Pacific Journal of Mathematics

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Volume 191 No. 2 December 1999

PACIFIC JOURNAL OF MATHEMATICS
Vol. 191, No. 2, 1999

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We study amalgamated free products of factors over their common Cartan subalgebras. We will show that the resulting amalgamated free product is a factor as long as given factors are non-type *I* and furthermore its (smooth) flow of weights is determined.

1. Introduction.

Amalgamated free products of von Neumann algebras were first used by S. Popa ([**26**]) to construct an irreducible inclusion of (non-AFD) type II_1 factors with an arbitrary (admissible) Jones index. Further investigation in this direction was made by K. Dykema ([**10**]) and F. Rădulescu ([**27**, **29**]) based on Voiculescu's powerful machine ([**40**, **41**, **44**]), and F. Boca ([**4**]) discussed the Haagerup approximation property, where only finite von Neumann algebras were dealt with. On the other hand, type III factors arising as free products (over $\mathbf{C}$) were studied by L. Barnett ([**3**]), K. Dykema ([**9**, **11**]), F. Rădulescu ([**28**]), and very recently by D. Shlyakhtenko ([**33**]). However, amalgamated free products in the type III setting have never been seriously investigated so far. The main purpose of the paper is to take a first step towards investigation on amalgamated free products in the type III setting.

A construction of amalgamated free products of arbitrary von Neumann algebras has never been (at least explicitly) given in the literature (see [**29**, **44**] in the type II_1 case), and hence we present such a construction in §2. Our construction requires (faithful) normal conditional expectations onto a common subalgebra, and the concept of bimodules is useful. We mainly study the amalgamated free product of non-type I factors A, B over their common Cartan subalgebra D:

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B),$$

where E_D^A, E_D^B are unique normal conditional expectations from A, B onto D. In §3 we summarize basic facts on Cartan subalgebras needed for later sections, and in §4 we show the existence of a faithful normal state φ on D satisfying

$$(A_{\varphi\circ E_D^A})' \cap M \subseteq A.$$

This of course shows that M is a factor. In the structure analysis on type III factors the continuous decomposition plays an important role. In §5 we compute the continuous decomposition of the amalgamated free product M (in terms of those of A and B), which enables us to determine the flow of weights of M.

Two appendices are given. In the first appendix an amalgamated free product version of what was shown in Barnett's paper [**3**] is obtained. In the second the modular operator and modular conjugation are determined for an amalgamated free product. In particular, we obtain a commutation theorem whose ordinary free product version (i.e., $D = \mathbf{C}1$) was pointed out in [**39**, **44**].

Finally we would like to point out that the amalgamated free product of von Neumann algebras over a common Cartan algebra can be captured as the groupoid von Neumann algebra associated with the "free product" of relevant measured equivalence relations (see [**19**]).

Acknowledgment. The author wishes to express his gratitude to H. Kosaki for suggesting that Theorem 5.1 might be true in the free product case and for a useful suggestion on Lemma II-A, to Y. Katayama for discussions on free group factors, to Y. Watatani and Y. Sekine for their constant warm encouragement, and finally to T. Hamachi for pointing out Corollary 5.6. He also wishes to express his gratitude to the referee for comments.

2. Amalgamated Free Product.

Let $(N_s)_{s\in S}$ be a family of σ-finite von Neumann algebras having a common von Neumann subalgebra N. Throughout this section we suppose that each inclusion $N_s \supseteq N$ has a faithful normal conditional expectation $E_s : N_s \to N$. We will construct the amalgamated free product of the family (N_s, E_s), $s \in S$, over N by the analogous way as in [**39**] (also [**2**]).

Let $(\mathcal{H}_s, N_s, J_s, \mathcal{P}_s^\natural)$ and $(L^2(N), N, J_N, \mathcal{P}_N^\natural)$ be standard forms of N_s and N respectively ([**1**, **6**, **14**]). The map: $\xi \in \mathcal{P}_N^\natural \mapsto (\omega_\xi \circ E_s)^{1/2} \in \mathcal{P}_s^\natural$, where $(\omega_\xi \circ E_s)^{1/2}$ is the implementing vector of $\omega_\xi \circ E_s$ in the natural cone, gives us the natural embedding of $(L^2(N), N, J_N, \mathcal{P}_N^\natural)$ into $(\mathcal{H}_s, N_s, J_s, \mathcal{P}_s^\natural)$ as a "sub-standard form". By this embedding we have the following:

(1) $L^2(N)$ is an N-N subbimodule of $\mathcal{H}_s$;
(2) The following matrix notation is valid:
$$J_s = \begin{bmatrix} J_N & \\ & J_s^\circ \end{bmatrix} \quad \left(\text{on } \begin{bmatrix} L^2(N) \\ \mathcal{H}_s^\circ \end{bmatrix}\right),$$
where the restriction J_s° of J_s to $\mathcal{H}_s^\circ := \mathcal{H}_s \ominus L^2(N)$ makes sense. (See [**35**, p. 317, Equation (8)].);

(3) A vector $\xi \in \mathcal{P}_N^\natural$ $(\subseteq \mathcal{P}_s^\natural)$ is an implementing vector of $\omega_\xi \circ E_s$.

We denote the kernel of E_s by N_s°, and introduce the operation $x \in N_s \mapsto x^\circ := x - E_s(x) \in N_s^\circ$.

Fix a faithful normal state φ on N and denote by ξ_0 its implementing vector in $\mathcal{P}_N^\natural$. By the above (3) the vector ξ_0 is also an implementing vector of the $\varphi \circ E_s$ in the natural cones.

Let

$$\mathcal{H} = L^2(N) \oplus \sum_{n\geq 1}^{\oplus} \left(\sum_{s_1\neq\cdots\neq s_n}^{\oplus} \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}^\circ \right),$$

where $\otimes_\varphi$ means Sauvageot's relative tensor product ([**31**]). Each of $L^2(N)$ and $\mathcal{H}_s^\circ$ is an N-N bimodule, and hence so is the above direct sum. The left and right action of N on $\mathcal{H}$ are denoted by λ and ρ respectively. We set

$$\mathcal{H}(s,\ell) = L^2(N) \oplus \sum_{n\geq 1}^{\oplus} \left(\sum_{\substack{s_1\neq\cdots\neq s_n \\ s_1\neq s}}^{\oplus} \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}^\circ \right),$$

$$\mathcal{H}(s,r) = L^2(N) \oplus \sum_{n\geq 1}^{\oplus} \left(\sum_{\substack{s_1\neq\cdots\neq s_n \\ s_n\neq s}}^{\oplus} \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}^\circ \right).$$

By [**31**, 2.4] we get the following N-N bimodule isomorphisms:

$$\mathcal{H}_s \otimes_\varphi \mathcal{H}(s,\ell) = (L^2(N) \oplus \mathcal{H}_s^\circ) \otimes_\varphi \mathcal{H}(s,\ell) \cong \mathcal{H},$$
$$\mathcal{H}(s,r) \otimes_\varphi \mathcal{H}_s = \mathcal{H}(s,r) \otimes_\varphi (L^2(N) \oplus \mathcal{H}_s^\circ) \cong \mathcal{H}.$$

Indeed they are given by the following unitary operators respectively:

$V_s : \mathcal{H}_s \otimes_\varphi \mathcal{H}(s,\ell) \to \mathcal{H}$ is defined by

$$V_s((n\xi_0) \otimes_\varphi \zeta) = n \cdot \zeta \quad \text{for } \zeta \in \mathcal{H}(s,\ell),\ n \in N,$$
$$V_s(\eta \otimes_\varphi (J_N n^* \xi_0)) = \eta \cdot n \quad \text{for } \eta \in \mathcal{H}_s^\circ, n \in N,$$
$$V_s(\eta \otimes_\varphi \zeta) = \eta \otimes_\varphi \zeta$$

for $\eta \in N_s^\circ \xi_0$ $(\subseteq \mathcal{H}_s^\circ)$ and $\zeta \in \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}^\circ$ $(s_1 \neq \cdots \neq s_n, s_1 \neq s)$;

$W_s : \mathcal{H}(s,r) \otimes_\varphi \mathcal{H}_s \to \mathcal{H}$ is defined by

$$W_s(\zeta \otimes_\varphi (J_N n^* \xi_0)) = \zeta \cdot n \quad \text{for } \zeta \in \mathcal{H}(s,r),\ n \in N,$$
$$W_s((n\xi_0) \otimes_\varphi \eta) = n \cdot \eta \quad \text{for } \eta \in \mathcal{H}_s^\circ, n \in N,$$

$$W_s(\zeta \otimes_\varphi \eta) = \zeta \otimes_\varphi \eta$$

for $\eta \in J_s N_s^\circ \xi_0$ $(\subseteq \mathcal{H}_s^\circ)$ and

$$\zeta \in \mathcal{H}_{s_1}^\circ \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}^\circ \ (s_1 \neq \cdots \neq s_n, s_n \neq s).$$

By using these unitary operators we define $*$-representations $\lambda_s : N_s \to \mathrm{End}_N(\mathcal{H}_N)$, $s \in S$, and anti$*$-representations $\rho_s : N_s \to {}_N\mathrm{End}({}_N\mathcal{H})$, $s \in S$, by the formulas:

$$\lambda_s(x) = V_s(x \otimes_\varphi \mathrm{id}_{\mathcal{H}(s,\ell)})V_s^*$$
$$\rho_s(x) = W_s(\mathrm{id}_{\mathcal{H}(s,r)} \otimes_\varphi (J_s x^* J_s))W_s^*$$

for $x \in N_s$.

Lemma 2.1.
(1) $\lambda_s|_N$ *coincides with the left action* λ *of* N *on* $\mathcal{H}$*;*
(2) $\rho_s|_N$ *coincides with the right action* ρ *of* N *on* $\mathcal{H}$*.*

For $x \in N_s$ we have

$$\lambda_s(x)\xi_0 = x\xi_0 \quad (\text{in } L^2(N) \oplus \mathcal{H}_s^\circ = \mathcal{H}_s),$$
$$\rho_s(x)\xi_0 = J_s x^* J_s \xi_0 \quad (\text{in } L^2(N) \oplus \mathcal{H}_s^\circ = \mathcal{H}_s)$$

so that λ_s, ρ_s are injective because ξ_0 is a separating vector for N_s in $\mathcal{H}_s$.

Let $\mathcal{N}_s$ and $\mathcal{N}$ be the sets of analytic elements in N_s and N for the modular actions $\sigma^{\varphi\circ E_s}$ and σ^φ respectively. In [**23**, §3] it was shown that $\mathcal{N}_s$ and $\mathcal{N}$ are dense, and from the construction of analytic elements there we observe that $\mathcal{N}_s^\circ := \mathcal{N}_s \cap N_s^\circ$ is also dense in N_s°. Thus, by [**31**, 2.2. Remarque.(a)] the (algebraic) direct sum

$$\mathfrak{A} = \mathcal{N}\xi_0 + \sum_{n\geq 1}\left(\sum_{s_1\neq\cdots\neq s_n} (\mathcal{N}_{s_1}^\circ\xi_0) \odot_\varphi \cdots \odot_\varphi (\mathcal{N}_{s_n}^\circ\xi_0)\right)$$

is dense in $\mathcal{H}$, where $\odot_\varphi$ is understood in the algebraic sense.

Lemma 2.2. *For* $s_1, s_2 \in S$*,* $x \in N_{s_1}$ *and* $y \in N_{s_2}$ *we have* $[\lambda_{s_1}(x), \rho_{s_2}(y)] = 0$*.*

Proof. Since λ_s, ρ_s $(s \in S)$ are normal (or σ-weakly continuous), we can assume that x, y are analytic elements (and so are x°, y°). We have

$$\begin{aligned}
[\lambda_{s_1}(x), \rho_{s_2}(y)] &= [\lambda(E_{s_1}(x)) + \lambda_{s_1}(x^\circ), \rho(E_{s_2}(y)) + \rho_{s_2}(y^\circ)] \\
&= [\lambda(E_{s_1}(x)), \rho(E_{s_2}(y))] + [\lambda(E_{s_1}(x)), \rho_{s_2}(y^\circ)] \\
&\qquad + [\lambda_{s_1}(x^\circ), \rho(E_{s_2}(y))] + [\lambda_{s_1}(x^\circ), \rho_{s_2}(y^\circ)] \\
&= [\lambda_{s_1}(x^\circ), \rho_{s_2}(y^\circ)].
\end{aligned}$$

Here the third equality comes from $\lambda_s(N_s) \subseteq \mathrm{End}_N(\mathcal{H}_N) = \rho(N)'$ and $\rho_s(N_s) \subseteq {}_N\mathrm{End}({}_N\mathcal{H}) = \lambda(N)'$. Thus it is sufficient to show

$$[\lambda_{s_1}(x^\circ), \rho_{s_2}(y^\circ)]\zeta = 0$$

for each simple tensor $\zeta \in \mathfrak{A}$. Since $\lambda_{s_1}(x^\circ)\mathfrak{A} \subseteq \mathfrak{A}$ and $\rho_{s_2}(y^\circ)\mathfrak{A} \subseteq \mathfrak{A}$, we get the above equation thanks to [**31**, 2.2. Remarque.(b)]. □

Let M be the von Neumann algebra generated by $\left\{\bigcup_{s\in S} \lambda_s(N_s)\right\}$, and $\psi = \omega_{\xi_0}|_M$. By [**31**, 2.2. Remarque.(a)]

$$\left(*\mathrm{alg}\left\{\bigcup_{s\in S}\rho_s(N_s)\right\}\right)\xi_0$$
$$= J_N N\xi_0 + \sum_{n\geq 1}\left(\sum_{s_1\neq\cdots\neq s_n}(J_s^\circ N_{s_1}^\circ\xi_0)\odot_\varphi\cdots\odot_\varphi(J_s^\circ N_{s_n}^\circ\xi_0)\right),$$

is a dense subspace of $\mathcal{H}$. Thus, by Lemma 2.2 the vector ξ_0 is cyclic and separating for M, and hence ψ is faithful.

