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***J*-APPROXIMATION OF COMPLEX PROJECTIVE SPACES
BY LENS SPACES**

I. DIBAG

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In this paper we study the group $J(L^k(n))$ of stable fibre homotopy classes of vector bundles over the lens space, $L^k(n) = S^{2k+1}/\mathbb{Z}_n$ where \mathbb{Z}_n is the cyclic group of order n . We establish the fundamental exact sequences and hence find the order of $J(L^k(n))$. We define a number N_k and prove that the inclusion-map $i : L^k(n) \rightarrow P_k(\mathbb{C})$ induces an isomorphism of $J(P_k(\mathbb{C}))$ with the subgroup of $J(L^k(n))$ generated by the powers of the realification of the Hopf-bundle iff n is divisible by N_k . This provides the discrete approximation to the continuous case.

0. Introduction.

Let p be a prime; $k, n \in \mathbb{Z}^+$ and $L^k(p^n) = S^{2k+1}/\mathbb{Z}_{p^n}$ be the lens space where \mathbb{Z}_{p^n} is the cyclic group of order p^n . $L^k(p^n)$ has the structure of a CW-complex $L^k(p^n) = \cup_{j=0}^{2k+1} e^j$ and its $2k$ -th skeleton,

$$L_0^k(p^n) = \{[z_0, \dots, z_k] \in L^k(p^n) : z_k \text{ is real } \geq 0\}.$$

In this paper we study the group $J(L^k(p^n))$, making use of the already established results in [10] and [12] on $\tilde{K}_{\mathbb{R}}(L^k(p^n))$. We first establish the exact sequences analogous to the ones proved in [4] for $J(P_k(\mathbb{C}))$. Define $\bar{L}^k(p^n) = \begin{cases} L_0^k(p^n) & \text{if } p \text{ is odd} \\ L^k(p^n) & \text{if } p = 2 \end{cases}$. The main difficulty is to prove the injectivity of the map $c^! : J(\bar{L}^{2k}(p^n)/\bar{L}^{2k-2}(p^n)) \rightarrow J(\bar{L}^{2k}(p^n))$, whereas the corresponding result, e.g. [4, Lemma 4.9], is trivial for complex projective spaces. We resolve this difficulty by using the transfer map $\tau : \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n)) \rightarrow \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^{n+1}))$ and to make the transfer map suitable for application, we prove a number of preliminary results in Section 2.2 concerning binomial expansions. This leads to Proposition 2.3.3 about the kernel of $(\psi_{\mathbb{R}}^t - 1)$ where t is an integer not divisible by p and $\psi_{\mathbb{R}}^t$ is the Adams operation and which plays a fundamental role in the proofs of Lemma 3.2.1 and Proposition 3.2.2 for the injectivity of $c^!$. Using the exact sequences we establish, we find in Proposition 3.3.4, the order of $J(\bar{L}^{2v}(p^n))$. Let $G(p, k, n)$ be the subgroup of $J(L^k(p^n))$ generated by the powers of the realification of the Hopf-bundle over $L^k(p^n)$

which coincides with $J(L^k(p^n))$, except for p odd and $k \equiv 0 \pmod{4}$ and for $p = 2$ and $n \geq 2$ in which case it is a subgroup of $J(L^k(p^n))$ of index 2. Let $i : L^k(p^n) \rightarrow P_k(\mathbb{C})$ be the inclusion-map. We define a number N_k in 2.2.9. The main result of the paper; e.g., Theorem 3.4.2 states that $i^!$ maps the p -summand, $J_p(P_k(\mathbb{C}))$ of $J(P_k(\mathbb{C}))$ isomorphically onto $G(p, k, n)$ iff n is greater or equal to the p -exponent of N_k . This provides the discrete approximation to the continuous case. We then conjecture in 3.4.4 a stronger version of this which involves the degree function on the J -groups.

Finally, we observe that the transfer map passes to the quotient and defines a map on the J -groups of the respective lens spaces. We prove in Proposition 3.5.3 that $\tau \circ i^!(x) = px$ for $\forall x \in G(p, k, n + 1)$.

The paper is self-contained as a whole. Only very elementary facts about the $\tilde{K}_{\mathbb{R}}$ -groups of lens spaces are used and everything concerning J -groups of lens spaces is developed from scratch.

1. $\tilde{K}_{\mathbb{R}}$ -groups of lens spaces.

1.1. Survey of results. Let p be a prime and $k, n \in \mathbb{Z}^+$. Let η be the complex Hopf-bundle over $L^k(p^n)$, $\mu = \eta - 1 \in \tilde{K}_{\mathbb{C}}(L^k(p^n))$ be its reduction and $w = r(\mu) \in \tilde{K}_{\mathbb{R}}(L^k(p^n))$ be the realification of μ . It is (essentially) shown in [10] and [12] that $\tilde{K}_{\mathbb{C}}(L^k(p^n))$ is generated multiplicatively by μ subject to the relations :

$$I. \mu^{k+1} = 0, \quad II. \psi_{\mathbb{C}}^{p^n}(\mu) = \mu\psi_{\mathbb{C}}^{p^n}(\mu) = \dots = \mu^{k-1}\psi_{\mathbb{C}}^{p^n}(\mu) = 0.$$

For p odd, $\tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ is generated multiplicatively by w subject to the relations :

$$I'. w^{[k/2]+1} = 0 \text{ and } II'. \text{ The realification of the relations II above.}$$

For $p = 2$ and $n \geq 2$, let ξ be the real line-bundle over $L^k(2^n)$ such that $c(\xi) = \eta^{2^{n-1}}$ where c is the complexification-map. Let $\lambda = \xi - 1 \in \tilde{K}_{\mathbb{R}}(L^k(2^n))$. Then $\tilde{K}_{\mathbb{R}}(L^k(2^n))$ is generated multiplicatively by w and λ subject to:

$$I'. w^{[k/2]+1} = 0 \text{ if } k \not\equiv 1 \pmod{4} \text{ and } 2w^{[k/2]+1} = w^{[k/2]+2} = 0 \text{ if } k \equiv 1 \pmod{4}$$

II'. The realification of relations II above. Relations II' in $\tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ for p odd and in $\tilde{K}_{\mathbb{R}}(L^k(2^n))$ for $p = 2$ are equivalent to the periodicity-relations: $\psi_{\mathbb{R}}^{s+p^n}(w) = \psi^s(w), \forall s \in \mathbb{Z}$ or to the single relation obtained by taking $s = -1$; i.e., $\psi_{\mathbb{R}}^{p^n-1}(w) - w = 0$ which by Proposition 1.1.6 is of the form: $p^n(p^n - 2)w + \sum_{j \geq 2} \alpha_j w^j = 0$ or upon multiplication by $w^{i-1} : (i \geq 1), p^n(p^n - 2)w^i + \sum_{j \geq 2} \alpha_j w^{i+j} = 0 (i \geq 1)$ which are equivalent to:

$p^n w^i + \sum_{j \geq 2} \beta_j w^{i+j} = 0$ for p odd and $2^{n+1} w^i + \sum_{j \geq 2} \beta_j w^{i+j} = 0$ for $p = 2$.
 III'. $2\lambda = \psi_{\mathbb{R}}^{2^{n-1}}(w)$ and IV'. $\lambda w = (\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)$.

For $k \equiv 0 \pmod{4}$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \mathbb{Z}_2 \oplus \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ if p is odd and $\tilde{K}_{\mathbb{R}}(L_0^k(2^n)) = \tilde{K}_{\mathbb{R}}(L^k(2^n))/\mathbb{Z}_2\langle 2^{n+k-2}w \rangle$ if $p = 2$ and for $k \not\equiv 0 \pmod{4}$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$.

Lemma 1.1.1. *Let p be a prime; $v, n \in \mathbb{Z}^+$, $n \geq 2$ if $p = 2$. Then*

$$\tilde{K}_{\mathbb{R}}(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n)) = \begin{cases} \mathbb{Z}_{p^n} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{2^{n+1}} & \text{if } p = 2 \end{cases}.$$

Lemma 1.1.2. *Let $v, n \in \mathbb{Z}^+$. Then*

$$\tilde{K}_{\mathbb{R}}(L^{4v+1}(p^n)/L^{4v}(p^n)) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p = 2 \end{cases}.$$

Lemma 1.1.3. *Let p be a prime, $v \in \mathbb{Z}^+$. Then*

$$\tilde{K}_{\mathbb{R}}(L^{4v+3}(p^n)/L^{4v+2}(p^n)) = 0.$$

Lemma 1.1.4. *Let p be an odd prime; $k, t \in \mathbb{Z}^+$ such that $(p, t) = 1$. Then $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1) = 0$ on $\tilde{K}_{\mathbb{R}}(L_0^k(p))$.*

Proof. By Fermat's Theorem, $t^{p-1} \equiv 1 \pmod{p}$ and thus $t^{\frac{p-1}{2}} \equiv \pm 1 \pmod{p}$. Let η, μ, w be defined as in 1.1. Then $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1)\mu = \eta^t \frac{p-1}{2} - \eta = \eta^{\pm 1} - \eta$. If we take realification of both sides and note that $r(\eta^{-1}) = r(\eta)$, we obtain $((\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1)w = 0$. □

Definition 1.1.5. For $m, k \in \mathbb{Z}^+$, define the even binomial coefficient $u_m(k) = \frac{k^2(k^2-1)\dots(k^2-(m-1)^2)}{\frac{1}{2}(2m)!}$. Note that $u_k(k) = 1$ and $u_m(k) = 0$ for $m > k$.

