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We explicitly describe cohomology of complete intersections in compact simplicial toric varieties.

In this paper we will study intersections of hypersurfaces in compact simplicial toric varieties \mathbf{P}_Σ . The main purpose is to relate naturally the Hodge structure of a complete intersection $X_{f_1} \cap \dots \cap X_{f_s}$ in \mathbf{P}_Σ to a graded ring. Originally this idea appears in [Gr], [St], [Dol], [PS]. The case of a hypersurface in a toric variety has been treated in [BC]. Also the Hodge structure of complete intersections in a projective space was described in [Te], [Ko], [L], [Di], [Na]. The common approach was to reduce studying of the Hodge structure on a complete intersection to studying of the Hodge structure on a hypersurface in a higher dimensional projective variety. This is the idea of a “Cayley trick”. About a Cayley trick in the toric context see [GKZ], [DK], [BB]. A special case of a complete intersection (when it is empty) in a complete simplicial toric variety was elaborated in [CCD]. The basic references on toric varieties are [F1], [O], [Da], [C].

The paper is organized as follows:

Section 1 establishes notation and studies cohomology of subvarieties in a complete simplicial toric variety. In Section 2 we describe a Cayley trick for toric varieties. In Section 3 we prove the main result where we relate the Hodge components $H^{d-s-p,p}(X_{f_1} \cap \dots \cap X_{f_s})$ in the middle cohomology group to homogeneous components of a graded ring. Section 4 treats a special case of complete intersections: a nondegenerate intersection.

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1. Quasi-smooth intersections.

We first fix some notation. Let M be a lattice of rank d , $N = \text{Hom}(M, \mathbf{Z})$ the dual lattice; $M_{\mathbf{R}}$ (resp. $N_{\mathbf{R}}$) denotes the \mathbf{R} -scalar extension of M (resp. of N). Let Σ be a rational simplicial complete d -dimensional fan in $N_{\mathbf{R}}$ [BC], \mathbf{P}_Σ a complete simplicial toric variety associated with this fan.

Such a toric variety can be described as a geometric quotient [C]. Let $S(\Sigma) = \mathbf{C}[x_1, \dots, x_n]$ be the polynomial ring over \mathbf{C} with variables x_1, \dots, x_n

corresponding to the integral generators e_1, \dots, e_n of the 1-dimensional cones of Σ . For $\sigma \in \Sigma$ let $\hat{x}_\sigma = \prod_{e_i \notin \sigma} x_i$, and let $B(\Sigma) = \langle \hat{x}_\sigma : \sigma \in \Sigma \rangle \subset S$ be the ideal generated by the \hat{x}_σ 's. This ideal gives the variety $Z(\Sigma) = \mathbf{V}(B(\Sigma)) \subset \mathbf{A}^n$. The toric variety $\mathbf{P} = \mathbf{P}_\Sigma$ will be a geometric quotient of $U(\Sigma) := \mathbf{A}^n \setminus Z(\Sigma)$ by the group $\mathbf{D} := \text{Hom}_{\mathbf{Z}}(A_{d-1}(\mathbf{P}), \mathbf{C}^*)$, where $A_{d-1}(\mathbf{P})$ is the Chow group of Weil divisors modulo rational equivalence.

Each variable x_i in the coordinate ring $S(\Sigma)$ corresponds to a torus-invariant irreducible divisor D_i of \mathbf{P} . As in [C], we grade $S = S(\Sigma)$ by assigning to a monomial $\prod_{i=1}^n x_i^{a_i}$ its degree $[\sum_{i=1}^n a_i D_i] \in A_{d-1}(\mathbf{P})$. A polynomial f in the graded piece S_α corresponding to $\alpha \in A_{d-1}(\mathbf{P})$ is said to be \mathbf{D} -homogeneous of degree α .

Let f_1, \dots, f_s be \mathbf{D} -homogeneous polynomials. They define a zero set $\mathbf{V}(f_1, \dots, f_s) \subset \mathbf{A}^n$, moreover $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is stable under the action of \mathbf{D} and hence descends to a closed subset $X \subset \mathbf{P}$, because \mathbf{P} is a geometric quotient.

Definition 1.1. We say that X is a *quasi-smooth intersection* if $\mathbf{V}(f_1, \dots, f_s) \cap U(\Sigma)$ is either empty or a smooth subvariety of codimension s in $U(\Sigma)$.

Remark 1.2. This notion generalizes a nonsingular complete intersection in a projective space. Notice that since the $(n-d)$ -dimensional group \mathbf{D} has only zero dimensional stabilizers [BC], X is of pure dimension $d-s$ or empty.

We can now relate this notion to a V -submanifold (see Definition 3.2 in [BC]).

Proposition 1.3. *If $X \subset \mathbf{P}$ is a closed subset of codimension s defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , then X is a quasi-smooth intersection if and only if X is a V -submanifold of \mathbf{P} .*

The proof of this is very similar to the proof of the Proposition 3.5 in [BC].