For $x \in N_s$ we have

$$\psi(\lambda_s(x)) = (\lambda_s(x)\xi_0|\xi_0)_{\mathcal{H}} = (E_s(x)\xi_0|\xi_0)_{L^2(N)} = \varphi\circ E_s(x), \tag{2.1}$$

and for $x_i^\circ \in N_{s_i}^\circ$ with $s_1 \neq \cdots \neq s_n$

$$\lambda_{s_1}(x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ)\xi_0 = (x_1^\circ\xi_0)\otimes_\varphi\cdots\otimes_\varphi(x_n^\circ\xi_0)$$

is orthogonal to ξ_0 so that

$$\psi(\lambda_{s_1}(x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ)) = 0. \tag{2.2}$$

For $x_i^\circ \in N_{s_i}^\circ,\ y_j^\circ \in N_{s_j'}^\circ$ with $s_1 \neq \cdots \neq s_m,\ s_1' \neq \cdots \neq s_n'$, we have, by the similar computations as in [**26**, p. 384] based on (2.1), (2.2),

$$\begin{aligned} &\psi(\lambda_{s_m}(x_m^\circ)\cdots\lambda_{s_1}(x_1^\circ)\lambda_{s_1'}(y_1^\circ)\cdots\lambda_{s_n'}(y_n^\circ)) \\ &= \delta_{m,n}\delta_{s_1,s_1'}\cdots\delta_{s_m,s_m'} \\ &\quad\cdot\varphi\circ E_{s_m}(x_m^\circ E_{s_{m-1}}(x_{m-1}^\circ\cdots E_{s_1}(x_1^\circ y_1^\circ)\cdots y_{m-1}^\circ)y_m^\circ). \end{aligned} \tag{2.3}$$

Lemma 2.3. *For $s \in S$, we have $\sigma_t^\psi\circ\lambda_s = \lambda_s\circ\sigma_t^{\varphi\circ E_s}$.*

Proof. By (2.1), (2.2) (or (2.3)), $\sigma_t^\varphi\circ E_s = E_s\circ\sigma_t^{\varphi\circ E_s}$ and $\varphi\circ\sigma_t^\varphi = \varphi$ there exists a unique ψ-preserving $*$-automorphism α_t on M satisfying

$$\alpha_t(\lambda_s(x)) = \lambda_s(\sigma_t^{\varphi\circ E_s}(x)) \quad \text{for } x \in N_s,$$

and $(\alpha_t)_{t\in\mathbf{R}}$ is a 1-parameter strong-operator continuous $*$-automorphism group.

For $x_i \in \mathcal{N}_{s_i}^\circ,\ y_j \in \mathcal{N}_{s'_j}^\circ$ with $s_1 \neq \cdots \neq s_n,\ s'_1 \neq \cdots \neq s'_m$, we have

$$\begin{aligned}
&\psi(\lambda_{s_m}(x_m^\circ)\cdots\lambda_{s_1}(x_1^\circ)\lambda_{s'_1}(y_1^\circ)\cdots\lambda_{s'_n}(y_n^\circ))\\
&= \delta_{m,n}\delta_{s_1,s'_1}\cdots\delta_{s_m,s'_m}\\
&\qquad \cdot\varphi\circ E_{s_m}(x_m^\circ E_{s_{m-1}}(x_{m-1}^\circ\cdots E_{s_1}(x_1^\circ y_1^\circ)\cdots y_{m-1}^\circ)y_m^\circ)\\
&= \delta_{m,n}\delta_{s_1,s'_1}\cdots\delta_{s_m,s'_m}\\
&\qquad \cdot\varphi\circ E_{s_m}(E_{s_{m-1}}(x_{m-1}^\circ\cdots E_{s_1}(x_1^\circ y_1^\circ)\cdots y_{m-1}^\circ)y_m^\circ\sigma_{-i}^{\varphi\circ E_{s_m}}(x_m^\circ))\\
&= \delta_{m,n}\delta_{s_1,s'_1}\cdots\delta_{s_m,s'_m}\\
&\qquad \cdot\varphi(E_{s_{m-1}}(x_{m-1}^\circ\cdots E_{s_1}(x_1^\circ y_1^\circ)\cdots y_{m-1}^\circ)E_{s_m}(y_m^\circ\sigma_{-i}^{\varphi\circ E_{s_m}}(x_m^\circ)))\\
&= \delta_{m,n}\delta_{s_1,s'_1}\cdots\delta_{s_m,s'_m}\\
&\qquad \cdot\varphi\circ E_{s_{m-1}}(x_{m-1}^\circ\cdots E_{s_1}(x_1^\circ y_1^\circ)\cdots(y_{m-1}^\circ)E_{s_m}(y_m^\circ\sigma_{-i}^{\varphi\circ E_{s_m}}(x_m^\circ))).
\end{aligned}$$

Here the first equality comes from (2.3) and the second equality comes from the K.M.S. condition: $\varphi_s(xy) = \varphi_s(y\sigma_{-i}^{\varphi_s}(x))$ for $x \in \mathcal{N}_s^\circ$, $y \in N_s$. Repeated use of the K.M.S. condition and (2.3) imply

$$\begin{aligned}
&\psi(\lambda_{s_m}(x_m^\circ)\cdots\lambda_{s_1}(x_1^\circ)\lambda_{s'_1}(y_1^\circ)\cdots\lambda_{s'_n}(y_n^\circ))\\
&= \psi(\lambda_{s'_1}(y_1^\circ)\cdots\lambda_{s'_n}(y_n^\circ)\alpha_{-i}(\lambda_{s_m}(x_m^\circ)\cdots\lambda_{s_1}(x_1^\circ))).
\end{aligned}$$

(In the above computations the fact: $\sigma_{-i}^{\varphi\circ E_s}(\mathcal{N}_s^\circ) = \mathcal{N}_s^\circ$ was used.) Therefore, we get $\alpha_t = \sigma_t^\psi$ by Takesaki's characterization of modular automorphisms ([**34**]) (and the Phragmén-Lindelöf theorem). □

The above argument is closely related to the arguments in [**26**, 3.1. Proposition] and [**3**, Lemma 1].

The above lemma implies $\sigma_t^\psi(\lambda(N)) = \lambda(\sigma_t^\varphi(N)) = \lambda(N)$, and hence there exists a unique faithful normal conditional expectation $E : M \to \lambda(N)$ conditioned by ψ thanks to Takesaki's theorem ([**35**]).

For $x_i^\circ \in N_{s_i}^\circ$ with $s_1 \neq \cdots \neq s_n$ and $n \in N$ we compute

$$\begin{aligned}
&(E(\lambda_{s_1}(x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ))\xi_0|\lambda(n)\xi_0)_{\mathcal{H}}\\
&= \psi(\lambda(n^*)E(\lambda_{s_1}(x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ)))\\
&= \psi(E(\lambda_{s_1}(n^*x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ))) && (\lambda_s|_N = \lambda)\\
&= \psi(\lambda_{s_1}(n^*x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ)) && (\psi\circ E = \psi)\\
&= 0 && (\text{by } n^*N_s^\circ \subseteq N_s^\circ \text{ and } (2.2)),
\end{aligned}$$

and hence we have

$$E(\lambda_{s_1}(x_1^\circ)\cdots\lambda_{s_n}(x_n^\circ)) = 0. \tag{2.4}$$

Therefore, the conditional expectation E satisfies the freeness relative to $\lambda(N)$. (See [**44**, §3.8].) By the similar computation as above we have

$$E(\lambda_s(x)) = \lambda(E_s(x)) \quad \text{for } x \in N_s. \tag{2.5}$$

Definition 2.4. We call the pair (M, E) the amalgamated free product of the (N_s, E_s) over N, and write

$$(M, E) = \mathop{*_N}_{s\in S} (N_s, E_s).$$

We will often identify $\lambda_s(x)$ with x itself. When no confusion is possible, we will denote the von Neumann algebra M by $\mathop{*_N}_{s\in S} N_s$.

The proposition below whose special case $N = \mathbf{C}1$ is in [**39**, **44**] guarantees that the pair (M, E) does not depend on the choice of φ. (It is possible to construct a relative tensor product in a functorial way as in [**7**, §B-δ] so that we can remove the dependency of the choice of φ from our construction of amalgamated free products.)

Proposition 2.5 (cf. [**42**, 1.3. Proposition])**.** *Let L be a von Neumann algebra and $\pi : N \to L$ be a $*$-isomorphism. Suppose that there exist normal $*$-isomorphisms $\pi_s : N_s \to L$ with $\pi_s|_N = \pi$ and a faithful normal conditional expectation $F : L \to \pi(N)$ such that:*

1. *L is generated by $\pi_s(N_s)$, $s \in S$;*
2. *$F \circ \pi_s = \pi \circ E_s$ for $s \in S$;*
3. *F satisfies the freeness relative to $\pi(N)$ ([**44**]).*

Then there exists a unique normal $*$-isomorphism $\Phi : L \to M$ such that $\Phi \circ \pi_s = \lambda_s$, $\Phi \circ F = E \circ \Phi$ for $s \in S$.

Proof. Equations (2.4), (2.5) and assumptions (2), (3) determine E and F on the dense $*$-subalgebras of M and L generated by $\lambda_s(N_s)$, $s \in S$, and $\pi_s(N_s)$, $s \in S$, respectively. (See computation (2.3).) Therefore, via the G.N.S. construction with respect to a fixed state on N we get the assertion. □

Here we briefly see the relationship between our amalgamated free products and the ones by S. Popa for finite von Neumann algebras ([**26**, 3.3 Definition]). Let P_1, P_2 be finite von Neumann algebras with normal normalized traces τ_1, τ_2 respectively, and assume that B is a common von Neumann subalgebra of P_1, P_2. We further suppose $\tau_1|_B = \tau_2|_B$. Let E_1, E_2 be the trace preserving conditional expectations from P_1, P_2 onto B respectively. Let

$$(P, \tau) = (P_1, \tau_1) *_B (P_2, \tau_2)$$

in Popa's sense. Here P is a finite von Neumann algebra generated by P_1, P_2 and τ is a faithful normal normalized trace satisfying $\tau|_{P_1} = \tau_1,\ \tau|_{P_2} = \tau_2$. By the construction in [**26**] the trace τ satisfies

$$\tau(x_1^\circ \cdots x_n^\circ) = 0$$

whenever $x_k^\circ \in P_{i_k}^\circ := \mathrm{Ker}\, E_{i_k}$ with $i_1 \neq \cdots \neq i_n$, $i_k \in \{1,2\}$. Note that there exists a (unique) trace-preserving conditional expectation $E : P \to B$, and it is easy to check that E satisfies the freeness relative to B and

$$E|_{P_i} = E_i \quad (i = 1, 2).$$

Therefore, by Proposition 2.5 the pair (P, E) is our amalgamated free product of (P_1, E_1) and (P_2, E_2) over B.

Let α_s, $s \in S$, be a $*$-automorphism on N_s, and suppose

$$\alpha_s|_N = \alpha_{s'}|_N \ (s \neq s'), \quad \alpha_s \circ E_s = E_s \circ \alpha_s.$$

Then Proposition 2.5 enables us to define the $*$-automorphism α on the amalgamated free product von Neumann algebra M $\left(= \underset{s\in S}{*_N} N_s \right)$ by

$$\alpha(x) = \alpha_s(x) \quad \text{for } x \in N_s,$$

and we write $\alpha = \underset{s\in S}{*_N} \alpha_s$.

Theorem 2.6. *For each faithful normal semi-finite weight ϕ on N we have*

$$\sigma_t^{\phi\circ E} = \underset{s\in S}{*_N} \sigma_t^{\phi\circ E_s} \quad \textit{for } \ t \in \mathbf{R}.$$

Proof. Lemma 2.3 shows

$$\sigma_t^{\varphi\circ E}(x) = \sigma_t^{\psi}(x) = \sigma_t^{\varphi\circ E_s}(x) \quad \text{for } x \in N_s.$$

Connes' cocycle Radon-Nikodym derivatives $[D\phi \circ E : D\varphi \circ E], [D\phi \circ E_s : D\varphi \circ E_s], [D\phi : D\varphi]$ ([**5**]) satisfy

$$[D\phi \circ E : D\varphi \circ E] = [D\phi : D\varphi], \quad [D\phi \circ E_s : D\varphi \circ E_s] = [D\phi : D\varphi],$$

and hence for $x \in N_s$ we have

$$\begin{aligned} \sigma_t^{\phi\circ E}(x) &= [D\phi \circ E : D\varphi \circ E]_t \ \sigma_t^{\varphi\circ E}(x) \ [D\phi \circ E : D\varphi \circ E]_t^* \\ &= [D\phi : D\varphi]_t \ \sigma_t^{\varphi\circ E_s}(x) \ [D\phi : D\varphi]_t^* \\ &= [D\phi \circ E_s : D\varphi \circ E_s]_t \ \sigma_t^{\varphi\circ E_s}(x) \ [D\phi \circ E_s : D\varphi \circ E_s]_t^* \\ &= \sigma_t^{\phi\circ E_s}(x). \end{aligned}$$

This completes the proof. □

3. Preliminaries on Cartan subalgebra.

3.1. Cartan Subalgebra, measured equivalence relation and the Krieger construction.

An abelian subalgebra D in a von Neumann algebra M is called a Cartan subalgebra ([**13**]) if the following conditions are satisfied:

(1) D is maximal abelian in M;

(2) There exists a (unique faithful) normal conditional expectation $E_D^M : M \to D$;
(3) The normalizer $\mathcal{N}_M(D) = \{u \in M : \ u \text{ is a unitary}, \ uDu^* = D\}$ generates M.

