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Summation formulae for q -series with independent bases are obtained and used to derive transformation and expansion of q -series involving independent bases.

1. Introduction.

The sum of the first $(n + 1)$ -terms of the non-terminating very-well-poised ${}_6\phi_5[q][1]$

$$(1.1) \quad \sum_{k=0}^n \frac{(1 - aq^{2k})}{(1 - a)} \frac{(a, b, c, \frac{a}{bc}; q)_k}{(q, \frac{aq}{b}, \frac{aq}{c}, bcq; q)_k} q^k = \frac{(aq, bq, cq, \frac{aq}{bc}; q)_n}{(\frac{aq}{b}, \frac{aq}{c}, bcq, q; q)_n}$$

follows from Jacksons q -analogue of Whipple's summation formula for a terminating very-well-poised balanced ${}_8\phi_7[q]$ (in [4, 2.6.2] setting $e = aq^{n+1}$). A bibasic analogue of (1.1) was obtained by Gasper [3].

$$(1.2) \quad \sum_{k=0}^n \frac{(1 - aq^k p^k)(1 - bp^k q^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, \frac{a}{bc}; q)_k}{(q, \frac{aq}{b}, q)_k (\frac{ap}{c}, bcp; p)_k} q^k \\ = \frac{(ap, bp; p)_n (cq, \frac{aq}{bc}; q)_n}{(q, \frac{aq}{b}; q)_n (\frac{ap}{c}, bcp; p)_n}, \quad n = 0, 1, 2, \dots$$

and he used it for obtaining quadratic and cubic summation and transformations formulae for q -hypergeometric series. A little later Gasper and Rahman [5] obtained a bilateral extension of Gasper's bibasic summation formula (1.2):

If m, n are non-negative integers, then

$$(1.3) \quad \sum_{k=-m}^n \frac{(1 - adp^k q^k)}{(1 - ad)} \frac{(1 - \frac{bp^k q^{-k}}{d})}{(1 - \frac{b}{d})} \frac{(a, b; p)_k}{(dq, \frac{adq}{b}; q)_k} \frac{(c, \frac{ad^2}{bc}; q)_k}{(\frac{adp}{c}, \frac{bcp}{d}; p)_k} q^k \\ = \frac{(1 - a)(1 - b)(1 - c)(1 - \frac{ad^2}{bc})}{d(1 - ad)(1 - \frac{b}{d})(1 - \frac{c}{d})(1 - \frac{ad}{bc})} \\ \times \left\{ \frac{(ap, bp; p)_n}{(dq, \frac{adq}{b}; q)_n} \frac{(cq, \frac{ad^2 q}{bc}; q)_n}{(\frac{adp}{c}, \frac{bcp}{d}; q)_n} - \frac{(\frac{c}{ad}, \frac{d}{bc}; p)_{m+1} (\frac{1}{d}, \frac{b}{ad}; q)_{m+1}}{(\frac{1}{c}, \frac{bc}{ad^2}; q)_{m+1} (\frac{1}{a}, \frac{1}{b}; p)_{m+1}} \right\}.$$

Jain and Verma [6] used transformations of q -hypergeometric series to obtain a summation formula involving three independent bases:

$$\begin{aligned}
 (1.4) \quad & \sum_{k=-m}^n \frac{(\beta; p)_k (c; q)_k (y; p)_k \left(\frac{\beta y c}{d^2}; \frac{pP}{q}\right)_k \left[\left(1 - \frac{\beta c y}{d} p^k P^k\right) \left(1 - \frac{y}{d} P^k q^{-k}\right) \left(1 - \frac{\beta}{d} p^k q^{-k}\right) \right] q^k}{(dq; q)_k \left(\frac{\beta c p}{d}; p\right)_k \left(\frac{\beta y}{d}; \frac{pP}{q}; \frac{pP}{q}\right)_k \left(\frac{c y P}{d^2}; P\right)_k} \\
 &= \frac{(1 - \beta)(1 - c)(1 - y)(1 - \frac{\beta c y}{d^2})}{(c - d)} \left\{ \frac{\left(\frac{d}{\beta c}; p\right)_{m+1} \left(\frac{1}{d}; q\right)_{m+1} \left(\frac{d}{c y}; P\right)_{m+1} \left(\frac{d}{\beta y}; \frac{pP}{q}\right)_{m+1}}{\left(\frac{1}{\beta}; p\right)_{m+1} \left(\frac{1}{c}; q\right)_{m+1} \left(\frac{1}{y}; P\right)_{m+1} \left(\frac{d^2}{\beta c y}; \frac{pP}{q}\right)_{m+1}} \right. \\
 &\quad \left. - \frac{(\beta p; p)_n (c q; q)_n (y P; P)_n \left(\frac{\beta c y}{d^2}; \frac{pP}{q}; \frac{pP}{q}\right)_n}{\left(\frac{\beta c p}{d}; p\right)_n (d q; q)_n \left(\frac{c y P}{d}; P\right)_n \left(\frac{\beta y}{d}; \frac{pP}{q}; \frac{pP}{q}\right)_n} \right\},
 \end{aligned}$$

which for $P = q$ reduces to the Gasper-Rahman’s summation formula (1.3). The proof of (1.4) could be given by considering

