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THE CONVERSE OF FATOU'S THEOREM FOR ZYGMUND MEASURES

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The converse of Fatou's theorem is true for positive measures but not for arbitrary measures. We prove that the converse holds for Zygmund (smooth) measures, being this result sharp in some sense. We also give an application to differentiation of positive singular measures in the little Zygmund class.

1. Introduction and statement of results.

Throughout this paper μ always will denote a complex Borel measure on $\mathbb{R}$ or a positive Borel measure such that $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty$. For each μ, let

$$u(x,y) = P[\mu](x,y) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{y}{y^2+(x-t)^2}\, d\mu(t)$$

denote its Poisson integral defined on the upper halfplane $\Pi^+ = \{(x,y) : y > 0\}$. Observe that $P[\mu](x,y) = P_y * \mu(x)$, where $P_y(x) = \frac{1}{\pi}\frac{y}{y^2+x^2}$. Analogously, if f is a bounded function we can also write $P[f](x,y) = P_y * f(x)$.

A classical Fatou theorem [**10**, p. 257] relates some differentiability properties of μ at $x \in \mathbb{R}$ to the asymptotic behaviour of $u(z)$ when z tends to x. In order to state it, we recall that the symmetric derivative of μ at x and the derivative of μ at x, are defined as

$$D_{\text{sym}}\mu(x) = \lim_{h\to 0^+} \frac{\mu((x-h,x+h))}{2h},$$
$$D\mu(x) = \lim_{\substack{t-s\to 0\\ s<x<t}} \frac{\mu((s,t))}{t-s},$$

provided that both limits exist.

The usual Stolz angle $\{(t,y) : |t-x| < \alpha y\}$ will be denoted by $\Delta_\alpha(x)$, where $0<\alpha<\infty$.

Theorem (Fatou). *Let u be the Poisson integral of a measure μ and let $L \in \mathbb{C}$.*

(a) *If* $D\mu(x_0) = L$, *then*

$$\lim_{z\to x_0, z\in\Delta_\alpha(x_0)} u(z) = L\,, \qquad \text{for all } \alpha > 0. \tag{1}$$

(b) *If* $D_{\rm sym}\mu(x_0) = L$, *then* $\lim\limits_{y\to 0^+} u(x_0, y) = L\,.$

When (1) holds we say that u has non-tangential limit L at x_0. The converses of the previous results are not true in general (see [**5**], and our Proposition 2 for an example). However, the positivity of μ is a tauberian condition which makes true the converses of (a) and (b), as it was proved by Loomis [**5**].

Theorem (Loomis)**.** *Suppose that* μ *is positive,* $0 \le L < \infty$*, and let* $u = P[\mu]$.

(i) *If* u *has non-tangential limit* L *at* x_0*, then* $D\mu(x_0) = L$.
(ii) *If* $\lim\limits_{y\to 0^+} u(x_0, y) = L$*, then* $D_{\rm sym}\mu(x_0) = L$.

Loomis obtained several proofs of these results. The most direct one uses an integral representation of positive harmonic functions on Π^+. Rudin [**9**], applying a version of Wiener's Tauberian theorem, generalized to higher dimensions the statement (ii), obtaining in this way another different proof. Moreover he gave an example of a positive measure ν such that $\lim\limits_{y\to 0^+} P[\nu](0, y) = \infty$ but $D_{\rm sym}\nu(0)$ does not exist; therefore the hypothesis $L < \infty$ is really needed in (ii).

In this paper we obtain another condition for which the converses of (a) and (b) hold. For simplicity we will denote by $|I|$ the Lebesgue measure of an interval I and by C certain absolute constants, not necessarily the same in each occurrence.

We need the following definition.

Definition. We say that a complex measure μ on $\mathbb{R}$ is a Zygmund measure if there exists a positive constant C such that

$$|\mu(I) - \mu(I')| \le C|I|,$$

for any two adjacent intervals I, I' of the same length.

Observe that if μ is a Zygmund measure then its distribution function, i.e. $f_\mu(x) = \mu((-\infty, x))$ when $x \in \mathbb{R}$, belongs to the Zygmund class Λ_*, that is, f_μ is bounded and

$$|f_\mu(x+h) + f_\mu(x-h) - 2f_\mu(x)| \le C|h|, \tag{2}$$

for all $x, h \in \mathbb{R}$. We recall that the Zygmund norm of $f \in \Lambda_*(\mathbb{R})$ is $\|f\|_* = \|f\|_\infty + A$, where A is the infimum of the constants C for which (2) holds.

Let us see as an important example that the measures $f\,dx$, $f \in \mathrm{BMO}\,(\mathbb{R})$, are Zygmund measures. Let I, I' be two adjacent intervals of the same length and write $\tilde{I} = I \cup I'$. Denoting by f_J the average $\frac{1}{|J|}\int_J f\,dx$, we already see that

$$\left|\int_I f\,dx - \int_{I'} f\,dx\right| \le \int_I |f - f_{\tilde{I}}|\,dx + \int_{I'} |f - f_{\tilde{I}}|\,dx$$
$$\le 2\int_{\tilde{I}} |f - f_{\tilde{I}}|\,dx \le C|\tilde{I}|,$$

as required by the definition.