The set of those partial isometries $v \in M$ such that $v^*v, vv^* \in D$ and $vDv^* = Dvv^*$ is denoted by $\mathcal{GN}_M(D)$ and called the normalizing groupoid (of the pair (M, D)).

Let $\mathcal{R}$ be a countable measured equivalence relation on a Lebesgue space (X, μ) with a 2-cocycle σ on $\mathcal{R}$ ([**12**]). A von Neumann algebra having a Cartan subalgebra can be constructed from the pair $(\mathcal{R}, \sigma)$ by the following way ([**13**]): The left counting measure μ_ℓ $(d\mu_\ell(x, y) = d\mu(y))$ on $\mathcal{R}$ gives the Hilbert space $\mathcal{H} = L^2(\mathcal{R}, \mu_\ell)$. For a "nice" function f on $\mathcal{R}$, the bounded convolution operator $\lambda(f)$ on $\mathcal{H}$ is defined by

$$(\lambda(f)\xi)(x, y) = \sum_{(x,z)\in\mathcal{R}} f(x, z)\xi(z, y)\sigma(x, z, y) \quad \text{for } \xi \in \mathcal{H}.$$

The von Neumann algebra, acting standardly on $\mathcal{H}$, generated by these $\lambda(f)$'s is denoted by $W^*(\mathcal{R}, \sigma)$. An element in $W^*(\mathcal{R}, \sigma)$ can be written as a convolution operator (in an extended sense) associated with a unique function ([**13**, Proposition 2.6]). Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal in $\mathcal{R}$. The distinguished abelian subalgebra $\mathcal{D}(\mathcal{R}, \sigma)$ consists of the $\lambda(f)$ with $\mathrm{supp}(f) \subseteq \Delta$, and it can be identified with $L^\infty(X, \mu)$. The map $E : \lambda(f) \mapsto \lambda(f \cdot \chi_\Delta)$ gives rise to a faithful normal conditional expectation from $W^*(\mathcal{R}, \sigma)$ onto $\mathcal{D}(\mathcal{R}, \sigma)$. Here $f \cdot \chi_\Delta$ is the point-wise product of f and the characteristic function χ_Δ of Δ. Then it can be checked that $\mathcal{D}(\mathcal{R}, \sigma)$ is a Cartan subalgebra of $W^*(\mathcal{R}, \sigma)$.

Conversely, every von Neumann algebra with separable predual having a Cartan subalgebra arises, up to $*$-isomorphism, from relevant pair of a countable measured equivalence relation and a 2-cocycle on it via the above construction. (See [**13**, Theorem 1].)

Let $\mathcal{R}$ and (X, μ) be as above. Then there exists a countable group G of non-singular transformations on (X, μ) satisfying $\mathcal{R} = \mathcal{R}_G := \{(gx, x) \in X^2 : \ x \in X, g \in G\}$. (See [**12**, Theorem 1].) It is known that for any 2-cocycle σ on $\mathcal{R}_G$ (i) $W^*(\mathcal{R}_G, \sigma)$ is a factor if and only if G acts ergodically on (X, μ) ([**13**, Proposition 2.9(2)]); (ii) $W^*(\mathcal{R}_G, \sigma)$ is of type III_λ $(0 \le \lambda \le 1)$ if and only if so is G ([**13**, Proposition 2.11]).

3.2. Lacunary or admissible measure and the corresponding state.
Let G be a countable ergodic group of non-singular transformations on a Lebesgue space (X, μ) and ν be a σ-finite measure equivalent to μ. The measure ν is called an admissible measure of G if there exists a ν-preserving ergodic subgroup H of the full group $[G]$. Here the full group $[G]$ is the set of non-singular (invertible) transformations on (X, μ) whose graphs are

contained in $\mathcal{R}_G$. The measure ν is also called a lacunary measure of G if there exists a positive constant $\delta(>0)$ such that if $g \in G$ satisfies $\log(d\nu \circ g/d\nu)(x) > 0$ for almost every $x \in X$ then $\log(d\nu \circ g/d\nu)(x) > \delta$.

Let M be a type III_λ factor $(0 < \lambda \leq 1)$ with separable predual having a Cartan subalgebra D. Then we can assume that $M = W^*(\mathcal{R}_G, \sigma)$ and $D = \mathcal{D}(\mathcal{R}_G, \sigma)$ for some pair of a type III_λ countable ergodic group G of non-singular transformations on a Lebesgue space (X, μ) and a 2-cocycle σ on $\mathcal{R}_G$. The transformation group G admits an admissible probability measure ν ([**16**, Proposition 17, §II-2]), and then we define the corresponding (faithful normal) state φ on $D = \mathcal{D}(\mathcal{R}, \sigma)$ by

$$\varphi(\lambda(f)) = \int_X f(x,x)d\nu(x).$$

By the assumption on ν there exists an ν-preserving ergodic subgroup H of the full group $[G]$, and then the map $u : \phi \in H \mapsto \lambda(\chi_{\Gamma(\phi)}) \in M = W^*(\mathcal{R}, \sigma)$, acting standardly on $L^2(\mathcal{R}_G, \nu_\ell)$, gives rise to a representation of H in $\mathcal{N}_M(D)$ satisfying the equation:

$$u(\phi)\lambda(f)u(\phi)^* = \lambda(f \circ \phi^{-1})$$

for $\lambda(f) \in D = \mathcal{D}(\mathcal{R}, \sigma)$ with the identification $f(x) = f(x,x)$. Here $\chi_{\Gamma(\phi)}$ is the characteristic function of the graph $\Gamma(\phi) = \{(\phi(x), x) \in X^2 \ : \ x \in X\}$. Then it is easy to check (i) H acts ergodically on D via $\phi \mapsto \mathrm{Ad}u(\phi)$; (ii) the image $u(H)$ is contained in the centralizer $M_{\varphi\circ E}$. Thus we have

$$(M_{\varphi\circ E})' \cap M \subseteq u(H)' \cap D' \cap M = u(H)' \cap D = \mathbf{C}1,$$

and hence $M_{\varphi\circ E}$ is a type II_1 factor.

Let M be a type III_0 factor with separable predual having a Cartan subalgebra D. Then the pair (M, D) arises from relevant pair of a type III_0 countable ergodic group G of non-singular transformations and a 2-cocycle on $\mathcal{R}_G$. The transformation group G admits a lacunary probability measure ν ([**16**, Lemma 16]), and let φ be the corresponding state on D. Then the modular operator associated with $\varphi \circ E$ is the multiplication operator of the Radon-Nikodym derivative $\delta_\nu(gx, x) = \frac{d\nu\cdot g}{d\nu}(x)$ ([**13**, Proposition 2.8]). Thus, by Definition, 1 is an isolated point in $\mathrm{Sp}(\Delta_{\varphi\circ E})$.

Summing up the above discussions, we get the following Proposition:

Proposition 3.1. *Let M be a factor with separable predual having a Cartan subalgebra D.*

(1) *If M is of type* III_λ $(0 < \lambda \leq 1)$, *there exists a faithful normal state φ on D such that $M_{\varphi\circ E}$ is a type* II_1 *factor.*
(2) *If M is of type* III_0, *there exists a faithful normal state φ on D such that* 1 *is an isolated point in* $\mathrm{Sp}(\Delta_{\varphi\circ E})$.

3.3. A left module decomposition for an inclusion having a common Cartan subalgebra.

In [**25**] S. Popa gave a special left module decomposition result for an inclusion of finite von Neumann algebras having a common Cartan subalgebra to prove Connes-Feldman-Weiss' theorem by using only operator algebra techniques. As was pointed out without a proof in the final remark there, the result remains valid for σ-finite arbitrary von Neumann algebras.

Let M be a von Neumann algebra having a Cartan subalgebra D. The unique (faithful normal) conditional expectation from M onto D is denoted by E_D. It is easy to check

$$E_D(vxv^*) = vE_D(x)v^* \quad \text{for } x \in M \tag{3.3.1}$$

for each $v \in \mathcal{GN}_M(D)$. Let $N(\subseteq M)$ be a von Neumann subalgebra containing D, and suppose that there exists a (unique faithful) normal conditional expectation $E_N : M \to N$. Fix a faithful normal state φ on D, and denote by ξ_0 an implementing vector of $\varphi \circ E_D$. For $v \in \mathcal{GN}_M(D)$ and $y \in N$ we compute

$$\begin{aligned}
&\varphi \circ E_D((yv)^*(x - E_N(xv^*)v)) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi \circ E_D(v^*y^*E_N(xv^*)v) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(v^*E_D(y^*E_N(xv^*))v) && \text{(by (3.3.1))} \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(v^*E_D(E_N(y^*xv^*))v) && \text{(since } y \in N) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(v^*E_D(y^*xv^*)v) && \text{(by } E_D \circ E_N = E_D) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(E_D(v^*y^*xv^*v)) && \text{(by (3.3.1))} \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(E_D(v^*y^*x)v^*v) && \text{(since } v^*v \in D) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(v^*vE_D(v^*y^*x)) && \text{(since } D \text{ is commutative)} \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(E_D(v^*vv^*y^*x)) && \text{(since } v^*v \in D) \\
&= \varphi \circ E_D(v^*y^*x) - \varphi(E_D(v^*y^*x)) && \text{(since } v^*vv^* = v^*) \\
&= 0.
\end{aligned}$$

Therefore, for each $v \in \mathcal{GN}_M(D)$ we have

$$P_{Nv}x\xi_0 = E_N(xv^*)v\xi_0 \quad \text{for } x \in N, \tag{3.3.2}$$

where P_{Nv} is the projection onto $\overline{Nv\xi_0}$.

The argument in [**17**, Lemma 2.2.] shows that for each $v \in \mathcal{GN}_M(D)$, there exists a unique projection $e \in D$ satisfying $E_N(v) = ev$, $e \leq vv^*$. Thus [**25**, 2.1. Lemma] is valid for an arbitrary von Neumann algebra having a Cartan subalgebra since its proof does not depend on the trace property. Combining (3.3.2) and the arguments in [**25**, p. 173-174] we get the following proposition:

Proposition 3.2 (S. Popa [**25**] in the finite von Neumann algebra setting)**.** *Let $M \supseteq N$ be an inclusion of σ-finite von Neumann algebras having a common Cartan subalgebra D. Suppose that there exists a (unique faithful) normal conditional expectation $E_N : M \to N$. Then, for each faithful normal state $\varphi \in D_*$, there exists a subset $\{v_i\}_{i\in I}$ of the normalizing groupoid $\mathcal{GN}_M(D)$ containing 1 such that*

$$\sum_{i\in I}^{\oplus} N v_i \xi_0 = M\xi_0 \quad \textit{in} \;\; L^2(M),$$

where ξ_0 is an implementing vector of $\varphi \circ E_D$.

4. Factoriality.

We will at first prove the following relative commutant property for amalgamated free products:

Proposition 4.1. *Let A, B be σ-finite von Neumann algebras having a common von Neumann subalgebra D. Suppose that there exist two faithful normal conditional expectations $E_D^A : A \to D$ and $E_D^B : B \to D$. Let*

$$\left(M, E_D^M\right) = \left(A, E_D^A\right) *_D \left(B, E_D^B\right)$$

be the amalgamated free product.

Let A_0 be a linear subspace of A with the unit 1 satisfying $E_D^A(A_0) \subseteq A_0$. Suppose that there exist a faithful normal state φ on D and a unitary $u \in A_{\varphi\circ E_D^A}$ such that:

(1) *For each $a \in A_0$, there exists a natural number $n(a) \in \mathbb{N}$ satisfying*

$$E_D^A(a^* u^n a) = 0$$

for $n \in \mathbb{Z}$ with $|n| \geq n(a)$.

(2) *The linear subspace $A_0\xi_0$ is dense in $L^2(A)$, where ξ_0 is an implementing vector of $\varphi \circ E$.*

Then we have

$$\left(A_{\varphi\circ E_D^A}\right)' \cap M \subseteq A.$$

Proof. The von Neumann algebra M acts standardly on

$$\mathcal{H} = L^2(D) \oplus L^2(A)^\circ \oplus L^2(B)^\circ \oplus \left(L^2(A)^\circ \otimes_\varphi L^2(B)^\circ\right) \oplus \cdots,$$

where $L^2(A)^\circ = L^2(A) \ominus L^2(D)$ and $L^2(B)^\circ = L^2(A) \ominus L^2(D)$, and there exists a vector $\xi_0 \in L^2(D)$ ($\subseteq \mathcal{H}$) such that ξ_0 is an implementing vector of $\varphi \circ E_D^M$ and

$$(\cdots a_1^\circ b_1^\circ \cdots a_n^\circ b_n^\circ \cdots)\xi_0 = \cdots (a_1^\circ \xi_0) \otimes_\varphi (b_1^\circ \xi_0) \otimes_\varphi \cdots \otimes_\varphi (a_n^\circ \xi_0) \otimes_\varphi (b_n^\circ \xi_0) \cdots,$$

for $a_j^\circ \in A^\circ$ and $b_j^\circ \in B^\circ$. (See §2.) Then, by [**31**, 2.2. Remarque.(a)] and assumption (2), the linear subspace

$$(D + \mathrm{span}\Lambda^\circ(A_0^\circ, B^\circ))\xi_0 = D\xi_0 + A_0^\circ\xi_0 + B^\circ\xi_0 + A_0^\circ\xi_0 \odot_\varphi B^\circ\xi_0 + \cdots$$

is dense in $\mathcal{H}$, where $\Lambda^\circ(A_0^\circ, B^\circ)$ is the set of alternating words in A_0° ($=A_0 \cap \mathrm{Ker}E_D^A$) and B° ($= \mathrm{Ker}E_D^B$).

Set $N = (A_{\varphi\circ E_D^A})' \cap M$. By Theorem 2.6 and Takesaki's theorem ([**35**]), there exists a unique $\varphi \circ E_D^M$-conditional expectation $E_N^M : M \to N$, and let e_N be the corresponding (Jones) projection.