The next result is a Lefschetz-type theorem.

Proposition 1.4. *Let $X \subset \mathbf{P}$ be a closed subset, defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , in a complete simplicial toric variety \mathbf{P} . If $f_1, \dots, f_s \in B(\Sigma)$, then the natural map $i^* : H^i(\mathbf{P}) \rightarrow H^i(X)$ is an isomorphism for $i < d-s$ and an injection for $i = d-s$. In particular, this is valid if X is an intersection of ample hypersurfaces.*

Proof. We can present $X = X_{f_1} \cap \dots \cap X_{f_s}$, where $X_{f_i} \subset \mathbf{P}$ is a hypersurface defined by f_i . As it was shown in the proof of the Proposition 10.8 [BC], if $f \in B(\Sigma)$ then $\mathbf{P} \setminus X_f = (\mathbf{A}^n \setminus \mathbf{V}(f))/\mathbf{D}(\Sigma)$ is affine, hence $H^i(\mathbf{P} \setminus X_f) = 0$ for $i > d$. We will prove by induction on s that $H^i(\mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_s})) = 0$ for $i > d+s-1$. Consider the Mayer-Vietoris sequence

$$\dots \rightarrow H^i(U \cap V) \rightarrow H^{i+1}(U \cup V) \rightarrow H^{i+1}(U) \oplus H^{i+1}(V) \rightarrow H^{i+1}(U \cap V) \rightarrow \dots$$

with $U = \mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_{s-1}})$, $V = \mathbf{P} \setminus X_{f_s}$. Notice that $U \cup V = \mathbf{P} \setminus (X_{f_1} \cap \dots \cap X_{f_s})$ and $U \cap V = \bigcup_{i=1}^{s-1} \mathbf{P} \setminus (X_{f_i} \cup X_{f_s}) = \mathbf{P} \setminus (X_{f_1 \cdot f_s} \cap \dots \cap X_{f_{s-1} \cdot f_s})$. So, using the induction and the above sequence, we obtain that $H^i(\mathbf{P} \setminus X) = 0$ for $i > d + s - 1$. As a consequence of this, X is nonempty unless $s > d$ because the dimension $h^{2d}(\mathbf{P}) = 1$. Since $\mathbf{P} \setminus X$ is a V-manifold, Poincaré duality implies that $H_c^i(\mathbf{P} \setminus X) = 0$ for $i \leq d - s$. Now the desired result follows from the long exact sequence of the cohomology with compact supports (X and \mathbf{P} are compact):

$$\dots \rightarrow H_c^i(\mathbf{P} \setminus X) \rightarrow H_c^i(\mathbf{P}) \rightarrow H_c^i(X) \rightarrow H_c^{i+1}(\mathbf{P} \setminus X) \rightarrow H_c^{i+1}(\mathbf{P}) \rightarrow \dots .$$

If X is an intersection of ample hypersurfaces defined by f_1, \dots, f_s , then Lemma 9.15 [BC] gives us that f_1, \dots, f_s belong to $B(\Sigma)$. \square

Corollary 1.5. *A quasi-smooth intersection $X = X_{f_1} \cap \dots \cap X_{f_s}$, defined by $f_1, \dots, f_s \in B(\Sigma)$, has pure dimension $d - s$.*

Since the dimension of $H^0(X, \mathbf{C})$ is the number of connected components of X , we obtain another important result.

Corollary 1.6. *An intersection $X_{f_1} \cap \dots \cap X_{f_s}$, defined by $f_1, \dots, f_s \in B(\Sigma)$, in a complete simplicial toric variety \mathbf{P}_Σ is connected provided $s < \dim \mathbf{P}_\Sigma$.*

Remark 1.7. If the polynomials f_1, \dots, f_s have ample degrees, then this corollary follows from a more general statement in [FL1] (see also [FL2] and [FH] for connectedness theorems).

2. “Cayley trick”.

We will explore a Cayley trick to reduce studying of the cohomology of quasi-smooth intersections to results already known for hypersurfaces.