Our main results are the following. Let L be a complex number.

Theorem 1. *Let μ be a Zygmund measure on $\mathbb{R}$. If u has non-tangential limit L at x_0 then $D\mu(x_0) = L$.*

Theorem 2. *Let μ be as above. If $\lim_{y\to 0^+} u(x_0, y) = L$, then $D_{\mathrm{sym}}\mu(x_0) = L$.*

The most general previous results of this kind were obtained by Brossard and Chevalier [**1**]. Before stating their results we need two definitions:

(a) The Poisson integral of μ, $u = P[\mu]$, verifies the hypothesis $(\mathcal{H})$ if and only if the function $P[|\mu|] - |P(\mu)|$ is bounded in $V \cap \Pi^+$, where V is a neighborhood of $(0,0)$.

(b) If μ is a measure the radial part of μ is defined by $\mu_{ra}(A) = (\mu(A) + \mu(-A))/2$, for each measurable set A.

The results are the following.

Theorem A (Brossard-Chevalier)**.** *Let μ be a measure. If $u = P[\mu]$ has non-tangential limit at the point $(0,0)$ and u satisfies the hypothesis $(\mathcal{H})$, then μ is derivable at* 0.

Theorem B (Brossard-Chevalier)**.** *Let μ be a measure and $u = P[\mu]$. If the radial part μ_{ra} of μ satisfies the hypothesis $(\mathcal{H})$ and if the function u has radial limit at the point $(0,0)$, then the measure μ has symmetric derivative at the point* 0.

For example, if μ is positive or $\mu = f dx$, where $f \in \mathrm{BMO}(\mathbb{R})$ then μ and μ_{ra} satisfy $(\mathcal{H})$. Previously Ramey and Ullrich [**7**] had proved that the above theorems in the case that μ is absolutely continuous with density in $\mathrm{BMO}(\mathbb{R})$. However there are radial Zygmund measures that do not satisfy the hypothesis $(\mathcal{H})$, as the Example at the end of Section 3 will show. Therefore it seems that our results are not consequence of Theorems A or B. We must say that the starting point of this research was Ullrich's paper [**12**], where a similar result was obtained for Bloch functions in the open unit disk $\mathbb{D}$. However, our theorems do not follow from his work. We will prove Theorem 2 following some of the arguments of Rudin [**9**].

Several natural questions arise from our stated results. One of them is whether Theorem 2 remains true when $L = \infty$. In contrast to the aforementioned example of Rudin of the measure ν, we have the next result.

Proposition 1. *Let μ be a real Zygmund measure. Then the following are equivalent.*

(i) $D_{\mathrm{sym}}\mu(x_0) = +\infty$.
(ii) $D\mu(x_0) = +\infty$.
(iii) $\lim_{y\to 0^+} u(x_0, y) = +\infty$.
(iv) *The function u has non-tangential limit $+\infty$ at x_0.*

As before, the case $f\,dx$ with $f \in$ BMO$\,(\mathbb{R})$ was considered in [**7**].

Another question is to what extent can be weakened the hypothesis of μ being a Zygmund measure. Since the modulus of continuity of a Zygmund function is $O(\delta \log \frac{1}{\delta})$ as $\delta \to 0$ [**13**, p. 44], then $|\mu(I)| \le C|I| \log(1/|I|)$, if $|I| \le 1/2$. According with this, the following proposition shows a certain sharpness of our main results.

Proposition 2. *Let $\varphi(x) = x \log \frac{1}{x}$ if $0 < x < 1/2$. Then there exists a real measure μ such that*

(i) $|\mu(I)| \le \varphi(|I|)$, *for any interval I with $|I| \le 1/2$,*
(ii) $P[\mu]$ *has non-tangential limit* 0 *at the origin,*
(iii) $D_{sym}\mu(0)$ *does not exist.*

The measure in the above proposition will be obtained by a modification of the Loomis' example given in [**5**].

We say that μ is a little Zygmund measure ($\mu \in \lambda_*$) if $f_\mu \in \lambda_*$, i.e. $f_\mu \in \Lambda_*$ and

$$\sup_x \frac{|f_\mu(x+h) + f_\mu(x-h) - 2f_\mu(x)|}{|h|} \to 0, \quad \text{as } h \to 0.$$

Assuming this strong hypothesis we have the following result.

Proposition 3. *Let μ be a little Zygmund measure. If $\lim_{y\to 0^+} u(x_0, y) = L$, then $D\mu(x_0) = L$.*

We finally apply the above results to prove:

Proposition 4. *Let $\mu \in \lambda_*$ be a positive measure, $\mu \ne 0$. Assume that μ is singular. Then for each $0 \le \alpha \le \infty$ the set $E_\alpha = \{x \in \mathbb{R} : D\mu(x) = \alpha\}$ has Hausdorff dimension* 1.

Examples of such kind of measures were given in [**4**].

2. Proof of Theorem 1.

First we recall an important result of Zygmund that will be used later, see [**11**, p. 146] and also [**13**, p. 263].