Let G be the unitary group of $A_{\varphi\circ E_D^A}$ and x be an element in M. Set $K_G(x) = \overline{co}^{\sigma-w}\{uxu^* : u \in G\}$, where $\overline{co}^{\sigma-w}$ is the σ-weak closure of the convex hull. Since $K_G(x)$ is σ-weakly compact, the convex set $K_G(x)\xi_0$ is closed in $\mathcal{H}$, and hence there exists a unique element $y_0 \in K_G(x)$ such that $\|y_0\xi_0\|_\mathcal{H} = \inf\{\|y\xi_0\|_\mathcal{H} \ : \ y \in K_G(x)\}$. Since G sits in $M_{\varphi\circ E_D^M}$, we have $\|uy_0u^*\xi_0\|_\mathcal{H} = \|y_0\xi_0\|_\mathcal{H}$ for every $u \in G$. Thus, by the uniqueness of y_0, we get $uy_0u^* = y_0$ for every $u \in G$ so that y_0 belongs to $G' \cap M$ ($= N$). By the bimodule property and the continuity of E_N^M we get $E_N^M(K_G(x)) = \{E_N^M(x)\}$, and then we have $E_N^M(x) = E_N^M(y_0) = y_0$. Therefore, $E_N^M(x)$ belongs to $K_G(x)$ and

$$\left\|E_N^M(x)\xi_0\right\|_\mathcal{H} = \inf\{\|y\xi_0\|_\mathcal{H} \ : \ y \in K_G(x)\}. \tag{4.1}$$

Claim. For each $x \in \Lambda^\circ(A_0^\circ, B^\circ) \setminus A_0^\circ$ we have $E_N^M(x) = 0$.

First, we prove that for each alternating word $x \in \Lambda^\circ(A_0^\circ, B^\circ) \setminus A_0^\circ$, there exists a natural number $n(x) \in \mathbb{N}$ such that

$$E_D^M(u^{kn(x)}x^*u^{-kn(x)}u^{\ell n(x)}xu^{-\ell n(x)}) = 0 \tag{4.2}$$

as long as $k \neq \ell$ in $\mathbb{Z}$. We may assume that there exist $y \in \Lambda^\circ(A_0^\circ, B^\circ)$ beginning and ending in B° and $a_1, a_2 \in A_0$ (admitting $a_1 = 1$ or $a_2 = 1$) with $x = a_1ya_2$. Set $n(x) = n(a_1)$, and hence by assumption we have $E_D^A(a_1^*u^na_1) = 0$ for every $|n| \geq n(x)$. Thus, for $k \neq \ell$ we have, by the freeness,

$$\begin{aligned}
&E_D^M\left(u^{kn(x)}x^*u^{-kn(x)}u^{\ell n(x)}xu^{-\ell n(x)}\right)\\
&= E_D^M\left(u^{kn(x)}(a_1ya_2)^*u^{-kn}u^{\ell n(x)}(a_1ya_2)u^{-\ell n(x)}\right)\\
&= E_D^M\left(\left(u^{kn(x)}a_2^*\right)y^*\left(a_1^*u^{(\ell-k)n(x)}a_1\right)y\left(a_2u^{-\ell n(x)}\right)\right)\\
&= 0.
\end{aligned}$$

We choose an arbitrary $x \in \Lambda^\circ(A_0^\circ, B^\circ) \setminus A_0^\circ$. By (4.1) and the assumption that u belongs to the centralizer of $\varphi \circ E_D^A$ (and hence that of $\varphi \circ E_D^M$ by

Theorem 2.6), it is sufficient to show that for each $\varepsilon(>0)$ there exists a natural number $m \in \mathbb{N}$ such that

$$(\|E_N^M(x)\xi_0\|_{\mathcal{H}} \le) \ \left\| \frac{1}{2m+1} \sum_{|k|\le m} u^{kn(x)} x u^{-kn(x)} \xi_0 \right\|_{\mathcal{H}} < \varepsilon.$$

By (4.2) we have

$$\begin{aligned}
&\left(u^{kn(x)} x u^{-kn(x)} \xi_0 | u^{\ell n(x)} x u^{-\ell n(x)} \xi_0\right)_{\mathcal{H}} \\
(4.3) \qquad &= \varphi\left(E_D^M\left(u^{\ell n(x)}(a_1 y a_2)^* u^{-\ell n(x)} u^{kn(x)} (a_1 y a_2) u^{-kn(x)}\right)\right) \\
&= 0
\end{aligned}$$

as long as $k \ne \ell \in \mathbb{Z}$. Then we compute

$$\begin{aligned}
&\left\| \frac{1}{2m+1} \sum_{|k|\le m} u^{kn(x)} x u^{-kn(x)} \xi_0 \right\|_{\mathcal{H}}^2 \\
&= \frac{1}{(2m+1)^2} \sum_{|k|\le m} \left\| u^{kn(x)} x u^{-kn(x)} \xi_0 \right\|_{\mathcal{H}}^2 \quad \text{(by (4.3))} \\
&= \frac{1}{(2m+1)^2} \sum_{|k|\le m} \|x\xi_0\|_{\mathcal{H}}^2 \qquad \text{(since } u \text{ belongs to } M_{\varphi\circ E_D^M}) \\
&= \frac{1}{(2m+1)} \|x\xi_0\|_{\mathcal{H}}^2.
\end{aligned}$$

Hence we are done by choosing m satisfying $m > ((\|x\xi_0\|_{\mathcal{H}}/\varepsilon)^2 - 1)/2$.

By the above claim, we get

$$(4.4) \qquad E_N^M(D + \mathrm{span}(\Lambda^\circ(A_0^\circ, B^\circ))) \subseteq A.$$

(Note that $E_N^M(a) \in K_G(a) \subseteq A$ for $a \in A$.) Let x be an element in N. Then there exists a sequence $\{x_k\}$ in $D + \mathrm{span}\Lambda^\circ(A_0^\circ, B^\circ)$ such that $\lim_{k\to\infty} x_k\xi_0 = x\xi_0$, and hence we have

$$x\xi_0 = E_N^M(x)\xi_0 = e_N(x\xi_0) = \lim_{k\to\infty} e_N(x_k\xi_0) = \lim_{k\to\infty} E_N^M(x_k)\xi_0.$$

Here $E_N^M(x_k)$ belongs to A by (4.4), and hence $x\xi_0$ belongs to $L^2(A) = L^2(D) \oplus L^2(A)^\circ$. By Theorem 2.6 and Takesaki's theorem ([**35**]) there exists a unique $\varphi \circ E_D^M$-conditional expectation $E_A^M : M \to A$, and the corresponding (Jones) projection is denoted by e_A. Since e_A is the projection onto $L^2(A) = L^2(D) \oplus L^2(A)^\circ$, we have

$$x\xi_0 = e_A(x\xi_0) = E_A^M(x)\xi_0.$$

Therefore, we get $x = E_A^M(x) \in A$ because ξ_0 is a separating vector for M. □

From now on, we assume that A, B are non-type I factors with separable preduals having a common Cartan subalgebra D. Let

$$\left(M, E_D^M\right) = \left(A, E_D^A\right) *_D \left(B, E_D^B\right)$$

be the amalgamated free product with respect to the unique normal conditional expectations $E_D^A : A \to D, E_D^B : B \to D$. Since no confusion arises, we often denote the von Neumann algebra M by $A *_D B$.

To show that the von Neumann algebra M is a factor, we must find a linear subspace of A (or B) satisfying the conditions in Proposition 4.1. To do this we need the following lemma:

Lemma 4.2 (S. Popa [**25**] in the type II_1 setting)**.** *There exist a faithful normal state φ on D, a unitary $u \in A_{\varphi\circ E_D^A} \cap \mathcal{N}_A(D)$ with $E_D^A(u^n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$ and a family $\{v_k\}_{k\in\mathbb{N}}$ of elements in $\mathcal{GN}_A(D)$ with $v_1 = 1$ such that*

$$\sum_{k\in\mathbb{N},\, n\in\mathbb{Z}}^{\oplus} Du^n v_k\xi_0 = A\xi_0 \quad \textit{in} \;\; L^2(A),$$

where ξ_0 is an implementing vector of $\varphi \circ E$. Of course, the same result holds for the inclusion $B \supseteq D$.

Proof. Suppose that we have chosen a faithful normal state φ on D and a unitary $u \in A_{\varphi\circ E_D^A} \cap \mathcal{N}_A(D)$ with $E_D^A(u^n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$. Then by applying Proposition 3.2 to the inclusions $A(\supseteq A_{\varphi\circ E_D^A}) \supseteq \{D, u\}'' \supseteq D$ we get a family of elements in $\mathcal{GN}_A(D)$ as in the lemma. Thus it is sufficient to show the existence of such a pair of a state and a unitary.

In the rest of the proof the type II_1, type II_∞, type III_λ and type III_0 cases are separately dealt with.

The type II_1 case ([**25**, Corollary 2.5]):

Suppose that A is of type II_1. Then we can choose a unitary $u \in \mathcal{N}_A(D)$ with $E_D^A(u^n) = 0$ for $n \in \mathbb{N}\setminus\{0\}$ by using the "array" construction technique (cf. [**21**, p. 166] in ergodic theory, [**24**, p. 278] in the finite von Neumann algebra setting). Set $\varphi = \tau|_D$ with the unique normalized trace τ on A, and then the equality $\varphi \circ E_D^A = \tau$ is obvious. Hence we are done.

The type II_∞ case:

Suppose that A is of type II_∞. Let $\{e_k\}_{k\in\mathbb{N}}$ be an orthogonal family of finite (in A) projections in D such that $e_k \sim e_{k'}$ and $\sum_{k=1}^{\infty} e_k = 1$. By the well-known Hopf equivalence theorem (cf. [**16**]) there exists a partial isometry $v \in \mathcal{GN}_A(D)$ with $v^*v = e_k$, $vv^* = e_{k'}$. Therefore, we get a system $\{e_{ij}\}_{i,j}$ in $\mathcal{GN}_A(D)$ of matrix units, and hence we have $A \supseteq D \cong e_{11}Ae_{11} \otimes B(\ell^2) \supseteq De_{11} \otimes \ell^\infty$. Then it can be easily checked that De_{11} is a Cartan subalgebra of $e_{11}Ae_{11}$. Set $A_0 := e_{11}Ae_{11}$ and $D_0 := De_{11}$. By the previous case we can choose a unitary $u_0 \in \mathcal{N}_{A_0}(D_0)$ with $E_{D_0}^{A_0}(u_0^n) = 0$ for

$n \in \mathbb{Z} \setminus \{0\}$. Set $u = u_0 \otimes 1 \in A_0 \otimes B(\ell^2)$, and hence we have $uDu^* = D$ and $E_D^A(u^n) = E_{D_0}^{A_0}(u_0^n) \otimes E_{\ell^\infty}^{B(\ell^2)}(1) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$, where $E_{D_0}^{A_0}$ is the unique faithful normal conditional expectation from A_0 onto D_0. Set $\varphi = (\tau_0|_{D_0}) \otimes \psi$ on $D_0 \otimes \ell^\infty = D$, where τ_0 is the unique normalized trace on A_0 and ψ is a faithful normal state on ℓ^∞. Then the unitary u is in $A_{\varphi\circ E_D^A} \cap \mathcal{N}_A(D)$. Hence we are done.

The type III_λ $(0 < \lambda \leq 1)$ case:

Suppose that A is of type III_λ $(0 < \lambda \leq 1)$. By Proposition 3.1 there exists a faithful normal state φ on D such that $A_{\varphi\circ E_D^A}$ is a type II_1 factor. Set $N = A_{\varphi\circ E_D^A}$, and then D is also a Cartan subalgebra of N. (See [**17**, 2.4. Remark].) Thus, by the type II_1 case we can choose a unitary $u \in \mathcal{N}_N(D)$ with $E_D^N(u^n) = 0$ (hence $E_D^A(u^n) = 0$) for $n \in \mathbb{Z} \setminus \{0\}$. Hence we are done.

The type III_0 case:

Suppose that A is of type III_0. By Proposition 3.1 there exists a faithful normal state φ on D such that 1 is an isolated point in $\mathrm{Sp}(\Delta_{\varphi\circ E_D^A})$. Set $N = A_{\varphi\circ E_D^A}$, and then by the proof of [**5**, Théorème 5.3.1.] the centralizer $(A \otimes B(\mathcal{H}))_{(\varphi\circ E_D^A)\otimes \mathrm{Tr}}$ is a type II_∞ von Neumann algebra. Since $\varphi \circ E_D^A$ is a normalized trace on N and $(A \otimes B(\mathcal{H}))_{(\varphi\circ E_D^A)\otimes \mathrm{Tr}} = N \otimes B(\mathcal{H})$, the von Neumann algebra N is of type II_1. Note that D is also a Cartan subalgebra of N ([**17**, 2.4. Remark]).

Let

$$N = \int_\Omega^\oplus N(\omega)d\mu(\omega) \supseteq D = \int_\Omega^\oplus D(\omega)d\mu(\omega),$$

be the simultaneous central decomposition of the inclusion $N \supseteq D$ with $\mathcal{Z}(N) = L^\infty(\Omega, \mu)$, and let

$$\mathcal{H} = \int_\Omega^\oplus \mathcal{H}(\omega)d\mu(\omega)$$

be the corresponding direct integral decomposition of $\mathcal{H} = L^2(N)$.

Then there exists a Borel subset Ω_0 with $\mu(\Omega \setminus \Omega_0) = 0$ such that $D(\omega)$ is a Cartan subalgebra of $N(\omega)$ for all $\omega \in \Omega_0$. (Cf. [**13**, Theorem 1].) Thus, for $\omega \in \Omega_0$ we can choose a unitary $u \in \mathcal{N}_{N(\omega)}(D(\omega))$ with $E_{D(\omega)}^{N(\omega)}(u^n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$ because $N(\omega)$ is a type II_1 factor. (See the type II_1 case.)

