gamma)(1), \ldots, \pi(\gamma)(j - 1)$ is again a positive standard generator of $B_{n-j+1}$.

Similarly, by removing from the braid $\gamma$ the strings starting in the $j - 1$ leftmost positions we obtain a braid $\bar{\gamma} \in B_{n-j+1}$.
Then the braid $\tilde{\gamma}\tilde{\sigma}\gamma^{-1} \in B_{n-j+1}$ is a conjugate of the positive standard generator $\tilde{\sigma}$. On the other hand, the braid $\tilde{\gamma}\tilde{\sigma}\gamma^{-1}$ is obtained from $\gamma\sigma_i\gamma^{-1}$ by removing the $j-1$ leftmost strings. Therefore it can be represented by a word containing the letter $\sigma_1^{-1}$, but not $\sigma_1$, thus contradicting Proposition 2.

**Added in proof.** Consider the monoid $\Pi = \{\pi \in B_n : \beta\pi > \beta \forall \beta \in B_n\}$ which is closed under conjugation by elements of $B_n$. I do not know a complete set of generators of $\Pi$. S. Orevkov has pointed out that, in addition to positive standard generators, $\Pi$ also contains all braids of the form $\alpha\sigma_1\sigma_2\ldots\sigma_n\sigma_{n-1}\sigma_n^{-1}\ldots\sigma_1$, where $\alpha$ is any braid not containing $\sigma_1^{\pm 1}$.

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A UNIFIED METHOD OF CONSTRUCTION OF ELLIPTIC CURVES WITH HIGH MORDELL–WEIL RANK

HIZURU YAMAGISHI

By using the twist theory, we reduce the problem of constructing elliptic curves of rank $n$ ($n \geq 1$) with generators to the problem of finding rational points on a certain variety $V_n$. By parametrizing all rational points on $V_n$ ($1 \leq n \leq 7$), we get all elliptic curves of at least rank $n$ ($n \leq 7$).

1. Introduction.

The purpose of this paper is to describe a unified method of construction of elliptic curves with given Mordell-Weil rank, and to show it is powerful enough to produce every known example in principle. In view of the fact that there is no general algorithm to give an elliptic curve with high rank, our method might shed a light on this area of active research.

The main ingredient in this paper is the twist theory as is developed in [3]. One of our main theorems says that the twists give us every elliptic curve with high Mordell-Weil rank. Once this is established, we naturally come to consider a variety $V_n$ parametrizing a family of elliptic curves with a given rank $n$, and we show that almost every rational point on this variety gives an elliptic curve with rank $\geq n$. Thus our method should provide us with every known example as a rational point on it. We will show that this is indeed the case.

In this paper, we parametrize the rational points on $V_n$ in each case of $n = 1$ to 7. As is mentioned above parametrizing the rational points on $V_n$ is equivalent to getting every elliptic curve of rank at least $n$ ($n \leq 7$). Moreover, the variety $V_n$ is expected to be useful for solving other problems. For example, in [5] it is used to construct a family of elliptic curves of rank 2 with given $j$-invariant. Moreover in [6], it is used to give a family of elliptic curves of rank 6 with a nontrivial rational two-torsion point. The point which is worthy of special mention is that we get very easily the equation of any elliptic curve which corresponds to a rational point on $V_n$. 

189
The present paper is organized as follows. In Section 2, we show every elliptic curve with generators comes from a twist. Then for each rank \( n \) we define the variety \( V_n \) mentioned above, which plays a very important role throughout the paper, and construct a generic elliptic curve with its generators as a generic fiber of a certain family of elliptic curves parametrized by \( V_n \). Furthermore we show that we can get a given elliptic curve by specializing this family at a certain rational point on \( V_n \). In Section 3, we focus our attention on the structure of \( V_n \). In each case of rank 1 to 7, we show \( V_n \) is rational and obtain a parametric representation of all rational points of \( V_n \). For the case of rank \( \geq 5 \), we define another variety which is birational to \( V_n \), and parametrize rational points on \( V_n \) in these cases using this variety. The concrete proof is given only for the case of rank 6, because it is the most typical and richest and the other cases are treated more easily. In Section 4, for a given elliptic curve with generators whose rank \( \leq 7 \), we specify the values of the parameters which are used in Section 3 to express the rational points of \( V_n \). As an application, we reconstruct the example of rank 7 due to Grunewald and Zimmert \[2\] in Section 5.