$$\beta_k = \frac{(\beta p; p)_k (c q; q)_k (y P; P)_k \left(\frac{\beta c y}{d^2}; \frac{pP}{q}; \frac{pP}{q}\right)_k}{\left(\frac{\beta c}{d}; p\right)_k (d q; q)_k \left(\frac{c y}{d}; P\right)_k \left(\frac{\beta y}{d}; \frac{pP}{q}; \frac{pP}{q}\right)_k}$$

and observing that

$$\begin{aligned}
 (1.5) \quad \Delta \beta_k &= (\beta p; p)_{k-1} (c q; q)_{k-1} (y P; P)_{k-1} \left(\frac{\beta y c}{d^2}; \frac{pP}{q}; \frac{pP}{q}\right)_{k-1} \\
 &\quad \times \frac{\left[\left(1 - \frac{\beta c y}{d^2} p^k P^k\right) \left(1 - \frac{y}{d} P^k q^{-k}\right) \left(1 - \frac{\beta}{d} p^k q^{-k}\right) \right]}{\left(\frac{\beta c p}{d}; p\right)_k (d q; q)_k \left(\frac{\beta y}{d}; \frac{pP}{q}; \frac{pP}{q}\right)_k \left(\frac{c y P}{d}; P\right)_k},
 \end{aligned}$$

and summing for k from $-m$ to n (m, n are non-negative integers) and using the usual convention:

$$(1.6) \quad \prod_{k=m}^n A_k = \begin{cases} A_m A_{m+1} \cdots A_n & m \leq n \\ 1 & m = n - 1 \\ (A_{n+1} A_{n+2} \cdots A_{m-1})^{-1} & m \geq n - 2. \end{cases}$$

Chu [2] obtained a generalization of Gasper-Rahman’s formula (after re-naming suitably the sequences so as to remove redundant sequences)

$$\begin{aligned}
 (1.7) \quad & \sum_{k=-m}^n \frac{(1 - \alpha a_k b_k) \left(b_k - \frac{a_k}{\alpha d}\right)}{(1 - \alpha a_0 b_0) \left(b_0 - \frac{a_0}{\alpha d}\right)} \frac{\prod_{j=0}^{k-1} \left[\left(1 - a_j\right) \left(1 - \frac{a_j}{d}\right) \left(1 - c b_j\right) \left(1 - \frac{\alpha^2 d}{c} b_j\right) \right]}{\prod_{j=1}^k \left[\left(1 - \alpha b_j\right) \left(1 - \alpha d b_j\right) \left(1 - \frac{\alpha a_j}{c}\right) \left(1 - \frac{c}{\alpha d} a_j\right) \right]} \\
 &= \frac{(1 - a_0) \left(1 - \frac{a_0}{d}\right) (1 - b_0 c) \left(1 - \frac{\alpha^2 d}{c} b_0\right)}{\alpha \left(1 - \alpha a_0 b_0\right) \left(1 - \frac{c}{\alpha}\right) \left(b_0 - \frac{a_0}{\alpha d}\right) \left(1 - \frac{\alpha d}{c}\right)}
 \end{aligned}$$

$$\begin{aligned} & \times \left\{ \prod_{j=1}^n \left[\frac{(1 - a_j)(1 - \frac{a_j}{d})(1 - b_j c)(1 - \frac{\alpha^2 d}{c} b_j)}{(1 - \alpha b_j)(1 - \alpha b_j d)(1 - \frac{\alpha a_j}{c})(1 - \frac{c}{\alpha d} a_j)} \right] \right. \\ & \left. - \prod_{j=-m}^0 \left[\frac{(1 - \alpha b_j)(1 - \alpha d b_j)(1 - \frac{\alpha a_j}{c})(1 - \frac{c}{\alpha d} a_j)}{(1 - a_j)(1 - \frac{a_j}{d})(1 - c b_j)(1 - \frac{\alpha^2 d}{c} b_j)} \right] \right\} \end{aligned}$$

where $\langle a_j \rangle$ and $\langle b_j \rangle$ are arbitrary sequences such that none of the terms in the denominators vanish. This reduces to the Gasper-Rahman summation formula on setting $a_k = ap^k$, $b_k = q^k$ and replacing α and d by d and a/b , respectively.

In this paper we obtain in §2 a generalization of Chu’s summation formula (1.7) involving four arbitrary sequences, which on specialization yields an extension of (1.1) to a summation formula with four independent bases p, q, P and Q and incorporating (1.7) as a special case. An expansion of the series $\sum_{n=0}^{\infty} A_n B_n (wx)^n$ into a series involving three independent bases is developed. A transformation of a series involving eight independent sequences is also developed. The note is concluded by obtaining in §3 some summation formulas which are different from the known ones by telescoping of series including q -Paff-Saalschütz’s summation formula for a terminating balanced $3\phi_2[q]$.

§2.

