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We give several necessary and sufficient conditions for an AH algebra to have its ideals generated by their projections. Denote by $\mathcal{C}$ the class of AH algebras as above and in addition with slow dimension growth. We completely classify the algebras in $\mathcal{C}$ up to a shape equivalence by a K-theoretical invariant. For this, we show first, in particular, that any C^*-algebra in $\mathcal{C}$ is shape equivalent to an AH algebra with slow dimension growth and real rank zero (generalizing so a result of Elliott-Gong); then, we use a classification result of Dadarlat-Gong. We prove that any AH algebra in $\mathcal{C}$ has stable rank one (i.e., in the unital case, that the set of the invertible elements is dense in the algebra), generalizing results of Blackadar-Dadarlat-Rørdam and of Elliott-Gong. Other nonstable K-theoretical results for C^*-algebras in $\mathcal{C}$ are also proved, generalizing results of Dadarlat-Némethi, Martin-Pasnicu and Blackadar.

1. Introduction.

An AH algebra is an amenable C^*-algebra of the form $A = \varinjlim(A_n, \Phi_{n,m})$, with $A_n = \oplus_{i=1}^{k_n} P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i}$, where $X_{n,i}$ are finite, connected CW complexes, k_n, $[n, i]$ are positive integers and $P_{n,i} \in C(X_{n,i}M_{[n,i]})$ are projections. The problem of finding suitable invariants for AH algebras was raised by Effros [**Eff**] (after the results obtained by Glimm [**Gl**], Dixmier [**Di**], Bratteli [**Br**] and especially after Elliott's classification of the AF algebras by their ordered K_0 group [**Ell 1**]) and is now a part of Elliott's project of classification of the separable, amenable C^*-algebras by invariants including K-theory (for details see e.g. [**Ell 6**]).

In this paper we shall deal with AH algebras B whose ideals are generated (as ideals) by their projections (here by an ideal we mean a closed, two-sided ideal). The study of these amenable C^*-algebras was suggested by Elliott and they provide a common generalization of two very important classes of algebras: The real rank zero AH algebras (i.e., with the property that any self-adjoint element can be approximated as close as we want by self-adjoint elements with finite spectrum [**BP**]) and the simple AH algebras.

Recently, successful classification results have been obtained for certain classes of AH algebras. On one hand is the remarkable classification of the real rank zero AH algebras with slow dimension growth (see [**Ell 3**], [**EGLP 1**, **EGLP 2**], [**EG 1**, **EG 2**], [**Da 2**], [**Go 1**, **Go 2**], [**DL 1**], [**DL 2**], [**Ei**] and [**DG**], the last paper containing the general classification result). On the other hand is the very remarkable classification obtained in [**Go 3**] and [**EGL**] for the simple, unital AH algebras with very slow dimension growth (also, see [**Ell 4**, **Ell 5**] and [**L**]). It would be important to generalize and unify these classification results for the above two classes of AH algebras. Our paper can be seen as the first step in this attempt. We believe that the results of this paper, combined with the techniques from [**Go 3**] and [**EGL**], will play an essential role in the classification of the AH algebras whose ideals are generated by projections and which have slow dimension growth. A particular class of AI algebras (i.e., inductive limits of sequences of C^*-algebras of the form $C([0,1],F)$, with F a finite dimension C^*-algebra) whose ideals are generated by projections has been classified in [**St**].

We have been able to give several necessary and sufficient conditions for an AH algebra to have its ideals generated by their projections (see Theorem 3.1 and Remark 3.2 b)). Let $\mathcal{C}$ be the class of AH algebras whose ideals are generated by their projections and with slow dimension growth. Also, we classified the C^*-algebras in $\mathcal{C}$ up to a shape equivalence by a K-theoretical invariant (see Theorem 2.15). To do it, we generalized first - relying heavily on part of Theorem 3.1, [**DN**] and [**EG 2**] - a result of Elliott-Gong ([**EG 2**]) proving, in particular, that any AH algebra in $\mathcal{C}$ is shape equivalent to an AH algebra with slow dimension growth and real rank zero (see Theorem 2.6); then, we used the classification result of the AH algebras with real rank zero and slow dimension growth obtained by Dadarlat-Gong in [**DG**].

We have been able also to obtain nonstable K-theoretical results for the algebras in $\mathcal{C}$. One of the main results of this paper says that any C^*-algebra in $\mathcal{C}$ has stable rank one (that means, in the unital case, that the set of the invertible elements is dense in the algebra [**R**]) (see Theorem 4.1). This Theorem extends a result of Blackadar-Dadarlat-Rørdam ([**BDR**]) (and hence, also one in [**DNNP**]) in the case when the C^*-algebra is simple and a result of Elliott-Gong in [**EG 2**] in the case when the AH algebra has real rank zero. It is important to mention that Theorem 4.1 contains more information than one might think at a first sight; e.g., it allows us to compute the real rank for the AH algebras in $\mathcal{C}$. Indeed, this theorem together with the inequality [**BP**], Proposition 1.2 imply that the real rank for a C^*-algebra in C could be 0 or 1. But the real rank zero case was characterized by us in [**P 2**]. Note that if we drop the slow dimension growth condition, then Theorem 4.1 is no more true. Indeed, a recent result of Villadsen [**V**] shows that for any positive integer n there is a simple AH algebra of stable rank n and which does not have slow dimension growth. Note also that the C^*-algebras A with stable rank one have many "nice" and important properties (e.g., cancellation and, in the unital case, the canonical map $U(A)/U_0(A) \to K_1(A)$ is a group isomorphism [**R**]).

We have proved also that for any AH algebra A in $\mathcal{C}, K_0(A)$ is weakly unperforated in the sense of Elliott [**Ell 2**] and, in the unital case, that if two projections p and q in A satisfy $\sigma(p) < \sigma(q)$ for any tracial state σ of A, then p is Murray-von Neumann equivalent to a proper subprojection of q (see Theorem 5.1). This theorem generalizes results of Dadarlat-Némethi [**DN**], Martin-Pasnicu [**MP**] and Blackadar [**Bl 2**].

We are grateful to Guihua Gong for useful discussions concerning the subject of our investigation.

2. The shape equivalence type.

In this section we shall describe the shape equivalence type for a large class of AH algebras by a K-theoretical invariant (see Theorem 2.15). The proof of Theorem 2.15 combines Theorem 2.6 (which is based on part of Theorem 3.1, [**DN**] and [**EG 2**]) with the classification result of the AH algebras with real rand zero and slow dimension growth obtained by Dadarlat-Gong in [**DG**].

We begin with some notations and definitions.

Notation 2.1. Let $\mathcal{H}$ be the set of homogeneus C^*-algebras of the form $\oplus_{i=1}^{k} P_i C(X_i, M_{n_i}) P_i$ where k and n_i are positive integers, X_i is a finite, connected CW complex and $P_i \in C(X_i, M_{n_i})$ is a projection.

Definition 2.2. An AH algebra is a C^*-algebra $A = \varinjlim(A_n, \Phi_{n,m})$, where $(A_n, \Phi_{n,m})$ is an inductive system of C^*-algebras $A_n \in \mathcal{H}$ and homomorphisms.

Definition 2.3 ([**Go 1**])**.** We recall the following definition from [**Go 1**]. An inductive system $(A_n, \Phi_{n,m})$ of C^*-algebras and homomorphisms will be said to have *slow dimension growth* if $A_n = \oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}, X_{n,i}$ are finite, connected CW complexes, $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ are projections, $[n,i]$ and k_n are positive integers and for any n, there is a positive integer M such that:

$$\lim_{m\to\infty} \min_{\substack{\dim X_{m,j} > M \\ \operatorname{rank}(\Phi_{n,m}^{i,j}(P_{n,i})) \neq 0}} \left\{ \frac{\operatorname{rank}(\Phi_{n,m}^{i,j}(P_{n,i}))}{\dim X_{m,j} + 1} \right\} = +\infty$$

(we use the convention that the minimum of the empty set is $+\infty$).

An AH algebra will be said to have *slow dimension growth* if there is an inductive system $(A_n, \Phi_{n,m})$ with slow dimension growth such that $A = \varinjlim(A_n, \Phi_{n,m})$.

Note that the above definition gives a more general slow dimension growth condition than that defined in [**EG 2**] and one of its advantages is that any system $(\oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}, \Phi_{n,m})$ with $\sup\limits_{n,i} \dim X_{n,i} < +\infty$ has slow dimension growth (take for any $n, M = \sup\limits_{n,i} \dim X_{n,i}$) even though the limit is not simple.