I would like to express my gratitude to Professor Fumio Hazama for his useful advice. And I also thank Professor Kenneth A. Ribet and Professor Robert Coleman for their stimulating conversation.

2. Generic case and its specialization.

In this section, we construct an elliptic curve with rank \( n \) defined over the function field of an algebraic variety. Let \( E \) be an elliptic curve over a field \( k \) of characteristic \( \neq 2 \) defined by the following equation

\[
E : y^2 = ax^3 + bx^2 + cx + d,
\]

and let \( f(x) \) be the right hand side of the equation of \( E \). Then we can express \( E^n \) by the simultaneous equation:

\[
y^2_i = f(x_i) \quad (i = 1, \ldots, n).
\]

Let \( \iota_i \) the involution on \( E^n \) defined by \( \iota_i((x_i, y_i)) = (x_i - y_i) \) \((i = 1, \ldots, n)\) and put \( V_n = E^n/\langle(\iota_1, \ldots, \iota_n)\rangle \), then the function field of \( V_n \) is the set of the invariant elements of the function field of \( E^n \) under the action of \( \langle(\iota_1, \ldots, \iota_n)\rangle \). Consequently,

\[
k(V_n) = k(E^n/\langle(\iota_1, \ldots, \iota_n)\rangle) = k(y_1y_2, \ldots, y_1y_n, x_1, \ldots, x_n).
\]

Since \( (y_1y_{i+1})^2 = f(x_1)f(x_{i+1}) \) holds for \( i = 1, \ldots, n - 1 \), we find that \( V_n \) is defined by

\[
y^2_i = f(x_1)f(x_{i+1}) \quad (i = 1, \ldots, n - 1).
\]
ELLIPTIC CURVES 191

(Here we rename \(y_1y_{i+1}\) as \(y_i\).) Let \(E_b\) be the twist of \(E\) by the quadratic extension \(k(E^n)/k(V_n)\). It is defined by the equation

\[f(x_1)y^2 = f(x)\]

(see [3, §4]). Let \(E_b(k(V_n))\) be the group of \(k(V_n)\)-rational points on \(E_b\).

**Theorem 2.1** (Hazama). If \(\text{End}_k(E) \cong \mathbb{Z}\), then the rank of \(E_b(k(V_n))\) is \(n\), and its generators are the following:

\[(x_1, 1) \left( x_{i+1}, \frac{y_i}{f(x_1)} \right) \quad (i = 1, \ldots, n-1).\]

Now, we can obtain a given elliptic curve with its generators by specializing the above twisted generic elliptic curve at a certain \(k\)-rational point on \(V_n\) as follows:

**Proposition 2.2.** Let \(E\) be a given elliptic curve defined by the following equation

\[E : y^2 = ax^3 + bx^2 + cx + d,\]

and let \((\alpha_i, \beta_i)\) \((i = 1, \ldots, n)\) be its independent generators. Let \(E_b\) be the twist of \(E\) by \(k(E^n)/k(V_n)\). Then \(E\) with these generators is obtained by specialization at the point \((x_1, \ldots, x_n, y_1, \ldots, y_{n-1}) = (\alpha_1, \ldots, \alpha_n, \beta_1\beta_2, \ldots, \beta_1\beta_n)\) on \(V_n\).

*Proof.* Put \(x_i = \alpha_i\) \((i = 1, \ldots, n)\). Then \(E_b : f(x_1)y^2 = f(x)\) is isomorphic to \(y^2 = f(x)\) by the map \((x, y) \mapsto (x, \beta_1 y)\). Here the generators of twisted elliptic curve become \((\alpha_1, 1), (\alpha_{i+1}, \frac{\beta_{i+1}}{\beta_i})\) \((i = 1, \ldots, n-1)\). Therefore they are mapped to \((\alpha_i, \beta_i)\) \((i = 1, \ldots, n-1)\). \(\square\)

3. The structure of the base space.

In this section, we investigate the structure of the variety \(V_n\) defined in Section 1 in each case.