Let L_1, \dots, L_s be line bundles on a complete d -dimensional toric variety $\mathbf{P} = \mathbf{P}_\Sigma$, and let $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$ be the projective space bundle associated to the vector bundle $E = L_1 \oplus \dots \oplus L_s$. Then the \mathbb{P}^{s-1} -bundle $\mathbf{P}(E)$ is a toric variety. The fan corresponding to it can be described as follows [O, p. 58]. Suppose that support functions h_1, \dots, h_s give rise to the isomorphism classes of line bundles $[L_1], \dots, [L_s] \in \text{Pic}(\mathbf{P})$, respectively. Introduce a \mathbf{Z} -module N' with a \mathbf{Z} -basis $\{n_2, \dots, n_s\}$ and let $\tilde{N} := N \oplus N'$ and $n_1 := -n_2 - \dots - n_s$. Denote by $\tilde{\sigma}$ the image of each $\sigma \in \Sigma$ under the \mathbf{R} -linear map $N_{\mathbf{R}} \rightarrow \tilde{N}_{\mathbf{R}}$ which sends $y \in N_{\mathbf{R}}$ to $y - \sum_{j=1}^s h_j(y)n_j$. On the other hand, let σ'_i be the cone in $N'_{\mathbf{R}}$ generated by $n_1, \dots, n_i, n_{i+1}, \dots, n_s$ and let Σ' be the fan in $N'_{\mathbf{R}}$ consisting of the faces of $\sigma'_1, \dots, \sigma'_s$. Then $\mathbf{P}(E)$ corresponds to the fan $\tilde{\Sigma} := \{\tilde{\sigma} + \sigma' : \sigma \in \Sigma, \sigma' \in \Sigma'\}$. From this description it is easy to see that if Σ is a complete simplicial fan then $\mathbf{P}(L_1 \oplus \dots \oplus L_s)$ is a complete simplicial toric variety. We see that the integral generators of

the 1-dimensional cones in $\tilde{\Sigma}$ are given by

$$\begin{aligned}\tilde{e}_i &= e_i - \sum_{1 \leq j \leq s} h_j(e_i) n_j, \quad i = 1, \dots, n, \\ \tilde{n}_1 &= -n_2 - \dots - n_s, \\ \tilde{n}_j &= n_j, \quad j = 2, \dots, s,\end{aligned}$$

where e_1, \dots, e_n are the integral generators of the 1-dimensional cones in Σ .

The homogeneous coordinate ring of $\mathbf{P}(E)$ is the polynomial ring

$$R = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_s],$$

where x_i corresponds to \tilde{e}_i and y_j corresponds to \tilde{n}_j . This ring has a grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Since \mathbf{P} is a normal variety, there is an embedding of the Picard group $\text{Pic}(\mathbf{P}) \hookrightarrow A_{d-1}(\mathbf{P})$. We want to show that if some polynomials $f_j \in S(\Sigma) = \mathbf{C}[x_1, \dots, x_n]$ have the property $\deg(f_j) = [L_j] \in \text{Pic}(\mathbf{P})$, then the polynomials $y_j f_j$ all have the same degree in R . This will allow us to consider a hypersurface defined by the homogeneous polynomial $F = \sum_{j=1}^s y_j f_j$.

Lemma 2.1. *Let $f_1, \dots, f_s \in S(\Sigma)$ be \mathbf{D} -homogeneous polynomials, such that $\deg(f_j) = [L_j]$ for some line bundles L_1, \dots, L_s . Then $F = \sum_{j=1}^s y_j f_j$ is homogeneous in R and its degree is the isomorphism class $[O_{\mathbf{P}(E)}(1)]$ of the canonical line bundle on $\mathbf{P}(E) = \mathbf{P}(L_1 \oplus \dots \oplus L_s)$.*

Proof. To prove that F is a homogeneous polynomial we will repeat the arguments in the proof of Lemma 3.5 in [CCD]. Let D_1, \dots, D_n be the torus-invariant divisors on $\mathbf{P} = \mathbf{P}_\Sigma$ corresponding to the 1-dimensional cones of the fan Σ . Then the pullback $\pi^* D_i$ is the torus-invariant divisor of $\mathbf{P}(E)$ corresponding to the cone generated by \tilde{e}_i . Also denote by D'_j the torus-invariant divisor corresponding to \tilde{n}_j . Let $\tilde{M} = M \oplus M'$ be the lattice dual to $\tilde{N} = N \oplus N'$ with $M' = \text{Hom}(N', \mathbf{Z})$ having $\{n_2^*, \dots, n_s^*\}$ as a basis dual to $\{n_2, \dots, n_s\}$. The divisor corresponding to the character $\chi^{n_j^*}$ is

$$\begin{aligned}\text{div}(\chi^{n_j^*}) &= \sum_{i=1}^n \langle n_j^*, \tilde{e}_i \rangle \pi^* D_i + \sum_{k=1}^s \langle n_j^*, \tilde{n}_k \rangle D'_k \\ &= \sum_{i=1}^n (h_1(e_i) - h_j(e_i)) \pi^* D_i - D'_1 + D'_j.\end{aligned}$$

Therefore, $[D'_j] + [\pi^* L_j]$ all have the same degree in the Chow group $A_{d+s-2}(\mathbf{P}(E))$, and, consequently, F is a homogeneous polynomial.