Theorem (Zygmund)**.** *Let u be a harmonic function on Π^+. Then $u = P[f]$ with $f \in \Lambda_*(\mathbb{R})$ if and only if* $\sup_x \left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| \le \frac{C}{y}$, *for any $y > 0$.*

The main ingredient in proving Theorem 1 will be Lemma 2 whose proof will need the following technical result concerning Poisson integrals of Zygmund functions. From now on, in the case that f is a continuous function, we call again $P[f]$ the continuous extension of the Poisson integral up to the boundary of Π^+.

Lemma 1. *Let $f \in \Lambda_*$ and $u = P[f]$. Then*

$$|u(x+t, y+s) + u(x-t, y-s) - 2u(x,y)| \le C|h|, \tag{3}$$

for any $h = (t,s)$ and (x,y) with $y \ge |s| \ge 0$.

Although this result is known among specialists in analytic function theory in $\mathbb{D}$, we have not found any reference. For the reader's convenience we have included a sketch of a direct proof, which does not involve any explicit properties of the harmonic conjugate of u. We will follow some ideas used in [**11**, pp. 143-147].

Proof of Lemma 1. By Zygmund's Theorem and Lemma 5 in [**11**, p. 145] we have

$$\left|\frac{\partial^2 u}{\partial y^2}(x,y)\right| \le \frac{C}{y}, \quad \left|\frac{\partial^3 u}{\partial y^3}(x,y)\right| \le \frac{C}{y^2} \quad \text{for all } x \in \mathbb{R},\ y > 0. \tag{4}$$

Also the following estimates hold:

$$\left|\frac{\partial^2 u}{\partial x^2}(x,y)\right| \le \frac{C}{y}, \qquad \left|\frac{\partial^3 u}{\partial y \partial x^2}(x,y)\right| \le \frac{C}{y^2}, \tag{5}$$

$$\left|\frac{\partial^3 u}{\partial y^2 \partial x}(x,y)\right| \le \frac{C}{y^2}, \qquad \left|\frac{\partial^2 u}{\partial y \partial x}(x,y)\right| \le \frac{C}{y},$$

for all $x \in \mathbb{R}$, $y > 0$.

The first and the second ones are consequence of (4) and the harmonicity of u. The fourth one is an integration of the third part of (5) and this one follows from the formula

$$\frac{\partial^3 u}{\partial y^2 \partial x}(x,y) = \left(\frac{\partial}{\partial x} P_{y_1} * \frac{\partial}{\partial y^2} u(\cdot, y - y_1)\right)(x), \text{ where } y_1 = y/2.$$

Fix $z = (x,y)$ and $h = (t,s)$. Write $\Delta_h^2 F(z) = F(z+h) + F(z-h) - 2F(z)$. If F has two continuous derivatives, then

$$\left|\Delta_h^2 F(z)\right| \le |h|^2 \max_{\alpha+\beta=2} \left\|\frac{\partial^2 F}{\partial x^\alpha \partial y^\beta}\right\|_L, \tag{6}$$

where $\| \, \|_L$ denotes the supremum norm on the segment L joining $z - h$ with $z + h$.

Now consider the identity

$$\begin{aligned} u(x,y) &= u(x,y+\rho) - \rho\frac{\partial u}{\partial y}(x,y+\rho) \\ &\quad + \int_0^\rho \tau \frac{\partial^2}{\partial y^2} u(x,y+\tau)\,d\tau, \quad \text{where } \rho \ge 0, \end{aligned} \tag{7}$$

which can be proved by noticing that the derivative with respect to ρ of the right-hand side of (7) vanishes. After this apply the identity (7) with $\rho = |h|$ to each term appearing in the definition of $\Delta_h^2 u(z)$ and get

$$\begin{aligned} \left|\Delta_h^2 u(z)\right| &\le \left|\Delta_h^2 u(x,y+\rho)\right| + \rho\left|\Delta_h^2 \frac{\partial u}{\partial y}(x,y+\rho)\right| \\ &\quad + \int_0^\rho \tau\left|\Delta_h^2 \frac{\partial^2 u}{\partial y^2}(x,y+\tau)\right| d\tau. \end{aligned} \tag{8}$$

The three terms of (8) can be bounded using (6), (5) and (4). This gives (3). □

Lemma 2. *Let μ be a Zygmund measure. There exists a positive constant C so that*

$$\left|\int_s^t P[\mu](x,y)\,dx - \mu((s,t))\right| \le Cy\log\frac{t-s}{y}, \tag{9}$$

for any $s < t$ and $0 < y \le (t-s)/2$.

Proof. For simplicity we denote by f the distribution function f_μ of μ. Let $s < t$ and $y > 0$ be fixed. By definition and Fubini's theorem one has

$$\int_s^t P[\mu](x,y)\,dx = \int_{-\infty}^\infty \omega(\rho)\,d\mu(\rho), \tag{10}$$

where

$$\omega(\rho) = \frac{1}{\pi}\left(\arctan\frac{\rho-s}{y} - \arctan\frac{\rho-t}{y}\right)$$

is the harmonic measure $\omega((\rho,y),[s,t],\Pi^+)$. Integration by parts yields

$$\int_s^t P[\mu](x,y)\,dx = -\int_{-\infty}^\infty \omega'(\rho)f(\rho)\,d\rho = P[f](t,y) - P[f](s,y). \tag{11}$$

