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If S is the graph of a minimal surface, then when given parametrically by the Weierstrass representation, the first two coordinate functions give a univalent harmonic mapping. In this paper, the starting point is a univalent harmonic mapping f of the unit disk U . A height function is defined on an appropriate Riemann surface over the range of f which satisfies the minimal surface equation away from the branch points. This height function is then used to obtain function theoretic information about f .

1. Introduction.

Let f be a univalent harmonic mapping of the unit disk U . By this it is meant not only that f is 1-1 and harmonic, but also that f is sense preserving.

Harmonic univalent mappings were first studied in connection with minimal surfaces by E. Heinz [H]. However, considerable interest in their function theoretic properties, quite apart from this connection, was generated by Clunie and Sheil-Small [CS-S].

Now, the Jacobian of $f(\zeta)$ is $J = |f_\zeta|^2 - |f_{\bar{\zeta}}|^2$, and f can be written

$$(1.1) \quad f = h + \bar{g}$$

where h and g are analytic in U . If $a(\zeta)$ is defined by

$$(1.2) \quad a(\zeta) = \overline{f_{\bar{\zeta}}(\zeta)}/f_\zeta(\zeta) = g'(\zeta)/h'(\zeta),$$

then $a(\zeta)$ is analytic and $|a(\zeta)| < 1$ in U . We shall refer to $a(\zeta)$ as the *analytic dilatation* as opposed to the usual dilatation $f_{\bar{\zeta}}/f_\zeta$ in the theory of quasiconformal mappings.

The case where $a(\zeta)$ is a finite Blaschke product is of special interest since this case arises in taking Fourier series of step functions [S-S]. Their function theoretic properties have been studied in [HS2] as well as in [S-S], and infinite Blaschke products have been considered in [L].

In the present paper we shall study a connection between harmonic mappings and the theory of minimal surfaces, and in §4 we use this to prove a special case of uniqueness for the Riemann mapping theorem of Hengartner

and Schober [HS1]. As we have shown elsewhere, uniqueness fails in general [W].

2. Definition of the height function and conjugate height function.

Using the Weierstrass representation [O, p. 63] we shall associate with f , a minimal surface given parametrically in a simply connected subdomain $N \subseteq U$ where $a(\zeta)$ does not have a zero of odd order.

With g and h as in (1.1) we define up to an additive constant, a branch of

$$(2.1) \quad F(\zeta) = 2i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)} \, d\zeta = 2i \int f_\zeta(\zeta)\sqrt{a(\zeta)} \, d\zeta.$$

Then, by (1.2) it follows that a branch of F can be defined in N , and for $\zeta \in N$,

$$(2.2) \quad \zeta \rightarrow (f(\zeta), \operatorname{Re} F(\zeta))$$

gives a parametric representation of a minimal surface. Here we have identified \mathbb{R}^2 with \mathbb{C} by $(x, y) \leftrightarrow (\operatorname{Re} f, \operatorname{Im} f)$.

Let \hat{U} be the Riemann surface of the function $\sqrt{a(\zeta)}$. Then \hat{U} has algebraic branch points corresponding to those points $\zeta \in U$ for which $a(\zeta)$ has a zero of odd order. Specifically, \hat{U} can be concretely described (the *analytic configuration* [Sp, 69-74]) in terms of function elements (α, F_α) where $\alpha \in U$, and F_α is a power series expansion of a branch of F in a neighborhood of α if $a(\zeta)$ does not have a zero of odd order at $\zeta = \alpha$, and F_α a power series in $\sqrt{\zeta - \alpha}$ otherwise. The mapping $p: (\alpha, F_\alpha) \rightarrow \alpha$ is the *projection* of the surface so realized. The mapping F may now be lifted to a mapping \hat{F} on \hat{U} .

By continuation, we may induce a mapping $\hat{U} \rightarrow \tilde{U}$ to a surface \tilde{U} with a real analytic structure defined in terms of elements (β, \tilde{F}_β) with $\beta \in f(U)$ by $\alpha = f^{-1}(\beta)$ and $\tilde{F}_\beta = F_\alpha \circ f^{-1}$. We again define a projection by $\pi: (\beta, \tilde{F}_\beta) \rightarrow \beta$.