The first notion of slow dimension growth for some simple AH algebras was introduced in [**BDR**]. For other extensions in the non-simple case see e.g. [**Goo**], [**MP**].

Notation 2.4. Let $\mathcal{S}$ be the category of separable C^*-algebras and homomorphisms.

Definition 2.5 ([**EK**], [**Bl 1**])**.** Two inductive systems $(A_n, \Phi_{n,m})$ and $(B_n, \Psi_{n,m})$ in $\mathcal{S}$ are said to be *shape equivalent* if there are positive integers $k_i < \ell_i < k_{i+1}$ $(i \geq 1)$ and homomorphisms $\xi_i : A_{k_i} \to B_{\ell_i}$ and $\eta_i : B_{\ell_i} \to A_{k_{i+1}}$ such that:

$$\eta_i \circ \xi_i \sim_h \Phi_{k_i, k_{i+1}} : A_{k_i} \to A_{k_{i+1}}$$

and

$$\xi_{i+1} \circ \eta_i \sim_h \Psi_{\ell_i, \ell_{i+1}} : B_{\ell_i} \to B_{\ell_{i+1}}$$

where $\sim_h$ means homotopy between homomorphisms.

The following theorem generalizes a result of Elliott and Gong in [**EG 2**].

Theorem 2.6. *Let $(A_n, \Phi_{n,m})$ be an inductive system in $\mathcal{S}$ with $A_n \in \mathcal{H}$, $n \geq 1$. Suppose that $(A_n, \Phi_{n,m})$ has slow dimension growth and that any ideal of its limit $A = \varinjlim (A_n, \Phi_{n,m})$ is generated by its projections.*

Then $(A_n, \Phi_{n,m})$ is shape equivalent to some inductive system $(B_n, \Psi_{n,m})$ in $\mathcal{S}$ with $B_n \in \mathcal{H}, n \geq 1$, such that $(B_n, \Psi_{n,m})$ has slow dimension growth and its limit $B = \varinjlim (B_n, \Psi_{n,m})$ has real rank zero.

Remarks 2.7. a) Observe that in the above Theorem 2.6 the AH algebra B can not be always simple. Indeed, let A be an AH algebra with real rank zero, stable rank one [**R**] which is not simple and which has slow dimension growth. Suppose that the AH algebra B given by Theorem 2.6 is in addition simple. The shape equivalence induces an isomorphism between $(K_0(A), K_0(A)_+)$ and $(K_0(B), K_0(B)_+)$. But since B is simple it follows that $(K_0(B), K_0(B)_+)$ is simple and hence $(K_0(A), K_0(A)_+)$ is also simple. But since A has real rank zero and stable rank one, by a general result in [**GL**] it follows that A is simple, a contradiction.

b) Observe that by [**Da 2**], [**Go 1**] and [**EG 2**], Theorem 2.2, the AH algebra B in the above Theorem is unique up to an isomorphism.

In the proof of the above theorem we shall need the following six lemmas. First, let us recall shortly some definitions. For a C^*-algebra $A = \oplus_{i=1}^k P_i C(X_i, M_{n_i}) P_i$, where $P_i \in C(X_i, M_{n_i})$ are nonzero projections and the X_i's are compact, connected spaces, define $SP(A) = \sqcup X_i$. If B is another C^*-algebra of the same type as A, $\Phi : A \to B$ is a homomorphism and $y \in SP(B)$, we want to define $SP(\Phi)_y$, the spectrum of Φ in the point y. The definition reduces practically (in a natural manner) (see e.g. [**EG 2**]) to the case when we have $\Phi : C(X) \to PC(Y, M_n)P$ and $y \in Y$. Since the map $C(X) \ni f \mapsto \Phi(f)(y) \in M_n$ is a finite dimensional $*$-representation, it follows that it is a direct sum of some irreducible $*$-representations of $C(X)$-given by the evaluation maps in $x_1(y), x_2(y), \cdots, x_m(y) \in X$- and of a zero $*$-representation of $C(X)$. Then, by definition $SP(\Phi)_y = \{x_1(y), x_2(y), \ldots, x_m(y)\}$, where we count multiplicity.

The first lemma is inspired by [**DNNP**], Proposition 2.1 (see also [**St**]).

Lemma 2.8. *Let* $A = \lim\limits_{\to}(A_n, \Phi_{n,m})$, *with*

$$A_n = \oplus_{i=1}^{k_n} A_n^i,\, A_n^i = P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i},$$

where $X_{n,i}$ *are finite, connected* CW *complexes and* $P_{n,i} \in C(X_{n,i}, M_{[n,i]})$ *are nonzero projections. Suppose that any ideal of* A *is generated by projections.*

Then, for any fixed n *and fixed* $F = \overline{F} \subset U = \overset{\circ}{U} \subset \sqcup_{i=1}^{k_n} X_{n,i} = SP(A_n)$ *there is* $m_o > n$ *such that for any* $m \geq m_o$ *any partial map* $\Phi_{n,m}^j : A_n \to A_m^j$ *satisfies either:*

$$SP(\Phi_{n,m}^j)_y \cap F = \phi \quad \textit{for all} \;\; y \in X_{m,j}$$

or

$$SP(\Phi_{n,m}^j)_y \cap U \neq \phi \quad \textit{for all} \;\; y \in X_{m,j}.$$

Proof. We may suppose that $n = 1$ and that $F \subset U \subset X_{1,1}$. Define $f \in C(X_{1,1})$ such that $f(x) = 1$ if $x \in F$ and $f(x) = 0$ if $x \notin U$. Consider $\tilde{f} = \oplus_{i=1}^{k_n} f_i \in \oplus_{i=1}^{k_n} A_n^i$ where $f_1 = f \cdot P_{1,1}$ and $f_i = 0$ if $i \neq 1$.

Define $G_k = \{y \in SP(A_k) : SP(\Phi_{1,k})_y \cap U = \phi\}, k \in \mathbb{N}$. But then each G_k is a closed subset of $SP(A_k)$ (see e.g. [**Da 1**], [**P 1**]). As in [**DNNP**] define $J_k = \{g \in A_k : g\,|_{G_k} = 0\},\;\; k \geq 1$. Obviously J_k is an ideal in A_k and let J be the ideal of A defined by $J_k,\;\; k \geq 1$. Define by G_k^j the component of G_k in $X_{k,j}$. It is not difficult to see that if we define $p_m = \oplus_{j=1}^{k_m} p_m^j \in \oplus_{j=1}^{k_m} A_m^j = A_m,\;\; m \geq 1$ by:

$$p_m^j = \begin{cases} P_{m,j} & \text{if } \; G_m^j = \phi \\ 0 & \text{if } \; G_m^j \neq \phi \end{cases}$$

then each p_m is a projection in $A_m, p_m \in J_m$ and:

$$p_m \cdot aeb = aeb \cdot p_m = aeb$$

for any projection e of J_m and any $a, b \in A_m$ (each $X_{m,j}$ is a connected space). Since, by hypothesis, J is generated by its projections, the above equalities imply that $(\Phi_{m,\infty}(p_m))_{m=1}^{\infty}$ is an approximate unit for J. Here $\Phi_{m,\infty} : A_m \to A = \varinjlim(A_n, \Phi_{n,k})$ is the canonical homomorphism. Since obviously $\widetilde{f} \in J_1 (f(x) = 0$ for any $x \in X_{1,1} \setminus U)$, these facts imply that there is $m_0 > n$ such that:

$$(1) \qquad \|p_m \Phi_{1,m}(\widetilde{f}) p_m - \Phi_{1,m}(\widetilde{f})\| < 1, \quad m \geq m_0.$$

Fix $m \geq m_0$. Then:

a) *Let j be such that $p_m^j = 0$.*

In this case, (1) implies that:

$$\|p_m^j \Phi_{1,m}^j(\widetilde{f}) p_m^j - \Phi_{1,m}^j(\widetilde{f})\| = \|\Phi_{1,m}^j(\widetilde{f})\| = \|\Phi_{1,m}^{1,j}(f_1)\| < 1.$$

But since $\|\Phi_{1,m}^{1,j}(f_1)\| = \sup_{y \in X_{m,j}} \|f_1 \mid_{SP(\Phi_{1,m}^{1,j})_y} \| = \sup_{y \in X_{m,j}} \|f \mid_{SP(\Phi_{1,m}^{1,j})_y} \|$, one can deduce that

$$SP(\Phi_{1,m}^{1,j})_y \cap F = \phi \text{ for all } y \in X_{m,j}$$

$(f(x) = 1$ if $x \in F)$.