Therefore, denoting $u = P[f]$, we get

$$\int_s^t P[\mu](x,y)\,dx - \mu((s,t)) = u(t,y) - u(s,y) - u(t,0) + u(s,0). \tag{12}$$

Now we claim that

$$\begin{aligned} &|u((1-\alpha)z + \alpha w) - (1-\alpha)u(z) - \alpha u(w)| \le C\varphi(\alpha)|z-w|, \\ &\qquad\qquad 0 \le \alpha \le 1, \text{ and } z,w \in \overline{\Pi^+}, \end{aligned} \tag{13}$$

where $\varphi(\alpha) = \alpha \log_2 \frac{1}{\alpha}$, if $0 \le \alpha \le 1/2$, and $\varphi(\alpha) = \varphi(1-\alpha)$, if $1/2 \le \alpha \le 1$. Inequality (13) is true for general functions in the Zygmund class on $\mathbb{R}^2$. To prove (13) it is enough, by continuity and symmetry, to show that

$$\begin{aligned}&\left|u\left(\left(1-\frac{m}{2^n}\right)z+\frac{m}{2^n}w\right)-\left(1-\frac{m}{2^n}\right)u(z)-\frac{m}{2^n}u(w)\right|\\&\le \frac{C}{2}\frac{m}{2^n}(n-\log_2 m)|z-w|,\end{aligned}$$

if $\frac{m}{2^n} \le \frac{1}{2}$ and $z, w \in \overline{\Pi^+}$. The previous inequality holds for $n = m = 1$ by Lemma 1. For general m and n it can be checked by an induction argument.

In order to estimate (12), keeping in mind the inequality (13), we add and subtract the terms $\frac{y}{t-s}u(s,t-s)$ and $\frac{y}{t-s}u(t,t-s)$ to the right-hand side. Then, the triangle inequality yields

$$\begin{aligned}&\left|\int_s^t P[\mu](x,y)\,dx-\mu((s,t))\right|\\&\le\left|\frac{y}{t-s}u(s,t-s)+\left(1-\frac{y}{t-s}\right)u(s,0)-u(s,y)\right|\\&\quad+\left|\frac{y}{t-s}u(t,t-s)+\left(1-\frac{y}{t-s}\right)u(t,0)-u(t,y)\right|\\&\quad+\frac{y}{t-s}\left|u(t,t-s)-u(t,0)-u(s,t-s)+u(s,0)\right|.\end{aligned}$$

The first two terms of the last inequality are controlled by (13). For the third one we argue as before, adding and subtracting the number $2u(\frac{t+s}{2}, \frac{t-s}{2})$ and afterwards applying again Lemma 1. Thus

$$\left|\int_s^t P[\mu](x,y)\,dx-\mu((s,t))\right| \le 2Cy\log\frac{t-s}{y}+Cy \le Cy\log\frac{t-s}{y}.$$

□

Proof of Theorem 1. Let x_0 be fixed and take $\epsilon > 0$. Consider $0 < \alpha < 1/2$ small enough so that $C\alpha\log(1/\alpha) < \epsilon$, where C is the constant in (9). By the hypothesis let $\delta > 0$ such that $|u(x,y) - L| < \epsilon$ if $(x,y) \in \Delta_{\frac{1}{\alpha}}(x_0)$ and $0 < y < \delta$. If $s < x_0 < t$, $(t-s)\alpha < \delta$ put $y = (t-s)\alpha$. Then

$$\left|\frac{1}{t-s}\int_s^t P[\mu](x,y)\,dx - L\right| < \epsilon.$$

Moreover, by Lemma 2, one has

$$\left|\frac{1}{t-s}\int_s^t P[\mu](x,y)\,dx - \frac{\mu((s,t))}{t-s}\right| < C\alpha\log\frac{1}{\alpha} < \epsilon.$$

Hence

$$\left|\frac{\mu((s,t))}{t-s} - L\right| < 2\epsilon, \qquad \text{if } t-s < \delta/\alpha \text{ and } s < x_0 < t.$$

□

Remark 1. Theorem 1 can be improved in the sense that it is possible to change the assumption that u has non-tangential limit at x_0 by the existence of the limit of u along two half-lines starting at x_0. This is the same improvement that Loomis made in his Theorem 1 in [**5**], for positive harmonic functions. The proof in our case, which will be omitted, is not difficult if we use his result and some of his arguments.

3. Proof of Theorem 2.

Following Rudin's scheme [**9**], we will prove Theorem 2 using Wiener's Tauberian theorem. However, since we are not dealing with a positive measure, we must introduce significant changes to his proof.

For the reader's convenience we recall some definitions and the statement of Wiener's theorem. We will work in the multiplicative group $\mathbb{R}^+$ of the positive real numbers with the Haar measure $d\tau = s^{-1}ds$. The convolution on $\mathbb{R}^+$ is defined as

$$f * g(r) = \int_0^\infty f(r/s)g(s)\,\frac{ds}{s},$$

and the Fourier transform $\hat{f}$ of $f \in L^1(d\tau)$ as

$$\hat{f}(y) = \int_0^\infty f(r)r^{-iy}\,\frac{dr}{r}, \quad y \in \mathbb{R}.$$

Wiener's theorem [**3**, p. 509] is the following.