The case of rank 1 is slightly different from the other cases, and can be treated easily. More precisely, the twisted elliptic curve

\[E_b : f(x_1)y^2 = ax^3 + bx^2 + cx + d,\]

has a rational point \((x_1, 1) \in E_b(k(x_1))\), and it follows from Theorem 2.1 that it is of rank one as an elliptic curve over \(k(x_1)\). But a generalization opposed to a specialization decreases the rank of an elliptic curve, hence \(E_b\) regarded as an elliptic curve over \(k(x_1, a, b, c, d)\) is of rank 1.

As is seen from this argument for the case of rank 1, it is natural to regard \(V_n\) defined by \((2)\) \((n \geq 2)\) as an algebraic variety defined over \(K = k(x_1, \ldots, x_n)\). Therefore \(V_n\) is a 3-dimensional subvariety in the projective \(n+2\) space \(\mathbb{P}^{n+2}\) with coordinates \((a, b, c, d, y_1, \ldots, y_{n-1})\), and \(E_b\) is regarded as a generic fiber of the elliptic fiber space defined by Equation \((3)\) over \(V_n\).
Case of rank 2. Our $V_2$ is defined by one quadratic equation with a rational point $P_1 (a, b, c, d, y_1) = (0, 0, 0, 1, 1)$, hence $V_2$ is $K$-rational and we can parametrize $K$-rational points on $V_2$ as follows:

**Theorem 3.1.** $V_2$ is rational, in fact, each $K$-rational point on $V_2$ is expressed as

$$((S + T)p_1, (S + T)p_2, (S + T)p_3, (S + T)p_4 - ST, -ST),$$

where $(p_1, p_2, p_3, p_4) \in \mathbb{P}^3$ and

$$S = p_1 x_1^3 + p_2 x_2^3 + p_3 x_1 + p_4, \quad T = p_1 x_2^3 + p_2 x_1^3 + p_3 x_2 + p_4.$$

Case of rank 3. We recall two results which will be frequently used later. The first of them is classical and well-known (see [1], for example), but in view of the fundamental role played by it, we recall its proof.

**Lemma 3.2.** Let $V$ be a complete intersection of $l$ quadrics in $\mathbb{P}^{l-1}$ defined over $k$. Suppose that it contains a linear subvariety $W$ $k$-isomorphic to $\mathbb{P}^{l-1}$. Then $V$ is $k$-rational.

**Proof.** There is a $k$-linear subvariety $L$ $k$-isomorphic to $\mathbb{P}^n$ such that the intersection of $W$ and $L$ is empty. For any point $P$ on $L(k)$, let $M$ be the variety spanned by $\{P\}$ and $W$, which is $k$-isomorphic to $\mathbb{P}^l$. Then we can express the intersection of the variety defined by $i$-th equation of $V$ and $M$ as the union of $W$ and a $k$-linear subvariety $W_i \cong \mathbb{P}^{l-1}$. Since $W_i$ $(i = 1, \ldots, l)$ are in $M$, we get a $k$-rational point $Q$ as intersection of $W_i$ $(i = 1, \ldots, l)$. The map $\varphi : L \to V$ defined by $\varphi(P) = Q$ gives a birational map from $L(\cong \mathbb{P}^{n-l})$ to $V$. \qed

The next result is from an elementary linear algebra:

**Lemma 3.3.** Let $N$ be a given $n \times (n + 1)$ matrix and we denote the matrix removed $i$-th column by $N_i$. The homogeneous equation

$$N \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \\ \lambda_{n+1} \end{pmatrix} = 0$$

in $\mathbb{P}^n$ has a unique solution if $N$ is of full rank. And then the solution is

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_i \\ \vdots \\ \lambda_{n+1} \end{pmatrix} = \begin{pmatrix} \det(N_1) \\ \vdots \\ (-1)^{i+1} \det(N_i) \\ \vdots \\ (-1)^{n+2} \det(N_{n+1}) \end{pmatrix}.$$
Now we show that $V_3$ is birational to $\mathbb{P}^3$. As $V_3$ is a $(2,2)$-intersection in $\mathbb{P}^5$ with coordinates $(a, b, c, d, y_1, y_2)$ and contains a line $W$ defined by the equations $x_1^2a + x_1b + c = 0$, $d = y_1 = y_2 = 0$, therefore we can apply Lemma 3.2 and 3.3. Note that $W$ is spanned by $S_1 = (1, 0, -x_1^2, 0, 0, 0)$, $S_2 = (0, 1, -x_1, 0, 0, 0)$. By a direct computation based on the map $\varphi$ in the proof of Lemma 3.2, we obtain the following:

**Theorem 3.4.** $V_3$ is rational. Every $K$-rational point on $V_3$ is given by

$$(\lambda + \nu p_1, \mu, -\lambda x_1^2 - \mu x_1, \nu p_2, \nu p_3, \nu p_4),$$

where $(p_1, p_2, p_3, p_4) \in \mathbb{P}^3$ and

$$(\lambda, \mu, \nu)
\begin{aligned}
= & \left( x_1^3 x_2 x_3 (x_3 - x_2)(x_1 x_3 - x_2 x_3 + x_1 x_2)p_1^2 \\
- & (x_3 - x_2)(x_1^4 - (x_2 + x_3)x_1^3 - x_1 x_2 x_3 (x_2 + x_3) + x_2^2 x_3^2) p_1 p_2 \\
+ & (x_3 - x_2)(x_2 - x_1 + x_3)p_2^2 - x_3(x_3 - x_1)p_3^2 + x_2(x_2 - x_1)p_4^2, \\
- & x_1^2 x_2 x_3 (x_3^2 - x_2^2)p_1^2 \\
- & x_1^2 (x_1 + x_2)(x_1 + x_3)(x_3 - x_2)(x_2 + x_3 - x_1)p_1 p_2 \\
- & (x_3^2 x_3^2 - x_1^2) - x_2(x_2^2 - x_1^2)p_2^2 + x_3(x_3^2 - x_1^2)p_3^2 - x_2(x_2^2 - x_1^2)p_4^2,
\end{aligned}
\begin{aligned}
x_2 x_3 (p_1 x_1^3 + p_2)(x_3 - x_1)(x_2 - x_1)(x_3 - x_2)) & \in \mathbb{P}^2.
\end{aligned}

Case of rank 4. In this case, $V_4$ contains the plane $W$ defined by the equations $x_1^2a + x_1^2b + x_1c + d = 0$, $y_1 = y_2 = y_3 = 0$. Hence by a similar argument to the one employed in the case of rank 3, we obtain the following:

**Theorem 3.5.** $V_4$ is rational. Every $K$-rational point on $V_4$ is expressed as

$$(\lambda + \rho p_1, \mu, \nu, -(x_1^2 + \mu x_1^2 + \nu x_1), \rho p_2, \rho p_3, \rho p_4),$$

where $(p_1, p_2, p_3, p_4) \in \mathbb{P}^3$ and

$$(\lambda, \mu, \nu, \rho)
\begin{aligned}
= & \left( -x_1^2 p_1^2 (x_3 - x_2)(x_4 - x_2)(x_4 - x_3) \\
\cdot & (x_1^2(x_2 + x_3 + x_4) - x_1(x_2 x_3 + x_3 x_4 + x_2 x_4) + x_2 x_3 x_4) \\
+ & p_2^2(x_3 - x_2)(x_4 - x_1)(x_4 - x_3) \\
- & (x_2 - x_1)((x_4 - x_1)(x_4 - x_2)p_3^2 - (x_3 - x_1)(x_3 - x_2)p_4^2), \\
x_1^6 p_1^2 (x_3 - x_2)(x_4 - x_2)(x_4 - x_3)(x_2 + x_3 + x_4) \\
- & p_3^2(x_3 - x_1)(x_4 - x_1)(x_4 - x_3)(x_3 + x_4) \\
+ & p_4^2(x_2 - x_1)(x_4 - x_1)(x_4 - x_2)(x_1 + x_2 + x_4)
\end{aligned}
\begin{aligned}
x & \in \mathbb{P}^2.
\end{aligned}
194 HIZURU YAMAGISHI

\[-p_1^2(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_1 + x_2 + x_3),\]

\[-x_1^6p_1^2(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)(x_2x_3 + x_3x_4 + x_2x_4)\]

\[+ p_1^2(x_2 - x_1)(x_4 - x_1)(x_4 - x_3)(x_1x_3 + x_3x_4 + x_1x_4)\]

\[-p_2^2(x_2 - x_1)(x_4 - x_1)(x_4 - x_2)(x_1x_2 + x_2x_4 + x_1x_4)\]

\[+ p_2^2(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_1x_2 + x_2x_3 + x_1x_3),\]

\[p_1x_1^3(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)).\]

Before proceeding to the case of rank \(\geq 5\), we state the following theorem. It can be proved by a similar argument to the one for [4, Theorem 2.1].