Proposition 1.1.6. *In $\tilde{K}_{\mathbb{R}}(P_{\infty}(\mathbb{C}))$, $\psi_{\mathbb{R}}^k(w) = \sum_{m=1}^k u_m(k)w^m$.*

Proof. This is [7, Theorem 5.2.4]. □

2. The transfer-map.

2.1. Properties of the transfer-map. Let H be a subgroup of the compact Lie group G of finite index. Then there exists an induction-homomorphism $i_! : R_F(H) \rightarrow R_F(G)$ ($F = \mathbb{R}, \mathbb{C}$) on the representation rings as defined in [6, Section 7]. Let P be the top space of a principal G -bundle. Using the induction-homomorphism, one defines a transfer-map, $\tau_F : K_F(P/H) \rightarrow K_F(P/G)$ for the fibration $f : P/H \xrightarrow{G/H} P/G$ which in turn induces $\tau_F : \tilde{K}_F(P/H) \rightarrow \tilde{K}_F(P/G)$. In the special case, $H = \mathbb{Z}_{p^n}$, $G = \mathbb{Z}_{p^{n+1}}$ where p

is a prime and $P = S^{2k+1}$, we obtain a transfer-map, $\tau_F : \tilde{K}_F(L^k(p^n)) \rightarrow \tilde{K}_F(L^k(p^{n+1}))$ and its restriction, $\tau_F : \tilde{K}_F(L_0^k(p^n)) \rightarrow \tilde{K}_F(L_0^k(p^{n+1}))$. We now list some fundamental properties of the transfer.

Proposition 2.1.1.

(i) *The transfer-map commutes with the complexification and realification maps, i.e., the following diagrams commute:*

$$\begin{array}{ccc}
 \tilde{K}_{\mathbb{R}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{R}}} & \tilde{K}_{\mathbb{R}}(L^k(p^{n+1})) & \tilde{K}_{\mathbb{C}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{C}}} & \tilde{K}_{\mathbb{C}}(L^k(p^{n+1})) \\
 \downarrow c & & \downarrow c & \downarrow r & & \downarrow r \\
 \tilde{K}_{\mathbb{C}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{C}}} & \tilde{K}_{\mathbb{C}}(L^k(p^{n+1})) & \tilde{K}_{\mathbb{R}}(L^k(p^n)) & \xrightarrow{\tau_{\mathbb{R}}} & \tilde{K}_{\mathbb{R}}(L^k(p^{n+1}))
 \end{array}$$

- (ii) *If $t \in \mathbb{Z}^+$ and $(p, t) = 1$ then $\psi_F^t \circ \tau_F = \tau_F \circ \psi_F^t$.*
- (iii) *$\tau_F \circ f^!(x) = \tau_F(1)x, \forall x \in \tilde{K}_F(L^k(p^{n+1}))$.*
- (iv) *$f^! \circ \tau_F(x) = px, \forall x \in \tilde{K}_F(L^k(p^n))$.*
- (v) *Let $F = \mathbb{C}$ and η_n and η_{n+1} be the Hopf-bundles over $L^k(p^n)$ and $L^k(p^{n+1})$ respectively. Then $\tau_{\mathbb{C}}(\eta_n^i) = \sum_{j \equiv i \pmod{p^n}} \eta_{n+1}^j$.*

Proof. (i) and (iv) follow immediately from the definition of the transfer-map as in [6, Section 7]. For (ii) and (iii) we refer the reader to [14, Lemma 2.2]. (v) is [3, Lemma 6.5.8]. □

Lemma 2.1.2. *Let $\mu \in \tilde{K}_{\mathbb{C}}(P_{\infty}(\mathbb{C}))$ and $w \in \tilde{K}_{\mathbb{R}}(P_{\infty}(\mathbb{C}))$ be the multiplicative generators and $r : \tilde{K}_{\mathbb{C}}(P_{\infty}(\mathbb{C})) \rightarrow \tilde{K}_{\mathbb{R}}(P_{\infty}(\mathbb{C}))$ be the realification-map. Then $r(\mu^k) = \sum_{i=\lfloor \frac{k+1}{2} \rfloor}^k a_i w^i$ ($a_i \in \mathbb{Z}$).*

Proof. This can be proved by induction on k using the relation $r(\psi_{\mathbb{C}}^k(\mu)) = \psi_{\mathbb{R}}^k(w)$. □

We shall now drop the subscript and write down τ for $\tau_{\mathbb{R}}$.

Proposition 2.1.3. *Let $k \in \mathbb{Z}^+$ and assume that $n \geq 2$ if $p = 2$. Then $\tau(w^k) = \sum_{i \geq 1} a_i w^{k+i-1}$ in $\tilde{K}_{\mathbb{R}}(L_0^k(p^{n+1}))$ where $a_1 = p$ and p/a_i for $2 \leq i \leq p$.*

Proof. It suffices to prove it for $k = 1$ since by (iii) of Proposition 2.1.1 $\tau(w^k) = \tau(w)w^{k-1}$.

That $a_1 = p$ follows from (iv) of Proposition 2.1.1.

For $p = 2$, $\tau_{\mathbb{C}}(1) = 1 + (1 + \mu)^{2^n} = 2 + \sum_{i=1}^{2^n-1} \binom{2^n}{i} \mu^i + \mu^{2^n}$, by (v) of Proposition 2.1.1, where $2/\binom{2^n}{i}$ for $1 \leq i \leq 2^n - 1$. $\tau_{\mathbb{C}}(\mu) = (\tau_{\mathbb{C}}(1))\mu = 2\mu + \sum_{i=1}^{2^n-1} \binom{2^n}{i} \mu^{i+1} + \mu^{2^n+1}$ by (iii) of Proposition 2.1.1.

We take realification of both sides and using Lemma 2.1.2 and commutativity of the second diagram in (i) of Proposition 2.1.1, we obtain

$\tau(w) = \sum_{i \geq 1} a_i w^i$ where $2/i$ for $2 \leq i \leq 2^{n-1}$ and since $n \geq 2$ this yields the result for $p = 2$.

For p odd, $\tau_{\mathbb{C}}(1) = 1 + (1 + \mu)^{p^n} + (1 + \mu)^{2p^n} + \dots + (1 + \mu)^{(p-1)p^n} = p + \sum_{i \geq 1} b_i \mu^i$.

For $1 \leq i \leq p^n - 1$, $b_i = \binom{p^n}{i} + \binom{2p^n}{i} + \dots + \binom{(p-1)p^n}{i}$ and hence p/b_i .

For $i = p^n$, $b_{p^n} = 1 + \binom{2p^n}{p^n} + \dots + \binom{(p-1)p^n}{p^n}$.

For $1 \leq s \leq p - 1$, $\binom{sp^n}{p^n} = s \prod_{m=1}^{p^n-1} \frac{sp^n - p^n + m}{m} \equiv s \pmod{p}$. Thus $b_{p^n} \equiv (1 + 2 + \dots + (p - 1)) \pmod{p} \equiv \frac{p(p-1)}{2} \equiv 0 \pmod{p}$, i.e., p/b_{p^n} .

For $p^n + 1 \leq i \leq 2p^n - 1$, $a_i = \binom{2p^n}{i} + \dots + \binom{(p-1)p^n}{i}$ and hence p/a_i . Thus p/b_i for $0 \leq i \leq 2p^n - 1$. $\tau_{\mathbb{C}}(\mu) = (\tau_{\mathbb{C}}(1))\mu = p\mu + \sum_{j \geq 1} b_j \mu^{j+1} = p\mu + \sum_{j \geq 2} b_{j-1} \mu^j$ $\tau(w) = \tau(r(\mu)) = r(\tau_{\mathbb{C}}(\mu)) = pw + \sum_{j \geq 2} b_{j-1} r(\mu^j)$ by the commutativity of the 2^{nd} -diagram in (i) of Proposition 2.1.1 $r(\mu^j) = \sum_{i=\lceil \frac{j+1}{2} \rceil}^j c_i^j w^i$ ($c_i^j \in \mathbb{Z}$) by Lemma 2.1.2.

Thus $\tau(w) = pw + \sum_{j \geq 2} \sum_{i=\lceil \frac{j+1}{2} \rceil}^j b_{j-1} c_i^j w^i = pw + \sum_{i \geq 1} a_i w^i$ where $a_i = \sum_{\lceil \frac{i+1}{2} \rceil \leq j \leq i} b_{j-1} c_i^j = \sum_{j=i}^{2i} b_{j-1} c_i^j$. Let $i \leq p^n$. Then in the second sum above, $j \leq 2i \leq 2p^n$ and $p|b_{j-1}$ by the first part of the proof. Hence $p|a_i$ for $2 \leq i \leq p^n$ and hence for $2 \leq i \leq p$. \square

2.2. Preliminaries on binomial expansions. Section 2.2 is a technical section aimed at proving Proposition 2.2.2.

If p is a prime and $n \in \mathbb{Z}^+$, $v_p(n)$ will denote the exponent of p in the prime factorization of n .