Now consider the following exact sequence [M]:

$$0 \rightarrow O_{\mathbf{P}(E)} \rightarrow \pi^* E^* \otimes O_{\mathbf{P}(E)}(1) \rightarrow T_{\mathbf{P}(E)} \rightarrow \pi^* T_{\mathbf{P}} \rightarrow 0,$$

where T_X denotes the tangent bundle, E^* is the dual bundle. From here we can compute the Chern class

$$\begin{aligned} c_1(\mathbf{T}_{\mathbf{P}(E)}) &= c_1(\pi^*\mathbf{T}_{\mathbf{P}}) + c_1(\pi^*E^* \otimes \mathcal{O}_{\mathbf{P}(E)}(1)) \\ &= \pi^*c_1(\mathbf{T}_{\mathbf{P}}) - \pi^*c_1(E) + s \cdot c_1(\mathcal{O}_{\mathbf{P}(E)}(1)). \end{aligned}$$

Hence, $s \cdot c_1(\mathcal{O}_{\mathbf{P}(E)}(1)) = \pi^*c_1(L_1) + \cdots + \pi^*c_1(L_s) + c_1(\mathbf{T}_{\mathbf{P}(E)}) - \pi^*c_1(\mathbf{T}_{\mathbf{P}})$. On the other hand, from the generalized Euler exact sequence [BC, §12] we get

$$0 \rightarrow \mathcal{O}_{\mathbf{P}}^{n-d} \rightarrow \bigoplus_{i=1}^n \mathcal{O}_{\mathbf{P}}(D_i) \rightarrow \mathbf{T}_{\mathbf{P}} \rightarrow 0.$$

This implies that $c_1(\mathbf{T}_{\mathbf{P}}) = [D_1] + \cdots + [D_n]$. Similarly we have $c_1(\mathbf{T}_{\mathbf{P}(E)}) = [\pi^*D_1] + \cdots + [\pi^*D_n] + [D'_1] + \cdots + [D'_s]$. Under the identification $\text{Pic}(\mathbf{P}(E)) \hookrightarrow A_{d+s-2}(\mathbf{P}(E))$ the first Chern class of a line bundle on $\mathbf{P}(E)$ is exactly its isomorphism class in the Picard group $\text{Pic}(\mathbf{P}(E))$. Therefore

$$s \cdot [\mathcal{O}_{\mathbf{P}(E)}(1)] = [\pi^*L_1] + \cdots + [\pi^*L_s] + [D'_1] + \cdots + [D'_s] = s \cdot ([\pi^*L_2] + [D'_2]).$$

It can be easily checked that D'_2 is a Cartier divisor on $\mathbf{P}(E)$. Hence all classes $[\mathcal{O}_{\mathbf{P}(E)}(1)]$, $[\pi^*L_2]$ and $[D'_2]$ lie in the Picard group $\text{Pic}(\mathbf{P}(E))$. But this group is free abelian, because $\mathbf{P}(E)$ is complete. So the above equality is divisible by s : $[\mathcal{O}_{\mathbf{P}(E)}(1)] = [\pi^*L_2] + [D'_2] = \text{deg}(F)$. \square

From now on we assume that $\mathbf{P} = \mathbf{P}_{\Sigma}$ is a complete simplicial toric variety and that $\text{deg}(f_j) \in \text{Pic}(\mathbf{P})$, $j = 1, \dots, s$. Denote by Y the hypersurface in $\mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$.

Lemma 2.2. *$X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection iff the hypersurface Y is quasi-smooth.*

Proof. $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection means that whenever $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$, the $\text{rank}(\frac{\partial f_j}{\partial x_i}(x))_{i,j} = s$. And Y is quasi-smooth iff $z = (x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j = 1, \dots, s$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \dots, n$, is nonzero.

So let $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$, then there is a cone $\tilde{\sigma} + \sigma' \in \tilde{\Sigma}$ with $\sigma \in \Sigma$, $\sigma' \in \Sigma'$, such that $\prod_{\tilde{e}_i \notin \tilde{\sigma}} x_i \prod_{\tilde{n}_j \notin \sigma'} y_j \neq 0$ where x_i, y_j are the coordinates of (x, y) . If $f_1(x) = \cdots = f_s(x) = 0$, then $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$ because $\prod_{e_i \in \sigma} x_i \neq 0$. And if $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i = 1, \dots, n$, is nonzero.

Conversely, suppose Y is quasi-smooth. Pick any $x \in \mathbf{V}(f_1, \dots, f_s) \setminus Z(\Sigma)$, then $(x, y) \in \mathbf{V}(F) \setminus Z(\tilde{\Sigma})$ for each $y = (y_1, \dots, y_s) \neq 0$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i , which means the $\text{rank}(\frac{\partial f_j}{\partial x_i}(x))_{i,j}$ is maximal. \square

3. Cohomology of quasi-smooth intersections.

Since a quasi-smooth intersection is a compact V-manifold (Proposition 1.3), the cohomology on it has a pure Hodge structure. Using Proposition 1.4 and the Poincaré duality, we can compute the cohomology of a quasi-smooth intersection except for the cohomology in the middle dimension $d - s$. So we introduce the following definition.

Definition 3.1. The *variable cohomology group* $H_{\text{var}}^{d-s}(X)$ is $\text{coker}(H^{d-s}(\mathbf{P}) \xrightarrow{i^*} H^{d-s}(X))$.

The variable cohomology group also has a pure Hodge structure.