We begin this section by proving the summation formula:

If $\langle u_k \rangle$, $\langle v_k \rangle$, $\langle w_k \rangle$ and $\langle z_k \rangle$ are arbitrary sequences such that none of the terms in the denominators vanish and M, N are non-negative integers then

$$\begin{aligned} (2.1) \quad & \sum_{k=-m}^n \frac{(1 - u_k v_k w_k z_k)(1 - \frac{w_k z_k}{u_k v_k})(1 - \frac{v_k z_k}{u_k w_j})(1 - \frac{u_k z_k}{v_k w_k})}{(1 - u_0 v_0 w_0 z_0)(1 - \frac{w_0 z_0}{u_0 v_0})(1 - \frac{v_0 z_0}{u_0 w_0})(1 - \frac{u_0 z_0}{v_0 w_0})} \\ & \times \frac{\prod_{j=0}^{k-1} [(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)](\frac{u_k v_k w_k}{z_k})}{\prod_{j=1}^k [(1 - \frac{v_j w_j z_j}{u_j})(1 - \frac{u_j w_j z_j}{v_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{u_j v_j w_j}{z_j})]} \\ & = \frac{(1 - u_0^2)(1 - v_0^2)(1 - w_0^2)(1 - z_0^2)}{(1 - u_0 v_0 w_0 z_0)(1 - \frac{w_0 z_0}{u_0 v_0})(1 - \frac{v_0 z_0}{u_0 w_0})(1 - \frac{u_0 z_0}{v_0 w_0})} \\ & \times \left\{ \prod_{j=1}^n \left[\frac{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)}{(1 - \frac{v_j w_j z_j}{u_j})(1 - \frac{u_j w_j z_j}{v_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{u_j v_j w_j}{z_j})} \right] \right\} \end{aligned}$$

$$- \prod_{j=-m}^0 \left[\frac{(1 - \frac{v_j w_j z_j}{u_j})(1 - \frac{u_j w_j z_j}{v_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{u_j v_j w_j}{z_j})}{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)} \right] \Bigg\}.$$

Proof. Let

$$\tau_k = \prod_{j=1}^k \left[\frac{(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2)}{(1 - \frac{v_j w_j z_j}{u_j})(1 - \frac{u_j w_j z_j}{v_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{u_j v_j w_j}{z_j})} \right].$$

Then by straight forward calculations, we get

$$\Delta\tau_k = \frac{\prod_{j=1}^{k-1} \left[(1 - u_j^2)(1 - v_j^2)(1 - w_j^2)(1 - z_j^2) \right]}{\prod_{j=1}^k \left[(1 - \frac{v_j w_j z_j}{u_j})(1 - \frac{u_j w_j z_j}{v_j})(1 - \frac{u_j v_j z_j}{w_j})(1 - \frac{u_j v_j w_j}{z_j}) \right] \left(\frac{z_k}{u_k v_k w_k} \right)} \times \left[(1 - u_k v_k w_k z_k) \left(1 - \frac{w_k z_k}{u_k v_k} \right) \left(1 - \frac{v_k z_k}{u_k w_k} \right) \left(1 - \frac{u_k z_k}{v_k w_k} \right) \right]$$

where $\Delta\tau_k = \tau_k - \tau_{k-1}$. Now summing with respect to k from $-m$ to n , and using the fact that $\sum_{k=-m}^n \tau_k = \tau_n - \tau_{-m-1}$ and keeping in mind (1.6), we get (2.1) on simplification.

By setting $u_j = \sqrt{a} p^j$, $v_j = \sqrt{c} q^j$, $w_j = \sqrt{b} P^j$, $z_j = d \sqrt{\frac{a}{bc}} Q^j$ in (2.1), we get a summation formula involving four independent bases:

$$(2.2) \quad \sum_{k=-m}^n \frac{(1 - adp^k q^k P^k Q^k)(c - \frac{dP^k Q^k}{p^k q^k})(1 - \frac{bp^k P^k}{dq^k Q^k})(1 - \frac{ad}{bc} \frac{p^k P^k}{q^k Q^k})q^{2k}}{(1 - ad)(c - d)(1 - \frac{b}{d})(1 - \frac{ad}{bc})} \times \frac{(a; p^2)_k (c; q^2)_k (b; P^2)_k (\frac{ad^2}{bc}; Q^2)_k}{(d \frac{qPQ}{p}; \frac{qPQ}{p})_k (\frac{ad}{c} \frac{pPQ}{q}; \frac{pPQ}{q})_k (\frac{ad}{b} \frac{pqQ}{P}; \frac{pqQ}{P})_k (\frac{bc}{d} \frac{pqP}{Q}; \frac{pqP}{Q})_k} = \frac{(1 - a)(1 - b)(1 - c)(1 - \frac{ad^2}{bc})}{(1 - ad)(c - d)(1 - \frac{b}{d})(1 - \frac{ad}{bc})} \times \left\{ \frac{(ap^2; p^2)_n (cq^2; q^2)_n (bP^2; P^2)_n (\frac{ad^2}{bc} Q^2; Q^2)_n}{(\frac{dqPQ}{p}; \frac{qPQ}{p})_n (\frac{ad}{c} \frac{pPQ}{q}; \frac{pPQ}{q})_n (\frac{ad}{b} \frac{pqQ}{P}; \frac{pqQ}{P})_n (\frac{bc}{d} \frac{pqP}{Q}; \frac{pqP}{Q})_n} - \frac{(\frac{1}{d}; \frac{qPQ}{p})_{m+1} (\frac{c}{ad}; \frac{pPQ}{q})_{m+1} (\frac{b}{ad}; \frac{pqQ}{P})_{m+1} (\frac{d}{bc}; \frac{pqP}{Q})_{m+1}}{(\frac{1}{a}; p^2)_{m+1} (\frac{1}{c}; q^2)_{m+1} (\frac{1}{b}; P^2)_{m+1} (\frac{bc}{ad}; Q^2)_{m+1}} \right\}.$$

Summation formula (2.2), on setting $Q = \frac{pP}{q}$ and replacing p^2, q^2, P^2, a, b, d by $p, q, P, \beta, y, \frac{cy}{d}$, respectively, reduces to the summation formula (1.4), which in turn incorporates (1.3) and (1.2) as special cases.