b) *Let j be such that $p_m^j = P_m^j$.*

By the definition of p_m it follows that $G_m^j = \phi$, which, by the definition of G_m, is equivalent to:

$$SP(\Phi_{1,m}^j)_y \cap U \neq \phi \text{ for all } y \in X_m^j.$$

□

Now we shall prove the following result:

Lemma 2.9. *Let $A = \varinjlim(A_n, \Phi_{n,m})$ be as in Lemma 2.8. Then, for any fixed n, i and $\delta > 0$ there is $m_0 > n$ such that the following is true:*

For any $F = \overline{F} \subset X_{n,i}$ and any $m \geq m_0$ we have that any partial map $\Phi_{n,m}^{i,j}$ satisfies either

$$SP(\Phi_{n,m}^{i,j})_y \cap F = \phi \text{ for all } y \in X_{m,j}$$

or

$$SP(\Phi_{n,m}^{i,j})_y \cap B_\delta(F) \neq \phi \text{ for all } y \in X_{m,j}.$$

(Here we use the standard notation $B_\delta(M) = \{x \in X_{n,i} : \text{dist}(x, M) < \delta\}$ for any subset M of $X_{n,i}$. Note that the result applies to any metric defining the topology of $X_{n,i}$ (of course, m_0 depends on the chosen metric).)

Proof. The proof will use the "test functions" introduced in [**Su**] and, of course, the above lemma. Let us denote $X = X_{n,i}$ and $H = \{\chi_{T,\frac{4}{\delta}} : T$ is a closed subset of $X\}$. We recall that for any closed subset ω of $X$and any $\epsilon > 0$ the continuous function $\chi_{\omega,\epsilon} : X \to \mathbb{R}$ is defined in [**Su**] by:

$$\chi_{\omega,\epsilon}(x) = \begin{cases} 1 & \text{if } x \in \omega \\ 1 - \epsilon \cdot \mathrm{dist}(x,\omega) & \text{if } \mathrm{dist}(x,\omega) \le \frac{1}{\epsilon} \\ 0 & \text{if } \mathrm{dist}(x,\omega) \ge \frac{1}{\epsilon}. \end{cases}$$

Notice that $\mathrm{supp}\,(\chi_{\omega,\epsilon}) = \overline{B_{\frac{1}{\epsilon}}(\omega)}$.

As observed in the proof of [**L**], Lemma 7.15, H is an equicontinuous family of functions and hence there is a finite subset $H_1 \subset H$ such that $\mathrm{dist}\,(h, H_1) < \frac{1}{8}$ for any $h \in H$ (see also [**Su**] for a direct proof). Using this, we shall prove now that:

$$(*) \qquad \begin{cases} \text{For any closed subset } C \text{ of } X \text{ there} \\ \text{is } f \in H_1 \text{ such that } f|_C > \frac{3}{4} \text{ and} \\ f|_{X\setminus B_{\frac{3\delta}{4}}(C)} < \frac{1}{8}. \end{cases}$$

Let $D = \overline{B_{\frac{\delta}{2}}(C)}$. By the above observation, for $\chi_{D,\frac{4}{\delta}}$ there is $f \in H_1$, say $f = \chi_{D_1,\frac{4}{\delta}}$ for some $D_1 = \overline{D_1} \subset X$ such that:

$$\text{(i)} \qquad |\chi_{D,\frac{4}{\delta}}(x) - \chi_{D_1,\frac{4}{\delta}}(x)| < \frac{1}{8}, x \in X.$$

Let $x \in C \subset D$. Then $(\chi_{D,\frac{4}{\delta}})(x) = 1$ and hence (i) implies:

$$|1 - \chi_{D_1,\frac{4}{\delta}}(x)| < \frac{1}{8} => f(x) = \chi_{D_1,\frac{4}{\delta}}(x) > \frac{3}{4}.$$

Now let $t \in X \setminus B_{\frac{3\delta}{4}}(C) \subset X \setminus B_{\frac{\delta}{4}}(D)$. Since $\chi_{D,\frac{4}{\delta}}(t) = 0$, (i) implies that:

$$f(t) < \frac{1}{8}.$$

Hence the above statement $(*)$ is proved.

Now, for any $h \in H_1$ let us introduce the notations:

$$F_h = \left\{x \in X : h(x) \ge \frac{3}{4}\right\}$$
$$U_h = \left\{x \in X : h(x) > \frac{1}{8}\right\}.$$

Obviously $F_h = \overline{F_h} \subset U_h = \overset{\circ}{U}_h \subset X$.

Applying Lemma 2.8 for the sets F_h and U_h corresponding to any h in the *finite* set H_1, we get a number $m_0 > n$ such that for all $m \geq m_0, j$ and $h \in H_1$, we have either

(ii) $$\begin{cases} SP(\Phi^{i,j}_{n,m})_y \cap F_h = \phi & \text{for all } \ y \in X_{m,j} \\ \quad \text{or} & \\ SP(\Phi^{i,j}_{n,m})_y \cap U_h \neq \phi & \text{for all } \ y \in X_{m,j}. \end{cases}$$

Fix a closed subset F of X. By $(*)$ there is $f \in H_1$ such that $f\mid_F > \frac{3}{4}$ and $f\mid_{X \setminus B_{\frac{3\delta}{4}}(F)} < \frac{1}{8}$.

Obviously:

(iii) $$F \subset F_f \subset U_f \subset B_\delta(F).$$

The conclusion follows now from (ii) and (iii). □

The following lemma is a generalization of [**EG 2**], Lemma 2.3.

Lemma 2.10. *Let $A = \varinjlim(A_n, \Phi_{n,m})$ be as in Lemma 2.8. Then, for any n, any finite subset $F^i_n \subset A^i_n \subset A_n$, any positive integer N and any $\epsilon > 0$ there is $m_0 > n$ such that any partial map $\Phi^{i,j}_{n,m}$ with $m \geq m_0$ satisfies either:*

a) rank $(\Phi^{i,j}_{n,m}(P_{n,i})) \geq N \cdot$ rank $(P_{n,i})$ *or*
b) $\Phi^{i,j}_{n,m}$ *is homotopic within the projection $\Phi^{i,j}_{n,m}(P_{n,i})$ to a homomorphism $\Psi^{i,j}_{n,m}$ with finite dimensional range and:*

$$\|\Phi^{i,j}_{n,m}(f) - \Psi^{i,j}_{n,m}(f)\| < \epsilon$$

for any $f \in F^i_n$.

Proof. Choose a $\delta > 0$ as in the proof of [**EG 2**], Lemma 2.3; that is such that any closed ball in $X_{n,i}$ of radius a is contractible for any $a \leq 2N\delta$ and such that whenever $x, x' \in X_{n,i}$ satisfy $d(x, x') \leq 2N\delta$ ($d(\cdot,\cdot)$ is the canonical metric on $X_{n,i}$) it follows that $\|f(x) - f(x')\| < \epsilon$ for all $f \in F^i_n$. Suppose that rank $(\Phi^{i,j}_{n,m}(P_{n,i})) < N \cdot$ rank $(P_{n,i})$.

Let $m_0 > n$ be the number obtained applying Lemma 2.9 for n, i and $\delta > 0$. We shall use the following standard notation: If $M \subset X$ and (X, d) is a metric space, then for any $\epsilon > 0, B_\epsilon(M) = \{x \in X : d(x, M) < \epsilon\}$. For any fixed $m \geq m_0$ and fixed $y_j \in X_{m,j}$ let $F = X_{n,i} \setminus B_\delta(SP(\Phi^{i,j}_{n,m})_{y_j})$. Obviously F is closed. By Lemma 2.9 it follows that either:

(i) $$SP(\Phi^{i,j}_{n,m})_y \cap F = \phi \quad \text{for all } \ y \in X_{m,j}$$

or

(ii) $$SP(\Phi^{i,j}_{n,m})_y \cap B_\delta(F) \neq \phi \quad \text{for all } \ y \in X_{m,j}.$$

But since for $y = y_j$ the relation from (ii) is false, it follows that we have (i), which implies that:

$$SP(\Phi^{i,j}_{n,m})_y \subset B_\delta(SP(\Phi^{i,j}_{n,m})_{y_j})$$

for all $y \in X_{m,j}$. Now the proof continues as in the proof of [**EG 2**], Lemma 2.3. □

The next lemma is a generalization of [**Go 1**], Lemma 2.23.

Lemma 2.11. *Suppose that A is the C^*-inductive limit of an inductive system*

$$\left(A_n = \oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}, \Phi_{n,m}\right)$$

with slow dimension growth. Assume that any ideal of A is generated by its projections.

