ON UNIVALENT HARMONIC MAPPINGS AND MINIMAL SURFACES

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If \( S \) is the graph of a minimal surface, then when given parametrically by the Weierstrass representation, the first two coordinate functions give a univalent harmonic mapping. In this paper, the starting point is a univalent harmonic mapping \( f \) of the unit disk \( U \). A height function is defined on an appropriate Riemann surface over the range of \( f \) which satisfies the minimal surface equation away from the branch points. This height function is then used to obtain function theoretic information about \( f \).

1. Introduction.

Let \( f \) be a univalent harmonic mapping of the unit disk \( U \). By this it is meant not only that \( f \) is \( 1 - 1 \) and harmonic, but also that \( f \) is sense preserving.

Harmonic univalent mappings were first studied in connection with minimal surfaces by E. Heinz [H]. However, considerable interest in their function theoretic properties, quite apart from this connection, was generated by Clunie and Sheil-Small [CS-S].

Now, the Jacobian of \( f(\zeta) \) is \( J = |f_\zeta|^2 - |f_\z\|^2 \), and \( f \) can be written

\[
f = h + g \tag{1.1}
\]

where \( h \) and \( g \) are analytic in \( U \). If \( a(\zeta) \) is defined by

\[
a(\zeta) = \frac{|f_\zeta(\zeta)|}{|f_\z\(\zeta)\|} = \frac{g'(\zeta)}{h'(\zeta)}, \tag{1.2}
\]

then \( a(\zeta) \) is analytic and \( |a(\zeta)| < 1 \) in \( U \). We shall refer to \( a(\zeta) \) as the analytic dilatation as opposed to the usual dilatation \( f_\zeta/f_\z\) in the theory of quasiconformal mappings.

The case where \( a(\zeta) \) is a finite Blaschke product is of special interest since this case arises in taking Fourier series of step functions [S-S]. Their function theoretic properties have been studied in [HS2] as well as in [S-S], and infinite Blaschke products have been considered in [L].

In the present paper we shall study a connection between harmonic mappings and the theory of minimal surfaces, and in \( \S 4 \) we use this to prove a special case of uniqueness for the Riemann mapping theorem of Hengartner
and Schober [HS1]. As we have shown elsewhere, uniqueness fails in general [W].

2. Definition of the height function and conjugate height function.

Using the Weierstrass representation [O, p. 63] we shall associate with \( f \), a minimal surface given parametrically in a simply connected subdomain \( N \subseteq U \) where \( a(\zeta) \) does not have a zero of odd order.

With \( g \) and \( h \) as in (1.1) we define up to an additive constant, a branch of

\[
(2.1) \quad F(\zeta) = 2i \int \sqrt{h'(\zeta)g'(\zeta)} \, d\zeta = 2i \int h'(\zeta)\sqrt{a(\zeta)} \, d\zeta = 2i \int f_\zeta(\zeta)\sqrt{a(\zeta)} \, d\zeta.
\]

Then, by (1.2) it follows that a branch of \( F \) can be defined in \( N \), and for \( \zeta \in N \),

\[
(2.2) \quad \zeta \rightarrow (f(\zeta), \Re F(\zeta))
\]

gives a parametric representation of a minimal surface. Here we have identified \( \mathbb{R}^2 \) with \( \mathbb{C} \) by \( (x, y) \leftrightarrow (\Re f, \Im f) \).

Let \( \hat{U} \) be the Riemann surface of the function \( \sqrt{a(\zeta)} \). Then \( \hat{U} \) has algebraic branch points corresponding to those points \( \zeta \in U \) for which \( a(\zeta) \) has a zero of odd order. Specifically, \( \hat{U} \) can be concretely described (the analytic configuration [Sp, 69-74]) in terms of function elements \( (\alpha, F_\alpha) \) where \( \alpha \in U \), and \( F_\alpha \) is a power series expansion of a branch of \( F \) in a neighborhood of \( \alpha \) if \( a(\zeta) \) does not have a zero of odd order at \( \zeta = \alpha \), and \( F_\alpha \) a power series in \( \sqrt{\zeta - \alpha} \) otherwise. The mapping \( p: (\alpha, F_\alpha) \rightarrow \alpha \) is the projection of the surface so realized. The mapping \( F \) may now be lifted to a mapping \( \hat{F} \) on \( \hat{U} \).