Theorem 3 (Wiener)**.** *Let $k \in L^1(d\tau)$ such that $\hat{k}(y) \neq 0$ for all $y \in \mathbb{R}$. Assume that there exists $M \in L^\infty(d\tau)$ such that*

$$\lim_{r\to 0}(M * k)(r) = L\hat{k}(0),$$

then $\lim_{r\to 0}(M * f)(r) = L\hat{f}(0)$ for all $f \in L^1(d\tau)$.

As in Rudin's paper, we define

$$M(t) = \frac{\mu((x_0 - t, x_0 + t))}{2t}, \quad k(t) = \frac{2t}{\pi(1+t^2)}. \tag{14}$$

A straightforward computation gives

$$\hat{k}(y) = \frac{2}{\pi}\int_0^\infty \frac{t^{-iy}}{1+t^2}\,dt = \frac{1}{\pi}\Gamma\left(\frac{1}{2} + i\frac{y}{2}\right)\Gamma\left(\frac{1}{2} - i\frac{y}{2}\right) = \frac{1}{\cosh\frac{\pi y}{2}} \neq 0,$$

for all $y \in \mathbb{R}$, and $\hat{k}(0) = 1$.

Now, by Fubini's theorem, one gets

$$M * k(r) = \frac{1}{\pi r}\int_{-\infty}^{\infty}\left(\int_0^{r/|x_0 - t|}\frac{s}{1+s^2}\,ds\right)d\mu(t).$$

The change of variables $y = s|x_0 - t|$ and again Fubini's theorem show that

$$M * k(r) = \frac{1}{r}\int_0^r u(x_0, y)\,dy. \tag{15}$$

Now we need to show that $M \in L^\infty$, this is given by the following lemma.

Lemma 3. *Let μ be a Zygmund measure. Then $P[\mu](x_0, \cdot)$ is bounded if and only if $M \in L^\infty$.*

Proof. Let $t > 0$ be fixed, then

$$\begin{aligned} \left|\frac{1}{2t}\int_{x_0-t}^{x_0+t} P[\mu](x,t)\,dx - P[\mu](x_0,t)\right| &\le \sup_x \left|\frac{\partial P[\mu]}{\partial x}(x,t)\right| t \\ &= \sup_x \left|\frac{\partial^2 P[f_\mu]}{\partial x^2}(x,t)\right| t \le C. \end{aligned} \tag{16}$$

To obtain (16) we have first applied the mean-value theorem, secondly the identity $P[\mu] = \frac{\partial P[f_\mu]}{\partial x}$, which is equivalent to (11), and finally the estimate of $\frac{\partial^2 P[f_\mu]}{\partial x^2}$ given in (5).

On the other hand, the inequality (9), with $y = t$, and (16) allows us to show that

$$\sup_{t>0} |M(t) - P[\mu](x_0, t)| \le C, \tag{17}$$

which gives the lemma. □

Next lemma will be crucial in the proof of Theorem 2.

Lemma 4. *Let μ be as usual and consider the function M defined in* (14). *For any $c > 0$ there exists a positive function f_c defined on $\mathbb{R}^+$ and depending only on c, such that:*

(i) $\hat{f}_c(0) = \int_0^\infty f_c(x)\frac{dx}{x} = 1.$
(ii) $(M * f_c)(r) = \frac{1}{2r}\int_{x_0-r}^{x_0+r} u(x, cr)\,dx,\ r > 0.$

Proof. Let us solve first the equation (ii). For this purpose we assume that $f_c \in L^1(d\tau)$. As in the proof of (15)

$$M * f_c(r) = \frac{1}{2r}\int_{-\infty}^{\infty}\left(\int_0^{r/|t-x_0|} f_c(s)\,ds\right) d\mu(t). \tag{18}$$

On the other hand, by (10)

$$\frac{1}{2r}\int_{x_0-r}^{x_0+r} P[\mu](x, cr)\,dx = \frac{1}{2r}\int_{-\infty}^{\infty} \omega((t, cr), [x_0 - r, x_0 + r], \Pi^+)\,d\mu(t). \tag{19}$$

Thus, by (18) and (19), it is enough to check if there exists a function f_c such that

$$\int_0^{r/|t-x_0|} f_c(s)\,ds = \omega((t, cr), [x_0 - r, x_0 + r], \Pi^+), \quad \text{for all } t \in \mathbb{R}.$$

By the invariance of the harmonic measure under translations, symmetries and homotheties the previous equality can be rewritten as

$$\int_0^x f_c(s)\,ds = \omega\left(\left(\frac{1}{x}, c\right), [-1,1], \Pi^+\right),$$

so

$$f_c(x) = \frac{d}{dx}\omega\left(\left(\frac{1}{x}, c\right), [-1,1], \Pi^+\right).$$

First observe that $f_c \geq 0$. So far we have not proved yet that $f_c \in L^1(d\tau)$. This will be consequence of the fact that computations involved in proving (18) and (19) are still true for $\mu = dx$ the Lebesgue measure. In this case $M = 1$ and $P[dx] = 1$, so (18) and (19) give (i). Hence $f \in L^1(d\tau)$ and therefore (ii) holds. $\square$