Definition 2.2.1. For $p^{n-1} \leq k \leq p^n - 1$, define $\Phi(k) = n + \lfloor \frac{p^n - k - 1}{p} \rfloor$.

If we arrange the integers in decreasing fashion from $k = p^n - 1$ to $k = p^{n-1}$ in blocks B_j of p consecutive integers then Φ is the step function which is constant on each block, increases by 1 with each increasing block and takes the value n on B_1 .

Proposition 2.2.2. Let $S_{n,p} = \sum_{j \geq p^{n-1}} c_j [\psi_{\mathbb{R}}^p(w)]^j$ in $\tilde{K}_{\mathbb{R}}(P_{\infty}(\mathbb{C}))$. If we expand $S_{n,p} = \sum_{k \geq p^{n-1}} a_k w^k$ then $v_p(a_k) \geq \Phi(k)$ ($p^{n-1} \leq k \leq p^n - 1$).

Proposition 2.2.2 is essential for the inductive proof of Proposition 2.3.1 which in turn is essential for the proof of Lemma 3.2.1 for the injectivity of the homomorphism c^1 . Proposition 2.3.1 asserts for p odd and t prime to p , the existence of a series in w^k that starts at w^j (for any j) and which belongs to $\text{Ker} [(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$, the exponents of whose coefficients have a lower-bound given by a certain function $\psi(j, k)$ which is attained for $k = j$. For $v_p(j) \leq n - 2$, the result follows by applying the transfer-map to the series we have in $\text{Ker} [(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^{n-1}))$ by the induction-hypothesis

and by applying Proposition 2.1.3 to the coefficients. The difficult case is the one for $j = p^{n-1}$ (the more general case, $v_p(j) \geq n - 1$ easily follows from this) and this is where Proposition 2.2.2 comes into play. For $j = p^{n-2}$, we have two series to compare; one that we have by the case $v_p(j) \leq n - 2$ and another one that we obtain by applying the homomorphism $f!$ induced by the p -th power map, $f : L_0^{2v}(p^n) \rightarrow L_0^{2v}(p^{n-1})$ to the series that we have by the induction-hypothesis for $j = p^{n-2}$. By noting that $f^!(w) = \psi_{\mathbb{R}}^p(w)$, the second-series is of the form $\sum_{k \geq p^{n-2}} b_k [\psi_{\mathbb{R}}^p(w)]^k$ which by Proposition 2.2.2 can be written as $\sum_{k \geq p^{n-2}} a_k w^k$ where $v_p(a_k) \geq \Phi(k)$. A lower-bound for the exponents of the coefficients of the first series is given by $\psi(p^{n-2}, k)$ which is attained for $k = p^{n-2}$. $\Phi(j) \geq \psi(j, j)$, in general and using the special case of this for $j = p^{n-2}$, we can subtract a scalar-multiple of the first series from the second to eliminate the term $w^{p^{n-2}}$ and the resulting series starts with the term $w^{p^{n-2}+1}$. If m is the exponent of the multiplying factor then $m + \psi(j, k) \geq \Phi(k)$ and an immediate consequence of the special-case of this inequality for $j = p^{n-2}$ is that the p^{n-1} -th coefficient of the resulting series is prime to p . We continue this process inductively until we knock off the terms $w^{p^{n-2}+1}, w^{p^{n-2}+2}, \dots, w^{p^{n-1}-1}$ and in the end, obtain a series in $\text{Ker}[(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$ that starts with the term $w^{p^{n-1}}$ and whose p^{n-1} -th coefficient is prime to p .

Lemma 2.2.3. *Let p be an odd prime, $m \in \mathbb{Z}^+$ and $u_m(p)$ the even binomial coefficient defined in 1.1.5. Then*

$$v_p(u_m(p)) = \begin{cases} 2 & \text{if } 1 \leq m \leq \frac{p-1}{2} \\ 1 & \text{if } \frac{p+1}{2} \leq m \leq p-1. \end{cases}$$

Observation 2.2.4. Let p be an odd prime, $n \in \mathbb{Z}^+$ and let $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}} = \sum_{k=p^{n-1}}^{p^n} a_k w^k$ in $\tilde{K}_{\mathbb{R}}(P_{\infty}(\mathbb{C}))$. Then

$$a_k = \sum_{\substack{s_1 + \dots + s_p = p^{n-1} \\ s_1 + 2s_2 + \dots + ps_p = k}} \frac{(p^{n-1})!}{s_1!s_2! \dots s_p!} \prod_{m=1}^{p-1} [u_m(p)]^{s_m}.$$

Proof. It is an immediate consequence of Proposition 1.1.6. □

Definition 2.2.5. Let p be an odd prime, $n \in \mathbb{Z}^+$ and $p^{n-1} \leq k \leq p^n - 1$. We let S_k denote the set of all sequences $s = (s_1, \dots, s_p)$ of non-negative integers such that $s_1 + \dots + s_p = p^{n-1}$ and $s_1 + 2s_2 + \dots + ps_p = k$. For $s \in S_k$, define $T(s) = \frac{(p^{n-1})!}{s_1!s_2! \dots s_p!}$ and $\theta(s) = T(s) \prod_{m=1}^{p-1} [u_m(p)]^{s_m}$. Observation 2.2.4 can be stated in an equivalent form, i.e.,

Observation 2.2.6. Under the hypothesis of Observation 2.2.4, $a_k = \sum_{s \in S_k} \theta(s)$.

Definition 2.2.7. Let p be an odd prime and $s \in S_k$. Define $e_p(s) = 2s_1 + \dots + 2s_{\frac{p-1}{2}} + s_{\frac{p+1}{2}} + \dots + s_{p-1}$.

We state the following Corollary to Lemma 2.2.3.

Corollary 2.2.8. $v_p(\theta(s)) = v_p(T(s)) + e_p(s)$.

Definition 2.2.9. For $k \in \mathbb{Z}^+$, define a number N_k by $v_p(N_k) = \sup_{1 \leq r \leq \lfloor \frac{k}{p-1} \rfloor} (1 + v_p(r))$. Let $N_{k,p}$ denote its p -component.

We now observe that [5, Lemma 6.1] can be proved under more general hypothesis; i.e.,

Lemma 2.2.10. *Let p be a prime, $n, k \in \mathbb{Z}^+$. If $v_p(n) \geq v_p(N_{k-1})$ then $v_p(\binom{n}{k}) = v_p(n) - v_p(k)$.*

Proof. Identical with that of [5, Lemma 6.1]. □

In the following, p is an odd prime and $n \in \mathbb{Z}^+$.

Definition 2.2.11. Let I_i be the closed interval, $I_i = [p^n - p^i + 1, p^n - p^{i-1}]$ in \mathbb{Z}^+ ($1 \leq i \leq n - 1$) and let $I_n = [p^{n-1}, p^n - p^{n-1}]$. Then $[p^{n-1}, p^n - 1] = \cup_{i=1}^n I_i$.

Lemma 2.2.12. *Let $s \in S_k$. Then $s_p \geq k - (p - 1)p^{n-1}$. If $k \in I_i$ then $s_p \geq p^{n-1} - p^i + 1$.*

Proof. $s_1 + s_2 + \dots + s_p = p^{n-1}$ and $s_1 + 2s_2 + \dots + ps_p = k$ and subtracting the first equation from the second yields $1 \cdot s_2 + 2s_3 + \dots + (p-2)s_{p-1} + (p-1)s_p = k - p^{n-1}$, or equivalently $s_2 + 2s_3 + \dots + (p-2)s_{p-1} + (p-2)s_p + p^{n-1} = k - s_p$. $LHS = (s_2 + \dots + s_p) + (s_3 + \dots + s_p) + \dots + (s_{p-1} + s_p) + p^{n-1} \leq (p-2)p^{n-1} + p^{n-1} = (p-1)p^{n-1}$. Thus, $k - s_p \leq (p-1)p^{n-1}$ or, equivalently, $s_p \geq k - (p-1)p^{n-1}$. If $k \in I_i$ then $k \geq p^n - p^i + 1$ and hence $s_p \geq k - (p-1)p^{n-1} \geq p^n - p^i + 1 - (p-1)p^{n-1} = p^{n-1} - p^i + 1$. □

Corollary 2.2.13. *Let $s \in S_k$ and $k \in I_i$. Then $v_p(T(s)) \geq n - i$.*

Proof. It follows from the second part of Lemma 2.2.12 that $v_p(s_p) \leq i - 1$. $T(s) = \binom{p^{n-1}}{s_p} \frac{(s_1 + \dots + s_{p-1})!}{s_1! \dots s_{p-1}!}$ and it follows from Lemma 2.2.10 that $v_p(\binom{p^{n-1}}{s_p}) = n - 1 - v_p(s_p) \geq n - 1 - (i - 1) = n - i$. □