**Theorem 3.6.** Let \(V_n\) be the algebraic variety over \(K\) defined by (2) where we regard \(a, b, c, d\) and \(y_i (i = 1, \ldots, n - 1, n \geq 5)\) as variables. Then \(V_n\) is \(K\)-birational to the variety \(\tilde{V}_n\) defined by the equations

\[
\begin{vmatrix}
0 & 1 & 2 & 3 & i \\
y_0^2 & y_1^2 & y_2^2 & y_3^2 & y_i^2
\end{vmatrix} = 0 \quad (i = 4, \ldots, n - 1)
\]

in \(\mathbb{P}^{n-1}\) with coordinate \((y_0, \ldots, y_{n-1})\), where we write

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_i \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_i^2 \\
\alpha_0^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_i^3 \\
y_0^2 & y_1^2 & y_2^2 & y_3^2 & y_i^2
\end{vmatrix} = 0 \quad (i = 4, \ldots, n - 1), \text{ and } \alpha_i = x_i \quad (i = 0, \ldots, n - 1).
\]

**Case of rank 5.** In this case, \(\tilde{V}_5\) in Theorem 3.6 is defined by one quadric equation with a rational point \(P_1 = (1, 1, 1, 1, 1)\). Hence we have the following theorem which is similar to Theorem 3.1:

**Theorem 3.7.** \(\tilde{V}_5\) is rational. Every \(K\)-rational point on \(\tilde{V}_5\) is expressed as

\[(2p_1S - T, 2p_2S - T, 2p_3S - T, 2p_4S - T, -T),\]

where \((p_1, p_2, p_3, p_4) \in \mathbb{P}^3\) and

\[
S = \begin{vmatrix}
0 & 1 & 2 & 3 \\
p_1 & p_2 & p_3 & p_4
\end{vmatrix}
\]

\[
\begin{vmatrix}
0 & 1 & 2 & 3 \\
p_1 & p_2 & p_3 & p_4
\end{vmatrix},
\]

**Case of rank 6.** In this case, \(\tilde{V}_6\) is a \((2, 2)\)-intersection in \(\mathbb{P}^5\) and contains a line \(W\) which is spanned by \(S_1 = (1, 1, 1, 1, 1, 1), S_2 = (\alpha_0, \ldots, \alpha_5)\). Therefore, we can apply Lemma 3.2. Let \(L\) be a linear subspace spanned by

\[
S_3 = (\alpha_4 - \alpha_5, \alpha_4 - \alpha_5, 0, 0, \alpha_5 - \alpha_0, \alpha_5 - \alpha_0),
\]

\[
S_4 = (0, \alpha_4 - \alpha_5, 0, 0, \alpha_5 - \alpha_1, \alpha_1 - \alpha_4),
\]

\[
S_5 = (0, 0, \alpha_4 - \alpha_5, 0, \alpha_5 - \alpha_2, \alpha_2 - \alpha_4),
\]

\[
S_6 = (0, 0, \alpha_4 - \alpha_5, \alpha_5 - \alpha_3, \alpha_3 - \alpha_4).
\]
For any point \( P = p_1S_3 + p_2S_4 + p_3S_5 + p_4S_6 \) on \( L \), the point on \( M \) is represented by the form \( \lambda S_1 + \mu S_2 + \nu P \). We denote the \( i \)-th coordinate of \( P \) by \( P(i - 1) \). Then the equation of \( W_1 \cap W_2 \) is easily seen to be

\[
N \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = 0,
\]

where \( N \) is a \( 2 \times 3 \)-matrix \((N_{ij})\) defined by

\[
\begin{align*}
N_{11} &= 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ P(0) & P(1) & P(2) & P(3) & P(4) \end{vmatrix}, \\
N_{12} &= 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ \alpha_0 P(0) & \alpha_1 P(1) & \alpha_2 P(2) & \alpha_3 P(3) & \alpha_4 P(4) \end{vmatrix}, \\
N_{13} &= \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ P(0)^2 & P(1)^2 & P(2)^2 & P(3)^2 & P(4)^2 \end{vmatrix}, \\
N_{21} &= 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 5 \\ P(0) & P(1) & P(2) & P(3) & P(5) \end{vmatrix}, \\
N_{22} &= 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 5 \\ \alpha_0 P(0) & \alpha_1 P(1) & \alpha_2 P(2) & \alpha_3 P(3) & \alpha_5 P(5) \end{vmatrix}, \\
N_{23} &= \begin{vmatrix} 0 & 1 & 2 & 3 & 5 \\ P(0)^2 & P(1)^2 & P(2)^2 & P(3)^2 & P(5)^2 \end{vmatrix}.
\]