Here, by [**18**, 14.1.23. Lemma] we can assume that $\{\mathcal{H}(\omega)\}_{\omega\in\Omega}$ is a constant field of the separable infinite dimensional Hilbert space $\mathcal{H}_0$. We define the set Γ of those elements $(\omega, u) \in \Omega_0 \times U(\mathcal{H}_0)$ such that:

(1) $u \in N(\omega)$;
(2) $uD(\omega)u^* = D(\omega)$;
(3) $E_{D(\omega)}^{N(\omega)}(u^n) = 0$ for every $n \in \mathbb{Z} \setminus \{0\}$,

where $U(\mathcal{H}_0)$ is the unitary group on $\mathcal{H}_0$ with the strong operator topology. We will prove that Γ is a Borel subset up to null set.

We define the subset Γ_0 of $\Omega_0 \times U(\mathcal{H}_0)$ by $\Gamma_0 = \{(\omega, u) \in \Omega_0 \times U(\mathcal{H}_0) : (\omega, u) \text{ satisfies } (1), (2)\}$. By the argument in [18, Proposition 14.1.24] there exists a Borel subset Ω_1 of Ω_0 with $\mu(\Omega \backslash \Omega_1) = 0$ such that $(\Omega_1 \times U(\mathcal{H}_0)) \cap \Gamma_0$ is Borel. Assume that ξ is a separating and cyclic vector for N in $\mathcal{H}$ and let

$$\xi = \int_\Omega^\oplus \xi(\omega) d\mu(\omega)$$

be the direct integral decomposition. Then by [18, 14.1.13. Lemma, 14.1.19. Lemma] we can choose a Borel subset Ω_2 of Ω_1 with $\mu(\Omega \setminus \Omega_2) = 0$ such that $\xi(\omega)$ is a cyclic and separating vector for $N(\omega)$ for all $\omega \in \Omega_2$. Let e be the (Jones) projection onto $\overline{D\xi}$ defined by $e(n\xi) = E_D^N(n)\xi$, and

$$e = \int_\Omega^\oplus e_\omega d\mu(\omega)$$

be the direct integral decomposition. By the construction, we have

$$\int_\Omega^\oplus e_\omega(n(\omega)\xi(\omega))d\mu(\omega) = e(n\xi) = E_D^N(n)\xi = \int_\Omega^\oplus E_{D(\omega)}^{N(\omega)}(n(\omega))\xi(\omega)d\mu(\omega)$$

so that by [18, 14.1.7. Remark] we get

$$e_\omega(x\xi(\omega)) = E_{D(\omega)}^{N(\omega)}(x)\xi(\omega), \ x \in N(\omega) \tag{4.5}$$

for almost every $\omega \in \Omega$. Thus, e_ω is the Jones projection associated with $E_{D(\omega)}^{N(\omega)}$. Let $\{\eta_j\}_j$ be an orthonormal basis of $\mathcal{H}_0$. By the argument in [18, Proposition 14.1.24] there exists a Borel subset Ω_3 of Ω_2 with $\mu(\Omega \setminus \Omega_3) = 0$ such that

$$(\omega, u) \in \Omega_3 \times U(\mathcal{H}_0) \mapsto (e_\omega u^k \xi(\omega)|\eta_j)_{\mathcal{H}_0}$$

is a Borel map for every $k \in \mathbb{Z}$ and j. Therefore, by using (4.5) it can be easily checked that the set

$$\Gamma_1 := \Gamma \cap (\Omega_3 \times U(\mathcal{H}_0)) = \{(\omega, u) \in \Gamma_0 \cap (\Omega_3 \times U(\mathcal{H}_0)) : (\omega, u) \text{ satisfies } (3)\}$$

is a Borel subset of $\Omega_3 \times U(\mathcal{H}_0)$.

For every $\omega \in \Omega_3$, $(\Gamma_1)_\omega := \{u \in U(\mathcal{H}_0) : (\omega, u) \in \Gamma_1\}$ is not empty. Therefore we get a measurable field $\omega \in \Omega_3 \to u_\omega$ of unitaries satisfying $(\omega, u_\omega) \in \Gamma_1$ thanks to the measurable-selection principle (cf. [18, 14.3.6. Theorem]). Define $u_\omega = 1$ for $\omega \in \Omega \setminus \Omega_3$, and set

$$u = \int_\Omega^\oplus u_\omega d\mu(\omega).$$

By the construction of the measurable field we have

$$E_D^N(u^n) = \int_\Omega^\oplus E_{D(\omega)}^{N(\omega)}(u_\omega^n)d\mu(\omega) = 0$$

for every $n \in \mathbb{Z} \setminus \{0\}$, and u belongs to $\mathcal{N}_N(D)$. Hence we are done. □

Theorem 4.3. *There exists a faithful normal state φ on D such that*

$$(A_{\varphi\circ E_D^A})' \cap M \subseteq A.$$

In particular, the von Neumann algebra M is a factor. Furthermore, if A is of type III_λ $(0 < \lambda \leq 1)$*, the above φ can be chosen in such a way that*

$$(A_{\varphi\circ E_D^A})' \cap M = \mathbf{C}1.$$

The analogous result holds for B.

Proof. By Lemma 4.2 there exists a faithful normal state φ on D, a unitary $u \in A_{\varphi\circ E_D^A} \cap \mathcal{N}_A(D)$ with $E_D^A(u^n) = 0$ for $n \in \mathbb{Z} \setminus \{0\}$ and a family $\{v_k\}_{k\in\mathbb{N}}$ of elements in $\mathcal{GN}_A(D)$ with $v_1 = 1$ such that

$$\sum_{k\in\mathbb{N},n\in\mathbb{Z}}^{\oplus} Du^n v_k \xi_0 = A\xi_0 \quad \text{in } L^2(A),$$

where ξ_0 is an implementing vector of $\varphi \circ E$.

Let A_0 be the linear span of elements $xu^n v_k$ for $n \in \mathbb{Z}$, $k \in \mathbb{N}$ and $x \in D$. For $x_1, x_2 \in D$, $k_1, k_2 \in \mathbb{N}$ and $m, n_1, n_2 \in \mathbb{Z}$ we compute

$$\begin{aligned}
&E_D^A((x_1u^{n_1}v_{k_1})^* u^m (x_2u^{n_2}v_{k_2}))\\
&= E_D^A(v_{k_1}^* u^{-n_1} x_1^* u^m x_2 u^{n_2} v_{k_2})\\
&= v_{k_1}^* v_{k_1} E_D^A(v_{k_1}^* u^{-n_1} x_1^* u^m x_2 u^{n_2} v_{k_2})\\
&\qquad\qquad\qquad\qquad (\text{since } v_{k_1}^* v_{k_1} v_{k_1}^* = v_{k_1}^*,\ v_{k_1}^* v_{k_1} \in D)\\
&= E_D^A(v_{k_1}^* u^{-n_1} x_1^* u^m x_2 u^{n_2} v_{k_2} v_{k_1}^* v_{k_1})\\
&\qquad\qquad\qquad\qquad (\text{since } D \text{ is commutative and } v_{k_1}^* v_{k_1} \in D)\\
&= v_{k_1}^* E_D^A(u^{-n_1} x_1^* u^m x_2 u^{n_2} v_{k_2} v_{k_1}^*) v_{k_1}\\
&\qquad\qquad\qquad\qquad (\text{by } (3.3.1))\\
&= v_{k_1}^* u^{-n_1} x_1^* u^{n_1} u^{m-n_1} x_2 u^{n_1-m} E_D^A(u^{m-n_1+n_2} v_{k_2} v_{k_1}^*) v_{k_1}\\
&\qquad\qquad\qquad\qquad (\text{since } u^{-n_1} x_1^* u^{n_1},\ u^{m-n_1} x_2 u^{n_1-m} \in D)\\
&= \delta_{k_1,k_2} \delta_{m,n_1-n_2} \times (v_{k_1}^* u^{-n_1} x_1^* u^{n_1} u^{m-n_1} x_2 u^{n_1-m} v_{k_1})\\
&\qquad\qquad\qquad\qquad (\text{by } (3.3.2)),
\end{aligned}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker delta. It is easy to check that the triple (φ, u, A_0) satisfies the conditions in Proposition 4.1 by using the above computation. Therefore, we get the first assertion.

The second assertion is clear from the proof of Lemma 4.2. □

Remark 4.4. The conditions (i) D is a Cartan subalgebra of B; (ii) B is a factor were not used in the above arguments.

Corollary 4.5. *If M $(= A *_D B)$ is of type* III_0, *then both of A and B must be of type* III_0.

Proof. First, we assume that A (or B) is of type II. The above theorem, in particular, shows

$$A' \cap M \subseteq A' \cap A = \mathbf{C}1.$$

By the assumption there exists a faithful normal (semi-finite) trace τ on A. It is well-known that the restriction of τ to D is semi-finite. Set $\psi := \tau|_D$, and then the centralizer $M_{\psi\circ E_D^M}$ contains A by Theorem 2.6. Thus we have

$$(M_{\psi\circ E_D^M})' \cap M \subseteq A' \cap M = \mathbf{C}1,$$

and hence M is not of type III_0 as is well-known.

Next, we assume that A (or B) is of type III_λ $(0 < \lambda \leq 1)$. In this case, the last part in the above theorem shows that M is not of type III_0 by the same reason as in the type II case. Therefore, we are done. $\square$

Here we obtain some type classification facts on our amalgamated free products.

The type II_1 case: Suppose that A (or B) is of type II_1, and further that M $(= A *_D B)$ is semi-finite. Since A is of type II_1, there exists a faithful normal state φ on D such that $\varphi \circ E_D^A$ is the unique normalized trace on A. Then, by the proof of Corollary 4.5 we have

$$\mathcal{Z}(M_{\varphi\circ E_D^M}) \subseteq (M_{\varphi\circ E_D^M})' \cap M = \mathbf{C}1.$$

On the other hand, there exists a positive non-singular self-adjoint operator H affiliated with M such that $\sigma_t^{\varphi\circ E_D^M} = \mathrm{Ad}H^{it}$ for $t \in \mathbf{R}$ because M is semi-finite. Then it is easy to show that H^{it} $(t \in \mathbf{R})$ belongs to $\mathcal{Z}(M_{\varphi\circ E_D^M}) = \mathbf{C}1$, and hence $\varphi \circ E_D^M$ is a normalized trace on M. Since M is not finite dimensional, M must be of type II_1.

The type II_∞ case: Suppose that A (or B) is of type II_∞. If M $(= A *_D B)$ is semifinite, then M must be of type II_∞ because there exists a faithful normal conditional expectation from M onto A (or B) thanks to Theorem 2.6.

The type III_λ case $(0 < \lambda \leq 1)$: Suppose that A (or B) is of type III_λ $(0 < \lambda \leq 1)$. Then there exists a faithful normal state φ on D such that $A_{\varphi\circ E_D^A}$ is a type II_1 factor by Proposition 3.1. By (the proof of) Theorem 4.3 we have

$$(M_{\varphi\circ E_D^M})' \cap M \subseteq (A_{\varphi\circ E_D^A})' \cap M = \mathbf{C}1,$$

and hence M $(= A *_D B)$ is not of type III_0 as is well-known. And Connes' T-set is

$$\mathrm{T}(M) = \{t \in \mathrm{T}(A) : \sigma_t^{\varphi\circ E_D^B} = \mathrm{id}\},$$

where $\mathrm{T}(A)$ is Connes' T-set of A. Therefore, M must be of type $\mathrm{III}_{\lambda^{1/n}}$ ($n \in \mathbb{N}$) or type III_1.

Example 4.6. Here we will give an example of a type III_λ factor ($0 < \lambda < 1$) arising as an amalgamated free product of two hyperfinite type II_1 factors over their common Cartan subalgebra.

Let

$$X = \prod_{n=1}^{\infty} \{0,1\}, \quad d\mu_\lambda = \prod_{n=1}^{\infty} \left\{ \frac{1}{1+\lambda^{1/2}}, \frac{\lambda^{1/2}}{1+\lambda^{1/2}} \right\},$$

and S_∞ be the group of finite permutations on (X, μ). We define the automorphism θ on (X, μ_λ) by

$$(\theta(\omega))_k = \begin{cases} \omega_k, & \text{if } k \geq 2, \\ \omega_1 + 1 \pmod 2, & \text{if } k = 1. \end{cases}$$

Then the Radon-Nikodym derivative $\frac{d\mu_\lambda \circ \theta}{d\mu_\lambda}$ looks like:

$$\frac{d\mu_\lambda \circ \theta}{d\mu_\lambda} = \begin{bmatrix} \lambda^{1/2} & \\ & \lambda^{-1/2} \end{bmatrix} \otimes 1 \otimes 1 \otimes \cdots \quad \in \prod_{n=1}^{\infty}{}^{\otimes} \begin{bmatrix} \mathbf{C} & \\ & \mathbf{C} \end{bmatrix} \cong L^\infty(X, \mu_\lambda).$$

Let $G = S_\infty$ and $\tilde{G} = \{\theta \circ g \circ \theta^{-1} : g \in G\}$, and it is well-known that they act ergodically on the measure space. It is obvious that μ_λ is a (unique) G-invariant probability measure and $\tilde{\mu}_\lambda := \mu_\lambda \circ \theta$ is a (unique) $\tilde{G}$-invariant probability measure. Therefore, the associated relations $\mathcal{R}_G$ and $\mathcal{R}_{\tilde{G}}$ (see §3.1) are of type II_1 and amenable.