It may be pointed out that (2.2) reduces to Chu's summation formula (1.7) on setting $u_j = \sqrt{a_j}$, $v_j = \sqrt{a_j}/\sqrt{d}$, $w_j = \sqrt{cb_j}$, $z_j = \alpha \sqrt{d/c} \sqrt{b_j}$.

Setting $m = 0$ in (2.2), replacing z_i by αz_i and setting $\alpha = \frac{u_0}{v_0 w_0 z_0}$, we get that

(2.3)

$$\begin{aligned} & \sum_{j=0}^n \frac{(\frac{v_0 w_0 z_0}{u_0} - u_j v_j w_j z_j)(\frac{v_0 w_0 z_0}{u_0} - \frac{w_j z_j}{u_j v_j})(\frac{v_0 w_0 z_0}{u_0} - \frac{v_j z_j}{u_j w_j})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_j z_j}{v_j w_j}) \frac{u_j v_j w_j}{z_j}}{(1 - u_0^2)(1 - v_0^2)(1 - w_0^2)(v_0^2 w_0^2 - u_0^2)} \\ & \times \frac{\prod_{i=0}^{j-1} [(1 - u_i^2)(1 - v_i^2)(1 - w_i^2)(\frac{v_0^2 w_0^2 z_0^2}{u_0^2} - z_i^2)]}{\prod_{i=1}^k [(\frac{v_0 w_0 z_0}{u_0} - \frac{v_i w_i z_i}{u_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i w_i z_i}{v_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i v_i z_i}{w_i})(\frac{u_0}{v_0 w_0 z_0} - \frac{u_i v_i w_i}{z_i})]} \\ & = \frac{z_0^3}{v_0 w_0 u_0^3} \\ & \times \prod_{i=1}^n \left[\frac{(1 - u_i^2)(1 - v_i^2)(1 - w_i^2)(\frac{v_0^2 w_0^2 z_0^2}{u_0^2} - z_i^2)}{(\frac{v_0 w_0 z_0}{u_0} - \frac{v_i w_i z_i}{u_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i w_i z_i}{v_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i v_i z_i}{w_i})(\frac{u_0}{v_0 w_0 z_0} - \frac{u_i v_i w_i}{z_i})} \right]. \end{aligned}$$

Next, using (2.3), the following transformation involving eight arbitrary sequences is obtained:

(2.4)

$$\begin{aligned} & \sum_{k=0}^n \prod_{i=1}^{n-k} \left[\frac{(1 - U_i^2)(1 - V_i^2)(1 - W_i^2)(\frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i^2)}{(\frac{V_0 W_0 Z_0}{U_0} - \frac{V_i W_i Z_i}{U_i})(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_i W_i Z_i}{V_i})(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_i V_i Z_i}{W_i})(\frac{U_0}{V_0 W_0 Z_0} - \frac{U_i V_i W_i}{Z_i})} \right] \\ & \times \frac{(\frac{v_0 w_0 z_0}{u_0} - u_k v_k w_k z_k)(\frac{v_0 w_0 z_0}{u_0} - \frac{w_k z_k}{u_k v_k})(\frac{v_0 w_0 z_0}{u_0} - \frac{v_k z_k}{u_k w_k})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_k z_k}{v_k w_k}) \frac{u_k v_k w_k}{z_k}}{(1 - u_0^2)(1 - v_0^2)(1 - w_0^2)(v_0^2 w_0^2 - u_0^2)} \\ & \times \frac{\prod_{i=0}^{k-1} [(1 - u_i^2)(1 - v_i^2)(1 - w_i^2)(\frac{v_0^2 w_0^2 z_0^2}{u_0^2} - z_i^2)]}{\prod_{i=1}^k [(\frac{v_0 w_0 z_0}{u_0} - \frac{v_i w_i z_i}{u_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i w_i z_i}{v_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i v_i z_i}{w_i})(\frac{u_0}{v_0 w_0 z_0} - \frac{u_i v_i w_i}{z_i})]} \\ & = \frac{V_0 W_0 U_0^3}{Z_0^3} \frac{z_0^3}{v_0 w_0 u_0^3} \\ & \times \prod_{i=1}^n \left[\frac{(1 - u_i^2)(1 - v_i^2)(1 - w_i^2)(\frac{v_0^2 w_0^2 z_0^2}{u_0^2} - z_i^2)}{(\frac{v_0 w_0 z_0}{u_0} - \frac{v_i w_i z_i}{u_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i w_i z_i}{v_i})(\frac{v_0 w_0 z_0}{u_0} - \frac{u_i v_i z_i}{w_i})(\frac{u_0}{v_0 w_0 z_0} - \frac{u_i v_i w_i}{z_i})} \right] \\ & \times \sum_{j=0}^n \frac{(\frac{V_0 W_0 Z_0}{U_0} - U_j V_j W_j Z_j)(\frac{V_0 W_0 Z_0}{U_0} - \frac{W_j Z_j}{U_j V_j})(\frac{V_0 W_0 Z_0}{U_0} - \frac{V_j Z_j}{U_j W_j})(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_j Z_j}{V_j W_j})}{(1 - U_0^2)(1 - V_0^2)(1 - W_0^2)(V_0^2 W_0^2 - U_0^2)} \end{aligned}$$