We shall refer to a point $\hat{\zeta} \in \hat{U}$ to be over ζ , if $p(\hat{\zeta}) = \zeta$, and $\tilde{z} \in \tilde{U}$ to be over z if $\pi(\tilde{z}) = z$.

The harmonic mapping $f: U \rightarrow f(U)$ lifts to a mapping $\hat{f}: \hat{U} \rightarrow \tilde{U}$ which is 1 - 1, onto, and satisfies the condition $\pi(\hat{f}(\hat{\zeta})) = f(p(\hat{\zeta}))$ for all $\zeta \in \hat{U}$. With these notations, we shall extend the meaning of (2.2). Thus

$$(2.3) \quad \hat{\zeta} \rightarrow (\hat{f}(\hat{\zeta}), \operatorname{Re} \hat{F}(\hat{\zeta}))$$

gives a parametric representation of a minimal surface in the sense that in a neighborhood of $\hat{\zeta} \in \hat{U} \setminus \mathcal{B}$ where \mathcal{B} is the branch set, that is, the points

above the zeros of a of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on $\tilde{U} \setminus \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} = \hat{f}^{-1}(\mathcal{B})$, as follows. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contains no zeros of a of odd multiplicity. Let $w = \varphi(x, y)$ be the nonparametric description of the minimal surface corresponding to (2.2), that is, for $\zeta \in f^{-1}(0)$ (cf. [HS3, p. 87]),

$$(2.4) \quad \begin{aligned} x &= \operatorname{Re} f(\zeta) & y &= \operatorname{Im} f(\zeta), \\ \varphi(x, y) &= \operatorname{Re} F(\zeta). \end{aligned}$$

Then, by continuation φ lifts to a function $\tilde{\varphi}$ on \tilde{U} which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set $\tilde{\mathcal{B}}$. We shall call $\tilde{\varphi}(z)$ a *height function* corresponding to f . Finally, we define a *conjugate height function* $\tilde{\psi}(z)$ by solving locally

$$(2.5) \quad \psi_y = \varphi_x/W, \quad \psi_x = -\varphi_y/W \quad \left(W = \sqrt{1 + \varphi_x^2 + \varphi_y^2} \right)$$

(cf. [F1, p. 344]) and lifting to $\tilde{U} \setminus \tilde{\mathcal{B}}$ as was done for φ . Let $\tilde{F} = \tilde{\varphi} + i\tilde{\psi}$. Then \tilde{F} is real analytic and locally quasiconformal on $\tilde{U} \setminus \tilde{\mathcal{B}}$, with dilatation whose magnitude is $(W - 1)/(W + 1)$. The fact that $\tilde{\psi}$ and \tilde{F} are well defined on $\tilde{U} \setminus \tilde{\mathcal{B}}$ follows from Theorem 1.

A glossary of terminology is given schematically in Figure 1.

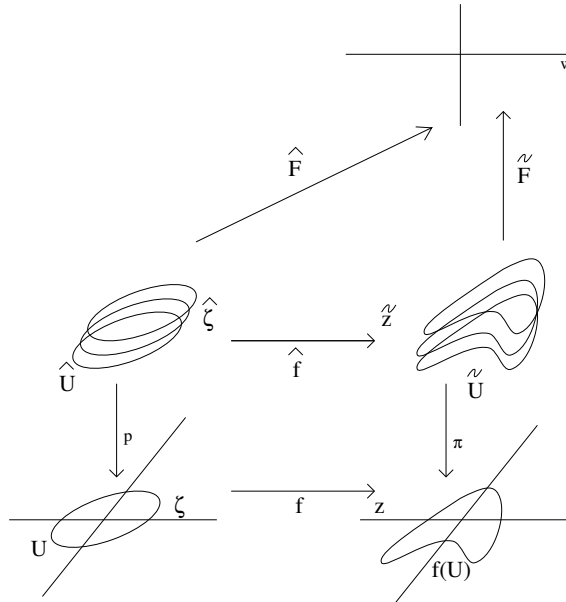


Figure 1.