Then, for any n, any finite subset $F_n^i \subset P_{n,i}C(X_{n,i}, M_{[n,i]})P_{n,i} \subset A_n$, any $\epsilon > 0$ and any integer N there is $m_0 > n$ such that each partial map $\Phi^{i,j}_{n,m}$ with $m \geq m_0$ satisfies either:

1) $\operatorname{rank}(\Phi^{i,j}_{n,m}(P_{n,i})) \geq N \cdot (\dim \quad X_{m,j} + 1)$ *or*
2) $\Phi^{i,j}_{n,m}$ *is homotopic within the projection $\Phi^{i,j}_{n,m}(P_{n,i})$ to a homomorphism $\Psi^{i,j}_{n,m}$ with finite dimensional range and:*

$$\|\Phi^{i,j}_{n,m}(f) - \Psi^{i,j}_{n,m}(f)\| < \epsilon$$

for any $f \in F_{n,i}$.

The proof of the above lemma follows easily from Lemma 2.10.

The next lemma is implicitely contained in the proof of [**EG 2**], Lemma 3.27:

Lemma 2.12 ([**EG 2**])**.** *Let $\Phi : PC(X, M_k)P \to QC(Y, M_\ell)Q$ be a unital homomorphism between homogeneous C^*-algebras. Let $\epsilon > 0$. Assume that X is a path connected space and that $\{x_1, x_2, \ldots, x_n\}$ is $\frac{\epsilon}{2}$-dense in X. Suppose that $\Psi : PC(X, M_k)P \to RC(Y, M_p)R$ is a unital homomorphism between homogenous C^*-algebras defined by:*

$$\Psi(f) = \begin{pmatrix} \Phi(f) & & & \\ & f(x_1) \otimes 1_{m_1} & & \\ & & \ddots & \\ & & & f(x_n) \otimes 1_{m_n} \end{pmatrix}$$

$f \in PC(X, M_k)P$, where $f(x) \in M_{\operatorname{rank}(P)} \cong P(x)M_kP(x)$ with $m_i \geq |SP(\Phi)_y|$, $y \in Y$ (we count with multiplicities). (Here 1_{m_i} is the identity in M_{m_i}.) Then:

$$SPV(\Psi) < \epsilon.$$

The following lemma is essentially contained in the proof of [**Go 1**], Lemma 4.6 (see also [**Da 2**], [**EG 2**]) and it is based on [**DN**].

Let $A = \oplus_{i=1}^{k} P_i M_{n_i}(C(X_i))P_i$, where X_i is a finite, connected CW complex and $P_i \in M_{n_i}(C(X_i))$ is a projection. Let $A' = \oplus_{i=1}^{k} M_{n_i}(C(X_i))$ and $\tilde{A}' = (\oplus_{i=1}^{k} M_{\ell n_i}(C(X_i))) \oplus (\oplus_{i=1}^{k} M_{n_i}) = (\oplus_{i=1}^{k} C^{'i}) \oplus (\oplus_{i=1}^{k} D^{'i})$, where ℓ is a positive integer. Let $\alpha' \oplus \beta' : A' \to \tilde{A}'$ be a unital homomorphism where $\alpha' : A' \to \oplus_{i=1}^{k} C^{'i}, \beta^{'} : A' \to \oplus_{i=1}^{k} D^{'i}$ are homomorphisms such that: $\alpha^{'i,j} = 0, \beta^{'i,j} = 0$ if $i \neq j, [\alpha^{'i,i}] = id \in kk(X_i, X_i)$ (see [**DN**]), $\alpha^{i,i}$ takes trivial projections to trivial projections and $\beta^{'i,i}(f) = f(x_0^i)$, where x_0^i is the base point of X_i. Let $(\alpha^{'} \oplus \beta^{'})(A) \subseteq \tilde{A} = (\oplus_{i=1}^{k} C^i) \oplus (\oplus_{i=1}^{k} D^i) = (\oplus_{i=1}^{k} Q_i M_{\ell n_i}(C(X_i))Q_i) \oplus (\oplus_{i=1}^{k} M_{\text{rank}\,(P_i)}) \subset \tilde{A}'$ where $Q_i = \alpha^{'i,i}(P_i)$ and define $\alpha \oplus \beta : A \to \tilde{A}$ by $(\alpha \oplus \beta)(a) = (\alpha' \oplus \beta')(a), \;\; a \in A$.

Lemma 2.13. *With the above notation, let $\Phi : A \to B$ be a homomorphism, where $B = \oplus_{j=1}^{\ell} B_j, B_j = R_j M_{m_j}(C(Y_j))R_j$, where Y_j is a finite, connected CW complex and $R_j \in M_{m_j}(C(Y_j))$ is a projection.*

Suppose that each partial map $\Phi^{i,j} : A_i \to B_j$ satisfies either

1) $\Phi^{i,j} : A_i \to \Phi^{i,j}(P_i)B_j\Phi^{i,j}(P_i)$ *is homotopic to a homomorphism $A_i \to \Phi^{i,j}(P_i)B_j\Phi^{i,j}(P_i)$ with finite dimensional range or*
2) $\text{rank}\,(\Phi^{i,j}(P_i)) \geq 3(\dim Y_j + 1)(\ell + 1) \cdot \text{rank}\,(P_i)$.

Then, there is a homomorphism $\gamma : \tilde{A} \to B$ such that $\gamma \circ (\alpha \oplus \beta)$ is homotopic to Φ and $\Phi(1_A) = (\gamma \circ (\alpha \oplus \beta))(1_A)$.

Proof. It follows from [**EG 2**], Lemma 2.13 and the proof of [**Go 1**], Lemma 4.6 (see also [**Da 2**]) and it is based on [**DN**]. □

Proof of Theorem 2.6. Suppose that A is the C^*-inductive limit of a system:

$$\left(A_n = \oplus_{i=1}^{k_n} P_{n,i} C(X_{n,i} M_{[n,i]}) P_{n,i}, \Phi_{n,m}\right)$$

with slow dimension growth. Combining Lemma 2.11 with Lemma 2.12 and Lemma 2.13 it follows that there is a sequence of positive integers $\ell_1 < \ell_2 < \cdots < \ell_n < \ell_{n+1} < ...$ and homomorphisms $\Psi_n : A_{\ell_n} \to A_{\ell_{n+1}}$ such that for any n:

a) $\Phi_{\ell_n, \ell_{n+1}}$ is homotopic to Ψ_n.
b) $SPV(\Psi_n) < 2^{-n}$.

Since the inductive system $(A_n, \Phi_{n,m})$ has slow dimension growth it follows obviously that $(A_{\ell_n}, \Phi_{\ell_n, \ell_{n+1}})$ has slow dimension growth and, by the above condition a), it follows easily that the system (A_{ℓ_n}, Ψ_n) has also slow dimension growth (note that the spaces $X_{n,i}$ are connected). If we denote $B = \varinjlim(A_{\ell_n}, \Psi_n)$ then b) implies as in the proofs of [**EG 2**], Corollary 2.25 and [**EG 2**], Remark 2.26 (see also the comment after [**Go 1**], Proposition 2.19) that the AH algebra B with slow dimension growth has real rank zero.

To conclude the proof observe that the systems $(A_n, \Phi_{n,m})$ and (A_{ℓ_n}, Ψ_n) are shape equivalent (see a)) (by unital homomorphisms in the unital case since if the $\Phi_{n,m}$'s are unital then the above Ψ_n's can be chosen unital). $\square$

Definition 2.14 ([**EK**], [**Bl 1**])**.** Two separable C^*-algebras A and B are said to be *shape equivalent* if there are two inductive systems $(A_n, \Phi_{n,m})$ and $(B_n, \Psi_{n,m})$ in $\mathcal{S}$ (the category of separable C^*-algebras and homomorphisms) which are shape equivalent (see Definition 2.5) and $A = \varinjlim(A_n, \Phi_{n,m}), B = \varinjlim(B_n, \Psi_{n,m})$.

If A and B are shape equivalent we shall write it in the following way:

$$Sh(A) = Sh(B).$$

The next theorem classifies completely, up to a shape equivalence, a large class of AH algebras with slow dimension growth. The invariant is the graded, preordered, scaled group $(\underline{K}(\cdot), \underline{K}(\cdot)^+, \sum(\cdot))$ together with the action of the Bockstein operations on $\underline{K}(\cdot)$ defined in ([**DG**], [**DL 1**]) (see also [**DL 2**], [**Ei**]), where:

$$\underline{K}(A) = K_*(A) \oplus \oplus_{p=2}^{\infty} K_*(A; \mathbb{Z}_p)$$

for any σ-unital C^*-algebra A.