$$\begin{aligned}
 & \times \frac{U_j V_j W_j}{Z_j} \\
 & \times \prod_{i=0}^{j-1} \left[(1 - U_i^2)(1 - V_i^2)(1 - W_i^2) \left(\frac{V_0^2 W_0^2 Z_0^2}{U_0^2} - Z_i^2 \right) \right. \\
 & \quad \left. \times \left(\frac{v_0 w_0 z_0}{u_0} - \frac{v_{n-i} w_{n-i} z_{n-i}}{u_{n-i}} \right) \left(\frac{v_0 w_0 z_0}{u_0} - \frac{u_{n-i} w_{n-i} z_{n-i}}{v_{n-i}} \right) \right] \\
 & \times \frac{1}{\prod_{i=0}^j \left[\left(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_i W_i Z_i}{U_i} \right) \left(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_i W_i Z_i}{V_i} \right) \left(\frac{V_0 W_0 Z_0}{U_0} - \frac{U_i V_i Z_i}{W_i} \right) \left(\frac{U_0}{V_0 W_0 Z_0} - \frac{U_i V_i W_i}{Z_i} \right) \right]} \\
 & \times \frac{\prod_{i=0}^{j-1} \left[\left(\frac{v_0 w_0 z_0}{u_0} - \frac{u_{n-i} v_{n-i} z_{n-i}}{w_{n-i}} \right) \left(\frac{u_0}{v_0 w_0 z_0} - \frac{u_{n-i} v_{n-i} w_{n-i}}{z_{n-i}} \right) \right]}{\prod_{i=0}^j \left[(1 - u_{1+n-i}^2)(1 - v_{1+n-i}^2)(1 - w_{1+n-i}^2) \left(\frac{v_0^2 w_0^2 z_0^2}{u_0^2} - z_{1+n-i}^2 \right) \right]}.
 \end{aligned}$$

Transformation (2.4) can be proved by expanding the first product on the left hand side by using (2.3) (with n replaced by $n - k$), interchanging the order of summations and evaluating the inner sum by using (2.3) once again and simplifying to get the right hand side of (2.4).

Transformation (2.4), on replacing $u_i, v_i, w_i, z_i, U_i, V_i, W_i, Z_i$ by $\sqrt{a} p^i, \sqrt{c} q^i, \sqrt{b} P^i, Q^i, \sqrt{A} \tilde{p}^i, \sqrt{C} \tilde{q}^i, \sqrt{B} \tilde{P}^i, \tilde{Q}^i$, respectively, reduces on some simplification to the following transformation of q -series involving eight independent bases:

$$\begin{aligned}
 (2.5) \quad & \sum_{k=0}^n \frac{(1 - ap^k q^k P^k Q^k) \left(\frac{q^k P^k Q^k}{p^k} - cq^{2k} \right) (1 - b \frac{p^k P^k}{q^k Q^k}) (1 - \frac{a}{bc} \frac{p^k Q^k}{q^k P^k})}{(1 - a)(1 - c)(1 - b)(1 - \frac{a}{bc})} \\
 & \times \frac{(a; p^2)_k (c; q^2)_k (b; P^2)_k (\frac{a}{bc}; Q^2)_k}{\left(\frac{qPQ}{p}; \frac{qPQ}{p} \right)_k \left(\frac{a}{c} \frac{pPQ}{q}; \frac{pPQ}{q} \right)_k \left(\frac{a}{b} \frac{pqQ}{P}; \frac{pqQ}{P} \right)_k (bc \frac{pqP}{Q}; \frac{pqP}{Q})_k} \\
 & \times \frac{(\tilde{q}^{-n} \tilde{P}^{-n} \tilde{Q}^{-n}; \frac{\tilde{q}\tilde{P}\tilde{Q}}{\tilde{p}})_k (\frac{C}{A} \frac{\tilde{p}^{-n} \tilde{P}^{-n} \tilde{Q}^{-n}}{\tilde{q}^{-n}}; \frac{\tilde{p}\tilde{P}\tilde{Q}}{\tilde{q}})_k (\frac{B}{A} \frac{\tilde{p}^{-n} \tilde{q}^{-n} \tilde{Q}^{-n}}{\tilde{P}^{-n}}; \frac{\tilde{p}\tilde{q}\tilde{Q}}{\tilde{P}})_k}{(\frac{\tilde{p}^{-2n}}{A}; \tilde{p}^2)_k (\frac{\tilde{q}^{-2n}}{C}; \tilde{q}^2)_k (\frac{\tilde{P}^{-2n}}{B}; \tilde{P}^2)_k (\frac{BC}{A} \tilde{Q}^{-2n}; \tilde{Q}^2)_k} \\
 & \times \left(\frac{\tilde{p}^{-n} \tilde{q}^{-n} \tilde{P}^{-n}}{BC \tilde{Q}^{-n}}; \frac{\tilde{p}\tilde{q}\tilde{P}}{\tilde{Q}} \right)_k \\
 & = \frac{(ap^2; p^2)_n (cq^2; q^2)_n (bP^2; P^2)_n (\frac{aQ^2}{bc}; Q^2)_n}{\left(\frac{qPQ}{p}; \frac{qPQ}{p} \right)_n \left(\frac{a}{c} \frac{pPQ}{q}; \frac{pPQ}{q} \right)_n \left(\frac{a}{b} \frac{pqQ}{P}; \frac{pqQ}{P} \right)_n (bc \frac{pqP}{Q}; \frac{pqP}{Q})_n}
 \end{aligned}$$