Theorem 1. *With the above notations, $\hat{F} = \tilde{F} \circ \hat{f} + C$ for some constant C .*

Proof. Let D be an open disk in $f(U)$ such that $f^{-1}(D)$ contain no zeros of odd multiplicities of a . We fix a branch of \sqrt{a} in $f^{-1}(D)$, and consider $\hat{\varphi}(\hat{\zeta}) + i\hat{\psi}(\hat{\zeta}) = \hat{F}(\hat{\zeta})$ for points in a component of \hat{U} over $f^{-1}(D)$, and $\tilde{\varphi}(\tilde{z}) + i\tilde{\psi}(\tilde{z}) = \tilde{F}(\tilde{z})$ for points in a component of \tilde{U} over D . Since we shall compute in local coordinates given by projection, to reduce notation in this proof, we shall subsequently write \hat{F} , $\hat{\varphi}$, $\hat{\psi}$ in place of $\hat{F} \circ p^{-1}$, $\hat{\varphi} \circ p^{-1}$, $\hat{\psi} \circ p^{-1}$, and \tilde{F} , $\tilde{\varphi}$, $\tilde{\psi}$ in place of $\tilde{F} \circ \pi^{-1}$, $\tilde{\varphi} \circ \pi^{-1}$, $\tilde{\psi} \circ \pi^{-1}$ respectively. With this notation, by (2.4) we have that

$$(2.6) \quad \hat{\varphi} = \tilde{\varphi} \circ f,$$

so it suffices to show that

$$(2.7) \quad \hat{\psi} = \tilde{\psi} \circ f + C.$$

The result then follows from continuation. □

In fact, since $\hat{\varphi} + i\hat{\psi}$ is analytic in $f^{-1}(D)$, it follows from (2.6) that to prove (2.7) it suffices to show that $\tilde{F} \circ f$ is analytic in $f^{-1}(D)$.

We first record the relationship between $a(\zeta)$ of (1.2) and $W(z)$ ($z = f(\zeta)$) of (2.5). This is given by [O, p. 105], [HS3, pp. 87-88] as

$$(2.8) \quad |a| = \frac{W - 1}{W + 1}.$$

Now,

$$(2.9) \quad (\tilde{F} \circ f)_{\bar{\zeta}} = \tilde{F}_z f_{\bar{\zeta}} + \tilde{F}_{\bar{z}} \overline{f_{\zeta}} = \tilde{F}_z f_{\bar{\zeta}} + \tilde{F}_{\bar{z}} \overline{f_{\zeta}}.$$

A simple computation using (2.5) gives

$$F_z = \frac{W + 1}{W} \varphi_z, \quad F_{\bar{z}} = \frac{W - 1}{W} \varphi_{\bar{z}}.$$

When used in (2.9) these give

$$(2.10) \quad (\tilde{F} \circ f)_{\bar{\zeta}} = \frac{W + 1}{W} \tilde{\varphi}_z f_{\bar{\zeta}} + \frac{W - 1}{W} \tilde{\varphi}_{\bar{z}} \overline{f_{\zeta}}.$$

Again, a direct computation gives

$$\tilde{\varphi}_z = \frac{\hat{\varphi}_{\zeta} \overline{f_{\zeta}} - \hat{\varphi}_{\bar{\zeta}} \overline{f_{\bar{\zeta}}}}{|f_{\zeta}|^2 - |f_{\bar{\zeta}}|^2}, \quad \tilde{\varphi}_{\bar{z}} = \frac{\hat{\varphi}_{\bar{\zeta}} f_{\zeta} - \hat{\varphi}_{\zeta} f_{\bar{\zeta}}}{|f_{\zeta}|^2 - |f_{\bar{\zeta}}|^2}.$$

When used in (2.10) this gives

$$(2.11) \quad (\tilde{F} \circ f)_{\bar{\zeta}} = \frac{1}{W(|f_{\zeta}|^2 - |f_{\bar{\zeta}}|^2)} \left(2\hat{\varphi}_{\zeta} f_{\bar{\zeta}} \overline{f_{\zeta}} + \hat{\varphi}_{\bar{\zeta}} |f_{\zeta}|^2 \left(W - 1 - \frac{|f_{\bar{\zeta}}|^2}{|f_{\zeta}|^2} (W + 1) \right) \right).$$

Now, by (1.2), (2.1), and (2.8) we have,

$$\hat{\varphi}_\zeta = ig'/\sqrt{a}, \hat{\varphi}_{\bar{\zeta}} = -i\bar{g}'/\sqrt{\bar{a}}, f_\zeta = g'/a, f_{\bar{\zeta}} = \bar{g}',$$

and

$$W - 1 - \frac{|f_{\bar{\zeta}}|^2}{|f_\zeta|^2}(W + 1) = W - 1 - |a|^2(W + 1) = 2(W - 1)/(W + 1).$$