Definition 2.2.14. For each $p^{n-1} \leq k \leq p^n - 1$, we define a unique special sequence $s^0(k)$ by $(s^0(k))_p = \lfloor \frac{k - p^{n-1}}{p-1} \rfloor$. Let $r = k - p^{n-1} - (p-1)(s^0(k))_p$. Then $0 \leq r \leq p - 2$. The remaining (possibly) non-zero indices of $s^0(k)$ are $(s^0(k))_{r+1} = 1$ if $r \geq 1$ and $(s^0(k))_1 = p^{n-1} - (s^0(k))_p - 1 + \delta_{r0}$ where δ_{r0} is the Kronecker-delta. If we arrange the integers in decreasing fashion from

$k = p^n - 1$ to $k = p^{n-1}$ in p^{n-1} blocks B_j of $(p-1)$ consecutive integers, then $(s^0(k))_p = p^{n-1} - j$ is constant on each block. If $B_j = (k_1, \dots, k_{p-1})$, $k_i = k_{i-1} + 1$, $k_i = p^n - j(p-1) + i - 1$ then the non-zero indices of $s^0(k_i)$ apart from $(s^0(k_i))_p$ are : $(s^0(k_i))_i = 1$ and

$$(s^0(k_i))_1 = \begin{cases} j & \text{if } i = 1 \\ j - 1 & \text{if } 2 \leq i \leq p - 1 \end{cases}.$$

Observation 2.2.15. If we arrange the integers in decreasing fashion from $k = p^n - 1$ to $k = p^{n-1}$ in $2p^{n-1}$ blocks of $\frac{p-1}{2}$ consecutive integers then $e_p(s^0(k))$ is constant on each block and increases by 1 with each increasing block and takes the value 1 on the first block.

Proof. Let $B_j^1 = (k_{\frac{p+1}{2}}, \dots, k_{p-1})$ and $B_j^2 = (k_1, \dots, k_{\frac{p-1}{2}})$. Then it is clear from the above and the definition of $e_p(s^0(k))$ that $e_p(s^0(k))$ is constant on B_i^j ($i = 1, 2$) and increases by 1 in passing from B_j^1 to B_j^2 and from B_j^2 to B_{j+1}^1 and takes the value 1 on B_1^1 . \square

Lemma 2.2.16. If $p^{n-1} \leq k \leq p^n - 1$ and $s \in S_k$ then $e_p(s) \geq e_p(s^0(k))$.

Proof. Define $u(s) = \sum_{i=1}^{\frac{p-1}{2}} s_i$ and $v(s) = \sum_{i=\frac{p+1}{2}}^{p-1} s_i$. Then by definition, $e_p(s) = 2u + v = 2(u + v + s_p) - v - 2s_p = 2p^{n-1} - v - 2s_p$. Hence:

1. $e_p(s) - e_p(s^0(k)) = [v(s^0(k)) - v(s)] + 2((s^0(k))_p - s_p) s_2 + 2s_3 + \dots + (p-1)s_p = k - p^{n-1} = r + (p-1)(s^0(k))_p$ where $0 \leq r \leq p-2$ and thus;

2. $s_2 + 2s_3 + \dots + (\frac{p-1}{2})s_{\frac{p+1}{2}} + \dots + (p-2)s_{p-1} = (p-1)((s^0(k))_p - s_p) + r$ LHS $\geq \sum_{i=\frac{p+1}{2}}^{p-1} (i+1)s_i \geq (\frac{p-1}{2}) \sum_{i=\frac{p+1}{2}}^{p-1} s_i = (\frac{p-1}{2})v(s)$ gives $v(s) \leq 2((s^0(k))_p - s_p) + \frac{2r}{p-1}$ and hence $v(s) \leq 2((s^0(k))_p - s_p) + [\frac{2r}{p-1}]$.

(i) If $r \geq \frac{p-1}{2}$, $(s^0(k))_{r+1} = 1$ and $v(s^0(k)) = 1$ and thus $v(s) \leq 2((s^0(k))_p - s_p) + 1$.

(ii) If $r \leq \frac{p-1}{2}$, $v(s^0(k)) = 0$ and thus $v(s) \leq 2((s^0(k))_p - s_p)$ and in either case, $v(s) \leq 2((s^0(k))_p - s_p) + v(s^0(k))$ and the result follows from 1 above. \square

Lemma 2.2.17. For $k \in I_i$, $n - i + e_p(s^0(k)) \geq \Phi(k)$.

Proof. It follows from Definition 2.2.1 in a straightforward way. \square

Corollary 2.2.18. Let $p^{n-1} \leq k \leq p^n - 1$ and $s \in S_k$. Then $v_p(\theta(s)) \geq \Phi(k)$.

Proof. It is an immediate consequence of Corollaries 2.2.8, 2.2.13 and Lemma 2.2.17. \square

Proof of Proposition 2.2.2. It suffices to prove that if $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}+j} = \sum_{k \geq p^{n-1}+j} a_k^j w^k$ then $v_p(a_k^j) \geq \Phi(k)$. $[\psi_{\mathbb{R}}^p(w)]^{p^{n-1}+j} = [\psi_{\mathbb{R}}^p(w)]^{p^{n-1}} [\psi_{\mathbb{R}}^p(w)]^j$

$= [\sum_{i \geq p^{n-1}} a_i w^i][\sum_{l \geq j} b_l w^l] = \sum_{k \geq p^{n-1}+j} a_k^j w^k$. $a_j^k = \sum_{i+l=k} a_i b_l$. By Observation 2.2.4 and Corollary 2.2.17, $v_p(a_i) \geq \Phi(i) \geq \Phi(k)$ for $i \leq k$ and thus $v_p(a_i b_l) \geq \Phi(k)$. Hence $v_p(a_k^j) \geq \Phi(k)$. \square

2.3. Kernel of $(\psi_{\mathbb{R}}^t - 1)$. In what follows t will be an integer not divisible by the prime p .

Proposition 2.3.1. *Let p be an odd prime and $j, n, t, v \in \mathbb{Z}^+$ such that $(p, t) = 1$. For $k \geq j$, define*

$$\psi(j, k) = \begin{cases} n - 1 - v_p(j) - \left\lfloor \frac{k-j}{p} \right\rfloor & \text{if } j \leq k \leq j + p(n - 1 - v_p(j)) - 1 \\ 0 & \text{if } k \geq j + p(n - 1 - v_p(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

- (i) $v_p(a_{j,j}) = \psi(j, j) = n - 1 - v_p(j)$.
- (ii) $v_p(a_{j,k}) \geq \psi(j, k)$.
- (iii) $\sum_{k \geq j} a_{j,k} w^k \in \text{Ker} [(\Psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$.

Proof. By induction on n .

For $n = 1$ it follows from Lemma 1.1.4.

Let $n > 1$ and assume it to be true for $n - 1$. Let $K_i = \text{Ker} [(\Psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1] \subseteq \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^i))$.

For $0 \leq v_p(j) \leq n - 2$, the result can be obtained by applying the transfer map $\tau_{n-1} : \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^{n-1})) \rightarrow \tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$ and by using the induction-hypothesis and Proposition 2.1.3 and by noting that τ_{n-1} maps K_{n-1} to K_n which is a consequence of (ii) of Proposition 2.1.1. For $j = p^{n-1}$, the p -th power map, $g : L_0^{2v}(p^n) \rightarrow L_0^{2v}(p^{n-1})$ factors through $L_0^{2v}(p^{n-1})$, i.e., there exists a map $f : L_0^{2v}(p^n) \rightarrow L_0^{2v}(p^{n-1})$ such that the following diagram commutes:

$$\begin{array}{ccc} L_0^{2v}(p^n) & \xrightarrow{f} & L_0^{2v}(p^{n-1}) \\ & \searrow g & \downarrow \\ & & L_0^{2v}(p^n) \end{array}$$

Thus, $f^!(w) = \psi_{\mathbb{R}}^p(w)$. By the induction-hypothesis, there exist $\sum_{s \geq p^{n-2}} b_{p^{n-2},s} w^s \in K_{n-1}$. Applying $f^!$ and by noting that $f^!$ maps K_{n-1} to K_n , we obtain $\sum_{s \geq p^{n-2}} b_{p^{n-2},s} [\psi_{\mathbb{R}}^p(w)]^s \in K_n$. By Proposition 2.2.2, there exist $c_k \in \mathbb{Z}$ ($k \geq p^{n-2}$) with $v_p(c_k) \geq \Phi(k)$ such that:

1. $\sum_{k \geq p^{n-2}} c_k w^k \in K_n$.

We now claim the following statement. For $p^{n-2} \leq j \leq p^{n-1}$, there exist $c_{j,k} \in \mathbb{Z} (k \geq 1)$ with $(p, c_{j,p^{n-1}}) = 1$, $v_p(c_{j,k}) \geq \Phi(k)$ such that $\sum_{k \geq j} c_{j,k} w^k \in K_n$.