Theorem 2.15. *Let A and A' be AH algebras with slow dimension growth and such that any of their ideals is generated by its projections.*

Then, the following are equivalent:

a) $Sh(A) = Sh(A')$.

b) *There is a graded isomorphism of ordered, scaled groups*

$$(\underline{K}(A), \underline{K}(A)^+, \textstyle\sum(A)) \cong (\underline{K}(A'), \underline{K}(A')^+, \textstyle\sum(A'))$$

which commutes with the Bockstein operations.

Proof. a) $\Rightarrow$ b). This implication follows easily from the definition of $(\underline{K}(\cdot), \underline{K}(\cdot)^+, \sum(\cdot))$ and from the definition of the Bockstein operations ([**DG**], [**DL 2**]) and it is true for more general C^*-algebras A and A'.

b) $\Rightarrow$ a). By Theorem 2.6, it follows that there are AH algebras B and B' with slow dimension growth and real rank zero such that:

(i) $$Sh(A) = Sh(B)$$

(ii) $$Sh(A') = Sh(B').$$

Using the hypothesis, the implication a) $\Rightarrow$ b) and [**DG**] it follows that:

$$B \cong B'$$

which obviously implies:

(iii) $$Sh(B) = Sh(B').$$

Results in [**Bl 1**] allow us to deduce from (i), (ii) and (iii) that:

$$Sh(A) = Sh(A').$$

$\square$

3. A characterization theorem.

We shall give now several characterizations of the AH algebras whose ideals are generated by their projections.

Theorem 3.1. *Let $A = \lim\limits_{\to}(A_n, \Phi_{n,m})$ be an AH algebra, with $A_n = \oplus_{i=1}^{k_n} A_n^i$, $A_n^i = P_{n,i}C(X_{n,i,}, M_{[n,i]})P_{n,i}$, where $X_{n,i}$ are connected, finite CW complexes and $P_{n,i} \in C(X_{n,i}M_{[n,i]})$ are projections. Then the following are equivalent:*

a) *Any ideal of A is generated by its projections.*

b) *For any fixed n and any fixed $F = \overline{F} \subset U = \overset{\circ}{U} \subset SP(A_n) = \sqcup_{i=1}^{k_n} X_{n,i}$ there is $m_0 > n$ such that for any $m \geq m_0$ any partial map $\Phi_{n,m}^j : A_n \to A_m^j$ satisfies either:*

$$SP(\Phi_{n,m}^j)_y \cap F = \phi \quad \textit{for all} \;\; y \in X_{m,j}$$

or

$$SP(\Phi_{n,m}^j)_y \cap U \neq \phi \quad \textit{for all} \;\; y \in X_{m,j}.$$

c) *For any fixed n, i and $\delta > 0$ there is $m_0 > n$ such that the following is true:*

For any $F = \overline{F} \subset X_{n,i}$ and any $m \geq m_0$ we have that any partial map $\Phi_{n,m}^{i,j}$ satisfies either:

$$SP(\Phi_{n,m}^{i,j})_y \cap F = \phi \quad \textit{for all} \;\; y \in X_{m,j}$$

or

$$SP(\Phi_{n,m}^{i,j})_y \cap B_\delta(F) \neq \phi \quad \textit{for all} \;\; y \in X_{m,j}.$$

(Here we used the standard notation $B_\delta(M) = \{x \in X_{n,i} : \mathrm{dist}\,(x, M) < \delta\}$ for any subset M of $X_{n,i}$.)

d) *Any ideal of A has a countable approximate unit consisting of projections.*

e) *For any ideal I of A we have:*

For any integer n, any $\epsilon > 0$ and any $x \in A_n \cap I$ there is $m > n$ and a projection $p \in A_m \cap I$ such that:

$$\|\Phi_{n,m}(x) - p \cdot \Phi_{n,m}(x)\| \leq \epsilon.$$

f) *For any ideal I of A we have:*

For any integer n, any $\epsilon > 0$ and any $x \in A_n \cap I$ there is $m > n$ and a projection $p \in A_m \cap I$ such that:

$$\|\Phi_{n,m}(x) - p \cdot \Phi_{n,m}(x) \cdot p\| \leq \epsilon.$$

(*Above we used the notation $A_k \cap I = \{y \in A_k : \Phi_{k,\infty}(y) \in I\}$.*)

Proof. a) $\Rightarrow$ b) follows from Lemma 2.8.

a) $\Rightarrow$ c) follows from Lemma 2.9.

b) $\Rightarrow$ d) Fix I an ideal in $A, n, \epsilon > 0$ and an element $x \in A_n \cap I$. Define:

$$F = \{t \in SP(A_n) : \|x(t)\| \geq \epsilon\}$$
$$U = \{t \in SP(A_n) : \|x(t)\| \neq 0\}.$$

Obviously $F = \overline{F} \subset U = \overset{\circ}{U} \subset SP(A_n)$. Then, by hypothesis, there is $m_0 > n$ such that for any $m \geq m_0$ any partial map $\Phi^j_{n,m} : A_n \to A^j_m$ satisfies either

(i) $$SP(\Phi^j_{n,m})_y \cap F = \phi \quad \text{for all } \; y \in X_{m,j}$$

or

(ii) $$SP(\Phi^j_{n,m})_y \cap U \neq \phi \quad \text{for all } \; y \in X_{m,j}.$$

Let $G_k = \overline{G}_k \subset SP(A_k)$ be such that $A_k \cap I = \{f \in A_k : f|_{G_k} = 0\}$, $k \geq 1$. Define $p_m = \oplus_{j=1}^{k_m} p^j_m \in \oplus_{j=1}^{k_m} A^j_m = A_m$ for any $m \geq 1$ by:

$$p^j_m = \begin{cases} P_{m,j} & \text{if } \; G^j_m = \phi \\ 0 & \text{if } \; G^j_m \neq \phi \end{cases}$$

(here, obviously, G^j_m is the component of G_m in $X_{m,j}$). It is clear that p_m is a projection in $A_m \cap I$. Denote for $m \geq m_0$:

$$d^j_m = \Phi^j_{n,m}(x) - p^j_m \Phi^j_{n,m}(x)$$

for any $1 \leq j \leq k_m$. We shall prove that $\|d^j_m\| \leq \epsilon$ for all j and all $m \geq m_0$. Let us fix now $m \geq m_0$.

1) *Let j be such that $p^j_m = P_{m,j}$.*

In this case $d^j_m = 0$ and hence $\|d^j_m\| = 0$.

2) *Let j be such that $p^j_m = 0$.*

Then $d^j_m = \Phi^j_{n,m}(x)$. We have two possibilities:

If (i) is true, then:

$$\|d^j_m\| = \|\Phi^j_{n,m}(x)\| = \sup_{y \in X_{m,j}} \left\| x \mid_{SP(\Phi^j_{n,m})_y} \right\| \leq \epsilon.$$

If (ii) is true, then, since obviously:

$$U \subset SP(A_n) \setminus G_n := G^c_n$$

it follows that:

$$SP(\Phi^j_{n,m})_y \cap G^c_n \neq \phi \quad \text{for all} \;\; y \in SP(A^j_m) = X_{m,j}.$$

But now it is not difficult to prove that if $z \in SP(A^j_m)$, then:

$$SP(\Phi^j_{n,m})_z \cap G^c_n \neq \phi \Rightarrow z \in SP(A^j_m) \setminus G^j_m := (G^j_m)^c.$$

(Indeed, let $t \in SP(\Phi^j_{n,m})_z \cap G^c_n$. Then there is $g \in A_n \cap I$ such that $g(t) \neq 0$. Hence:

$$\|\Phi^j_{n,m}(g)(z)\| = \left\| g \mid_{SP(\Phi^j_{n,m})_z} \right\| > 0$$

since $t \in SP(\Phi^j_{n,m})_z$. But $g \in A_n \cap I$ and $\Phi_{n,m}(A_n \cap I) \subset A_m \cap I$. All these things imply that $z \in (G^j_m)^c$.)