Substituting into (2.11) we obtain

$$\begin{aligned} (\tilde{F} \circ f)_{\bar{\zeta}} &= \frac{1}{W(|f_\zeta|^2 - |f_{\bar{\zeta}}|^2)} \left(\frac{2ig'(\bar{g}')^2}{\sqrt{a\bar{a}}} - \frac{2i\bar{g}'|g'|^2}{\sqrt{\bar{a}}|a|^2} \left(\frac{W - 1}{W + 1} \right) \right) \\ &= 0. \end{aligned}$$

Thus, $\tilde{F} \circ f$ is analytic and (2.7) follows.

3. The height function corresponding to Poisson integrals of step functions.

Let \mathcal{P} be a polygon with vertices c_1, \dots, c_n given cyclically, and in order induced by a positive orientation of $\partial\mathcal{P}$. Let f be the Poisson integral of a step function on ∂U having values c_1, \dots, c_n and suppose that f is then a univalent harmonic mapping, $f: U \rightarrow \mathcal{P}$. If \mathcal{P} is convex, for example, this will always be the case [C], [K]. The analytic dilatation $a(\zeta)$ for such mappings were studied in [HS2] and [S-S]. In general, $a(\zeta)$ is a Blaschke product of order at most $n - 2$, and of order precisely $n - 2$ if \mathcal{P} is convex [S-S, pp. 469, 473].

We shall now explore the boundary behavior of height functions corresponding to such mappings. The prototype for this is Scherk's minimal surface over the square $-\pi/2 < x < \pi/2, -\pi/2 < y < \pi/2$, given by

$$(3.1) \quad \psi(x, y) = \log(\cos x / \cos y)$$

which tends to $+\infty$ and $-\infty$ over alternate sides. It seems remarkable that this type of behavior persists in general for height functions corresponding to all such f described above.

Theorem 2. *Let \mathcal{P} be a polygon having vertices c_1, \dots, c_n given cyclically, and ordered by a positive orientation on $\partial\mathcal{P}$. Let f be a univalent harmonic mapping of U such that f is the Poisson integral of a step function having the ordered sequence c_1, \dots, c_n as its values. Then the analytic dilatation $a(\zeta)$ of f is a finite Blaschke product of order at most $n - 2$, $f(U) = \mathcal{P}$, and if φ is a height function for f , then φ tends to $+\infty$ or $-\infty$ at points over the open segments making up the sides of \mathcal{P} . If \mathcal{P} is convex, then $+\infty$ and $-\infty$ alternate on adjacent sides.*

Proof. That $a(\zeta)$ is a Blaschke product of order at most $n - 2$ and $f(U) = \mathcal{P}$ follow from general properties of Poisson integrals [S-S, p. 469], [HS2, p. 203]. \square

Let $f = h + \bar{g}$ as in (1.1). Then we may write h' and g' in the form [S-S, pp. 460-461]

$$h'(\zeta) = \sum_{k=1}^n \frac{\alpha_k}{\zeta - \zeta_k}, \quad g'(\zeta) = - \sum_{k=1}^n \frac{\overline{\alpha_k}}{\zeta - \zeta_k},$$

where $\alpha_k \neq 0$, $k = 1, \dots, n$.

With F as in (2.1), we are then interested in the branches of

$$(3.2) \quad F(\zeta) = -2 \int \sqrt{\sum_{k=1}^n \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^n \frac{-\overline{\alpha_k}}{\zeta - \zeta_k}} d\zeta$$

as $\zeta \rightarrow \zeta_k$, $k = 1, \dots, n$. The cluster sets for the nontangential approaches to points over the ζ_k give the points lying over the open segments making up the sides of \mathcal{P} .