Then by Lemma 3.3,

\[
\begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} \det(N_{11}) \\ -\det(N_{21}) \\ \det(N_{31}) \end{pmatrix},
\]

where \( N_i \) is the \( 2 \times 2 \)-matrix with the \( i \)-th column removed from \( N \). Hence the point on \( \tilde{V}_6 \) which corresponds to \( P \) is \( \lambda S_1 + \mu S_2 + \nu P \), where \( (\lambda, \mu, \nu) = \left( \det(N_{11}), -\det(N_{21}), \det(N_{31}) \right) \). Hence we have the following:

**Theorem 3.8.** \( \tilde{V}_6 \) is rational. Every \( K \)-rational point on \( \tilde{V}_6 \) is expressed as

\[
\begin{align*}
\lambda + \mu \alpha_0 + \nu p_1(\alpha_4 - \alpha_5), \\
\lambda + \mu \alpha_1 + \nu p_2(\alpha_4 - \alpha_5), \\
\lambda + \mu \alpha_2 + \nu p_3(\alpha_4 - \alpha_5), \\
\lambda + \mu \alpha_3 + \nu p_4(\alpha_4 - \alpha_5), \\
\lambda + \mu \alpha_4 + \nu \sum_{i=0}^{3} p_{i+1}(\alpha_5 - \alpha_i), \\
\lambda + \mu \alpha_5 + \nu \sum_{i=0}^{3} p_{i+1}(\alpha_i - \alpha_4)
\end{align*}
\]

where \( (p_1, p_2, p_3, p_4) \in \mathbb{P}^3 \),

\[
(\lambda, \mu, \nu) = (N_{12}N_{23} - N_{13}N_{22}, -N_{11}N_{23} + N_{13}N_{21}, N_{11}N_{22} - N_{12}N_{21})
\]

with \( N_{ij} \) as above.
Case of rank 7. In this case, $\bar{V}_7$ is a $(2, 2, 2)$-intersection in $\mathbb{P}^6$. Unfortunately, this is not rational and only unirational [1]. To remedy this situation, we consider

$$E : y^2 = ax^4 + bx^3 + cx^2 + dx + e$$

instead of (1). Let $f(x)$ be the right hand side of the equation of $E$ and $V_7$ the variety defined by the equation

$$y_i^2 = f(x_i)f(x_{i+1}) \quad (i = 1, \ldots, 6).$$

Repeating a similar argument to the one given above, we obtain the variety which is birational to $V_7$ defined by the equation,

$$\begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ Y_0^2 & Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 & Y_5^2 \end{vmatrix} = 0 \quad (i = 5, 6, 7).$$

We call this $\bar{V}_7$. This is in $\mathbb{P}^7$ and its dimension is 4. Note that $\bar{V}_7$ contains a plane $W$ spanned by $S_1 = (1, \ldots, 1)$, $S_2 = (\alpha_0, \ldots, \alpha_7)$, $S_3 = (\alpha_0^2, \ldots, \alpha_7^2)$. Hence we are in the same situation as the case of rank 6. Thus we have the following theorem which can be proved similarly:

**Theorem 3.9.** $\bar{V}_7$ is rational. Every $K$-rational point on $\bar{V}_7$ is expressed as

$$(\lambda + \mu a_0 + \nu a_0^2, \lambda + \mu a_1 + \nu a_1^2, \lambda + \mu a_2 + \nu a_2^2),$$

$$\lambda + \mu a_3 + \nu a_3^2 + \rho p_1, \lambda + \mu a_4 + \nu a_4^2 + \rho p_2, \lambda + \mu a_5 + \nu a_5^2 + \rho p_3,$$

$$\lambda + \mu a_6 + \nu a_6^2 + \rho p_4, \lambda + \mu a_7 + \nu a_7^2 + \rho p_5),$$

where $(p_1, p_2, p_3, p_4, p_5) \in \mathbb{P}^4$ and

$$(\lambda, \mu, \nu) = (\det(N_1), -\det(N_2), \det(N_3), -\det(N_4))$$

with $N_i \ (i = 1, \ldots, 4)$ as following:

$N_{11} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & p_1 & p_2 & p_3 \end{vmatrix}$,

$N_{12} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \alpha_3 p_1 & \alpha_4 p_2 & \alpha_5 p_3 \end{vmatrix}$,

$N_{13} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \alpha_2^2 p_1 & \alpha_2^2 p_2 & \alpha_2^2 p_3 \end{vmatrix}$,

$N_{14} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & p_1^2 & p_2^2 & p_3^2 \end{vmatrix}$,

$N_{21} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & p_1 & p_2 & p_4 \end{vmatrix}$,

$N_{22} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \alpha_3 p_1 & \alpha_4 p_2 & \alpha_6 p_4 \end{vmatrix}$,

$N_{23} = 2 \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & \alpha_2^2 p_1 & \alpha_2^2 p_2 & \alpha_2^2 p_4 \end{vmatrix}$,
Remark 3.10. In the case of $n \leq 4$, one can write down immediately the defining equation of elliptic curve which corresponds to a point on $V_n$. In the case of $n \geq 5$, let $P = (\bar{y}_0, \ldots, \bar{y}_{n-1})$ be a rational point on $\bar{V}_n$. Then, the defining equation of the elliptic curve which corresponds to $P$ is

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 \\
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & x \\
\alpha_0^2 & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & x^2 \\
\alpha_0^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & x^3 \\
\bar{y}_0 & \bar{y}_1 & \bar{y}_2 & \bar{y}_3 & y^2
\end{vmatrix} = 0.
\]

This is obtained by tracing the birational map between $V_n$ and $\bar{V}_n$.

4. The value of the parameter which corresponds to a given elliptic curve.

In view of Proposition 2.2, every elliptic curve with rank $n$ should correspond to a point of $V_n$. In this section, we give the values of the parameters for this point of $V_n$ ($2 \leq n \leq 7$). Let $E$ be an elliptic curve defined by the following equation

\[y^2 = ax^3 + bx^2 + cx + d\]

with independent generators $(x_i, y_i)$ ($i = 1, \ldots, r$, $2 \leq r \leq 7$). Then we have the values of parameters as follows:

\[
N_{24} = \begin{vmatrix}
0 & 1 & 2 & 3 & 4 & 6 \\
0 & 0 & 0 & p_1^2 & p_2^2 & p_4^2
\end{vmatrix},
\]
\[
N_{31} = 2 \begin{vmatrix}
0 & 1 & 2 & 3 & 4 & 7 \\
0 & 0 & 0 & p_1 & p_2 & p_3
\end{vmatrix},
\]
\[
N_{32} = 2 \begin{vmatrix}
0 & 1 & 2 & 3 & 4 & 7 \\
0 & 0 & 0 & \alpha_3 p_1 & \alpha_4 p_2 & \alpha_7 p_5
\end{vmatrix},
\]
\[
N_{33} = 2 \begin{vmatrix}
0 & 1 & 2 & 3 & 4 & 7 \\
0 & 0 & 0 & \alpha_3^2 p_1 & \alpha_4^2 p_2 & \alpha_7^2 p_5
\end{vmatrix},
\]
\[
N_{34} = \begin{vmatrix}
0 & 1 & 2 & 3 & 4 & 7 \\
0 & 0 & 0 & p_1^2 & p_2^2 & p_5^2
\end{vmatrix}.
\]
\[ r = 2 \]
\[ (p_1, p_2, p_3, p_4) = (a, b, c, d - y_1 y_2), \]

\[ r = 3 \]
\[ (p_1, p_2, p_3, p_4) = \left( \frac{ax^2 + bx + c}{x^3}, d, y_1 y_2, y_1 y_3 \right), \]

\[ r = 4 \]
\[ (p_1, p_2, p_3, p_4) = \left( \frac{y_1}{x^3}, y_2, y_3, y_4 \right), \]

\[ r = 5 \]
\[ (p_1, p_2, p_3, p_4) = (y_1 - y_5, y_2 - y_5, y_3 - y_5, y_4 - y_5), \]