Proof. By induction on j . For $j = p^{n-2}$ this follows from 1 above.

Let $p^{n-2} < j \leq p^{n-1}$ and assume it to be true for $(j - 1)$. Since $0 \leq v_p(j - 1) \leq n - 2$, by the first part of the proof, there exist coefficients $a_{j-1,k} \in \mathbb{Z} (k \geq j - 1)$ with $v_p(a_{j-1,j-1}) = \psi(j - 1, j - 1) = n - 1 - v_p(j - 1)$ and $v_p(a_{j-1,k}) \geq \psi(j - 1, k)$ such that:

2. $\sum_{k \geq j-1} a_{j-1,k} w^k \in K_n$.

By the induction-hypothesis, there exist coefficients $c_{j-1,k} \in \mathbb{Z} (k \geq j - 1)$ with $(p, c_{j-1,p^{n-1}}) = 1$, $v_p(c_{j-1,k}) \geq \Phi(k)$ such that

3. $\sum_{k \geq j-1} c_{j-1,k} w^k \in K_n$.

Define $m = v_p(c_{j-1,j-1}) - v_p(a_{j-1,j-1}) \geq \Phi(j - 1) - \psi(j - 1, j - 1) \geq 0$ $a_{j-1,j-1} = p^{v_p(a_{j-1,j-1})} \alpha_{j-1}$ and $c_{j-1,j-1} = p^{v_p(c_{j-1,j-1})} \gamma_{j-1}$ where $(p, \alpha_{j-1}) = (p, \gamma_{j-1}) = 1$. Multiply Equation 2 by $p^m \gamma_{j-1}$ and 3 by $-\alpha_{j-1}$ and add up the resulting equations to obtain:

5. $\sum_{k \geq j} c_{j,k} w^k \in K_n$.

Let $\Delta\psi(j - 1, k)$ and $\Delta\Phi(k)$ be the respective increases in $\psi(j - 1, k)$ and $\Phi(k)$ from $j - 1$ to k . $\psi(j - 1, k)$ and $\Phi(k)$ are constant on each p -block of consecutive (increasing) integers starting with $j - 1$ and p^{n-2} respectively and decrease by 1 with each increasing block. Thus $\Delta\psi(j - 1, k) \geq \Delta\Phi(k)$ $m + \psi(j - 1, j - 1) \geq \Phi(j - 1)$ and $m + \psi(j - 1, k) = m + \psi(j - 1, j - 1) + \Delta\psi(j - 1, k) \geq \Phi(j - 1) + \Delta\Phi(k) = \Phi(k)$. Hence $v_p(p^m \gamma_{j-1} a_{j-1,k}) \geq m + \psi(j - 1, k) \geq \Phi(k)$ and also $v_p(-\alpha_{j-1} c_{j-1,k}) = v_p(c_{j-1,k}) \geq \Phi(k)$ and thus $v_p(c_{j,k}) \geq \Phi(k)$.

- (i) By the induction-hypothesis, $(p, \alpha_{j-1} c_{j-1,p^{n-1}}) = 1$ and if:
 - a) $j - 1 > p^{n-1} - p$ then $\Phi(j - 1) \geq n + 1$ and $\psi(j - 1, j - 1) \leq n$ and thus $m \geq \Phi(j - 1) - \psi(j - 1, j - 1) \geq 1$;
 - b) $j - 1 = p^{n-1} - p$, then $\Phi(j - 1) = n$ and $\psi(j - 1, j - 1) = n - 1$ and thus $m \geq \Phi(j - 1) - \psi(j - 1, j - 1) = n - (n - 1) = 1$;
 - c) $j - 1 \leq p^{n-1} - p - 1$ then $v_p(a_{j-1,p^{n-1}}) \geq \psi(j - 1, p^{n-1}) \geq 1$.

In all three cases,

- (ii) $p/p^m \gamma_{j-1} a_{j-1,p^{n-1}}$.

We deduce from (i) and (ii) above that $(p, a_{j,p^n}) = 1$ and this proves the statement.

We deduce from the special case of the statement for $j = p^{n-1}$ that there exist coefficients $a_{p^{n-1},k} (k \geq p^{n-1})$ with $(p, a_{p^{n-1},p^{n-1}}) = 1$ such that $\sum_{k \geq p^{n-1}} a_{p^{n-1},k} w^k \in K_n$.

More generally, for $v_p(j) \geq n - 1$, let $j = p^{n-1}j'$. Then by what we have already proved and since K_n is an ideal in $\tilde{K}_{\mathbb{R}}(L_0^{2v}(p^n))$,

$$\left(\sum_{k \geq p^{n-1}} a_{p^{n-1},k} w^k \right) \left(\sum_{l \geq j'} a_{j',l} w^l \right) \in K_n$$

and hence the result. □

We now extend this to $p = 2$. We replace $L_0^k(p^n)$ for odd p by $L^k(2^n)$ for $p = 2$. Let t be an odd integer. Here $L^k(4)$ plays the role of $L_0^k(p)$, p odd, and the analogous result to Lemma 1.1.4 is that $\psi_{\mathbb{R}}^t - 1 = 0$ in $\tilde{K}_{\mathbb{R}}(L^k(4))$. We, necessarily, assume $n \geq 2$ and consider the sequence of transfer-maps, $\tilde{K}_{\mathbb{R}}(L^k(4)) \rightarrow \dots \rightarrow \tilde{K}_{\mathbb{R}}(L^k(2^n))$. The analogue of Proposition 2.3.1 is:

Proposition 2.3.2. *Let $t, j, v, n \in \mathbb{Z}^+$, t odd, $n \geq 2$ and define*

$$\psi(j, k) = \begin{cases} n - 2 - v_2(j) - \lfloor \frac{k-j}{2} \rfloor & \text{if } j \leq k \leq j + 2(n - 2 - v_2(j)) - 1 \\ 0 & \text{if } k \geq j + 2(n - 2 - v_2(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

- (i) $v_2(a_{j,j}) = \psi(j, j) = n - 2 - v_2(j)$;
- (ii) $v_2(a_{j,k}) \geq \psi(j, k)$ for $k \geq j$;
- (iii) $\sum_{k \geq j} a_{j,k} w^k \in \text{Ker}(\psi_{\mathbb{R}}^t - 1) \subseteq \tilde{K}_{\mathbb{R}}(L^{2v}(2^n))$.

Proof. Almost identical with that of Proposition 2.3.1. □

Let p be a prime, $n \in \mathbb{Z}^+$ and let G_{p^n} be the multiplicative group of units in \mathbb{Z}_p^n .

$$G_{p^n} = \begin{cases} \mathbb{Z}_{p^{n-1}(p-1)} & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} & \text{if } p = 2, \text{ where the first summand is generated by } -1. \end{cases}$$

Proposition 2.3.3. *Let p be a prime, $t \in \mathbb{Z}^+$ such that $(t, p) = 1$ and that t is a generator of G_{p^2} if p is odd and a generator of $G_8/\{\mp 1\}$ if $p = 2$. Define*

$$n_p = \begin{cases} 1 & \text{if } p = 2 \\ \frac{p-1}{2} & \text{if } p \text{ is odd} \end{cases} \quad \text{and} \quad \epsilon_p = \frac{3 + (-1)^p}{2}.$$

Let $n, v, j \in \mathbb{Z}^+$ and assume that $j \equiv 0 \pmod{n_p}$. For $k \geq j$, define

$$\psi_p(j, k) = \begin{cases} n - \epsilon_p - v_p(j) - \lfloor \frac{k-j}{p} \rfloor & \text{if } j \leq k \leq j + p(n - \epsilon_p - v_p(j)) - 1 \\ 0 & \text{if } k \geq j + p(n - \epsilon_p - v_p(j)). \end{cases}$$

Then there exist $a_{j,k} \in \mathbb{Z}$ such that:

- (i) $v_p(a_{j,j}) = \psi_p(j, j) = n - \epsilon_p - v_p(j)$;
- (ii) $v_p(a_{j,k}) \geq \psi_p(j, k)$;
- (iii) $\sum_{k \geq j} a_{j,k} w^k \in \text{Ker}(\psi_{\mathbb{R}}^t - 1) \subseteq \tilde{K}_{\mathbb{R}}(\bar{L}^{2v}(p^n))$.