Thus, take a vertex ζ_j , and an open segment l_j of $\partial\mathcal{P}$ corresponding to it. Then, as $\zeta \rightarrow \zeta_j$,

$$\sum_{k=1}^n \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^n \frac{-\overline{\alpha_k}}{\zeta - \zeta_k} = \frac{|\alpha_j|^2}{(\zeta - \zeta_j)^2} (1 + o(1)),$$

and hence, by (3.2), a branch of F satisfies

$$(3.3) \quad F(\zeta) = \pm 2|\alpha_j| \log(\zeta - \zeta_j) + o(1)$$

as $\zeta \rightarrow \zeta_j$, for a fixed branch of the log. Suppose the fixed branch of (3.3) has minus sign, and let $\phi(z) = \operatorname{Re} F \circ f^{-1}(z)$ be a corresponding branch in \mathcal{P} for points near the corresponding side l_j . Now suppose \mathcal{P} is convex and $F(\zeta)$ is analytically continued to an adjacent point, say ζ_{j+1} , so that ϕ is then continued to a corresponding side l_{j+1} having common endpoint c_j with l_j . Since $\phi \rightarrow -\infty$ as $z \rightarrow l_j$, it remains to show that $\phi \rightarrow +\infty$ as $z \rightarrow l_{j+1}$. This effect has been noted for minimal surfaces [JS], and can be accomplished by a simple barrier argument. I thank Professor Finn for pointing this out.

Let $0 < \beta < \pi$ be the angle in \mathcal{P} between l_j and l_{j+1} . Suppose that $\phi \rightarrow -\infty$ on both open segments l_j and l_{j+1} . Since ϕ satisfies the minimal surface equation, ϕ can only tend to $-\infty$ over line segments [O, p. 102]. Since we make no assumption at the common endpoint c_j , in order to get a contradiction we must show that $\phi \rightarrow -\infty$ at c_j as well. We may assume that $c_j = (\pi/2, 0)$, and l_j, l_{j+1} make the angle β symmetrically with respect to the x axis, opening toward the origin. Let $0 < \varepsilon < (\pi/2) \cot(\beta/2)$ be small enough so that the isosceles triangle N formed by the sector and the line $x = \pi/2 - \varepsilon$ has the given branch of F single valued. Then, two of

the sides of N are contained in the segments l_j and l_{j+1} , and the third is $x = \pi/2 - \varepsilon$, $-\delta < y < \delta$, where $\delta = \varepsilon \tan(\beta/2)$. If ψ is the height function for Scherk's surface given by (3.1), then for any $M > 0$, clearly

$$(3.4) \quad \phi(x, y) < -\psi(x - \pi + \varepsilon, y) - M$$

on $\partial N \setminus \{c_j\}$. By the extended maximum principle [F1, pp. 342-343], it follows that (3.4) holds throughout N . Since $M > 0$ was arbitrary, it follows that $\phi \equiv -\infty$ on N , a contradiction. Thus $\phi = +\infty$ on l_{j+1} .

4. An application to the Riemann mapping theorem.

One of the most basic results in the theory of univalent harmonic mappings is the Riemann mapping theorem of Hengartner and Schober [HS1].

Theorem A. *Let D be a bounded simply connected domain whose boundary is locally connected. Fix $w_0 \in D$, and let $a(\zeta)$ be analytic in U , with $a(U) \subseteq U$. Then there exists a univalent harmonic mapping f with the following properties.*

- a) f maps U into D and $f(0) = w_0$, $f_z(0) > 0$.
- b) f satisfies the equation $\overline{f_{\bar{z}}} = af_{\zeta}$.
- c) Except for a countable set $E \subseteq \partial U$, the unrestricted limit $f^*(e^{it}) = \lim_{\zeta \rightarrow e^{it}} f(\zeta)$ exists and belongs to ∂D .
- d) The one sided limits $\lim_{\tau \rightarrow t^+} f^*(e^{i\tau})$, $\lim_{\tau \rightarrow t^-} f^*(e^{i\tau})$ through values of $e^{i\tau} \notin E$ exist and belong to ∂D ; for $e^{it} \notin E$ they are equal and for $e^{it} \in E$ they are different.
- e) The cluster set of f at $e^{it} \in E$ is the straight line segment joining the left and right limits in d).

If in Theorem A, the set D is convex, and $a(\zeta)$ is a finite Blaschke product, one can say more [HS2, p. 203], [S-S, p. 473].