Remark 2. Some calculations show that actually

$$f_c(x) = \frac{4cx}{\pi(c^2x^2 + (x-1)^2)(c^2x^2 + (x+1)^2)}, \quad x > 0.$$

Proof of Theorem 2. By (15) and by the hypothesis $\lim\limits_{y\to 0^+} u(x_0, y) = L$, one has

$$\lim_{r\to 0^+} (M * k)(r) = \lim_{r\to 0^+} \frac{1}{r}\int_0^r u(x_0, y)\,dy = L = L\hat{k}(0).$$

Now, all the hypotheses of Wiener's theorem are satisfied, then using also Lemma 4, we get

$$\lim_{r\to 0^+} \frac{1}{2r}\int_{x_0-r}^{x_0+r} u(x, cr)\,dx = \lim_{r\to 0^+} (M * f_c)(r) = L\hat{f}_c(0) = L, \tag{20}$$

for any $c > 0$.

Given $\epsilon > 0$, let $0 < c \leq 1/2$ be such that $Cc\log(2/c) < 2\epsilon$, where C is the constant in (9). For this c, by (20), there exists $\delta > 0$ such that $|(M * f_c)(r) - L| < \epsilon$ if $0 < r < \delta$. On the other hand by (ii) in Lemma 4 and (9)

$$|M * f_c(r) - M(r)| = \left|\frac{1}{2r}\int_{x_0-r}^{x_0+r} u(x, cr)\,dx - \frac{\mu((x_0 - r, x_0 + r))}{2r}\right| \leq \epsilon.$$

Then by the triangle inequality $|M(r) - L| < 2\epsilon$ if $r < \delta$. $\square$

Remark 3. If μ is a Zygmund measure and $J \subset I$ are two intervals, then

$$\left|\frac{\mu(I)}{|I|} - \frac{\mu(J)}{|J|}\right| \leq C\frac{|I|}{|J|}\varphi(|J|/|I|), \tag{21}$$

where φ is the function that appears in (13). Inequality (21) is a consequence of (13).

Example. We are going to construct a radial Zygmund measure μ for which the function $P[\mu]$ has non-tangential limit zero at 0 but does not satisfy hypothesis $(\mathcal{H})$.

We will sketch the construction. First we consider the Kahane measure μ_K [**4**], which is a positive singular Zygmund measure. For this measure, if I is the 4-adic interval of length 4^{-n} which contains the point 1/3, one has

$$\mu_K(I) = (n+1)|I|\,. \tag{22}$$

Let ν be the following Zygmund measure supported on $[0,1]$,

$$\nu = \mu_K + \log|3x-1| 1_{[0,1]}(x)/\log 4\,dx.$$

Since μ_K is singular, the definition of ν implies that $D|\nu|(1/3) = +\infty$.

We claim that

$$\sup_{\delta>0} \frac{|\nu(\frac{1}{3}-\delta, \frac{1}{3}+\delta)|}{2\delta} < \infty. \tag{23}$$

In order to check (23), let J be the 4-adic interval of length 4^{-n} containing the point 1/3. By computations and taking into account (22), one has that $\nu(J)/|J|$ does not depend on n. Fix $\delta > 0$ and $I = (1/3-\delta, 1/3+\delta)$. Let J be the largest 4-adic interval of length 4^{-n} which contains 1/3 and $J \subset I$. Taking into account that $|J|/|I| \geq 3/16$, inequality (21) gives (23).

Let $\mu_j = \nu \circ T_j$ where $T_j(x) = 2^{|j|+1}(\operatorname{sgn}(j)x - 1/2^{|j|+1})$ if $j \neq 0$ and consider the radial Zygmund measure

$$\mu = \sum_{j\neq 0} \frac{1}{4^{|j|}} \mu_j.$$

By the definition one has

$$D|\mu|\left(\frac{4}{3\cdot 2^j}\right) = +\infty \quad \text{if } \ j \geq 2.$$

Using this fact and inequality (17) and (23) we conclude that

$$\lim_{y\to 0} P[|\mu|]\left(\frac{4}{3\cdot 2^j} + iy\right) = +\infty \qquad \text{and}$$

$$\limsup_{y\to 0} \left| P[\mu]\left(\frac{4}{3\cdot 2^j} + iy\right)\right| \leq C, \quad j \geq 2.$$

Thus, $P[\mu]$ does not satisfy $(\mathcal{H})$.

4. Proof of the additional results.

We begin by proving Proposition 1, which corresponds to the case $L = +\infty$.

Proof of Proposition 1. For simplicity put $x_0 = 0$ and recall that f_μ is the distribution function of μ.

(i) $\Rightarrow$ (ii). This is a consequence of the following identity:

$$\frac{f_\mu(t)-f_\mu(0)}{t}=\frac{f_\mu(t)-f_\mu(-t)}{2t}+\frac{f_\mu(t)+f_\mu(-t)-2f_\mu(0)}{2t}. \tag{24}$$

(ii) $\Rightarrow$ (iii), (iv). It will be not necessary to apply a possible generalization of Fatou's theorem, is enough to use Lemma 2 in the following way. Fix $\alpha\geq 1$. The same type of arguments made to obtain (16) yield now

$$\left|\frac{1}{2t}\int_{-t}^{t}P[\mu](x,t/\alpha)\,dx-P[\mu](x',t/\alpha)\right|\leq C\quad\text{if } |x'|\leq t.$$

By (9)

$$\left|\frac{1}{2t}\int_{-t}^{t}P[\mu](x,t/\alpha)\,dx-\frac{\mu((-t,t))}{2t}\right|\leq\frac{C}{\alpha}\log 2\alpha. \tag{25}$$

Since $\lim\limits_{t\to 0^+}\frac{\mu((-t,t))}{2t}=+\infty$, the two previous inequalities show that $P[\mu](z)$ goes to $+\infty$ when $z\to 0$ within $\Delta_\alpha(0)$.