Proposition 3.2. *Let $X = X_{f_1} \cap \dots \cap X_{f_s}$ be a quasi-smooth intersection of ample hypersurfaces. Then there is an exact sequence of mixed Hodge structures*

$$0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P}) \rightarrow H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{\text{var}}^{d-s}(X) \rightarrow 0,$$

where $[X] \in H^{2s}(\mathbf{P})$ is the cohomology class of X .

Proof. Consider the Gysin exact sequence:

$$(1) \quad \dots \rightarrow H^{i-2s}(X) \xrightarrow{i_!} H^i(\mathbf{P}) \rightarrow H^i(\mathbf{P} \setminus X) \rightarrow H^{i-2s+1}(X) \xrightarrow{i_!} H^{i+1}(\mathbf{P}) \rightarrow \dots$$

Since i^* is Poincaré dual to the Gysin map $i_!$, it follows that $H_{\text{var}}^{d-s}(X)$ is isomorphic to the kernel of $i_! : H^{d-s}(X) \rightarrow H^{d+s}(\mathbf{P})$. So we get an exact sequence

$$H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(\mathbf{P}) \rightarrow H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{\text{var}}^{d-s}(X) \rightarrow 0.$$

Now we use a commutative diagram

$$\begin{array}{ccc} H^{d-s-1}(X) & \xrightarrow{i_!} & H^{d+s-1}(\mathbf{P}) \\ i^* \uparrow & \nearrow \cup[X] & \\ H^{d-s-1}(\mathbf{P}) & & \end{array}$$

By Proposition 1.4 i^* is an isomorphism in this diagram, so it suffices to prove that the Gysin map $i_!$ is injective in the above diagram.

Lemma 3.3. *If $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a quasi-smooth intersection of ample hypersurfaces, then the Gysin map $H^{d-s-1}(X) \xrightarrow{i_!} H^{d+s-1}(\mathbf{P})$ is injective.*

Proof. Since the odd dimensional cohomology of a complete simplicial toric variety vanishes [F1, pp. 92-94] and $i^* : H^{d-s-1}(\mathbf{P}) \rightarrow H^{d-s-1}(X)$ is an isomorphism by Proposition 1.4, it follows that $H^{d-s-1}(X) = H^{d-s-1}(\mathbf{P}) = H^{d+s-1}(\mathbf{P}) = 0$ when $d + s - 1$ is odd. So by the Gysin exact sequence (1) it is enough to show that $H^{d+s-2}(\mathbf{P} \setminus X) = 0$ when $d + s - 2$ is odd. To prove this we use the Cayley trick again. Let Y be the hypersurface defined by

$F = \sum_{j=1}^s y_j f_j$. Then the natural map $\mathbf{P}(E) \setminus Y \rightarrow \mathbf{P} \setminus X$, induced by the projection $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$, is a \mathbf{C}^{s-1} bundle in the Zariski topology. Notice that $\mathbf{P} \setminus X$ is simply connected, because \mathbf{P} is simply connected [F1, p. 56] and X has codimension at least 2 in \mathbf{P} . Hence, the Leray-Serre spectral sequence implies that $H^i(\mathbf{P}(E) \setminus Y) = H^i(\mathbf{P} \setminus X)$ for $i \geq 0$. We have that $H^{d+s-2}(\mathbf{P}(E)) = 0$ for $d+s-2$ odd and Y is quasi-smooth by Lemma 2.2. So from the Gysin exact sequence

$$H^{d+s-2}(\mathbf{P}(E)) \rightarrow H^{d+s-2}(\mathbf{P}(E) \setminus Y) \rightarrow H^{d+s-3}(Y) \xrightarrow{j_!} H^{d+s-1}(\mathbf{P}(E))$$

(here the Gysin map $j_!$ is induced by the inclusion $j : Y \hookrightarrow \mathbf{P}(E)$) it follows that we need to show injectivity of $j_! : H^{d+s-3}(Y) \rightarrow H^{d+s-1}(\mathbf{P}(E))$. Consider the commutative diagram

$$\begin{array}{ccc} H^{d+s-3}(Y) & \xrightarrow{j_!} & H^{d+s-1}(\mathbf{P}(E)) \\ j^* \uparrow & \nearrow_{\cup[Y]} & \\ H^{d+s-3}(\mathbf{P}(E)) & & \end{array}$$

where $[Y] \in H^2(\mathbf{P}(E))$ is the cohomology class of Y . The canonical line bundle $O_{\mathbf{P}(E)}(1)$ is ample [H, III, §1], whence by Lemma 2.1, Y is ample. So by Proposition 10.8 [BC] $j^* : H^{d+s-3}(\mathbf{P}(E)) \rightarrow H^{d+s-3}(Y)$ is an isomorphism and by Hard Lefschetz $\cup[Y] : H^{d+s-3}(\mathbf{P}(E)) \rightarrow H^{d+s-1}(\mathbf{P}(E))$ is injective. Thus, from the above diagram the lemma follows. \square

Definition 3.4. For a nonzero polynomial $F \in R = \mathbf{C}[x_1, \dots, x_n, y_1, \dots, y_s]$ the *Jacobian ring* $R(F)$ denotes the quotient of R by the ideal generated by the partial derivatives $\frac{\partial F}{\partial y_j}$, $j = 1, \dots, s$, $\frac{\partial F}{\partial x_i}$, $i = 1, \dots, n$.