Theorem B. *Let f be as in Theorem A with D bounded and convex, and $a(\zeta)$ a Blaschke product of order $n - 2$. Then $f(U)$ is a polygon with n vertices all of which lie on ∂D .*

We shall prove uniqueness in the case $a(\zeta) = \zeta^n$ and D convex. The case of uniqueness when $D = U$ and $a(\zeta) = \zeta$ was done in [HS2, p. 204].

The proof involves a combinatorial argument with the level sets of the height function. Such arguments are often useful in the theory of partial differential equation, and in particular the minimal surface equation [F1], [FO], [JS], [Se].

Theorem 3. *The solution $f(\zeta)$ to the Riemann mapping theorem above with D convex and*

$$(4.1) \quad a(\zeta) = \zeta^{n-2}$$

is unique for each $n = 3, 4, \dots$

Proof. Let f_1 and f_2 be Riemann mappings corresponding to D . We may assume $f_1(0) = f_2(0) = 0$. Let Δ be a disk centered at 0, and contained in $f_1(U) \cap f_2(U)$.

If n is even, then $\hat{U} = U$ and if n is odd \hat{U} is a two sheeted cover of U with branch point over 0. Similarly, if \tilde{U}_1 corresponds to $f_1(U)$ and \tilde{U}_2 to $f_2(U)$, then \tilde{U}_1 and \tilde{U}_2 are one or two sheeted according as n is even or odd. We consider the case where n is odd. The even case goes the same way, but is simpler since one can bypass discussion of Riemann surfaces.

Let $\varphi_j, \psi_j, \tilde{\varphi}_j, \tilde{\psi}_j, \tilde{F}_j, \tilde{U}_j, \pi_j, \quad j = 1, 2$ be the quantities of §2 defined for f_1 and f_2 respectively. We may assume that $\tilde{F}_1(\tilde{0}) = \tilde{F}_2(\tilde{0}) = 0$. If $\tilde{\Delta}$ represents the Riemann surface of \sqrt{z} over Δ , then we may consider $\tilde{\Delta} \subseteq \tilde{U}_1$ and $\tilde{\Delta} \subseteq \tilde{U}_2$, so that \tilde{F}_1 and \tilde{F}_2 may both be considered as defined for all $\tilde{z} \in \tilde{\Delta}$. For brevity of notation, we shall write \tilde{F} for $\tilde{F} \circ \pi^{-1}$.

Since the analytic dilatation for $f_1(\zeta)$ and $f_2(\zeta)$ is 0 when $\zeta = 0$, it follows from (1.2), (4.1), and a) of Theorem A, that

$$(4.2) \quad f_j(\zeta) = c_j \zeta(1 + o(1)) \quad (\zeta \rightarrow 0, \quad c_j > 0, \quad j = 1, 2).$$

Then, from (2.1), (4.1), (4.2), and Theorem 1 we may take determinations of \tilde{F}_1 and \tilde{F}_2 in $\tilde{\Delta}$ so that

$$(4.3) \quad \tilde{\varphi}_j(z) + i\tilde{\psi}_j(z) = \tilde{F}_j(z) = d_j z^{n/2}(1 + o(1)) \quad (j = 1, 2 \quad z \rightarrow 0)$$

with $d_1, d_2 > 0$ and $z^{n/2}$ is some fixed branch.

Having thus fixed branches in (4.3) we may then take a constant $\lambda > 0$ such that

$$(4.4) \quad \tilde{F}_1(z) - \lambda \tilde{F}_2(z/\lambda) = Cz^{\frac{p+2}{2}}(1 + o(1)) \quad (z \rightarrow 0)$$

for some constant C and integer $p \geq n$. We suppose $\lambda \geq 1$; otherwise we interchange \tilde{F}_1 and \tilde{F}_2 . Now, the change from $F(z)$ to $\lambda F(z/\lambda)$ corresponds to replacing f by λf . Then the analytic dilatation is unchanged, and following the change in (2.1) it gives the parametrization $\zeta \rightarrow (\lambda f(\zeta), \operatorname{Re} \lambda F(\zeta))$.