By continuation, we may induce a mapping \( \hat{U} \rightarrow \hat{U} \) to a surface \( \tilde{U} \) with a real analytic structure defined in terms of elements \( (\beta, \tilde{F}_\beta) \) with \( \beta \in f(U) \) by \( \alpha = f^{-1}(\beta) \) and \( \tilde{F}_\beta = F_\alpha \circ f^{-1} \). We again define a projection by \( \pi: (\beta, \tilde{F}_\beta) \rightarrow \beta \).

We shall refer to a point \( \hat{\zeta} \in \hat{U} \) to be over \( \zeta \), if \( p(\hat{\zeta}) = \zeta \), and \( \tilde{z} \in \tilde{U} \) to be over \( z \) if \( \pi(\tilde{z}) = z \).

The harmonic mapping \( f: U \rightarrow f(U) \) lifts to a mapping \( \hat{f}: \hat{U} \rightarrow \hat{U} \) which is \( 1-1 \), onto, and satisfies the condition \( \pi(\hat{f}(\hat{\zeta})) = f(p(\hat{\zeta})) \) for all \( \zeta \in U \).

With these notations, we shall extend the meaning of (2.2). Thus

\[
(2.3) \quad \hat{\zeta} \rightarrow (\hat{f}(\hat{\zeta}), \Re \hat{F}(\hat{\zeta}))
\]

gives a parametric representation of a minimal surface in the sense that in a neighborhood of \( \hat{\zeta} \in \hat{U} \setminus \mathcal{B} \) where \( \mathcal{B} \) is the branch set, that is, the points
above the zeros of \( a \) of odd order, then (2.2) is the same as (2.3) computed in terms of local coordinates given by projection.

We may also define the surface nonparametrically on \( \hat{U} \setminus \hat{B} \), where \( \hat{B} = \hat{f}(B) \), as follows. Let \( D \) be an open disk in \( f(U) \) such that \( f^{-1}(D) \) contains no zeros of \( a \) of odd multiplicity. Let \( w = \varphi(x, y) \) be the nonparametric description of the minimal surface corresponding to (2.2), that is, for \( \zeta \in f^{-1}(0) \) (cf. \([HS3, p. 87]\)),

\[
\begin{align*}
(2.4) \quad x &= \Re f(\zeta) \quad y = \Im f(\zeta), \\
\varphi(x, y) &= \Re F(\zeta).
\end{align*}
\]

Then, by continuation \( \varphi \) lifts to a function \( \hat{\varphi} \) on \( \hat{U} \) which satisfies the minimal surface equation when computed in local coordinates given by projection off the branch set \( \hat{B} \). We shall call \( \hat{\varphi}(z) \) a height function corresponding to \( f \). Finally, we define a conjugate height function \( \hat{\psi}(z) \) by solving locally

\[
(2.5) \quad \psi_y = \varphi_x/W, \quad \psi_x = -\varphi_y/W \quad \left(W = \sqrt{1 + \varphi_x^2 + \varphi_y^2}\right)
\]

(cf. \([F1, p. 344]\)) and lifting to \( \hat{U} \setminus \hat{B} \) as was done for \( \varphi \). Let \( \hat{F} = \hat{\varphi} + i\hat{\psi} \).

Then \( \hat{F} \) is real analytic and locally quasiconformal on \( \hat{U} \setminus \hat{B} \), with dilatation whose magnitude is \((W - 1)/(W + 1)\). The fact that \( \hat{\psi} \) and \( \hat{F} \) are well defined on \( \hat{U} \setminus \hat{B} \) follows from Theorem 1.

A glossary of terminology is given schematically in Figure 1.
Theorem 1. With the above notations, $\hat{F} = \tilde{F} \circ \hat{f} + C$ for some constant $C$.