$$\begin{aligned} & \times \frac{(\tilde{q}\tilde{P}\tilde{Q}; \tilde{q}\tilde{P}\tilde{Q})_n (A/C \tilde{p}\tilde{P}\tilde{Q}; \tilde{p}\tilde{P}\tilde{Q})_n (A/B \tilde{p}\tilde{q}\tilde{Q}; \tilde{p}\tilde{q}\tilde{Q})_n (BC \tilde{p}\tilde{q}\tilde{P}; \tilde{p}\tilde{q}\tilde{P})_n}{(a\tilde{p}^2; \tilde{p}^2)_n (C\tilde{q}^2; \tilde{q}^2)_n (B\tilde{P}^2; \tilde{P}^2)_n (A\tilde{Q}^2; \tilde{Q}^2)_n} \\ & \times \sum_{j=0}^n \frac{(1 - A\tilde{p}^j \tilde{q}^j \tilde{P}^j \tilde{Q}^j) (\frac{\tilde{q}^j \tilde{P}^j \tilde{Q}^j}{\tilde{p}^j} - C\tilde{q}^{2j}) (1 - \frac{B\tilde{p}^j \tilde{P}^j}{\tilde{q}^j \tilde{Q}^j}) (1 - \frac{A}{BC} \frac{\tilde{p}^j \tilde{Q}^j}{\tilde{q}^j \tilde{P}^j})}{(1 - A)(1 - C)(1 - B)(1 - \frac{A}{BC})} \\ & \times \frac{(A; \tilde{p}^2)_j (C; \tilde{q}^2)_j (B; \tilde{P}^2)_j (\frac{A}{BC}; \tilde{Q}^2)_j}{(\frac{\tilde{q}\tilde{P}\tilde{Q}}{\tilde{p}}; \frac{\tilde{q}\tilde{P}\tilde{Q}}{\tilde{p}})_j (\frac{A}{C} \frac{\tilde{p}\tilde{P}\tilde{Q}}{\tilde{q}}; \frac{\tilde{p}\tilde{P}\tilde{Q}}{\tilde{q}})_j (\frac{A}{B} \frac{\tilde{p}\tilde{q}\tilde{Q}}{\tilde{P}}; \frac{\tilde{p}\tilde{q}\tilde{Q}}{\tilde{P}})_j (BC \frac{\tilde{p}\tilde{q}\tilde{P}}{\tilde{Q}}; \frac{\tilde{p}\tilde{q}\tilde{P}}{\tilde{Q}})_j} \\ & \times \frac{(\frac{q^{-n} P^{-n} Q^{-n}}{p^{-n}}; \frac{qPQ}{P})_j (\frac{c}{a} \frac{p^{-n} P^{-n} Q^{-n}}{q^{-n}}; \frac{pPQ}{q})_j (\frac{b}{a} \frac{p^{-n} q^{-n} Q^{-n}}{P^{-n}}; \frac{pqQ}{P})_j}{(\frac{p^{-2n}}{a}; p^2)_j (\frac{q^{-2n}}{c}; q^2)_j (\frac{P^{-2n}}{b}; P^2)_j (\frac{bcQ^{-2n}}{a}; Q^2)_j} \\ & \times \left(\frac{p^{-n} q^{-n} P^{-n}}{bcQ^{-n}}; \frac{pqP}{Q} \right)_j. \end{aligned}$$

Transformation (2.5) is a generalization of Gasper’s [4, Ex. 3.21] quad-basic transformation (from which Gasper deduces a transformation of a half-poised $_{10}\phi_9$ into another half-poised $_{10}\phi_9$ [4, Ex. 3.24]) to which it reduces on setting $Q = q$, $P = p$, $\tilde{P} = \tilde{p}$, $\tilde{Q} = \tilde{q}$ and then replacing p^2 , q^2 , \tilde{p}^2 and \tilde{q}^2 by p, q, P and Q , respectively.