Remark 3.5. If $F = y_1 f_1 + \dots + y_s f_s$ is as in Lemma 2.1 with $f_j \in S_{\alpha_j}$, then $R(F)$ carries a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E))$. Moreover, there are canonical isomorphisms $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus A_d(\mathbf{P}) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$ ([F2]). With respect to this bigrading of the Chow group $A_{d+s-2}(\mathbf{P}(E))$ we have that $\deg(F) = (0, 1)$, $\deg(f_j) = (\alpha_j, 0)$, $\deg(y_j) = (-\alpha_j, 1)$, which is very similar to the case when \mathbf{P} is a projective space.

We now can state the main result.

Theorem 3.6. *Let \mathbf{P} be a d -dimensional complete simplicial toric variety, and let $X \subset \mathbf{P}$ be a quasi-smooth intersection of ample hypersurfaces defined by $f_j \in S_{\alpha_j}$, $j = 1, \dots, s$. If $F = y_1 f_1 + \dots + y_s f_s$, then for $p \neq \frac{d+s-1}{2}$, we have a canonical isomorphism*

$$R(F)_{(d+s-p)\beta-\beta_0} \cong H_{\text{var}}^{p-s, d-p}(X)$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s)$, $\beta = \deg(F) = \deg(f_j) + \deg(y_j)$. In the case $p = \frac{d+s-1}{2}$ there is an exact sequence

$$0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P}) \rightarrow R(F)_{\frac{d+s+1}{2}\beta-\beta_0} \rightarrow H_{\text{var}}^{\frac{d-s-1}{2}, \frac{d-s+1}{2}}(X) \rightarrow 0.$$

Proof. Since $H^i(\mathbf{P})$ vanishes for i odd and has a pure Hodge structure of type (p, p) for i even, from Proposition 3.2 we get $\mathrm{Gr}_F^p H^{d+s-1}(\mathbf{P} \setminus X) \cong H_{var}^{p-s, d-p}(X)$ if $p \neq \frac{d+s-1}{2}$, and in case $p = \frac{d+s-1}{2}$ the following sequence

$$\begin{aligned} 0 \rightarrow H^{d-s-1}(\mathbf{P}) \xrightarrow{\cup[X]} H^{d+s-1}(\mathbf{P}) \\ \rightarrow \mathrm{Gr}_F^{\frac{d+s-1}{2}} H^{d+s-1}(\mathbf{P} \setminus X) \rightarrow H_{var}^{\frac{d-s-1}{2}, \frac{d-s+1}{2}}(X) \rightarrow 0 \end{aligned}$$

is exact.

Now use the isomorphism of mixed Hodge structures $H^i(\mathbf{P} \setminus X) \cong H^i(\mathbf{P}(E) \setminus Y)$ and by the Theorem 10.6 [BC] the desired result follows. \square

4. Cohomology of nondegenerate intersections.

In this section we consider a special case of quasi-smooth intersections.

Definition 4.1. A closed subset $X = X_{f_1} \cap \dots \cap X_{f_s}$, defined by \mathbf{D} -homogeneous polynomials f_1, \dots, f_s , is called a *nondegenerate intersection* if $X_{f_{j_1}} \cap \dots \cap X_{f_{j_k}} \cap \mathbf{T}_\tau$ is a smooth subvariety of codimension k in \mathbf{T}_τ for any $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$ and $\tau \in \Sigma$. (Here \mathbf{T}_τ denotes the torus in \mathbf{P}_Σ associated with a cone $\tau \in \Sigma$.)

We will show how to define a nondegenerate intersection in terms of the polynomials f_1, \dots, f_s . For $\sigma \in \Sigma$, let $U_\sigma = \{x \in \mathbf{A}^n : \hat{x}_\sigma \neq 0\}$. We know that \mathbf{P}_Σ has an affine toric open cover by $\mathbf{A}_\sigma = U_\sigma / \mathbf{D}(\Sigma)$, $\sigma \in \Sigma$ [BC]. Also $\mathbf{T}_\tau = (U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma) / \mathbf{D}(\Sigma)$. Notice that $U_\tau \setminus \cup_{\gamma \prec \tau} U_\gamma = \{x \in \mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$ is a torus. So each \mathbf{T}_τ is a quotient of a torus by a D -subgroup, because \mathbf{D} is diagonalizable [BC].