Proof. Let $D$ be an open disk in $f(U)$ such that $f^{-1}(D)$ contain no zeros of odd multiplicities of $a$. We fix a branch of $\sqrt{a}$ in $f^{-1}(D)$, and consider $\hat{\varphi}(\zeta) + i\hat{\psi}(\zeta) = \hat{F}(\zeta)$ for points in a component of $\hat{U}$ over $f^{-1}(D)$, and $\tilde{\varphi}(\tilde{z}) + i\tilde{\psi}(\tilde{z}) = \tilde{F}(\tilde{z})$ for points in a component of $\tilde{U}$ over $D$. Since we shall compute in local coordinates given by projection, to reduce notation in this proof, we shall subsequently write $\hat{F}$, $\hat{\varphi}$, $\hat{\psi}$ in place of $\hat{F} \circ p^{-1}$, $\hat{\varphi} \circ p^{-1}$, $\hat{\psi} \circ p^{-1}$, and $\tilde{F}$, $\tilde{\varphi}$, $\tilde{\psi}$ in place of $\tilde{F} \circ \pi^{-1}$, $\tilde{\varphi} \circ \pi^{-1}$, $\tilde{\psi} \circ \pi^{-1}$ respectively. With this notation, by (2.4) we have that

$$(2.6) \hat{\varphi} = \tilde{\varphi} \circ f,$$

so it suffices to show that

$$(2.7) \hat{\psi} = \tilde{\psi} \circ f + C.$$

The result then follows from continuation. □

In fact, since $\hat{\varphi} + i\hat{\psi}$ is analytic in $f^{-1}(D)$, it follows from (2.6) that to prove (2.7) it suffices to show that $\tilde{F} \circ f$ is analytic in $f^{-1}(D)$.

We first record the relationship between $a(\zeta)$ of (1.2) and $W(z)$ ($z = f(\zeta)$) of (2.5). This is given by [O, p. 105], [HS3, pp. 87-88] as

$$(2.8) |a| = \frac{W - 1}{W + 1}.$$

Now,

$$(2.9) (\hat{F} \circ f)_{\zeta} = \hat{F}_{\tilde{z}} f_{\zeta} + \hat{F} z_{\zeta} = \hat{F}_{\tilde{z}} f_{\zeta} + \hat{F} z_{\zeta}. f_{\zeta}.$$

A simple computation using (2.5) gives

$$F_{\tilde{z}} = \frac{W + 1}{W} \varphi_{\tilde{z}}, \quad F_{\zeta} = \frac{W - 1}{W} \varphi_{\zeta}.$$

When used in (2.9) these give

$$(2.10) (\hat{F} \circ f)_{\zeta} = \frac{W + 1}{W} \varphi_{\tilde{z}} f_{\zeta} + \frac{W - 1}{W} \varphi_{\zeta} f_{\zeta}.$$

Again, a direct computation gives

$$\hat{\varphi}_{\tilde{z}} = \frac{\hat{\varphi} f_{\tilde{z}} (f_{\zeta}) - \hat{\varphi} f_{\zeta} (f_{\tilde{z}})}{|f_{\zeta}|^2 - |f_{\tilde{z}}|^2} , \quad \hat{\varphi}_{\zeta} = \frac{\hat{\varphi} f_{\zeta} (f_{\tilde{z}}) - \hat{\varphi} f_{\tilde{z}} (f_{\zeta})}{|f_{\zeta}|^2 - |f_{\tilde{z}}|^2}.$$

When used in (2.10) this gives

$$(2.11) (\hat{F} \circ f)_{\zeta} = \frac{1}{W(|f_{\tilde{z}}|^2 - |f_{\zeta}|^2)} \left( 2 \hat{\varphi} f_{\zeta} (f_{\tilde{z}}) + \hat{\varphi} f_{\zeta} |f_{\zeta}|^2 \left( W - 1 - \frac{|f_{\tilde{z}}|^2}{|f_{\zeta}|^2} (W + 1) \right) \right).$$
Now, by (1.2), (2.1), and (2.8) we have,
\[ \hat{\psi}_\zeta = ig'/\sqrt{a}, \quad \hat{\psi}_\zeta = -i\hat{g}'/\sqrt{a}, \quad f_\zeta = g'/a, \quad f_\zeta = \hat{g}, \]
and
\[ W - 1 - \frac{|f_\zeta|^2}{|f_{\bar{\zeta}}|^2}(W + 1) = W - 1 - |a|^2(W + 1) = 2(W - 1)/(W + 1). \]
Substituting into (2.11) we obtain
\[ (\tilde{F} \circ f)_\zeta = \frac{1}{W(|f_\zeta|^2 - |f_{\bar{\zeta}}|^2)} \left( \frac{2ig'(g')^2}{\sqrt{aa}} - \frac{2i\hat{g}'|g'|^2}{\sqrt{a}a^2} \left( \frac{W - 1}{W + 1} \right) \right) \]
\[ = 0. \]
Thus, \( \tilde{F} \circ f \) is analytic and (2.7) follows.