Next, we obtain an expansion of $\sum_{r=0}^{\infty} A_r B_r (xw)^r / (q; q)_r$ terms of q -series having three independent bases. In the summation formula (2.2), setting $m = 0$, $c = q^{-2n}$, where n is a non-negative integer and letting $d \rightarrow 1$, we get

$$\begin{aligned} & \sum_{j=0}^n \frac{(1 - ap^j q^j P^j Q^j) (\frac{q^j P^j Q^j}{p^j} - q^{2j-2n}) (1 - \frac{bp^j P^j}{q^j Q^j}) (1 - \frac{a}{b} \frac{p^j Q^j}{q^{j-2n} P^j})}{(1 - a)(1 - q^{-2n})(1 - b)(1 - \frac{a}{b} q^{2n})} \\ & \times \frac{(a; p^2)_j (q^{-2n}; q^2)_j (b; P^2)_j (\frac{a}{b} q^{2n}; Q^2)_j}{(\frac{qPQ}{p}; \frac{qPQ}{p})_j (\frac{apPQ}{q^{1-2n}}; \frac{pPQ}{q})_j (\frac{a}{b} \frac{pqQ}{P}; \frac{pqQ}{P})_j (\frac{bpq^{1-2n}P}{Q}; \frac{pqP}{Q})_j} = \delta_{n,0}. \end{aligned}$$

Replacing a and b by $ap^r q^r P^r Q^r$ and $bp^r q^{-r} P^r Q^{-r}$ respectively, where r is a non-negative integer, setting $Q = \frac{pq}{P}$ and then replacing p^2 , q^2 , P^2 , n by p, q, P, m , respectively, we get

$$\begin{aligned} (2.6) \quad \delta_{m,0} &= \sum_{j=0}^m \frac{(1 - ap^{j+r} q^{j+r}) (1 - bP^{j+r} q^{-j-r}) (1 - \frac{a}{b} p^{r+j} q^{2r+m} P^{-r-j}) q^j}{(1 - ap^r q^r) (1 - bP^r q^{-r}) (1 - \frac{a}{b} p^r q^{2r+m} P^{-r})} \\ & \times \frac{(ap^r q^r; p)_j (q^{-m}; q)_j (bP^r q^{-r}; P)_j (\frac{a}{b} p^r q^{2r+m} P^{-r}; \frac{pq}{P})_j}{(q; q)_j (ap^{1+r} q^{r+m}; p)_j (\frac{a}{b} p^{r+1} q^{1+2r} P^{-r-1}; \frac{pq}{P})_j (bq^{-m-r} P^{r+1}; P)_j}. \end{aligned}$$

But, we know that

$$\begin{aligned}
 & B_r x^r \\
 &= \sum_{m=0}^{\infty} \frac{(1 - \frac{a}{b} p^{r+m} q^{2r+2m} P^{-r-m}) (\frac{a}{b} p^r q^{2r} P^{-r}; \frac{pq}{P})_m (apq^r; p)_r (bPq^{-r}; P)_r}{(1 - \frac{a}{b} p^r q^{2r} P^{-r}) (q; q)_m (apq^{r+m}; p)_r (bPq^{-m-r}; P)_r} \\
 & \times \{q^{-rm} B_{r+m} C_{r,m} x^{r+m} \delta_{m,0}\},
 \end{aligned}$$

where $\langle B_r \rangle$ and $\langle C_{r,m} \rangle$ are arbitrary sequences of complex numbers such that $C_{r,0} = 1$ for $r = 0, 1, 2, \dots$. Substituting for $\delta_{m,0}$ from (2.6), interchanging the order of summation and setting $j = n - r$ and $m = n + k - r$, we get

$$\begin{aligned}
 B_r x^r &= \sum_{k=0}^{\infty} \sum_{n=r}^{\infty} (-)^n (1 - ap^n q^n) (apq^r; p)_{n-1} (1 - bP^n q^{-n}) (bPq^{-r}; P)_{n-1} \\
 & \times \frac{(1 - \frac{a}{b} p^{n+k} q^{2n+2k} P^{-n-k}) x^{n+k} (q^{-n}; q)_r}{(apq^{n+k}; p)_n (bPq^{-n-k}; P)_n (q; q)_n (q; q)_k} \\
 & \left\{ \left(1 - \frac{a}{b} p^n q^{n+k+r} P^{-n}\right) \left(\frac{a}{b} \frac{p^{n+1} q^{n+r+1}}{P^{n+1}}; \frac{pq}{P}\right)_{k-1} \right. \\
 & \left. \times \left(\frac{a}{b} p^{r+1} q^{n+k+r+1} P^{-r-1}; \frac{pq}{P}\right)_{n-r-1} B_{n+k} C_{r,n+k-r} q^{n(r-k-n) + \frac{n^2}{2} + \frac{n}{2}} \right\}.
 \end{aligned}$$

Multiplying both sides by $\frac{A_r w^r}{(q; q)_r}$ and summing from $r = 0$ to ∞ and interchanging the order of summation on the right hand side, we get