Combining the last relations we get:

$$SP(A^j_m) = X_{m,j} \subset (G^j_m)^c \Leftrightarrow G^j_m = \phi \Leftrightarrow p^j_m = P_{m,j}.$$

Hence, we are in the case 1) when $\|d^j_m\| = 0$. This ends the proof of b) $\Rightarrow$ d).

d) $\Rightarrow$ e), e) $\Rightarrow$ a), d) $\Rightarrow$ f), c) $\Rightarrow$ b) and f) $\Rightarrow$ a) are obvious. The proof of the Theorem is completed. □

Remarks 3.2. a) Let us note that part c) in Theorem 3.1 is independent of the metric defining the topology of $X_{n,i}$.

b) Theorem 3.1 remains true if the spaces $X_{n,i}$ are arbitrary compact, connected, metrizable topological spaces (the proof is the same).

4. Stable rank one.

Our objective is the following result:

Theorem 4.1. *Let A be an AH algebra with slow dimension growth and such that any ideal of A is generated by its projections.*

Then A has stable rank one ($tsr\,(A) = 1$).

Note that if in this theorem we drop the slow dimension growth condition, then the result is not true in general, as follows from [**V**]. The above theorem generalizes results of Dadarlat-Nagy-Némethi-Pasnicu [**DNNP**], Blackadar-Dadarlat-Rørdam [**BDR**], when A is simple and of Elliott-Gong [**EG 2**] when A has real rank zero. For the proof of the above theorem we shall need Theorem 3.1, Lemma 2.11 and the following lemmas:

Lemma 4.2. *Let X be a compact Hausdorff space and let a be a non-invertible element of $A = C(X, M_n)$. Let $F = \{x \in X : a(x)$ is not invertible in $M_n\}$ and let $\epsilon > 0$.*

Then, there are $s, F_i = \overline{F}_i \subset U_i = \overset{\circ}{U}_i \subset X, b_i \in A$ and projections $p_i, q_i \in A$ $(1 \leq i \leq s)$ such that:

1) $F = \cup_{i=1}^{s} F_i$.
2) $p_i(x), q_i(x) \neq 0, x \in X,\ 1 \leq i \leq s$.
3) $b_i(x)p_i(x) = q_i(x)b_i(x) = 0, x \in U_i,\ 1 \leq i \leq s$.
4) $\|a - b_i\| < \epsilon,\ 1 \leq i \leq s$.

Proof. Let $x \in F$. Since det $a(x) = 0$, it is not difficult to deduce that there are $x \in U_x = \overset{\circ}{U}_x \subset X, b_x \in A$ and projections $p_x, q_x \in C(X, M_n)$ such that:

$$p_x(t), q_x(t) \neq 0, t \in X$$

$$b_x(t)p_x(t) = q_x(t)b_x(t) = 0, \quad t \in U_x$$

$$\|a - b_x\| < \epsilon.$$

Choose $x \in W_x = \overset{\circ}{W}_x$ such that $\overline{W_x} \subset U_x$.

We make such a construction for any $x \in F$. Since $F \subset \cup_{x \in F} W_x$ and F is compact, it follows that there are $x_1, x_2, \ldots, x_s \in F$ such that:

$$F \subset \cup_{i=1}^{s} W_{x_i} (\subset \cup_{i=1}^{s} \overline{W_{x_i}})$$

and hence

$$F = \cup_{i=1}^{s} (F \cap \overline{W_{x_i}}).$$

Denote $F_i = F \cap \overline{W_{x_i}}$, $1 \leq i \leq s$. Since $F_i = \overline{F_i}$ and $F_i \subset \overline{W_{x_i}} \subset U_{x_i}$ for each i, we may define $U_i = U_{x_i}, b_i = b_{x_i}, p_i = p_{x_i}, q_i = q_{x_i} (1 \leq i \leq s)$ and the proof is over. □

Lemma 4.3. *Let $\Phi : A = C(X, M_n) \to B = C(Y, M_m)$ be a homomorphism, where X, Y are compact Hausdorff spaces. Let $a \in A$ and let $\mathcal{W}$ be an open cover of X such that for every $W \in \mathcal{W}$ there are positive elements $p_W, q_W \in A$ such that for any $x \in W, p_W(x)$ and $q_W(x)$ are non-zero projections and:*

$$a(x)p_W(x) = q_W(x)a(x) = 0.$$

Then, there is an open cover $\mathcal{Z}$ of Y such that for every $Z \in \mathcal{Z}$ there are positive elements $P_Z, Q_Z \in \Phi(A)$ such that for any $t \in Z, P_Z(t)$ and $Q_Z(t)$ are projections with

$$\Phi(a)(t)P_Z(t) = Q_Z(t)\Phi(a)(t) = 0$$

and

$$\operatorname{rank} P_Z(t),\ \ \operatorname{rank} Q_Z(t) \geq |SP(\Phi)_t| = \frac{\operatorname{rank} \Phi(1_A)}{\operatorname{rank} 1_A}$$

(*count multiplicities*).

Proof. Let $t \in Y$. Suppose that $y_1, y_2, \ldots, y_k$ are the distinct elements of $SP(\Phi)_t$. Choose $W_1, W_2, \ldots, W_k \in \mathcal{W}$ such that $y_i \in W_i$ and choose also open sets U_i with $y_i \in U_i \subset W_i$ such that:

$$\overline{U_i} \cap (\overline{\cup_{j \neq i} U_j}) = \phi, \quad 1 \leq i \leq k.$$

Let Z be an open neighborhood of t in Y such that:

$$SP(\Phi)_t \subset \cup_{i=1}^k U_i, \quad t \in Z.$$

Let $\chi_i : X \to [0,1]$ be continuous maps such that $\chi_i(x) = 1$ for any $x \in U_i$ and $\chi_i(x) = 0$ for any $x \in \cup_{j \neq i} U_j$ $(1 \leq i \leq k)$.

Define $P_Z, Q_Z \in B$ by:

$$P_Z = \Phi\left(\sum_{i=1}^k \chi_i p_{W_i}\right)$$
$$Q_Z = \Phi\left(\sum_{i=1}^k \chi_i q_{W_i}\right).$$

Obviously, $0 \leq P_Z, Q_Z \in \Phi(A)$ and for any $t \in Z$ it is clear that $P_Z(t)$ and $Q_Z(t)$ are projections $\left(\text{since } \sum_{i=1}^k \chi_i p_i \text{ and } \sum_{i=1}^k \chi_i q_i \text{ are projections on } SP(\Phi)_t\right)$ and:

$$\Phi(a)(t)P_Z(t) = Q_Z(t)\Phi(a)(t) = 0$$

$\Bigg($since if $x \in SP(\Phi)_t \subset \cup_{i=1}^k U_i \Rightarrow x \in U_j$ for some j and then:

$$\begin{aligned} a(x) \cdot \left(\sum_{i=1}^k \chi_i p_{W_i}\right)(x) &= a(x)\chi_j(x)p_{W_j}(x) = a(x)p_{W_j}(x) = 0 \\ &= q_{W_j}(x)a(x) = \left(\sum_{i=1}^k \chi_i p_{W_i}\right)(x) \cdot a(x)\Bigg) \end{aligned}$$

and also:

$$\begin{aligned} \operatorname{rank} P_Z(t) &= \sum_{i=1}^k \left(\sum_{x \in SP(\Phi)_t \cap U_i} \operatorname{rank} p_{W_i}(x)\right) \\ &\geq \sum_{i=1}^k |SP(\Phi)_t \cap U_i| = |SP(\Phi)_t| = \frac{\operatorname{rank} \Phi(1_A)}{\operatorname{rank} 1_A} \end{aligned}$$

and similarly:

$$\operatorname{rank} Q_Z(t) \geq |SP(\Phi)_t|$$

(count multiplicities). $\square$

The next lemma is the Selection principle from [**DNNP**]:

Lemma 4.4 ([**DNNP**], Proposition 3.2)**.** *Let X be a Hausdorff compact space, let $k' \geq k \geq 1$ be integers, let $\mathcal{W}$ be an open cover of X and assume that for each $W \in \mathcal{W}$ there is given a continuous projection valued map $p_W : W \to M_n$ such that* rank $p_W(x) \geq k'$ *for $x \in W$. If $\dim(X) \leq 2(k'-k)-1$ then there is a continuous projection valued map $p : X \to M_n$ such that for $x \in X$:*

$$\operatorname{rank} p(x) \geq k$$

$$p(x) \leq \vee\{p_W(x) : W \in \mathcal{W}, x \in W\}.$$

Lemma 4.5 ([**DNNP**], Lemma 3.3)**.** *Let B be a unital C^*-algebra and let*

$$k \geq \max(tsr\,(B), csr\,(B)).$$

Then for any positive integer m and any $a \in M_m(B)$, the matrix $\begin{pmatrix} a & 0 \\ a & 0_k \end{pmatrix}$ belongs to the closure of $GL(m+k, B)$.