\[ r = 6 \]
\[ p_j = \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_5 - x_{i+1}) \right) 
\]
\[ + \left( \sum_{i=1}^{6} (x_5 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_6 - x_{i+1}) \right) 
\]
\[ - \left( \sum_{i=1}^{6} (x_5 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_6 - x_{i+1}) \right) 
\]
\[ + \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_5 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_6 - x_{i+1}) \right) 
\]
\[ + x_j \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_5 - x_{i+1}) \right) 
\]
\[ - \left( \sum_{i=1}^{6} (x_5 - x_{i+1}) \right) \left( \sum_{i=1}^{6} y_{i+1} (x_6 - x_{i+1}) \right) 
\]
\[ + y_j \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} (x_5 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_6 - x_{i+1}) \right) 
\]
\[ - \left( \sum_{i=1}^{6} (x_6 - x_{i+1}) \right) \left( \sum_{i=1}^{6} x_{i+1} (x_5 - x_{i+1}) \right), \]

\((j = 1, \ldots, 4)\).

In the case \( r = 7 \), let \( E \) be an elliptic curve defined by the following equation

\[ y^2 = ax^4 + bx^3 + cx^2 + dx + e \]
with independent generators \((x_i, y_i)\) \((i = 1, \ldots, 8)\). Then the values of the parameters are given by the formula

\[
p_i = (x_3 - x_2)(y_1(x_2 x_3 - x_3 x_{i+3} - x_2 x_{i+3} + x_{i+3}^2) - y_{i+3} x_2 x_3) - (x_3 - x_1)(y_2(x_1 x_3 - x_3 x_{i+3} - x_1 x_{i+3} + x_{i+3}^2) - y_{i+3} x_1 x_3) + (x_2 - x_1)(y_3(x_1 x_2 - x_2 x_{i+3} - x_2 x_{i+3} + x_{i+3}^2) - y_{i+3} x_1 x_2),
\]

\((i = 1, \ldots, 5)\).

5. Examples.

In the previous sections, we have given a parametric representation of \(V_n\) \((1 \leq n \leq 7)\). Therefore we will get as many elliptic curves with specified rank as we like by specialization at rational points on \(V_n\). As an example, we give the values of parameters which enable one to obtain the elliptic curve with rank 7 in \([2, \text{Corollary C}]\). The elliptic curve is

\[
y^2 = x^3 - 1717730532x + 27401746395780
\]

with generators

\((24144, 56466), (23562, 97182), (23736, 50022), (24840, 245430), (25422, 404082), (23793, 34119), (26121, 596187)\).

The values of the parameters are

\[
(p_1, p_2, p_3, p_4, p_5) = (47822467248393469632, 66206014691795224675, 104521834162171114920, 76522282208132178600, 101559585548776675320),
\]

\[
(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = \left(\frac{1}{234}, -\frac{1}{348}, -\frac{1}{174}, \frac{1}{930}, \frac{1}{1512}, -\frac{1}{117}, \frac{1}{2211}, 0\right).
\]

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Explicit Cayley triples in real forms of $E_7$  
Dragomir Ž. Đoković  
1

On the index formula for singular surfaces  
B. Fedosov, B.-W. Schulze and N. Tarkhanov  
25

Ordering the braid groups  
R. Fenn, M. T. Greene, D. Rolfsen, C. Rourke and B. Wiest  
49

Numerical semigroups generated by intervals  
P. A. García-Sánchez and J. C. Rosales  
75

A Selberg integral formula and applications  
Loukas Grafakos and Carlo Morpurgo  
85

Ascent and descent for finite sequences of commuting endomorphisms  
Luzius Grünenfelder  
95

Topology versus Chern numbers for complex 3-folds  
Claude LeBrun  
123

Cohomology of complete intersections in toric varieties  
Anvar R. Mavlyutov  
133

Connected sums of self-dual manifolds and equivariant relative smoothings  
Henrik Pedersen and Yat Sun Poon  
145

Some summations of $q$-series by telescoping  
M. V. Subbarao and A. Verma  
173

Dehornoy's ordering of the braid groups extends the subword ordering  
Bert Wiest  
183

A unified method of construction of elliptic curves with high Mordell–Weil rank  
Hizuru Yamagishi  
189