Set $A = W^*(\mathcal{R}_G)$ and $B = W^*(\mathcal{R}_{\tilde{G}})$, which are AFD type II_1 factors and contain $D := L^\infty(X, \mu_\lambda)$ as a common Cartan subalgebra. (See §3.1.) The states induced from the measures μ_λ and $\tilde{\mu}_\lambda$ are denoted by φ and $\tilde{\varphi}$ respectively. Let $A *_D B$ be the amalgamated free product with respect to the unique normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$. Then by Corollary 4.5 it is not of type III_0, and Connes' T-set $\mathrm{T}(A *_D B)$ is

$$\mathrm{T}(A *_D B) = \left\{ t \in \mathbf{R} : \sigma_t^{\varphi \circ E_D^B} = \mathrm{id} \right\}.$$

Notice

$$\sigma_t^{\varphi \circ E_D^B} = \mathrm{Ad}\left(\frac{d\mu_\lambda \circ \theta}{d\mu_\lambda} \right)^{-it} \circ \sigma_t^{\tilde{\varphi} \circ E_D^B} = \mathrm{Ad}\left(\frac{d\mu_\lambda \circ \theta}{d\mu_\lambda} \right)^{-it},$$

and hence we have

$$\mathrm{T}(A *_D B) = \frac{2\pi}{\log \lambda} \mathbb{Z}.$$

Therefore, $A *_D B$ is of type III_λ.

By the similar method as above we can construct a type III_1 factor arising as an amalgamated free product of two hyperfinite type II_1 factors over their common Cartan subalgebra and also type III_λ factors ($0 < \lambda \leq 1$) arising as amalgamated free products of hyperfinite type II_∞ factors over their common Cartan subalgebras.

5. Continuous decomposition And Flow of Weights.

In this section, we will determine the flows of weights ([**8**]) of amalgamated free products of non-type I factors over their common Cartan subalgebras.

We will at first obtain a theorem on continuous decomposition for general amalgamated free products, which is closely related to [**29**, Remark 16]. Here we would like to emphasize that it is a byproduct of our formulation of amalgamated free products of von Neumann algebras.

Assume that A, B are σ-finite von Neumann algebras and D is a common von Neumann subalgebra with two faithful normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$.

Fix a faithful normal state φ on D, and we set

$$\tilde{A} = A \rtimes_{\sigma^{\varphi\circ E_D^A}} \mathbf{R} \supseteq \tilde{D} = D \rtimes_{\sigma^\varphi} \mathbf{R} \subseteq \tilde{B} = B \rtimes_{\sigma^{\varphi\circ E_D^B}} \mathbf{R}.$$

Here, note that the above inclusions do not depend on the choice of φ thanks to Connes' cocycle Radon-Nikodym theorem ([**5**]). By Takesaki's theorem ([**35**]) there exist two faithful normal conditional expectations $\hat{E_D^A} : \tilde{A} \to \tilde{D}$ and $\hat{E_D^B} : \tilde{B} \to \tilde{D}$ such that

$$\hat{E_D^A}|_A = E_D^A, \quad \hat{E_D^B}|_B = E_D^B.$$

It is easy to check

$$\hat{E_D^A} \circ \theta_t^A = \theta_t^D \circ \hat{E_D^A}, \quad \hat{E_D^B} \circ \theta^B = \theta_t^D \circ \hat{E_D^B},$$

$$\theta_t^A|_{\tilde{D}} = \theta_t^B|_{\tilde{D}}.$$

Here θ^A, θ^B and θ^D are the dual actions. Further details can be found in [**20**, 2.1].

Theorem 5.1. *Let*

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

be the amalgamated free product. Then we have

$$\left(\tilde{M}, \hat{E_D^M}\right) \cong \left(\tilde{A}, \hat{E_D^A}\right) *_{\tilde{D}} \left(\tilde{B}, \hat{E_D^B}\right),$$

and $\theta_t^M = \theta_t^A *_{\tilde{D}} \theta_t^B$ *via the isomorphism. Here* $\tilde{M}$ *is the crossed product of* M *by the modular action* $\sigma^{\varphi\circ E_D^M}$ *with the dual action* θ^M*.*

Lemma 5.2. *Assume that N is a σ-finite von Neumann algebra with a continuous action α of $\mathbf{R}$. Let $N \rtimes_\alpha \mathbf{R}$ $(= \{N, \lambda(\mathbf{R})\}'')$ be the continuous crossed product acting on $L^2(N) \otimes L^2(\mathbf{R})$. For an element $x = \sum_{k=1}^n x_k \lambda(t_k) \in N \rtimes_\alpha \mathbf{R}$ with $t_i \neq t_j$ $(i \neq j)$, we have*

$$x = \sum_{k=1}^n x_k \lambda(t_k) = 0 \Longleftrightarrow x_k = 0 \quad k = 1, \dots, n.$$

Proof. Let ξ be a separating and cyclic vector in $L^2(N)$ of N. To show the lemma, one should hit x against the vector $\xi \otimes \chi_{[-\varepsilon,\varepsilon]}$ with $0 < \varepsilon < \min\{|t_i - t_j| : i \neq j \in \{1, \dots, n\}\}$. □

Proof of Theorem 5.1. We define two subalgebras $\tilde{A}_0$ and $\tilde{B}_0$ of $\tilde{A}$ and $\tilde{B}$ by

$$\tilde{A}_0 = \mathrm{span}\{a\lambda(t) : a \in A, t \in \mathbf{R}\}, \quad \tilde{B}_0 = \mathrm{span}\{b\lambda(t) : b \in B, t \in \mathbf{R}\}$$

respectively. These subalgebras are obviously dense in the σ-weak topology.

For an element $x = \sum_{k=1}^n a_k \lambda(t_k) \in \tilde{A}_0$ with $t_i \neq t_j$ $(i \neq j)$, we have

$$\hat{E_D^A}(x) = \sum_{k=1}^n E_D^A(a_k)\lambda(t_k)$$

so that $\hat{E_D^A}(x) = 0$ implies $E_D^A(a_k) = 0$ for every $k = 1, 2, \cdots, n$ by the previous lemma. The analogous fact also holds for $\tilde{B}_0$.

By the similar argument as in [44, Proposition 2.5.7] it is sufficient to check the freeness of $\tilde{A}_0, \tilde{B}_0$ in the $D \rtimes \mathbf{R}$-probability space $(\tilde{M}, \hat{E_D^M})$. (We must also check $\hat{E_D^A}(\tilde{A}_0) \subseteq \tilde{A}_0$ and $\hat{E_D^B}(\tilde{B}_0) \subseteq \tilde{B}_0$. However these are obvious in our case.) Hence it is sufficient to show that the expectation values of the following four type alternating words are always zero:

$$\tilde{A}_0^\circ \cdot \cdots \tilde{A}_0^\circ, \quad \tilde{A}_0^\circ \cdot \cdots \tilde{B}_0^\circ, \quad \tilde{B}_0^\circ \cdot \cdots \tilde{A}_0^\circ, \quad \tilde{B}_0^\circ \cdot \cdots \tilde{B}_0^\circ,$$

where $\tilde{A}_0^\circ = \tilde{A}_0 \cap \mathrm{Ker}\hat{E_D^A}$ and $\tilde{B}_0^\circ = \tilde{B}_0 \cap \mathrm{Ker}\hat{E_D^B}$.

Let $x_1^\circ x_2^\circ \cdots x_n^\circ$ be an alternating word in $\tilde{A}_0^\circ \cdot \cdots \tilde{A}_0^\circ$, and hence

$$x_1^\circ = \sum a_k^{(1)} \lambda(t_k) \quad \text{with } E_D^A\left(a_k^{(1)}\right) = 0,\ t_i \neq t_j\ (i \neq j),$$
$$x_2^\circ = \sum b_k^{(2)} \lambda(t_k) \quad \text{with } E_D^B\left(b_k^{(2)}\right) = 0,\ t_i \neq t_j\ (i \neq j),$$
$$\dots$$
$$x_n^\circ = \sum a_k^{(n)} \lambda(t_k) \quad \text{with } E_D^A\left(a_k^{(n)}\right) = 0,\ t_i \neq t_j\ (i \neq j).$$

We have

$$\begin{aligned}&\hat{E_D^M}(x_1^\circ x_2^\circ \cdots x_n^\circ)\\ &= \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} \hat{E_D^M}\left(a_{k_1}^{(1)}\lambda(t_{k_1}) b_{k_2}^{(2)}\lambda(t_{k_2}) \cdots a_{k_n}^{(n)}\lambda(t_{k_n})\right)\end{aligned}$$

$$= \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} \hat{E_D^M}\left(a_{k_1}^{(1)}\sigma_{t_{k_1}}^{\varphi\circ E_D^M}\left(b_{k_2}^{(2)}\right)\cdots\sigma_{(t_{k_1}+\cdots)}^{\varphi\circ E_D^M}\left(a_{k_n}^{(n)}\right)\right) \cdot\lambda(t_{k_1}+\cdots+t_{k_n})$$
$$= \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} E_D^M\left(a_{k_1}^{(1)}\sigma_{t_{k_1}}^{\varphi\circ E_D^B}\left(b_{k_2}^{(2)}\right)\cdots\sigma_{(t_{k_1}+\cdots)}^{\varphi\circ E_D^A}\left(a_{k_n}^{(n)}\right)\right) \cdot\lambda(t_{k_1}+\cdots+t_{k_n}),$$

and each coefficient is zero because A, B are free relative to D and $\sigma_t^{\varphi\circ E_D^A}(A^\circ) = A^\circ$, $\sigma_t^{\varphi\circ E_D^B}(B^\circ) = B^\circ$. Here the second equality comes from $\mathrm{Ad}\lambda(t)|_M = \sigma_t^{\varphi\circ E_D^M}$ and the third equality comes from Theorem 2.6. Another cases can be proved similarly.

Finally the equation on dual actions can be easily proved, so that details are left to the reader. □

Remark 5.3. (1) The above result is valid for a general crossed product by an action with the condition before Theorem 2.6 and moreover for the crossed product by the minimal action of the compact quantum group $SU_q(n)$ (or an arbitrary compact Kac algebra) constructed in [**37**].

(2) The above theorem is useful also in the context of free products (i.e., amalgamated free products over $\mathbf{C}$), see [**38**]. Indeed, the theorem says that the continuous core of the free product $(A,\phi)*(B,\psi)$ can be written as the amalgamated free product of the ones of A, B over $\mathbf{C}\rtimes\mathbf{R}$ ($\cong L^\infty(\mathbf{R})$). (Added Oct. 28, 1998.)

(3) Let A, B be σ-finite type III_λ factors with the same λ and ϕ, ψ be faithful normal states on them respectively. If the modular automorphisms of ϕ, ψ satisfy $\sigma_{t_0}^\phi = \mathrm{Id} = \sigma_{t_0}^\psi$ with $t_0 = \frac{-2\pi}{\log\lambda}$, then the discrete core of the free product $(A,\phi)*(B,\psi)$ can be written as the amalgamated free product of the ones of A, B over $\mathbf{C}\rtimes(\mathbf{R}/t_0\mathbb{Z})$. (Added Oct. 28, 1998.)

From now on, we further assume that A, B are non-type I factors with separable preduals and D is their common Cartan subalgebra. It is easy to check that $\tilde{D}$ is also a common Cartan subalgebra of $\tilde{A}, \tilde{B}$. (See [**32**, Lemma 1].) By Lemma 4.2 we can choose a faithful normal state φ on D and a unitary $u \in \mathcal{N}_A(D)\cap A_{\varphi\circ E_D^A}$ with $E_D^A(u^n) = 0$ for $n\in\mathbb{Z}\setminus\{0\}$. Notice $\tilde{D} = D\rtimes_{\mathrm{id}}\mathbf{R} = D\otimes\lambda(\mathbf{R})''$, and hence we define the state $\tilde{\varphi} := \varphi\otimes\psi$, where ψ is a faithful normal state on $\lambda(\mathbf{R})''$. For $x = \sum_k a_k\lambda(t_k)\in\tilde{A}_0$ we have:

$$\tilde{\varphi}\circ\hat{E_D^A}(ux) = \sum_k \tilde{\varphi}(E_D^A(ua_k)\lambda(t_k)) = \sum_k \varphi\circ E_D^A(ua_k)\psi(\lambda(t_k))$$
$$= \sum_k \varphi\circ E_D^A(a_k u)\psi(\lambda(t_k)) \qquad (\text{since } u\in A_{\varphi\circ E_D^A})$$

$$
\begin{aligned}
&= \sum_k \tilde{\varphi}(E_D^A(a_k u)\lambda(t_k)) = \sum_k \tilde{\varphi}\circ \hat{E_D^A}(a_k u\lambda(t_k)) \\
&= \sum_k \tilde{\varphi}\circ \hat{E_D^A}(a_k\lambda(t_k)u) \qquad\qquad (\text{since } u\lambda(t)=\lambda(t)u) \\
&= \tilde{\varphi}\circ \hat{E_D^A}(xu).
\end{aligned}
$$

Since $\tilde{A}_0$ is σ-weakly dense in $\tilde{A}$, the unitary u belongs to $\tilde{A}_{\tilde{\varphi}\circ\hat{E_D^A}}$. Furthermore, we have $\hat{E_D^A}(u^n) = E_D^A(u^n) = 0$ for $n \in \mathbb{Z}\setminus\{0\}$. Thus, by applying Proposition 3.2 to the inclusions $\tilde{A} \supseteq \{\tilde{D}, u\}'' \supseteq \tilde{D}$ we get the same left module decomposition result for $\tilde{A} \supseteq \tilde{D}$ as in Lemma 4.2. Of course the analogous result holds for $\tilde{B} \supseteq \tilde{D}$. Therefore, by Proposition 4.1 (see the proof of Theorem 4.3) we have

$$\tilde{A}' \cap \tilde{M} \subseteq \tilde{A} \quad \text{and} \quad \tilde{B}' \cap \tilde{M} \subseteq \tilde{B}.$$

Since $\tilde{D}$ is a common Cartan subalgebra of $\tilde{A}, \tilde{B}$, the centers $\mathcal{Z}(\tilde{A}), \mathcal{Z}(\tilde{B})$ are contained in $\tilde{D}$ so that we get the following theorem:

Theorem 5.4. *The center $\mathcal{Z}(\tilde{M})$ is $\mathcal{Z}(\tilde{A}) \cap \mathcal{Z}(\tilde{B})$ $(\subseteq \tilde{D})$.*

Remark 5.5. By [**36**, Theorem 8.5] we have

$$\mathcal{Z}(M) = \mathcal{Z}(\tilde{M})^{\theta^M} = \mathcal{Z}(\tilde{A})^{\theta^A} \cap \mathcal{Z}(\tilde{B})^{\theta^B} = \mathcal{Z}(A) \cap \mathcal{Z}(B) = \mathbf{C}1,$$

and this gives us another (indirect) proof of the factoriality result in Theorem 4.3.