Lemma 4.2. *Let $T = (\mathbf{C}^*)^n / G$ be the quotient of a torus by a D -subgroup G . Suppose that $X \subset (\mathbf{C}^*)^n$ is an invariant subvariety with respect to the action of G . Then the geometric quotient X/G is smooth iff X is smooth.*

Proof. By the structure theorem of a D -group [Hu, §16.2] we can assume that $(\mathbf{C}^*)^n = G^\circ \times (\mathbf{C}^*)^k$, where $G^\circ \cong (\mathbf{C}^*)^{n-k}$ is the identity component of G , and $G = G^\circ \times H$ for some finite subgroup H in $(\mathbf{C}^*)^k$. Now it suffices to show the Lemma if G is a torus or a finite group. If $G = G^\circ$ then $X = (\mathbf{C}^*)^{n-k} \times p(X)$, where by $p(X)$ we mean the projection of X onto $(\mathbf{C}^*)^k$. Notice that $p(X) \cong X/G$, hence X is smooth iff X/G is smooth. In the case $G = H$ is a finite group it can be easily checked that $X \rightarrow X/G$ is an unramified cover [Sh, p. 346]. So X and X/G are smooth simultaneously. \square

From this Lemma it follows that $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection iff $\mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ is a smooth subvariety of codimension k in the torus $V_\tau = \{x \in \mathbf{A}^n : \hat{x}_\tau \neq 0, x_i = 0 \text{ if } \rho_i \subset \tau\}$.

As in Section 2 we can consider the hypersurface $Y \subset \mathbf{P}(E)$ defined by $F = \sum_{j=1}^s y_j f_j$. \square

Lemma 4.3. $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection iff Y is a nondegenerate hypersurface.

Proof. As shown above, $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection if the rank $(\frac{\partial f_j}{\partial x_i}(x))_{\substack{j \in \{j_1, \dots, j_k\} \\ i \in \{i: e_i \notin \tau\}}} = k$ for all $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$, $\tau \in \Sigma$ and $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$. Similarly Y is nondegenerate iff $z = (x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$, $\tilde{\tau} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$ (recall the definition of $\mathbf{P}(E)$ associated with $\tilde{\Sigma}$ in the Section 2) implies that one of the partial derivatives $\frac{\partial F}{\partial y_j}(z) = f_j(x)$, $j \in \{j : \tilde{n}_j \notin \tau'\}$, $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tilde{\tau}\}$, is nonzero.

Let $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$, where $\tilde{\tau} + \tau' \in \tilde{\Sigma}$ with $\tau \in \Sigma$, $\tau' \in \Sigma'$. Then $\prod_{\tilde{e}_i \notin \tilde{\tau}} x_i \prod_{\tilde{n}_j \notin \tau'} y_j \neq 0$ and $x_i = 0$ if $\tilde{e}_i \in \tilde{\tau}$, $y_j = 0$ if $\tilde{n}_j \in \tau'$. If $f_j(x) = 0$ for all $j \in \{j : \tilde{n}_j \notin \tau'\}$, then $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ where $\{j_1, \dots, j_k\} = \{j : \tilde{n}_j \notin \tau'\}$. So if $X = X_{f_1} \cap \dots \cap X_{f_s}$ is a nondegenerate intersection, one of the partial derivatives $\frac{\partial F}{\partial x_i}(z) = \sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x)$, $i \in \{i : \tilde{e}_i \notin \tilde{\tau}\}$, is nonzero.

Conversely, suppose Y is nondegenerate. Take any $x \in \mathbf{V}(f_{j_1}, \dots, f_{j_k}) \cap V_\tau$ with $\tau \in \Sigma$, $\{j_1, \dots, j_k\} \subset \{1, \dots, s\}$. Then $(x, y) \in \mathbf{V}(F) \cap V_{\tilde{\tau} + \tau'}$ for each $y \in V_{\tau'} = \{y \in \mathbf{A}^s : y_j \neq 0 \text{ if } \tilde{n}_j \notin \tau', y_j = 0 \text{ if } \tilde{n}_j \in \tau'\}$ where τ' is the cone generated by the complement of $\{\tilde{n}_{j_1}, \dots, \tilde{n}_{j_k}\}$ in the set $\{\tilde{n}_1, \dots, \tilde{n}_s\}$. Therefore $\sum_{j=1}^s y_j \frac{\partial f_j}{\partial x_i}(x) \neq 0$ for some i , which means the rank $(\frac{\partial f_j}{\partial x_i}(x))_{i \in \{i: e_i \notin \tau\}}^{j \in \{j_1, \dots, j_k\}} = k$. \square

Since a nondegenerate hypersurface is quasi-smooth [BC], Lemma 2.2 shows that a nondegenerate intersection is quasi-smooth.

Definition 4.4 ([BC]). Given a polynomial $f \in S = \mathbf{C}[x_1, \dots, x_n]$, we get the ideal quotient $J_1(f) = \langle x_1 \partial f / \partial x_1, \dots, x_n \partial f / \partial x_n \rangle : x_1 \cdots x_n$ (see [CLO, p. 193]) and the ring $R_1(f) = S/J_1(f)$.