(iii) $\Rightarrow$ (iv). Proceeding as in (16) one has

$$|P[\mu](x,t/\alpha)\,dx-P[\mu](0,t/\alpha)|\leq C,\quad\text{if } |x|\leq t.$$

Then (iv) holds.

Finally taking $\alpha=1$ in (25) one obtains (iv) $\Rightarrow$ (i). $\square$

Proof of Proposition 2. In [**5**], Loomis considered the following example. For $n\geq 1$, let $I_n=[2^{-n}-a_n,2^{-n}+a_n]$, where $0\leq a_n\leq 2^{-n-3}$. Let f be the continuous function vanishing outside $\bigcup_{n=1}^{\infty}I_n$, such that $f(2^{-n})=2^{-n}$ and is linear on each interval of the form $[2^{-n}-a_n,2^{-n}]$, $[2^{-n},2^{-n}+a_n]$. This function f is clearly of bounded variation, so we can consider the corresponding measure μ such that $\mu([x,y])=f(y)-f(x)$. Loomis was able to show that: If $\sum_{n=1}^{\infty}2^n a_n<\infty$, then $u=P[\mu]$ has non-tangential limit 0 at the origin and $D_{\mathrm{sym}}\mu(0)$ does not exist.

Now we modify this example in an appropriate way. Let $I_n=[2^{-n^2}-a_n,2^{-n^2}+a_n]$ where a_n will be chosen later and take μ constructed as before. In order to prove the statement (i), we consider first that $I=[2^{-n^2}-a_n,2^{-n^2}]$, therefore (i) holds in this case by choosing $a_n=\varphi^{-1}(2^{-n^2})$. The fact that φ is increasing reduces the validity of (i) to the case when I is included in some $[2^{-n^2}-a_n,2^{-n^2}]$. Then

$$\frac{\mu(I)}{|I|}=\frac{\varphi(a_n)}{a_n}\leq\frac{\varphi(|I|)}{|I|},$$

so (i) is true for all I. Since

$$\varphi^{-1}(x)\leq\frac{x}{\log\frac{1}{x}},\quad 0<x\leq\frac{1}{e},$$

we obtain that $\sum_{n=1}^{\infty} 2^{n^2} a_n < \infty$. Hence, for any $\alpha > 0$, $\lim P[\mu](z) = 0$ if $z \to 0$ within $\Delta_\alpha(0)$.

Since $\mu([0, 2^{-n^2}]) = 2^{-n^2}$ and $\mu([0, 2^{-n^2} - a_n]) = 0$ we conclude that $D_{\mathrm{sym}}\mu(0)$ does not exist. □

Remark 4. Actually Proposition 2 holds in a more general setting in the sense that φ of (i) can be replaced by any continuous increasing function ϕ such that $\phi(0) = 0$ and $\frac{\phi(t)}{t} \uparrow \infty$ as $t \to 0^+$. The proof is the same as before taking $I_n = [\phi(a_n) - a_n, \phi(a_n) + a_n]$ with the sequence (a_n) is chosen in such a way that

$$\sum_{n \geq 1} \phi(a_n) < \infty, \quad \sum_{n \geq 1} \frac{a_n}{\phi(a_n)} < \infty, \quad \phi(a_{n+1}) + a_{n+1} < \phi(a_n) - a_n.$$

Now we are going to consider the case when μ is a little Zygmund measure.

Proof of Proposition 3. As before put $x_0 = 0$ and assume that $\lim_{y \to 0^+} P[\mu](0, y) = L$. By Theorem 2 we know that $D_{\mathrm{sym}}\mu(0) = L$. The little Zygmund condition applied to (24) gives $D\mu(0) = L$. □

We will give a proof of Proposition 4 using complex variables techniques. The main ingredient is the following theorem of Rohde. Recall that $f \in \mathcal{B}_0$ (little Bloch space) if f is analytic in $\mathbb{D}$ and

$$\lim_{\delta \to 0} \sup_{|z| \geq 1 - \delta} (1 - |z|^2)|f'(z)| = 0.$$

See [**6**, Chap. 4] for more information.