Proof of Theorem 4.1. We shall prove first the theorem in the case when $A = \varinjlim(A_n, \Phi_{n,m}), A_n = \oplus_{i=1}^{k_n} A_n^i, A_n^i = C(X_{n,i}, M_{[n,i]})$, where $k_n,\ [n,i]$ are positive integers and $X_{n,i}$ are connected, finite CW complexes.

Fix a non-invertible element a in some A_n^i and fix also $\epsilon > 0$. Define $F = \{x \in SP(A_n^i) : a(x) \text{ is not invertible}\} = \{x \in SP(A_n^i) : \det a(x) = 0\}$. By Lemma 4.2 there are $s, F_\ell = \overline{F_\ell} \subset U_\ell = \overset{\circ}{U}_\ell \subset SP(A_n^i), b_\ell \in A_n^i$ and projections $p_\ell, q_\ell \in A_n^i,\ \ 1 \leq \ell \leq s$ such that $F = \cup_{\ell=1}^{s} F_\ell$ and:

$$p_\ell(x), q_\ell(x) \neq 0,\ x \in SP(A_n^i), \quad 1 \leq \ell \leq s \tag{1}$$

$$b_\ell(x) \cdot p_\ell(x) = q_\ell(x) \cdot b_\ell(x) = 0, \quad x \in U_\ell, \quad 1 \leq \ell \leq s \tag{2}$$

$$\|a - b_\ell\| < \epsilon, \quad 1 \leq \ell \leq s. \tag{3}$$

By Theorem 3.1, there is $m_0 > n$ such that for any $m \geq m_0$ and any $1 \leq \ell \leq s$ any partial map $\Phi_{n,m}^{i,j} : A_n^i \to A_m^j \ \ (1 \leq j \leq k_m)$ satisfies either:

$$SP(\Phi_{n,m}^{i,j})_y \cap F_\ell = \phi \quad \text{for all} \ \ y \in X_{m,j}$$

or

$$SP(\Phi_{n,m}^{i,j})_y \cap U_\ell \neq \phi \quad \text{for all} \ \ y \in X_{m,j}.$$

Using this and arguing by contradiction it is easy to see that for any $m \geq m_0$ and for any j there is $1 \leq k \leq s$ such that either:

$$SP(\Phi_{n,m}^{i,j})_y \cap F = \phi \quad \text{for any} \ \ y \in SP(A_m^j) \tag{4}$$

or

$$SP(\Phi_{n,m}^{i,j})_y \cap U_k \neq \phi \quad \text{for any} \ \ y \in SP(A_m^j). \tag{5}$$

Fix such m, j and k.

If (4) is true, then it is easy to see that $\Phi^{i,j}_{n,m}(a)$ is invertible in $\Phi^{i,j}_{n,m}(1_{A^i_n})A^j_m\Phi^{i,j}_{n,m}(1_{A^i_n})$.

Let us assume now that (5) is true. Then, using also (1) and (2) and working as in the proof of [**DNNP**], Lemma 2.2, we get an open cover $\mathcal{W}$ of $SP(A^j_m)$ such that for every $W \in \mathcal{W}$ there are positive elements p_W, q_W of A^j_m such that for any $x \in W, p_W(x)$ and $q_W(x)$ are non-zero projections with:

$$\Phi^{i,j}_{n,m}(b_k)(x)p_W(x) = q_W(x)\Phi^{i,j}_{n,m}(b_k)(x) = 0.$$

Denote $c_j = \Phi^{i,j}_{n,m}(b_k)$. By Lemma 2.11 we have that there is $r > m$ such that any partial map $\Phi^{j,s}_{m,r}$ satisfies either:

$$\begin{cases} \|\Phi^{j,s}_{m,r}(c_j) - \Psi^{j,s}_{m,r}(c_j)\| < \epsilon \\ \text{where } \ \Psi^{j,s}_{m,r} : A^j_m \to \Phi^{j,s}_{m,r}(1_{A^j_m})A^s_r\Phi^{j,s}_{m,r}(1_{A^j_m}) \\ \text{is some homomorphism with finite dimensional range} \end{cases} \tag{6}$$

or:

$$\operatorname{rank}\Phi^{j,s}_{m,r}(1_{A^j_m}) \geq \operatorname{rank} 1_{A^j_m} \cdot \left(\frac{3}{2}\dim\, X_{r,s} + 2\right). \tag{7}$$

Fix s and r. If (6) is true then, obviously, $\Phi^{j,s}_{m,r}(c_j)$ can be approximated within ϵ by an invertible element of $\Phi^{j,s}_{m,r}(1_{A^j_m})A^s_r\Phi^{j,s}_{m,r}(1_{A^j_m})$.

If (7) is true, then, by Lemma 4.3 it follows that there is an open cover $\mathcal{Z}$ of $SP(A^s_r)$ such that for any $Z \in \mathcal{Z}$ there are positive elements $P_Z, Q_Z \in \Phi^{j,s}_{m,r}(A^j_m)$ with the property that for any $t \in Z, P_Z(t), Q_Z(t)$ are projections with:

$$\Phi^{j,s}_{m,r}(c_j)(t)P_Z(t) = Q_Z(t)\Phi^{j,s}_{m,r}(c_j)(t) = 0$$

$$\operatorname{rank} P_Z(t),\ \ \operatorname{rank} Q_Z(t) \geq \frac{3}{2}\dim\, X_{r,s} + 2.$$

Using now Lemma 4.4 (the Selection principle) it follows that there are projections p and q in A^s_r such that:

$$\operatorname{rank} p, \operatorname{rank} q \geq \dim\, X_{r,s} + \frac{3}{2} \tag{8}$$

$$\Phi^{j,s}_{m,r}(c_j)p = q\Phi^{j,s}_{m,r}(c_j) = 0 \tag{9}$$

$$p, q \leq \Phi^{j,s}_{m,r}(1_{A^j_m}). \tag{10}$$

Using (8) and stability results for vector bundles (see [**H**]) and passing eventually to subprojections it follows that there are trivial projections $\overline{p}$ and $\overline{q}$ in A^s_r (i.e., unitarily equivalent to a constant projection in $C(X_{r,s}, M_{[r,s]}) = A^s_r$) such that $\overline{p} \leq p, \overline{q} \leq q$ and:

$$\dim\, X_{r,s} + 1 \geq \operatorname{rank}\overline{p} = \operatorname{rank}\overline{q} \geq \left[\frac{\dim\, X_{r,s} + 1}{2}\right] + 1. \tag{11}$$

Since by [**N**] and [**R**]:

$$\left[\frac{\dim\, X_{r,s}+1}{2}\right]+1\geq \max\{tsr\, C(X_{r,s}), csr\, C(X_{r,s})\}$$

it follows that:

$$\operatorname{rank}\overline{p}=\operatorname{rank}\overline{q}\geq \max\{tsr\, C(X_{r,s}), csr\, C(X_{r,s})\}.$$

Since $\overline{p}$ and $\overline{q}$ are trivial projections, (10) and (11) imply that:

$$[\overline{p}]=[\overline{q}]\in K_0(A_r^s)=K_0(\Phi_{m,r}^{j,s}(1_{A_m^j})A_r^s\Phi_{m,r}^{j,s}(1_{A_m^j})). \tag{12}$$

We have by (11):

$$\operatorname{rank}\overline{p}=\operatorname{rank}\overline{q}\geq \left\langle\frac{\dim\, X_{r,s}}{2}\right\rangle$$

($\langle\cdot\rangle$ denotes the least integer $\geq$) and by (7) and (11):

$$\begin{aligned}&\operatorname{rank}(\Phi_{m,r}^{j,s}(1_{A_m^j})-\overline{p})=\operatorname{rank}\Phi_{m,r}^{j,s}(1_{A_m^j})-\operatorname{rank}\overline{p}\\ &\geq\frac{3}{2}\dim\, X_{r,s}+2-(\dim\, X_{r,s}+1)=\frac{\dim\, X_{r,s}}{2}+1\geq\left\langle\frac{\dim\, X_{r,s}}{2}\right\rangle.\end{aligned}$$

Using also (12), (10) and stability results for vector bundles in [**H**] it follows that:

$$\overline{p}=u\overline{q}u^*\ \text{ for some unitary } u \text{ in }\ \Phi_{m,r}^{j,s}(1_{A_m^j})A_r^s\Phi_{m,r}^{j,s}(1_{A_m^j}). \tag{13}$$

Since $\Phi_{m,r}^{j,s}(c_j)\overline{p}=\overline{q}\Phi_{m,r}^{j,s}(c_j)=0$ (see (9)), using (13) we get that:

$$\left(u^*\Phi_{m,r}^{j,s}(c_j)\right)\overline{p}=\overline{p}\left(u^*\Phi_{m,r}^{j,s}(c_j)\right)=0.$$

Since $\overline{p}, u^*\Phi_{m,r}^{j,s}(c_j)\in\Phi_{m,r}^{j,s}(1_{A_m^j})A_r^s\Phi_{m,r}^{j,s}(1_{A_m^j})$, using Lemma 4.5 as in the past part of the proof of [**DNNP**], Theorem 3.6, we can conclude that:

$$\begin{aligned}u^*\Phi_{m,r}^{j,s}(c_j)\in\overline{GL(\Phi_{m,r}^{j,s}(1_{A_m^j})A_r^s\Phi_{m,r}^{j,s}(1_{A_m^j}))}\Leftrightarrow&\\ \Phi_{m,r}^{j,s}(c_j)\in\overline{GL(\Phi_{m,r}^{j,s}(1_{A_m^j})A_r^s\Phi_{m,r}^{j,s}(1_{A_m^j}))},&\end{aligned}$$

(see (13)).