By the similar argument as in the proof of Lemma 4.2 we can prove the same left module decomposition result for $\tilde{A}$ (or $\tilde{B}$) as in Lemma 4.2 in the case that A (or B) is a von Neumann algebra with separable predual having no type I direct summand. Therefore, the above theorem (also $\mathcal{Z}(A *_D B) = \mathcal{Z}(A) \cap \mathcal{Z}(B)$) still holds under this assumption.

Let (X_A, F_t^A), (X_B, F_t^B) be (smooth) flows of weights of A, B respectively. Set $X_D := Y \times \mathbf{R}$ and $F_t^D(y,s) := (y, s+t)$ for $(y,s) \in Y \times \mathbf{R}$ under the identification $D = L^\infty(Y)$ with a Lebesgue space Y, and then there exist two factor maps

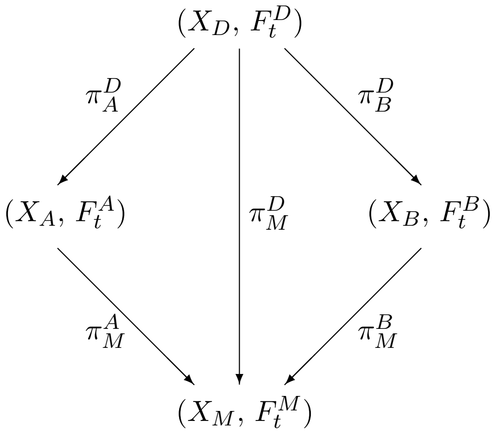
$$\pi_A^D : (X_D, F_t^D) \longrightarrow (X_A, F_t^A), \quad \pi_B^D : (X_D, F_t^D) \longrightarrow (X_B, F_t^B).$$

Let (X_M, F_t^M) be the (smooth) flow of weights of M. By the above theorem there exist three factor maps

$$\pi_M^A : (X_A, F_t^A) \longrightarrow (X_M, F_t^M), \quad \pi_M^B : (X_B, F_t^B) \longrightarrow (X_M, F_t^M),$$
$$\pi_M^D : (X_D, F_t^D) \longrightarrow (X_M, F_t^M).$$

Here we clarify the effect on taking of amalgamated free product over a common Cartan subalgebra in the flow (of weights) level.

Corollary 5.6. *The flow* (X_M, F_t^M) *is determined as the* (*unique*) *maximal flow satisfying the following commutative diagram:*



Proof. This immediately follows from Theorem 5.4. □

The above corollary gives us the type classification results below.

Corollary 5.7.

(1) *If either* A *or* B *is of type* III_1*, then* M *is of type* III_1*;*
(2) *Let* A *be of type* III_λ *and* B *be of type* III_μ*. If* $\log\lambda$ *and* $\log\mu$ *are rationally independent, then* M *is of type* III_1*.*

Corollary 5.8. *Let* A *be a non-type* I *factor with separable predual having a Cartan subalgebra* D*. The amalgamated free product von Neumann algebra* $M = A *_D A$ *and* A *have always the same flow of weights.*

Appendix I. A relative commutant property.

Let A, B be σ-finite von Neumann algebras having a common von Neumann subalgebra D. We suppose that there exist faithful normal conditional expectations $E_D^A : A \to D$, $E_D^B : B \to D$, and let

$$(M, E_D^M) = (A, E_D^A) *_D (B, E_D^B)$$

be the amalgamated free product. Suppose the following condition:

Condition I-A. There exist a faithful normal state φ on D and three unitaries $a \in A_{\varphi\circ E_D^A}$ and $b, c \in B_{\varphi\circ E_D^B}$ such that

(A.1)
$$\mathrm{Ad}a \circ E_D^A = E_D^A \circ \mathrm{Ad}a,\ \ \mathrm{Ad}b \circ E_D^B = E_D^B \circ \mathrm{Ad}b,\ \ \mathrm{Ad}c \circ E_D^B = E_D^B \circ \mathrm{Ad}c,$$

(A.2)
$$E_D^A(a) = E_D^B(b) = E_D^B(c) = E_D^B(b^*c) = 0.$$

Lemma I-B. *Assume that M acts standardly on a Hilbert space $\mathcal{H}$. Under the above condition we have*

$$0 \in \overline{co}^{\mathrm{s.o}}\{ux^\circ u^* : u \in G(a,b,c)\} \quad \text{for } x^\circ \in \mathrm{Ker}E_D^M.$$

Here $G(a,b,c)$ is the group generated by a, b, c in the unitary group of M and $\overline{co}^{\mathrm{s.o}}$ denotes the closure of the convex hull in the strong operator topology.

Proof. Let ξ_0 be an implementing vector of $\varphi \circ E_D^M$. For a given $\varepsilon > 0$ and $\zeta \in \mathcal{H}$ we can choose a non-zero $y' \in M'$ satisfying

$$\|y'\xi_0 - \zeta\|_{\mathcal{H}} < \frac{\varepsilon}{4\|x^\circ\|}.$$

We then choose $y^\circ \in M$ (which depends only on ε and ζ) such that y° is a finite linear combination of alternating words in $\mathrm{Ker}E_D^A, \mathrm{Ker}E_D^B$, and

$$\|(x^\circ - y^\circ)\xi_0\|_{\mathcal{H}} < \frac{\varepsilon}{4\|y'\|}.$$

We set $u_{k,j} = (ba)^k cac(ab)^j$ (belonging to $G(a,b,c)$). By D. Avitzour-K. McClanahan's result ([**22**, Lemma 3.8.], [**2**, Proposition 3.1.]) there exist $n_0, j_0 \in \mathbb{N}$ (which depend only on y° and ε) such that

$$\left\| \frac{1}{n} \sum_{k=1}^{n} u_{k,j} y^\circ u_{k,j}^* \right\| < \frac{\varepsilon}{4\|y'\|}$$

as long as $n \geq n_0, j \geq j_0$.

By Theorem 2.6 $G(a,b,c)$ sits in $M_{\varphi \circ E_D^M}$, and we estimate

$$\begin{aligned}
&\left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} x^\circ u_{k,j}^* \right) \xi_0 \right\|_{\mathcal{H}} \\
&\leq \left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} (x^\circ - y^\circ) u_{k,j}^* \right) \xi_0 \right\|_{\mathcal{H}} + \left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} y^\circ u_{k,j}^* \right) \xi_0 \right\|_{\mathcal{H}} \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \|u_{k,j}(x^\circ - y^\circ) u_{k,j}^* \xi_0\|_{\mathcal{H}} + \left\| \frac{1}{n} \sum_{k=1}^{n} u_{k,j} y^\circ u_{k,j}^* \right\| \|\xi_0\|_{\mathcal{H}} \\
&\leq \frac{1}{n} \sum_{k=1}^{n} \|(x^\circ - y^\circ)\xi_0\|_{\mathcal{H}} + \left\| \frac{1}{n} \sum_{k=1}^{n} u_{k,j} y^\circ u_{k,j}^* \right\| \\
&< \frac{\varepsilon}{4\|y'\|} + \frac{\varepsilon}{4\|y'\|} = \frac{\varepsilon}{2\|y'\|},
\end{aligned}$$

and hence we have

$$\left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} x^\circ u_{k,j}^* \right) \zeta \right\|_{\mathcal{H}}$$

$$\begin{aligned}
&\le \left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} x^{\circ} u_{k,j}^{*} \right) (\zeta - y'\xi_0) \right\|_{\mathcal{H}} + \left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} x^{\circ} u_{k,j}^{*} \right) y'\xi_0 \right\|_{\mathcal{H}} \\
&\le \|x^{\circ}\| \, \|\zeta - y'\xi_0\|_{\mathcal{H}} + \|y'\| \left\| \left(\frac{1}{n} \sum_{k=1}^{n} u_{k,j} x^{\circ} u_{k,j}^{*} \right) \xi_0 \right\|_{\mathcal{H}} \\
&< \|x^{\circ}\| \frac{\varepsilon}{2\|x^{\circ}\|} + \|y'\| \frac{\varepsilon}{2\|y'\|} = \varepsilon
\end{aligned}$$

for $n \ge n_0, j \ge j_0$. Hence we are done. □

Let a, b, c be as in Condition I-A. For $x \in \{a, b, c\}' \cap M$ we set $x^{\circ} = x - E_D^M(x)$. By (A.1) and [**22**, Proposition 3.5. (3)] we have

$$uE_D^M(x)u^* = E_D^M(uxu^*) = E_D^M(x)$$

for every $u \in G(a, b, c)$, and hence x° belongs to $G(a, b, c)'$.

On the other hand, by Lemma I-B we have

$$0 \in \overline{co}^{\mathrm{s.o}}\{ux^{\circ}u^* : u \in G(a, b, c)\}.$$

Therefore, we get $x^{\circ} = 0$, i.e., $x = E_D^M(x) \in D$.

Consequently, we get the following proposition:

Proposition I-C. *Under Condition I-A we have $\{a, b, c\}' \cap M \subseteq D$.*

The special case $D = \mathbf{C}1$ of the result can be found in [**3**, Corollary 4].

In the previous sections we have not touched the type I case. Suppose that A, B are factors and D is a common Cartan subalgebra, and further that either A or B is of type I_n with $n = 2, \dots, \infty$. Then $A \supseteq D \subseteq B$ is isomorphic to

$$\begin{cases} M_n(\mathbf{C}) \supseteq \mathcal{D}_n \subseteq M_n(\mathbf{C}), & \text{if } n \text{ is finite}, \\ B(\ell^2(\mathbb{N})) \supseteq \ell^{\infty}(\mathbb{N}) \subseteq B(\ell^2(\mathbb{N})), & \text{if } n \text{ is infinite}, \end{cases}$$

where $\mathcal{D}_n$ is the diagonals in $M_n(\mathbf{C})$.

First, suppose $3 \le n < \infty$. Condition I-A can be easily checked (use $M_n(\mathbf{C}) = \mathcal{D}_n \rtimes \mathbb{Z}_n$ for example) so that M is a factor thanks to Proposition I-C. (Note that this factoriality fact also follows from the classical argument by using I.C.C.-property.) Furthermore, by using Theorem 2.6 we easily see that M has a faithful normal normalized trace. Therefore, M is a type II_1 factor.

Next, suppose $n = \infty$. Then, by the above we may assume that $A = B = B(\ell^2(\mathbb{N}))$ and $D = \ell^{\infty}(\mathbb{N})$. We choose mutually orthogonal and equivalent (in $A = B$) three projections p_1, p_2, p_3 in D with $p_1 + p_2 + p_3 = 1$, and hence

$$(A \supseteq D) \;\cong\; (p_1 A p_1 \otimes M_3(\mathbf{C}) \supseteq D p_1 \otimes \mathcal{D}_3),$$

where $\mathcal{D}_3$ is the diagonals in $M_3(\mathbf{C})$. Condition I-A is clear by the presence of $M_3(\mathbf{C})$ so that M ($\cong A *_D A$) is a factor by Proposition I-C. Let φ be the faithful normal semi-finite weight on D constructed from the counting measure on $\mathbb{N}$, and then the weight $\varphi \circ E_D^M$ is tracial thanks to Theorem 2.6. Furthermore M contains the von Neumann subalgebra $N = (\mathbf{C}1 \otimes M_3(\mathbf{C})) *_{(\mathbf{C}1\otimes\mathcal{D}_3)} (\mathbf{C}1 \otimes M_3(\mathbf{C}))$. Let ψ be the tensor product state of a faithful normal state on Dp_1 and the restriction of the normalized trace on $M_3(\mathbf{C})$ to $\mathcal{D}_3$. Then it is easy to see that N is the range of the faithful normal conditional expectation conditioned by $\psi \circ E_D^M$. Thus M is not of type I, and then M is a type II_∞ factor.

Remark I-D.

(1) In [**10**] K. Dykema gave deep analysis on amalgamated free products of multi-matrix algebras. Indeed, the above factoriality fact for $3 \le n < \infty$ was obtained there.

(2) When $n = 2$, the amalgamated free product von Neumann algebra M is not a factor. (See [**10**, p. 1575].)

(3) A more natural and detailed analysis for amalgamated free products of type I von Neumann algebras over their common Cartan subalgebras from the view-point of groupoids will be given by H. Kosaki ([**19**]).

Appendix II. Modular Theory for amalgamated free products.

Keeping the assumptions and notations in §2, we here develop modular theory for amalgamated free products. More precisely, we will compute the modular operator $\Delta_{\varphi\circ E}$ ($= \Delta_{\xi_0}$) and modular conjugation J^M ($= J_{\xi_0}$), and as a consequence the commutation theorem for amalgamated free products will be obtained.