Remark 4.5. If $F = \sum_{j=1}^s y_j f_j \in R$ is as in Lemma 2.1, then $R_1(F) = R/J_1(F)$ has a natural grading by the Chow group $A_{d+s-2}(\mathbf{P}(E)) \cong A_{d-1}(\mathbf{P}) \oplus \mathbf{Z}$.

Theorem 4.6. Let $X = X_{f_1} \cap \dots \cap X_{f_s}$ be a nondegenerate intersection of ample hypersurfaces given by $f_j \in S_{\alpha_j}$, $j = 1, \dots, s$. If $F = \sum_{j=1}^s y_j f_j \in R$, then there is a canonical isomorphism

$$H_{\text{var}}^{p-s, d-p}(X) = R_1(F)_{(d+s-p)\beta - \beta_0},$$

where $\beta_0 = \deg(x_1 \cdots x_n \cdot y_1 \cdots y_s)$, $\beta = \deg(F)$.

Proof. First we will show that there is an isomorphism of Hodge structures $H_{\text{var}}^{d-s}(X)(1-s) \cong H_{\text{var}}^{d+s-2}(Y)$. Let $\varphi : Y \rightarrow \mathbf{P}$ be the composition of the inclusion $j : Y \hookrightarrow \mathbf{P}(E)$ and the projection $\pi : \mathbf{P}(E) \rightarrow \mathbf{P}$. As in [Te], consider the following morphism of the Leray spectral sequences

$$\begin{array}{ccc} E_2^{p,q} = H^p(\mathbf{P}, R^q\pi_*\mathbf{C}) & \Rightarrow & H^{p+q}(\mathbf{P}(E)) \\ & \downarrow & \downarrow \\ {}'E_2^{p,q} = H^p(\mathbf{P}, R^q\varphi_*\mathbf{C}) & \Rightarrow & H^{p+q}(Y). \end{array}$$

Since

$$\varphi^{-1}(X) = \begin{cases} \mathbb{P}^{s-1} & \text{if } x \in X, \\ \mathbb{P}^{s-2} & \text{if } x \notin X, \end{cases}$$

we have that (see [Go, p. 202], [De])

$$R^q\varphi_*\mathbf{C} = \begin{cases} \mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q < 2s-2, \\ \mathbf{C}_X(1-s) & \text{if } q = 2s-2, \\ 0 & \text{otherwise.} \end{cases}$$

Also we have

$$R^q\pi_*\mathbf{C} = \begin{cases} \mathbf{C}_{\mathbf{P}}(-\frac{q}{2}) & \text{if } q \text{ is even and } 0 \leq q \leq 2s-2, \\ 0 & \text{otherwise.} \end{cases}$$

The first spectral sequence degenerates at E_2 , because for p or q odd $E_r^{p,q}$ vanishes. The second spectral sequence also degenerates at E_2 :

$$h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(\mathbf{P}) = \sum_{p+q=l} \dim {}'E_2^{p,q} \geq \sum_{p+q=l} \dim {}'E_{\infty}^{p,q} = h^l(Y).$$

To show the degeneracy of $'E_2^{p,q}$ it suffices to show that the above inequality is an equality. From Proposition 10.8 [BC] and Proposition 3.2 we get

$$\begin{aligned} h^{d+s-2}(Y) &= h^{d+s-2}(\mathbf{P}(E)) + h^{d+s-1}(\mathbf{P}(E) \setminus Y) \\ &\quad - h^{d+s-1}(\mathbf{P}(E)) + h^{d+s-3}(\mathbf{P}(E)), \\ h^{d-s}(X) &= h^{d-s}(\mathbf{P}) + h^{d+s-1}(\mathbf{P} \setminus X) - h^{d+s-1}(\mathbf{P}) + h^{d-s-1}(\mathbf{P}). \end{aligned}$$

Hence, using the spectral sequence $E_2^{p,q}$, we can easily compute the Hodge numbers of $\mathbf{P}(E)$ and check that $h^{l-2s-2}(X) + \sum_{q=0}^{2s-4} h^{l-q}(\mathbf{P}) = h^l(Y)$ for $l = d + s - 2$. Using Proposition 1.4, we can similarly show the above equality for $l \neq d + s - 2$ as well. So the spectral sequence $'E_2^{p,q}$ degenerates at E_2 . Since $E_2^{d+s-2-q,q} = {}'E_2^{d+s-2-q,q}$ for $q \neq 2s-2$ and, by Proposition 1.4, $E_2^{d-s,2s-2} \hookrightarrow {}'E_2^{d-s,2s-2}$, we get an isomorphism of Hodge structures (for details see [Te]):

$$H_{\text{var}}^{d+s-2}(Y) \cong {}'E_2^{d-s,2s-s} / E_2^{d-s,2s-2} \cong H_{\text{var}}^{d-s}(X)(1-s).$$

Now we only need to apply Theorem 11.8 [BC] to finish the proof. \square

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