Let $\varphi_3, \psi_3, \tilde{\varphi}_3, \tilde{\psi}_3$ correspond to $f_3 = \lambda f_2$ so that $f_3(U)$, is nothing more than $f_1(U)$ dilated by the constant $\lambda \geq 1$, and (4.5) becomes

$$(4.5) \quad \tilde{F}_1(\tilde{z}) - \tilde{F}_3(\tilde{z}) = Cz^{\frac{p+2}{2}}(1 + o(1)) \quad (z \rightarrow 0).$$

Case 1. $C = 0$ for every p . Since $\tilde{F}_1(z^2) - \tilde{F}_3(z^2)$ is real analytic, then $\tilde{F}_1 \equiv \tilde{F}_3$. Thus, in particular $\lambda = 1$ and $f_1(U) = f_3(U) = \mathcal{P}$. In order to show that $f_1 \equiv f_3$ we use the subordination principle of [BHH, p. 170]. Briefly, since \mathcal{P} is a convex polygon by Theorem B, and $(f_1)_z(0), (f_3)_z(0) > 0$, we may apply the argument principle in [BHH, p. 170] to

$$G(z) = (f_3)_z(0)f_1(z) - (f_1)_z(0)f_3(z)$$

to deduce that $(f_1)_z(0) = (f_3)_z(0)$. Then, another application of the argument principle as in [BHH] to $G_\varepsilon(z) = (1 + \varepsilon)f_1(z) - f_3(z)$ ($\varepsilon \rightarrow 0$) shows that $f_1 \equiv f_3$.

Case 2. $C \neq 0$ for some $p \geq n$. In this case, near the origin on $\tilde{\Delta}$, by (4.5) there are $2p + 4$ level curves $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ emanating from $\tilde{0}$. Between the level curves, $\tilde{\varphi}_1 - \tilde{\varphi}_3$ alternates in sign. In order to analyze the component sets between the level sets, we must modify f_3 .

Let η_1, η_2, \dots be homeomorphisms of $|\zeta| = 1$ onto the boundary of λD , which converge to the (step function) boundary values of f_3 , and let $f_3^{(n)}$, $n = 1, 2, \dots$ their corresponding Poisson integrals so that $f_3^{(n)} \rightarrow f_3$ uniformly on compact subsets of U .

The level sets of $\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0$ create $2p + 4$ disjoint component open sets $O_1, O_2, \dots, O_{2p+4}$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3 > 0$ in O_{2j-1} and $\tilde{\varphi}_1 - \tilde{\varphi}_3 < 0$ in O_{2j} for $j = 1, \dots, p + 2$. These components alternate in position around the origin.

For $\varepsilon > 0$ we can find nonempty components at $O_1(\varepsilon), O_2(\varepsilon), \dots, O_{2p+4}(\varepsilon)$ where $\tilde{\varphi}_1 - \tilde{\varphi}_3^{(n)} > \varepsilon$ in $O_{2j-1}(\varepsilon)$, $\tilde{\varphi}_1 - \tilde{\varphi}_3^{(n)} = \varepsilon$ on $\partial O_{2j-1}(\varepsilon)$, $\tilde{\Delta} \cap O_{2j-1}(\varepsilon) \subseteq O_{2j-1}$, $j = 1, \dots, 2p$, and analogous statements hold for $O_{2j}(\varepsilon)$, $j = 1, \dots, p + 2$.

Now, $f_3^{(j)}(U) = \lambda D$, so by the maximum principle for solutions to the minimal surface equation, the level sets forming the boundaries of the $O_j(\varepsilon)$'s must extend to points over the boundary of $\mathcal{P} = f_1(U)$. As in [FO, pp. 357-358], we observe that since \tilde{F}_1 is $\pm\infty$ over the sides of \mathcal{P} by Theorem 2, if a component $O_j(\varepsilon)$ has a boundary point over an interior point of a side of \mathcal{P} , then the boundary must contain that side. Since, by Theorem B, \mathcal{P} has n sides, then $\tilde{\mathcal{P}} = \pi_1^{-1}(\mathcal{P})$ has $2n$ sides. This implies that there are at most $2n$ sets $O_j(\varepsilon)$ whose boundaries have interior points over $\partial\mathcal{P}$. If $O_j(\varepsilon)$ were a component whose boundary contained no points over $\partial\mathcal{P}$, then its boundary could only be interior points over \mathcal{P} , or vertices. As pointed out in [FO, p. 358], this is impossible by a theorem of Finn [F1, pp. 342-343]. Thus, $2p + 4 \leq 2n$. Since $p \geq n$, we obtain a contradiction and the theorem is proved. □

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