Let Δ_φ and $\Delta_{\varphi\circ E_s}$ be the modular operators associated with φ and $\varphi \circ E_s$ respectively. Notice that the following matrix notation is valid:

$$\Delta_{\varphi\circ E_s} = \begin{bmatrix} \Delta_\varphi & \\ & \Delta_s^\circ \end{bmatrix} \quad \left(\text{on} \quad \begin{bmatrix} L^2(N) \\ \mathcal{H}_s^\circ \end{bmatrix} \right),$$

where the restriction Δ_s° of $\Delta_{\varphi\circ E_s}$ to $\mathcal{H}_s^\circ$ makes sense. (See [**35**, p. 317, Equation (9)].)

For non-zero integers ℓ, m we define the operator $\Delta_0^{\ell/m}$ on $\mathcal{H}$ by

$$\mathcal{D}\left(\Delta_0^{\ell/m}\right) = \mathfrak{A},$$

$$\Delta_0^{\ell/m} = \Delta_\varphi^{\ell/m} + \sum_{n\ge1} \left(\sum_{s_1\ne\cdots\ne s_n} (\Delta_s^\circ)^{\ell/m} \odot_\varphi \cdots \odot_\varphi (\Delta_s^\circ)^{\ell/m} \right),$$

where $\odot_\varphi$ is understood in the natural sense.

We also define the (conjugate-linear) operator J_0 by

$$\mathcal{D}(J_0) = \mathfrak{A},$$
$$J_0\zeta = J_N\zeta,$$
$$J_0(\zeta_1 \otimes_\varphi \cdots \otimes_\varphi \zeta_n) = (J_{s_n}\zeta_n) \otimes_\varphi \cdots \otimes_\varphi (J_{s_1}\zeta_1)$$

for $\zeta \in L^2(N)$ and $\zeta_1 \otimes_\varphi \cdots \otimes_\varphi \zeta_n \in (\mathcal{H}_{s_1} \otimes_\varphi \cdots \otimes_\varphi \mathcal{H}_{s_n}) \cap \mathfrak{A}$.

Lemma II-A. *The operator Δ_0 is essentially self-adjoint and its closure is a positive self-adjoint operator. Furthermore, we have $\overline{\Delta_0^{1/2}} = (\overline{\Delta_0})^{1/2}$.*

Proof. For non-zero integers ℓ, m and $x_i, y_i \in \mathcal{N}_{s_i}^\circ$ with $s_1 \neq \cdots \neq s_n$ we compute

$$\begin{aligned}
&(((\Delta_{s_1}^\circ)^{\ell/m}(x_1\xi_0)) \otimes_\varphi \cdots \otimes_\varphi ((\Delta_{s_n}^\circ)^{\ell/m}(x_n\xi_0))|(y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0))_\mathcal{H}\\
&= ((\sigma_{-i\ell/m}^{\varphi\circ E_{s_1}}(x_1)\xi_0) \otimes_\varphi \cdots \otimes_\varphi (\sigma_{-i\ell/m}^{\varphi\circ E_{s_n}}(x_n)\xi_0)|(y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0))_\mathcal{H}\\
&= (\lambda_{s_1}(\sigma_{-i\ell/m}^{\varphi\circ E_{s_1}}(x_1)) \cdots \lambda_{s_n}(\sigma_{-i\ell/m}^{\varphi\circ E_{s_n}}(x_n))\xi_0|\lambda_{s_1}(y_1) \cdots \lambda_{s_n}(y_n)\xi_0)_\mathcal{H}\\
&= \varphi \circ E_{s_n}(y_n^* \cdots E_{s_1}(y_1^*\sigma_{-i\ell/m}^{\varphi\circ E_{s_1}}(x_1)) \cdots \sigma_{-i\ell/m}^{\varphi\circ E_{s_n}}(x_n)).
\end{aligned}$$

For each $x \in \mathcal{N}_s$, $y \in \mathcal{N}$, and $z \in \mathbf{C}$ we have

$$\begin{aligned}
&\varphi \circ E_{s_n}(y_n^* \cdots E_{s_1}(y_1^*\sigma_{-i\ell/m}^{\varphi\circ E_{s_1}}(x_1)) \cdots \sigma_{-i\ell/m}^{\varphi\circ E_{s_n}}(x_n))\\
&= \varphi \circ E_{s_n}(\sigma_{i\ell/m}^{\varphi\circ E_{s_n}}(y_n^*) \cdots E_{s_1}(\sigma_{i\ell/m}^{\varphi\circ E_{s_1}}(y_1^*)x_1) \cdots x_n)\\
&= \varphi \circ E_{s_n}(\sigma_{-i\ell/m}^{\varphi\circ E_{s_n}}(y_n)^* \cdots E_{s_1}(\sigma_{-i\ell/m}^{\varphi\circ E_{s_1}}(y_1)^*x_1) \cdots x_n),
\end{aligned}$$

and hence we get

$$\begin{aligned}
&(\Delta_0^{\ell/m}((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0))|(y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0))_\mathcal{H}\\
&= (((\Delta_{s_1}^\circ)^{\ell/m}(x_1\xi_0)) \otimes_\varphi \cdots \otimes_\varphi ((\Delta_{s_n}^\circ)^{\ell/m}(x_n\xi_0))|(y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0))_\mathcal{H}\\
&= ((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0)|((\Delta_{s_1}^\circ)^{\ell/m}(y_1\xi_0)) \otimes_\varphi \cdots \otimes_\varphi ((\Delta_{s_n}^\circ)^{\ell/m}(y_n\xi_0)))_\mathcal{H}\\
&= ((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0)|\Delta_0^{\ell/m})(y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0))_\mathcal{H}.
\end{aligned}$$

Therefore, $\Delta_0^{\ell/m}$ is well-defined and symmetric.

Next, we choose elements $x_i \in \mathcal{N}_{s_1}^\circ$ of exponential type with $s_1 \neq \cdots \neq s_n$, i.e., there exist $\beta_i, \delta_i > 0$ such that $\|\sigma_z^{\varphi\circ E_{s_i}}(x_i)\| \leq \beta_i \exp(\delta_i|\mathrm{Im} z|)$ for every $z \in \mathbf{C}$. Then we estimate

$$\begin{aligned}
&\|(\Delta_0^{\ell/m})^k((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0))\|_\mathcal{H}\\
&= \varphi \circ E_{s_n}(\sigma_{-ik\ell/m}^{\varphi\circ E_{s_n}}(x_n)^* \cdots E_{s_1}(\sigma_{-ik\ell/m}^{\varphi\circ E_{s_1}}(x_1)^*\sigma_{-ik\ell/m}^{\varphi\circ E_{s_1}}(x_1)) \cdots\\
&\qquad\qquad \sigma_{-ik\ell/m}^{\varphi\circ E_{s_n}}(x_n))^{1/2}
\end{aligned}$$

$$\le \|\varphi\| \|\sigma^{\varphi\circ E_{s_1}}_{-ik\ell/m}(x_1)\| \cdots \|\sigma^{\varphi\circ E_{s_n}}_{-ik\ell/m}(x_n)\|$$
$$\le \|\varphi\| \beta_1 \exp(\delta_1 k\ell/m) \cdots \beta_n \exp(\delta_n k\ell/m)$$
$$\le \|\varphi\| (\beta_1 \cdots \beta_n) \exp\left(\frac{\ell}{m}(\delta_1 + \cdots \delta_n)\right)^k,$$

and hence

$$\sum_{k=0}^{\infty} \frac{1}{k!} \|(\Delta_0^{\ell/m})^k((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0))\|_{\mathcal{H}}$$
$$\le \sum_{k=0}^{\infty} \frac{1}{k!} \|\varphi\| (\beta_1 \cdots \beta_n) \exp\left(\frac{\ell}{m}(\delta_1 + \cdots \delta_n)\right)^k$$
$$\le \|\varphi\| (\beta_1 \cdots \beta_n) \exp\left(\exp\left(\frac{\ell}{m}(\delta_1 + \cdots \delta_n)\right)\right) \ (< +\infty).$$

The set of elements in $\mathcal{N}_s^\circ$ of exponential type is dense ([**15**, Lemma 4.2.]) so that $\Delta_0^{\ell/m}$ is essentially self-adjoint thanks to Nelson's analytic vector theorem (cf. [**30**, X.6]).

Since $\Delta_0 = (\Delta_0^{1/2})^2 \subseteq (\overline{\Delta_0^{1/2}})^2$, we have $\overline{\Delta_0} \subseteq (\overline{\Delta_0^{1/2}})^2$. Since the both sides are self-adjoint, we get $\overline{\Delta_0} = (\overline{\Delta_0^{1/2}})^2$ by taking the adjoints of the both sides. Thus $\overline{\Delta_0}$ is positive, and by the same argument $\overline{\Delta_0^{1/2}}$ is also positive. Therefore, we have $(\overline{\Delta_0})^{1/2} = \overline{\Delta_0^{1/2}}$ by the uniqueness of square root. □

Lemma II-B. *J_0 can be extended to a conjugate-linear involutive unitary.*

Proof. For $x_i, y_i \in \mathcal{N}_{s_i}$ with $s_1 \neq \cdots \neq s_n$ we compute

$$\begin{aligned}
&(J_0((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0)) | J_0((y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0)))_{\mathcal{H}} \\
&= ((J_{s_n}(x_n\xi_0)) \otimes_\varphi \cdots \otimes_\varphi (J_{s_1}(x_1\xi_0)) | (J_{s_n}(y_n\xi_0)) \otimes_\varphi \cdots \otimes_\varphi (J_{s_1}(y_1\xi_0)))_{\mathcal{H}} \\
&= \varphi \circ E_{s_1}(\sigma^{\varphi\circ E_{s_1}}_{i/2}(y_1) \cdots E_{s_n}(\sigma^{\varphi\circ E_{s_n}}_{i/2}(y_n)\sigma^{\varphi\circ E_{s_n}}_{-i/2}(x_n^*)) \cdots \sigma^{\varphi\circ E_{s_1}}_{-i/2}(x_1^*)) \\
&= \varphi \circ E_{s_1}(y_1 \cdots E_{s_n}(y_n \sigma^{\varphi\circ E_{s_n}}_{-i}(x_n^*)) \cdots \sigma^{\varphi\circ E_{s_1}}_{-i}(x_1^*)) \\
&= \varphi \circ E_{s_1}(x_1^* y_1 E_{s_2}(\cdots E_{s_n}(y_n \sigma^{\varphi\circ E_{s_n}}_{-i}(x_n^*)) \cdots)) \\
&= \varphi(E_{s_1}(x_1^* y_1) E_{s_2}(\cdots E_{s_n}(y_n \sigma^{\varphi\circ E_{s_n}}_{-i}(x_n^*)) \cdots) \\
&= \varphi \circ E_{s_2}(E_{s_1}(x_1^* y_1) \cdots E_{s_n}(y_n \sigma^{\varphi\circ E_{s_n}}_{-i}(x_n^*)) \cdots).
\end{aligned}$$

Here the second equality comes from the equation: $J_s(x\xi_0) = \sigma^{\varphi\circ E_s}_{-i/2}(x^*)\xi_0$ for $x \in \mathcal{N}_s$ and the fourth equality comes from the K.M.S. condition. Repeated use of the K.M.S condition implies

$$(J_0((x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0)) | J_0((y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0)))_{\mathcal{H}}$$

$$= \varphi \circ E_{s_n}(x_n^* \cdots E_{s_1}(x_1^* y_1) \cdots y_n)$$
$$= ((y_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (y_n\xi_0)|(x_1\xi_0) \otimes_\varphi \cdots \otimes_\varphi (x_n\xi_0))_{\mathcal{H}}$$

so that J_0 is isometric. And $J_0\mathfrak{A} = \mathfrak{A}$ and $J_0^2 = 1_{\mathfrak{A}}$ are obvious, and hence J_0 can be extended to a conjugate-linear involutive unitary. □

Set $\Delta = \overline{\Delta_0}$ and $J = \overline{J_0}$. By direct computations we easily check that the S-operator $S_{\xi_0}^M$ satisfies $S_{\xi_0}^M|_{\mathfrak{A}} = J_0\Delta_0^{1/2}$. It is clear that $\mathfrak{A}$ forms a common core for $S_{\xi_0}^M$ and Δ so that we have $S_{\xi_0}^M = J\Delta^{1/2}$. Thus we get $\Delta = \Delta_{\xi_0}$, $J = J_{\xi_0}$ by the uniqueness of the polar decomposition of $S_{\xi_0}^M$. Therefore, we get the following proposition:

Proposition II-C. *The modular operator $\Delta_{\varphi\circ E_D^M} = \Delta_{\xi_0}$ and the modular conjugation $J^M = J_{\xi_0}$ are Δ and J respectively.*

By simple computations we have $J\lambda_s(x)^*J = \rho_s(x)$ for $x \in N_s$ so that Tomita's theorem ([**34**]) implies the following commutation theorem:

Theorem II-D. *The commutant M' on $\mathcal{H}$ is given by $\left\{\bigcup_{s\in S} \rho_s(N_s)\right\}''$.*

Appendix III. (Added October 28, 1998.)

Here, we would like to mention some results obtained after the submission: Let M be the amalgamated free product of non-type I factors A, B over a common Cartan subalgebra D, and $\tilde{M}, \tilde{A}, \tilde{B}, \tilde{D}$ be the continuous cores as in §5. We have known that $\mathcal{Z}(\tilde{M}) = \mathcal{Z}(\tilde{A}) \cap \mathcal{Z}(\tilde{B}) \subseteq \tilde{D}$. Based on this together with a result of D. Voiculescu ([**43**]) we can show the following: If M is of type III, then almost every type II_∞ factor appearing in the central decomposition of $\tilde{M}$ is not of the form: $L(\mathbb{F}_r)\otimes B(\mathcal{H})$ for $r > 1$ with possibly $r = \infty$. This implies that if M is of type III_λ, $0 < \lambda < 1$, then the discrete core of M is also not of the above-mentioned form. The details with further investigation will be presented elsewhere.

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Received January 15, 1998 and revised November 6, 1998.

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