Proof. For $p = 2$ it reduces to the statement of Proposition 2.3.2. For p odd, it follows from Proposition 2.3.1 that there exist $b_{j,k} \in \mathbb{Z}$ such that:

- (i) $v_p(b_{j,j}) = \psi_p(j, j) = n - 1 - v_p(j)$.
- (ii) $v_p(b_{j,k}) \geq \psi_p(j, k)$.
- (iii)

$$\begin{aligned} 0 &= \left[(\psi_{\mathbb{R}}^t)^{\frac{p-1}{2}} - 1 \right] \left(\sum_{k \geq j} b_{j,k} w^k \right) \\ &= (\psi_{\mathbb{R}}^t - 1) \left(1 + \psi_{\mathbb{R}}^t + \dots + (\psi_{\mathbb{R}}^t)^{\frac{p-5}{2}} + (\psi_{\mathbb{R}}^t)^{\frac{p-3}{2}} \right) \left(\sum_{k \geq j} b_{j,k} w^k \right) \\ &= (\psi_{\mathbb{R}}^t - 1) \left(\sum_{k \geq j} a_{j,k} w^k \right) \quad \text{i.e.,} \\ \sum_{k \geq j} a_{j,k} w^k &= \left[1 + \psi_{\mathbb{R}}^t + \dots + (\psi_{\mathbb{R}}^t)^{\frac{p-5}{2}} + (\psi_{\mathbb{R}}^t)^{\frac{p-3}{2}} \right] \left(\sum_{k \geq j} b_{j,k} w^k \right) \\ a_{j,j} &= \left[1 + (t^{2j} - 1) + (t^{4j} - 1) + \dots + (t^{(p-3)j} - 1) \right] b_{j,j}. \end{aligned}$$

Since $2j \equiv 0 \pmod{(p-1)}$, it follows from [1, Lemma 2.12] that

$$v_p(t^{2mj} - 1) = 1 + v_p(2mj) \geq 1 \quad \left(1 \leq m \leq \frac{p-3}{2} \right)$$

i.e., p divides all the terms inside the bracket except the first one and thus the bracket is not divisible by p . Hence

$$v_p(a_{j,j}) = v_p(b_{j,j}) = \psi_p(j, j) = n - 1 - v_p(j).$$

If $[\psi_{\mathbb{R}}^{tm}(w^s)]_k$ denotes the coefficient of w^k in the expansion of $\psi_{\mathbb{R}}^{tm}(w^s)$ then

$$a_{j,k} = b_{j,k} + \sum_{1 \leq m \leq \frac{p-3}{2}} \sum_{s \geq j} b_{j,s} [\psi_{\mathbb{R}}^{tm}(w^s)]_k.$$

$v_p(b_{j,s}) \geq \psi_p(j, s) \geq \psi_p(j, k)$ and hence $v_p(a_{j,k}) \geq \psi_p(j, k)$. □

3. J -Groups of Lens spaces.

3.1. J -triviality.

Lemma 3.1.1. *Let $k, n \in \mathbb{Z}^+$, p and q be distinct primes such that q is a generator of G_{p^n} if p is odd and of the summand $\mathbb{Z}_{2^{n-2}}$ if $p = 2$ and $u \in \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n))$. Then $J(u) = 0$ in $J(\bar{L}^k(p^n))$ iff there exists $x \in \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n))$ such that $u = (\psi_{\mathbb{R}}^q - 1)x$.*

Proof. It follows from the Adams conjecture (for an elementary proof see [9]), [2, Theorem 1.1] and the fact that $\tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n))$ is a p -group that J -trivial bundles over $\bar{L}^k(p^n)$ are finite linear combinations of the form:

1. $\sum_{(k,p)=1} (\psi_{\mathbb{R}}^k - 1)y$.

If $k = p_1 \cdots p_r$ for prime, p_i ($1 \leq i \leq r$) then:

2. $(\psi_{\mathbb{R}}^k - 1)x = (\psi_{\mathbb{R}}^{p_1} - 1)\psi^{p_2 \cdots p_r}(x) + (\psi_{\mathbb{R}}^{p_2} - 1)\psi^{p_3 \cdots p_r}(x) + \cdots + (\psi_{\mathbb{R}}^{p_r} - 1)\psi^{p_1 \cdots p_{r-1}}(x)$

and hence we may, without loss of generality, assume that in 1, k runs over the set of complementary primes to p . Let $k = q'$ be such a prime. Thus $q' \equiv \pm q^m \pmod{p^n}$ for some $m \in \mathbb{Z}^+$. Hence if η is the Hopf-bundle over $\bar{L}^k(p^n)$, $\eta^{q'} = \eta^{\pm q^m}$ i.e., $\psi_{\mathbb{C}}^{q'}(\mu) = \psi_{\mathbb{C}}^{\pm q^m}(\mu)$ and taking realifications yields $\psi_{\mathbb{R}}^{q'}(w) = \psi_{\mathbb{R}}^{q^m}(w)$ and thus, $(\psi_{\mathbb{R}}^{q'} - 1)w^i = (\psi_{\mathbb{R}}^{q^m} - 1)w^i = (\psi_{\mathbb{R}}^q - 1)x$ by 2 above.

Also for $p = 2$, $n \geq 2$ and if λ is the reduction of the canonical line-bundle over $\bar{L}^k(p^n)$ as defined in Section 1.1, then $(\psi_{\mathbb{R}}^q - 1)\lambda = 0$ for q odd. \square

In his solution of the vector-field problem, Adams has (essentially) proved that $J(P^n) = \tilde{K}_{\mathbb{R}}((P^n))$. We now extend his result.

Corollary 3.1.2. $J(L^k(4)) = \tilde{K}_{\mathbb{R}}(L^k(4))$.

Proof. Assume that $n = 2k$ is even. (i) If $q = 4m + 1$, $\eta^q = \eta$ and hence $(\psi_{\mathbb{C}}^q - 1)\mu = 0$. (ii) If $q = 4m - 1$, $\eta^q = \eta^{-1}$ and hence $(\psi_{\mathbb{C}}^q - \psi_{\mathbb{C}}^{-1})\mu = 0$. $r[(\psi_{\mathbb{C}}^q - 1)\mu] = r[(\psi_{\mathbb{C}}^q - \psi_{\mathbb{C}}^{-1})\mu] = (\psi_{\mathbb{R}}^q - 1)w$ and hence $(\psi_{\mathbb{R}}^q - 1)w = 0$ in either case. Also $(\psi_{\mathbb{R}}^q - 1)\lambda = 0$. Thus $(\psi_{\mathbb{R}}^q - 1) = 0$ for q odd and the result follows from Lemma 3.1.1. \square

3.2. Injectivity of the map, $c^1 : J(\bar{L}^{2v}(p^n))/\bar{L}^{2v-2}(p^n) \rightarrow J(\bar{L}^{2v}(p^n))$.

Lemma 3.2.1. *Let p be a prime; $i, n, s, t, v \in \mathbb{Z}^+$ such that $(p, t) = 1$ and $sw^v = (\psi_{\mathbb{R}}^t - 1)(\sum_{j=i}^v m_j w^j)$ in $\tilde{K}_{\mathbb{R}}(\bar{L}^{2v}(p^n))$ for $1 \leq i \leq v$ and $m_j \in \mathbb{Z}$ ($i \leq j \leq v$). Then there exist $n_j \in \mathbb{Z}$ ($i + 1 \leq j \leq v$) such that $sw^v = (\psi_{\mathbb{R}}^t - 1)(\sum_{j=i+1}^v n_j w^j)$.*

Proof. $sw^v = m_i(t^{2i} - 1)w^i + \sum_{j=i+1}^v m'_j w^j$.

(i) Let p be odd and $2i \not\equiv 0 \pmod{p-1}$. It follows from Section 1.1 that $p^n / m_i(t^{2i} - 1)$ and from [1, Lemma 2.12] that p does not divide $(t^{2i} - 1)$. Hence p^n / m_i . We deduce from Section 1.1 that $m_i w^i = \sum_{j=i+1}^v \alpha_j w^j$ and we put $n_j = m_j + \alpha_j$ ($i + 1 \leq j \leq v$).

(ii) Let p be odd and $2i \equiv 0 \pmod{p-1}$. It follows from Section 1.1 that $v_p(m_i(t^{2i} - 1)) \geq n$ and from [1, Lemma 2.12] that $v_p(t^{2i} - 1) = 1 + v_p(2i)$. Thus, $v_p(m_i) \geq n - 1 - v_p(2i) = n - 1 - v_p(i) = \psi_p(i, i)$ where $\psi_p(i, j)$ is as defined in Proposition 2.3.3.

(iii) Let $p = 2$. It follows from Section 1.1 that $v_2(m_i(t^{2i} - 1)) \geq n + 1$ and from [1, Lemma 2.12] that $v_2(t^{2i} - 1) = 2 + v_2(2i)$. Thus $v_2(m_i) \geq n + 1 -$

$2 - v_2(2i) = n - 2 - v_2(i) = \psi_2(i, i)$. It follows from Proposition 2.3.3 that in both cases (ii) and (iii), there exist $\beta_j \in \mathbb{Z}$ ($i \leq j \leq v$) with $\beta_i = m_i$ such that $\sum_{j=i}^v \beta_j w^j \in \text{Ker}(\psi_{\mathbb{R}}^t - 1)$. Hence $(\psi_{\mathbb{R}}^t - 1)m_i w^i = -(\psi_{\mathbb{R}}^t - 1)(\sum_{j=i+1}^v \beta_j w^j)$ and we put $n_j = m_j - \beta_j$. \square

Proposition 3.2.2. *Let p be a prime and $n, v \in \mathbb{Z}^+$ and $c : \overline{L}^{2v}(p^n) \rightarrow \overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)$. Then the induced homomorphism*

$$c^! : J\left(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)\right) \rightarrow J\left(\overline{L}^{2v}(p^n)\right)$$

is injective.