$$\begin{aligned}
 & \sum_{r=0}^{\infty} A_r B_r \frac{(xw)^r}{(q; q)_r} \\
 &= \sum_{n=0}^{\infty} \frac{(1 - ap^n q^n) (1 - bP^n q^{-n}) (-x)^n q^{\binom{n}{2} + n}}{(q; q)_n} \\
 & \times \sum_{k=0}^{\infty} \frac{(1 - \frac{a}{b} \frac{p^{n+k} q^{2n+2k}}{P^{n+k}}) x^k B_{n+k}}{(q; q)_k (apq^{n+k}; p)_n (bPq^{-n-k}; P)_n} \\
 & \times \sum_{r=0}^{\infty} \frac{(q^{-n}; q)_r (apq^r; p)_{n-1} (bPq^{-r}; P)_{n-1}}{(q; q)_r} \\
 & \times \left\{ \left(1 - \frac{a}{b} \frac{p^n q^{n+k+r}}{P^r}\right) \left(\frac{a}{b} \frac{p^{n+1} q^{n+r+1}}{P^{n+1}}; \frac{pq}{P}\right)_{k-1} \right. \\
 & \left. \times \left(\frac{a}{b} \frac{p^{r+1} q^{n+k+r+1}}{P^{r+1}}; \frac{pq}{P}\right)_{n-r-1} \right\} C_{r,n+k-r} w^n q^{n(r-k-n)} A_r,
 \end{aligned}$$

which is a generalization of Gasper's bibasic expansion formula [4, (3.7.6)] to which it reduces on setting $P = p$. It may be noted that on setting $P = p$ the

terms in $\{\dots\}$ of the above expression combine to yield $(\frac{a}{b}q^{n+r+1}; q)_{n+k-r-1}$ as in the bibasic expansion formula of Gasper [4, (3.7.6)].

§3.

All the summation formulae proved so far are for one generalization of (1.1), a very-well-poised q -series. We next derive a summation formula which gives the sum of a balanced series.

Let $\langle x_i \rangle, \langle y_i \rangle$ and $\langle z_i \rangle$ be arbitrary sequences and a an indeterminate so that none of the terms in the denominators vanish and m, n are non-negative integers. Then

$$\begin{aligned}
 & \sum_{k=-m}^n \frac{(1 - \frac{y_k}{az_k})(1 - \frac{x_k}{az_k})z_k}{(1 - \frac{y_0}{az_0})(1 - \frac{x_0}{az_0})} \frac{\prod_{i=0}^{k-1} [(1 - x_i)(1 - y_i)]}{\prod_{i=1}^k [(1 - az_i)(1 - \frac{x_i y_i}{az_i})]} \\
 &= \frac{(1 - x_0)(1 - y_0)}{a(1 - \frac{y_0}{az_0})(1 - \frac{x_0}{az_0})} \\
 (3.1) \quad & \times \left\{ \prod_{i=1}^n \left[\frac{(1 - x_i)(1 - y_i)}{(1 - az_i)(1 - \frac{x_i y_i}{az_i})} \right] - \prod_{i=-m}^0 \left[\frac{(1 - az_i)(1 - \frac{x_i y_i}{az_i})}{(1 - x_i)(1 - y_i)} \right] \right\}.
 \end{aligned}$$

For proving (3.1) we consider

$$\tau_k = \prod_{i=1}^k \left[\frac{(1 - x_i)(1 - y_i)}{(1 - az_i)(1 - \frac{x_i y_i}{az_i})} \right].$$

Then by straight forward calculations we find that

$$\Delta \tau_k = az_k \left(1 - \frac{y_k}{az_k}\right) \left(1 - \frac{x_k}{az_k}\right) \frac{\prod_{i=1}^{k-1} [(1 - x_i)(1 - y_i)]}{\prod_{i=1}^k [(1 - az_i)(1 - \frac{x_i y_i}{az_i})]},$$

which on summing over k from $-m$ to n , gives (3.1) after using (1.6).

In view of this it is natural to look for a telescoping proof of the q -Paff-Salschütz summation formula [4, (1.7.2)]

$$(3.2) \quad S_n \equiv {}_3\phi_2 \left[\begin{matrix} a, b, q^{-n}; q, q \\ c, \frac{ab}{c} q^{1-n} \end{matrix} \right] = \frac{(\frac{c}{a}, \frac{c}{b}; q)_n}{(c, \frac{c}{ab}; q)_n}.$$

To this end define for non-negative integers n and r

$$F(n, r) = \frac{(a, b, q^{-n}; q)_r q^r}{(q, c, \frac{ab}{c} q^{1-n}; q)_r}$$

$$\text{and } G(n, r) = \frac{(a, b; q)_{r+1}(q^{-n}; q)_r}{(q, c, \frac{ab}{c}q^{1-n}; q)_r}.$$

Notice that $S_0 = 1$, $G(n, n + 1) = G(0, -1) = 0$. By straightforward calculations we can verify that

$$\begin{aligned} \left(1 - \frac{c}{a}q^n\right) \left(1 - \frac{c}{b}q^n\right) F(n, r) - (1 - cq^n) \left(1 - \frac{cq^n}{ab}\right) F(n + 1, r) \\ - \frac{c}{ab}q^n [G(n, r) - G(n, r - 1)] = 0. \end{aligned}$$

Summing over r from 0 to $n + 1$ and using $G(n, n + 1) = 0 = G(0, -1)$, we get

$$\left(1 - \frac{c}{a}q^n\right) \left(1 - \frac{c}{b}q^n\right) S_n = (1 - cq^n) \left(1 - \frac{c}{ab}\right) S_{n+1}$$

which yields

$$S_n = \frac{(\frac{c}{a}, \frac{c}{b}; q)_n}{(c, \frac{c}{ab}; q)_n} S_0.$$

By using $S_0 = 1$, we get (3.2).

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