In conclusion, A has stable rank one (see (3)).

Let us consider now the general case, when:

$$A=\varinjlim(A_n,\Phi_{n,m})$$

where $A_n=\oplus_{i=1}^{k_n}P_{n,i}C(X_{n,i},M_{[n,i]})P_{n,i}$ with $X_{n,i}$ finite, connected CW complexes and $P_{n,i}\in C(X_{n,i}M_{[n,i]})$ projections. Then, as in [**EG 2**], 4.24, we use the proof of [**EG 2**], Lemma 2.13 to construct an AH algebra

$\tilde{A} = \varinjlim(\tilde{A}_n, , \tilde{\Phi}_{n,m})$, where $\tilde{A}_n$ is a direct sum of matrix algebras over the same CW complexes $X_{n,i}, A \subset \tilde{A}$ and:

$$A = \overline{\cup_{n=1}^{\infty} P_n \tilde{A} P_n}$$

for some increasing sequence of projections $P_1 \leq P_2 \leq \ldots \leq P_n \leq \ldots$. In fact, we may suppose that A_n is a corner subalgebra of $\tilde{A}_n$ and that $\Phi_{n,m} = \tilde{\Phi}_{n,m|A_n}$. It is clear now that $\tilde{A} = \varinjlim(\tilde{A}_n, \tilde{\Phi}_{n,m})$ has slow dimension growth and that by Theorem 3.1 it follows that any ideal of $\tilde{A}$ is generated by its projections (since any ideal of A has the same property). On the other hand, due to the special form of the connecting homomorphisms $\tilde{\Phi}_{n,m}$, one can easily construct an isomorphsim between $A \otimes \mathcal{K}$ and $\tilde{A} \otimes \mathcal{K}$ ($\mathcal{K}$ = the compact operators acting on a separable, infinite dimensional Hilbert space). Since $tsr\,(\tilde{A}) = 1$ (by the first part of the proof), a theorem of Rieffel [**R**], Theorem 3.6 implies that $tsr\,(A) = 1$. □

5. Other nonstable K-theoretical results.

In this section we shall prove other nonstable K-theory results for the AH algebras considered in Theorem 4.1.

Theorem 5.1. *Let A be an AH algebra with slow dimension growth and such that any ideal is generated by its projections.*

Then:

a) $K_0(A)$ *is weakly unperforated in the sense of Elliott* ([**Ell 2**]).

b) *If furthermore, all the connecting homomorphisms in the inductive system with slow dimension growth whose limit is A are unital then A has* (SC) *(see* [**P 2**]*), i.e., if p and q are projections in A such that $\tau(p) < \tau(q)$ for any tracial state τ of A, then p is Murray-von Neumann equivalent to a proper subprojection of q.*

The above theorem generalizes results of Dadarlat-Némethi [**DN**], Martin-Pasnicu [**MP**] and Blackadar [**Bl 2**].

To prove the above theorem we need, among other things, the following result:

Proposition 5.2 (see [**P 2**] and also [**MP**])**.** *Let $A = \varinjlim(A_n, \Phi_n)$ and $B = \varinjlim(B_n, \Psi_n)$, where A_n, B_n are arbitrary unital C^*-algebras and the connecting homomorphisms Φ_n, Ψ_n are unital.*

Suppose that there is an EP-commutative diagram (see [**MP**, 2.3]*) with unital homomorphisms α_n and $\beta_n (n \geq 1)$:*

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{\Phi_1} & A_2 & \xrightarrow{\Phi_2} & A_3 & \xrightarrow{\Phi_3} & A_4 & \xrightarrow{\Phi_4} & \cdots \\ \alpha_1\searrow & \beta_1\nearrow & \alpha_2\searrow & \beta_2\nearrow & \alpha_3\searrow & \beta_3\nearrow & & & \\ & B_1 & \xrightarrow{\Psi_1} & B_2 & \xrightarrow{\Psi_2} & B_3 & \xrightarrow{\Psi_3} & \cdots & \end{array}$$

(*note that this happens if e.g., the above diagram is commutative at the level of homotopy*).

Then:

a) $T(A) \neq \phi \Leftrightarrow T(B) \neq \phi$.

b) A *has* $(SC) \Leftrightarrow B$ *has* (SC).

If furthermore the above diagram is a stably EP-commutative diagram (*that is, after taking the tensor product with* M_n *for any* n*, it is still an EP-commutative diagram*) *then:*

c) A *has cancellation* $\Leftrightarrow B$ *has cancellation.*

d) $K_0(A)$ *is weakly unperforated in the sense of Elliott* $\Leftrightarrow K_0(B)$ *is weakly unperforated in the sense of Elliott.*

a), b) and c) appeared in [**P 2**], Proposition 2.5. d) follows from the fact that the preordered groups $(K_0(A), K_0(A)_+)$ and $(K_0(B), K_0(B)_+)$ are isomorphic. Obviously, we need in fact the homomorphisms to be unital in the diagram only for a) and b).

Proof of Theorem 5.1. The proof is inspired by [**P 2**] and [**MP**]. Assume that A is the C^*-inductive limit of an inductive system:

$$(A_n = \oplus_{i=1}^{k_n} A_n^i, \Phi_{n,m})$$

$A_n^i = P_{n,i} C(X_{n,i}, M_{[n,i]}) P_{n,i}$, with slow dimension growth. Using Lemma 2.11 it follows that there is a sequence of positive integers $\ell_1 < \ell_2 < ... < \ell_n < ...$ and homomorphisms $\Psi_n : A_{\ell_n} \to A_{\ell_{n+1}}$ such that $\Phi_{\ell_n, \ell_{n+1}}$ is homotopic to Ψ_n and each partial map

$$(\Psi_n)^{i,j} : A_{\ell_n}^i \to (\Psi_n)^{i,j}(1_{A_{\ell_n}^i}) A_{\ell_{n+1}}^j (\Psi_n)^{i,j}(1_{A_{\ell_n}^i})$$

induced by Ψ_n satisfies either:

$$\operatorname{rank}\left((\Psi_n)^{i,j}(P_{\ell_n,i})\right) \geq (\dim \ \ X_{\ell_{n+1},j} + 1)\operatorname{rank}\left(P_{\ell_n,i}\right)$$

or

$$(\Psi_n)^{i,j} \ \text{ has finite dimensional range.}$$

If we denote $B = \varinjlim(A_{\ell_n}, \Psi_n)$, then obviously A and B are shape equivalent (and, moreover, if the $\Phi_{n,m}$'s are unital then all the Ψ_n's above can be chosen also unital).

Now, by Proposition 5.2, it is enough to prove that $K_0(B)$ is weakly unperforated in the sense of Elliott and that (in the unital case) B has (SC). But this can be shown using stability results for vector bundles ([**H**]) as in [**P 2**] (see also [**MP**]). □

Remarks 5.3. We know that any AH algebra with slow dimension growth and real rank zero is isomorphic to an AH algebra over spaces of dimension ≤ 3 and with real rank zero ([**Da 2**], [**Go 1**]). Therefore, using also Theorem 2.6 and Proposition 5.2, the proof of the above Theorem reduces to the proof of the fact that any AH algebra over spaces of dimension ≤ 3 and with real rank zero has the required properties. But this is a standard fact. We prefered the other proof of Theorem 5.1 beause it is more "elementary".

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