Proof. Let $c^!J(sw^v) = 0$ in $J(\overline{L}^{2v}(p^n))$. Let q be a prime which is a generator of G_{p^n} . We claim the following:

Statement. For each $1 \leq i \leq v$, there exist $m_j \in \mathbb{Z}$ ($i \leq j \leq v$) such that $sw^v = (\psi_{\mathbb{R}}^q - 1)(\sum_{j=i}^v m_j w^j)$.

Proof. By induction on i .

For $i = 1$, it follows from Lemma 3.1.1, Section 1.1 and the fact that for $p = 2$, $(\psi_{\mathbb{R}}^q - 1)\lambda = 0$. Let $i > 1$ and assume it to be true for $i - 1$. Then it is true for i by Lemma 3.2.1. This proves the Statement and the Proposition follows from the special case of the statement for $i = v$. \square

Corollary 3.2.3. *We have an exact sequence,*

$$0 \rightarrow J(\overline{L}^{2v}(p^n)/\overline{L}^{2v-2}(p^n)) \xrightarrow{c^!} J(\overline{L}^{2v}(p^n)) \xrightarrow{i^!} J(\overline{L}^{2v-2}(p^n)) \rightarrow 0.$$

Proof. The exactness of the four terms on the right follows from [1, Theorem 3.12], [2, Theorem 1.1] and the Adams conjecture. The injectivity of $c^!$ follows from Proposition 3.2.2. \square

Lemma 3.2.4. $c^! : J(L^{4v+1}(p^n)/L^{4v}(p^n)) \rightarrow J(L^{4v+1}(p^n))$ *is injective.*

Proof. By Lemma 1.1.2, $\tilde{K}_{\mathbb{R}}(L^{4v+1}(2^n)/L^{4v}(2^n)) = \mathbb{Z}_2$ and generator maps to w^{2v+1} . The proof is identical with that of Proposition 3.2.2. \square

Corollary 3.2.5. *The following sequence is exact,*

$$0 \rightarrow J(L^{4v+1}(p^n)/L^{4v}(p^n)) \xrightarrow{c^!} J(L^{4v+1}(p^n)) \xrightarrow{i^!} J(L^{4v}(p^n)) \rightarrow 0.$$

Proof. Identical with that of Corollary 3.2.3. \square

3.3. Order of $J(\overline{L}^k(p^n))$.

Definition 3.3.1. We define as in [1, Section 2] numbers $m(t)$ by: For p odd,

$$v_p(m(t)) = \begin{cases} 0 & \text{if } t \not\equiv 0 \pmod{p-1} \\ 1 + v_p(t) & \text{if } t \equiv 0 \pmod{p-1}. \end{cases}$$

For $p = 2$,

$$v_2(m(t)) = \begin{cases} 1 & \text{if } t \not\equiv 0 \pmod{2} \\ 2 + v_2(t) & \text{if } t \equiv 0 \pmod{2}. \end{cases}$$

Definition 3.3.2. Let p be a prime and $v, n \in \mathbb{Z}^+$. Define

$$e(p, v, n) = \begin{cases} p^{\min(n, v_p(m(2v)))} & \text{if } p \text{ is odd} \\ 2^{\min(n+1, v_2(m(2v)))} & \text{if } p = 2. \end{cases}$$

Note that $e(p, v, n) = 1$ if $v \not\equiv 0 \pmod{p-1}$.

For $v \equiv 0 \pmod{p-1}$, $e(p, v, n) = p^{\epsilon_p + (\min(n, 1+v_p(2v)))}$ where

$$\epsilon_p = \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{if } p = 2. \end{cases}$$

Lemma 3.3.3. Let p be a prime; $v, n \in \mathbb{Z}^+$, $n \geq 2$ if $p = 2$. Then $J(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n)) = \mathbb{Z}_{e(p,v,n)}$.

Proof. By Lemma 1.1.1,

$$\tilde{K}_{\mathbb{R}}(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n)) = \begin{cases} \mathbb{Z}_{p^n} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{2^{n+1}} & \text{if } p = 2 \end{cases}$$

and is generated by w^v . By [2, Theorem 1.1] and the Adams conjecture, $J(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n)) = \tilde{K}_{\mathbb{R}}(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n))/W$ where $W = \cap_f W_f$ where W_f is the subgroup generated by

$$\sum_{k \in \mathbb{Z}^+} a_k k^{f(k)} (\psi_{\mathbb{R}}^k - 1) w^v = \sum_{k \in \mathbb{Z}^+} a_k k^{f(k)} (k^{2v} - 1) w^v.$$

Let K_p be the principal ideal in \mathbb{Z} generated by p^n if p is odd and by 2^{n+1} if $p = 2$. Let $\phi_p : \mathbb{Z} \rightarrow \mathbb{Z}/K_p = \tilde{K}_{\mathbb{R}}(B_{4v}(\mathbb{Z}_{p^n})/B_{4v-4}(\mathbb{Z}_{p^n}))$ be the surjection. Define $W'_f = \phi_p^{-1}(W_f)$ and $W' = \cap_f W'_f = \phi_p^{-1}(W)$. Let $h(f, 2v)$ be the highest common divisor of the integers $k^{f(k)}(k^{2v} - 1)$. Then W'_f is the principal ideal generated by $h(f, 2v)$ and by [1, Theorem 2.7], W_f is the principal ideal generated by $m(2v)$.

$$\begin{aligned} J(\bar{L}^{2v}(p^n)/\bar{L}^{2v-2}(p^n)) &= (\mathbb{Z}/K_p)/W = (\mathbb{Z}/K_p)/(W'/W' \cap K_p) \\ &= (\mathbb{Z}/K_p)/((W' + K_p)/K_p) = \mathbb{Z}/(W' + K_p) \end{aligned}$$

and $W' + K_p$ is the principal ideal generated by $e(p, v, n)$. □

Proposition 3.3.4. Let p be a prime and $v, n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then

$$\left| J(\bar{L}^{2v}(p^n)) \right| = \begin{cases} \prod_{v'=1}^v e(p, v', n) & \text{if } p \text{ is odd} \\ 2 \prod_{v'=1}^v e(2, v', n) & \text{if } p = 2. \end{cases}$$

Proof. It follows by induction from Corollary 3.2.3. □

Definition 3.3.5. Let p be a prime and $k, n \in \mathbb{Z}$. Define $G(p, k, n)$ and $G_0(p, k, n)$ to be subgroups of $J(L^k(p^n))$ and $J(L_0^k(p^n))$ generated by the powers of w respectively.

Lemma 3.3.6. For p odd,

$$J(L^k(p^n)) = \begin{cases} \mathbb{Z}_2 \oplus J(L_0^k(p^n)) & \text{if } k \equiv 0 \pmod{4} \\ J(L_0^k(p^n)) & \text{otherwise.} \end{cases}$$

Proof. This is [13, Proposition 1.3]. □

Corollary 3.3.7. For p odd, $G(p, k, n) = G_0(p, k, n)$.

Corollary 3.3.8. $G(p, k, n) = J(L^k(p^n))$ for p odd and $k \not\equiv 0 \pmod{4}$ and is a subgroup of index 2 if either p is odd and $k \equiv 0 \pmod{4}$ or $p = 2$.

We now state the following Corollary to Proposition 3.3.4.

Corollary 3.3.9. $|G(p, 2v, n)| = \prod_{v'=1}^v e(p, v', n)$.

Proposition 3.3.10. Let p be a prime; $v, n \in \mathbb{Z}^+$. Then

$$|J(L^{4v+1}(p^n))| = \begin{cases} |J(L^{4v}(p^n))| & \text{if } p \text{ is odd} \\ 2 |J(L^{4v}(2^n))| & \text{if } p = 2. \end{cases}$$

Proof. It follows from Lemma 1.1.2 and the fact that

$$J(L^{4v+1}(\mathbb{Z}_2)/L^{4v}(\mathbb{Z}_2)) = J(P^{8v+2}/P^{8v}) = \mathbb{Z}_2 \quad \text{that}$$

$$J(L^{4v+1}(p^n)/L^{4v}(p^n)) = \begin{cases} 0 & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

The result follows from this and Corollary 3.2.5. □

Proposition 3.3.11. $J(L^{4v+3}(p^n)) = J(L^{4v+2}(p^n))$.

Proof. It follows from [1, Theorem 3.12] that there is an exact sequence,

$$J(L^{4v+3}(p^n)/L^{4v+2}(p^n)) \xrightarrow{c^1} J(L^{4v+3}(p^n)) \xrightarrow{i^1} J(L^{4v+2}(p^n)) \rightarrow 0.$$

By Lemma 1.1.3, $\tilde{K}_{\mathbb{R}}(L^{4v+3}(p^n)/L^{4v+2}(p^n)) = 0$ and hence

$$J(L^{4v+3}(p^n)/L^{4v+2}(p^n)) = 0.$$

□

3.4. Approximation to complex projective spaces by lens spaces.

Let $i : L^k(p^n) \rightarrow P_k(\mathbb{C})$ be the inclusion. Let $J_p(P_k(\mathbb{C}))$ denote the p -summand of $J(P_k(\mathbb{C}))$.