Theorem. *Let f be an inner function in the little Bloch space which is not a finite Blaschke product. Then for each $|w| < 1$ there exists a set $E_w \subset \partial\mathbb{D} = \mathbb{T}$ of Hausdorff dimension one such that $\lim_{r \to 1} f(re^{i\theta}) = w$ for all $e^{i\theta} \in E_w$.*

Proof of Proposition 4. Let μ be a little Zygmund measure on $\mathbb{R}$. Let us consider the functions

$$b(w) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{t - w} - \frac{t}{1 + t^2} \right) d\mu(t),$$

and

$$f(z) = \exp(-b(T(z))), \quad z \in \mathbb{D},$$

where $T(z) = i\frac{1+z}{1-z}$ maps conformally $\mathbb{D}$ onto Π^+. Since μ is singular, one has $D\mu(x) = 0$ for almost all $x \in \mathbb{R}$. Then by the identity $\operatorname{Re} b(x + iy) = P[\mu](x, y) \geq 0$ and Fatou's theorem we obtain $\lim_{y \to 0} \operatorname{Re} b(x + iy) = 0$ a.e. $x \in \mathbb{R}$ and $|f(z)| \leq 1$, $z \in \mathbb{D}$. But $b \circ T$ is normal [**6**, p. 71] and has radial limits almost everywhere on $\mathbb{T}$. Since radial limits coincide with angular limits for normal functions [**6**, p. 76], we have

$$\lim_{r\to 1} \operatorname{Re} b(T(re^{i\theta})) = 0 \quad \text{ a.e. } e^{i\theta} \in \partial\mathbb{D}.$$

This implies $\lim_{r\to 1} |f(re^{i\theta})| = 1$ a.e. $e^{i\theta} \in \mathbb{T}$. Hence f is a nonconstant singular inner function.

The most technical part is to show that $f \in \mathcal{B}_0$. This is the analogue of a well-known result saying that the function

$$g(z) = \int_T \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\nu(\theta), \quad z \in \mathbb{D},$$

belongs to $\mathcal{B}_0$ if ν is a little Zygmund measure on $\mathbb{T}$ (see [**6**, Sect. 7.2]). We believe that it is enough to sketch the main steps of the proof that $f \in \mathcal{B}_0$.

First of all observe that it is enough to show that $b \circ T \in \mathcal{B}_0$. By computations we see that

$$|(b\circ T)'(z)|\,(1-|z|^2) = |b'(T(z))|\, 2 \operatorname{Im} T(z).$$

By the definition of $\mathcal{B}_0$ it suffices to show that for any sequence (w_n), $w_n \in \Pi^+$ such that $w_n \to \mathbb{R}\cup\{\infty\}$ then

$$|b'(w_n)| \operatorname{Im} w_n \to 0 \quad \text{ as } n\to\infty. \tag{26}$$

Some standard arguments on subsequences show that the possible (w_n) can be reduced to one of the following cases.

Case (a): $0 < C \le \operatorname{Im} w_n$ for any n and $|w_n| \to \infty$ as $n\to\infty$.

Case (b): $\operatorname{Im} w_n \to 0$ as $n \to \infty$.

The first case is handled using the Lebesgue dominated convergence theorem in the integral representation

$$b'(w) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{1}{(t-w)^2}\, d\mu(t).$$

For the second one, let us consider the analytic function

$$g(w) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \left(\frac{1}{t-w} - \frac{t}{1+t^2}\right)\phi(t)\, dt = u(w) + i\tilde{u}(w),$$

where $\phi(t) = \mu((-\infty, t))$ belongs to λ_* and where $\tilde{u}$ denotes the harmonic conjugate of u. An integration by parts yields

$$b = -g' - \int_{-\infty}^{\infty} \frac{1-t^2}{(1+t^2)^2}\phi(t)\, dt,$$

hence

$$|b'(w_n)| \operatorname{Im} w_n = |g''(w_n)| \operatorname{Im} w_n. \tag{27}$$

Since $u = \operatorname{Re} g = P[\phi]$ where $\phi \in \lambda_*$, an accurate estimate of $\frac{\partial^2 u}{\partial y^2}$, following [**11**, p. 146], gives

$$\sup_x \left| \frac{\partial^2 u}{\partial y^2}(x,y) \right| y \to 0 \quad \text{as } y \to 0. \tag{28}$$

On the other hand $\tilde{u} = P[H\phi]$, where $H\phi$ denotes the Hilbert transform of ϕ (see [**2**, p. 109]). But it is well known that $H\phi \in \lambda_*$, then as before

$$\sup_x \left| \frac{\partial^2 \tilde{u}}{\partial y^2}(x,y) \right| y \to 0 \quad \text{as } y \to 0. \tag{29}$$

Using (28), (29), (27) and the identity

$$|g''| = \left| \frac{\partial^2 u}{\partial x^2} + i \frac{\partial^2 \tilde{u}}{\partial x^2} \right|,$$

we obtain (26). This ends the proof of $f \in \mathcal{B}_0$.

Given $0 < \alpha \leq \infty$, let E_w the set such that $\lim_{r \to 1} f(re^{i\theta}) = w = e^{-\alpha}$, if $e^{i\theta} \in E_w$. By Rohde's theorem the Hausdorff dimension of E_w is one, so $T(E_w)$ has also dimension one, because T is locally Lipschitz. As argued before, at each point $x \in T(E_w)$ one has

$$\lim_{y \to 0^+} \operatorname{Re} b(x + iy) = \alpha.$$

Proposition 3 yields $D\mu(x) = \alpha$.

Since radial limits of b coincide with angular limits, we can conclude the previous proof using Theorem 1 instead of Theorem 2. $\square$

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