Observation 3.4.1. $i^!$ maps $J_p(P_k(\mathbb{C}))$ onto $G(p, k, n)$.

Theorem 3.4.2. $i^!$ maps $J_p(P_k(\mathbb{C}))$ isomorphically onto $G(p, k, n)$ iff $n \geq v_p(N_k)$.

Proof. (i) Let $k = 2v$ be even. By Corollary 3.3.9,

$$|G(p, k, n)| = \prod_{v'=1}^v e(p, v', n).$$

It follows from the proof of [4, Lemma 6.1] that there is a short exact sequence,

$$0 \rightarrow J(P_{2v}(\mathbb{C})/P_{2v-2}(\mathbb{C})) \rightarrow J(P_{2v}(\mathbb{C})) \rightarrow J(P_{2v-2}(\mathbb{C})) \rightarrow 0,$$

and from [4, Lemma 5.3] that $J(P_{2v}(\mathbb{C})/P_{2v-2}(\mathbb{C})) = \mathbb{Z}_{m(2v)}$ and hence by induction that $|J_p(P_{2v}(\mathbb{C}))| = \prod_{v'=1}^v m_p(2v')$ where $m_p(2v')$ is the p -component of $m(2v')$. Thus $|G(p, k, n)| = |J_p(P_k(\mathbb{C}))|$ iff $e(p, v', n) = m(2v')$ for all $1 \leq v' \leq v$ iff $e(p, v', n) = m_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$ iff $n \geq 1 + v_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$, and putting $2v' = r(p-1)$, iff $n \geq 1 + v_p(r)$ for all $1 \leq r \leq \lfloor \frac{2v}{p-1} \rfloor$, i.e., iff $n \geq v_p(N_k)$.

(ii) $k = 4v + 1$.

$$|J_p(P_k(\mathbb{C}))| = \begin{cases} |J_p(P_{4v}(\mathbb{C}))| & \text{if } p \text{ is odd} \\ 2|J_2(P_{4v}(\mathbb{C}))| & \text{if } p = 2 \end{cases}$$

by [4, Lemma 6.2] and

$$|G(p, k, n)| = \begin{cases} |G(p, 4v, n)| & \text{if } p \text{ is odd} \\ 2|G(2, 4v, n)| & \text{if } p = 2 \end{cases}$$

by Proposition 3.3.10. The result follows from (i) above and the fact that $N_k = N_{4v}$.

(iii) $k = 4v + 3$.

$$|J_p(P_k(\mathbb{C}))| = |J_p(P_{4v+2}(\mathbb{C}))|$$

by [4, Lemma 6.2] and $|G(p, k, n)| = |G(p, 4v + 2, n)|$ by Proposition 3.3.11 and the result follows from (i) above and the fact that $N_k = N_{4v+2}$. \square

Corollary 3.4.3. Let $i : L^k(m) \rightarrow P_k(\mathbb{C})$ be the inclusion. Then $i^!$ maps $J(P_k(\mathbb{C}))$ isomorphically onto the subgroup of $J(L^k(m))$ generated by w iff N_k/m .

Stable co-degrees of vector-bundles enables us as in [8, Section 4] or [9, Definition 1.1.4] to define a degree-function q on $J(X)$; e.g., a function $q : J(X) \rightarrow \mathbb{Z}^+$ such that $u = 0$ in $J(X)$ iff $q(u) = 1$. The degree-function imposes on $J(X)$ an additional structure other than the usual algebraic structure. We now conjecture a stronger version of Theorem 3.4.5.

Conjecture 3.4.4. *Let p be a prime. $n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then the map $i^! : J_p(P_k(\mathbb{C})) \rightarrow J(L^k(p^n))$ is a q -isometry iff $n \geq v_p(N_k)$.*

3.5. The transfer map on the J -groups.

Let $\tau : \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n)) \rightarrow \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^{n+1}))$ be the transfer-map defined on the $\tilde{K}_{\mathbb{R}}$ -groups.

Lemma 3.5.1. τ passes to the quotient and defines $\tau : J(\bar{L}^k(p^n)) \rightarrow J(\bar{L}^k(p^{n+1}))$.

Proof. Let q be a prime which is a generator of both G_{p^n} and $G_{p^{n+1}}$ if p is odd and of the summands $\mathbb{Z}_{2^{n-2}}$ and $\mathbb{Z}_{2^{n-1}}$ if $p = 2$. By Lemma 3.1.1, J -trivial bundles on $\bar{L}^k(p^n)$ are of the form $(\psi_{\mathbb{R}}^q - 1)x$, $x \in \tilde{K}(\bar{L}^k(p^n))$. By (ii) of Proposition 2.1.1, $\tau \circ (\psi_{\mathbb{R}}^q - 1)x = (\psi_{\mathbb{R}}^q - 1) \circ \tau(x)$ is J -trivial on $\bar{L}^k(p^{n+1})$. □

Corollary 3.5.2. *The transfer map $\tau : \tilde{K}_{\mathbb{R}}(L^k(p^n)) \rightarrow \tilde{K}_{\mathbb{R}}(L^k(p^{n+1}))$ passes to the quotient and defines $\tau : J(L^k(p^n)) \rightarrow J(L^k(p^{n+1}))$.*

Proof. The case $p = 2$ is already proved in Lemma 3.5.1. For p odd and $k \not\equiv 0 \pmod{4}$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ and it also follows from Lemma 3.5.1. For p odd and $k \equiv 0 \pmod{4}$, $\tilde{K}_{\mathbb{R}}(L^k(p^n)) = \mathbb{Z}_2 \oplus \tilde{K}_{\mathbb{R}}(L_0^k(p^n))$ where the first summand is generated by u and $\tau(u) = u$. By Lemma 3.3.6, $J(L^k(p^n)) = \mathbb{Z}_2 \oplus J(L_0^k(p^n))$ where the first summand is generated by $J(u)$. Hence J -trivial elements on $\tilde{K}_{\mathbb{R}}(L^k(p^n))$ are of the form x where $i^!J(x) = 0$ in $J(L_0^k(p^n))$. Hence $i^![\tau(J(x))] = \tau(i^!J(x)) = 0$ by Lemma 3.5.1. Since $i^!$ is an isomorphism on the 2^{nd} -summand, $\tau(J(x)) = 0$ and hence $J(\tau(x)) = 0$. □

Proposition 3.5.3. *Let $i : L^k(p^n) \rightarrow L^k(p^{n+1})$ be the \mathbb{Z}_p -fibration and $\tau : J(L^k(p^n)) \rightarrow J(L^k(p^{n+1}))$ be the transfer-map. Then $\tau(i^!(x)) = px \forall x \in G(p, k, n + 1)$.*

Proof. By Corollary 3.3.7, $G(p, k, n) = G_0(p, k, n)$ for p odd and hence we shall assume without loss of generality that $\tau : \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^n)) \rightarrow \tilde{K}_{\mathbb{R}}(\bar{L}^k(p^{n+1}))$. We let

$$\bar{G}(p, k, n) = \begin{cases} G_0(p, k, n), & p \text{ odd} \\ G(p, k, n), & p \text{ even.} \end{cases}$$

By (i) and (v) of Proposition 2.1.1,

$$\begin{aligned} \tau(\psi_{\mathbb{R}}^{p^i}(w)) &= \sum_{s=0}^{p-1} \psi_{\mathbb{R}}^{p^i+sp^n}(w) \quad (0 \leq i \leq n-1) \\ &= \sum_{s=0}^{p-1} \psi_{\mathbb{R}}^{p^i} \circ \psi_{\mathbb{R}}^{1+sp^{n-i}}(w). \end{aligned}$$

$J(\overline{L}^k(p^n))$ is a p -group and $(1 + sp^{n-i})$ is prime to p and it follows from (ii) of [9, Proposition 2.3.3] that $\psi_{\mathbb{R}}^{1+sp^{n-i}}(w) = w$ and hence:

- $\tau(\psi_{\mathbb{R}}^{p^i}(w)) = p\psi_{\mathbb{R}}^{p^i}(w) \quad (0 \leq i \leq n-1).$

The group $i'\overline{G}(p, k, n+1) = \overline{G}(p, k, n)$ is generated by $\{\psi_{\mathbb{R}}^m(w) : 0 \leq m \leq p^n - 1\}$. Let p^i ($1 \leq i \leq n-1$) be the p -primary component of m . It follows from (ii) of [9, Proposition 2.3.3] that $\psi_{\mathbb{R}}^m(w) = \psi_{\mathbb{R}}^{p^i}(w)$ in $J(\overline{L}^k(p^n))$. Hence the group $\overline{G}(p, k, n)$ is generated by $\{\psi_{\mathbb{R}}^{p^i}(w) : 1 \leq i \leq n-1\}$. The result follows from this and Equation 1 above. \square

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BILKENT UNIVERSITY

ANKARA

TURKEY

E-mail address: dibag@fen.bilkent.edu.tr