3. The height function corresponding to Poisson integrals of step functions.

Let \( \mathcal{P} \) be a polygon with vertices \( c_1, \ldots, c_n \) given cyclically, and in order induced by a positive orientation of \( \partial \mathcal{P} \). Let \( f \) be the Poisson integral of a step function on \( \partial U \) having values \( c_1, \ldots, c_n \) and suppose that \( f \) is then a univalent harmonic mapping, \( f: U \to \mathcal{P} \). If \( \mathcal{P} \) is convex, for example, this will always be the case \([C], [K]\). The analytic dilatation \( a(\zeta) \) for such mappings were studied in \([HS2]\) and \([S-S]\). In general, \( a(\zeta) \) is a Blaschke product of order at most \( n - 2 \), and of order precisely \( n - 2 \) if \( \mathcal{P} \) is convex \([S-S], \text{pp. 469, 473}\).

We shall now explore the boundary behavior of height functions corresponding to such mappings. The prototype for this is Scherk’s minimal surface over the square \(-\pi/2 < x < \pi/2, -\pi/2 < y < \pi/2\), given by
\[ \psi(x, y) = \log(\cos x/\cos y) \]
which tends to \(+\infty\) and \(-\infty\) over alternate sides. It seems remarkable that this type of behavior persists in general for height functions corresponding to all such \( f \) described above.

**Theorem 2.** Let \( \mathcal{P} \) be a polygon having vertices \( c_1, \ldots, c_n \) given cyclically, and ordered by a positive orientation on \( \partial \mathcal{P} \). Let \( f \) be a univalent harmonic mapping of \( U \) such that \( f \) is the Poisson integral of a step function having the ordered sequence \( c_1, \ldots, c_n \) as its values. Then the analytic dilatation \( a(\zeta) \) of \( f \) is a finite Blaschke product of order at most \( n - 2 \), \( f(U) = \mathcal{P} \), and if \( \varphi \) is a height function for \( f \), then \( \varphi \) tends to \(+\infty\) or \(-\infty\) at points over the open segments making up the sides of \( \mathcal{P} \). If \( \mathcal{P} \) is convex, then \(+\infty\) and \(-\infty\) alternate on adjacent sides.
Proof. That $a(\zeta)$ is a Blaschke product of order at most $n - 2$ and $f(U) = \mathcal{P}$ follow from general properties of Poisson integrals [S-S, p. 469], [HS2, p. 203]. □

Let $f = h + g$ as in (1.1). Then we may write $h'$ and $g'$ in the form [S-S, pp. 460-461]

$$h'(\zeta) = \sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k}, \quad g'(\zeta) = -\sum_{k=1}^{n} \frac{\overline{\alpha_k}}{\zeta - \zeta_k},$$

where $\alpha_k \neq 0, \ k = 1, \ldots , n$.

With $F$ as in (2.1), we are then interested in the branches of

(3.2) $$F(\zeta) = -2 \int \frac{\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^{n} \frac{-\overline{\alpha_k}}{\zeta - \zeta_k}}{1 + o(1)} d\zeta$$

as $\zeta \to \zeta_k, \ k = 1, \ldots , n$. The cluster sets for the nontangential approaches to points over the $\zeta_k$ give the points lying over the open segments making up the sides of $\mathcal{P}$.

Thus, take a vertex $\zeta_j$, and an open segment $l_j$ of $\partial \mathcal{P}$ corresponding to it. Then, as $\zeta \to \zeta_j$,

$$\sum_{k=1}^{n} \frac{\alpha_k}{\zeta - \zeta_k} \sum_{k=1}^{n} \frac{-\overline{\alpha_k}}{\zeta - \zeta_k} = |\alpha_j|^2 \left(1 + o(1)\right),$$

and hence, by (3.2), a branch of $F$ satisfies

(3.3) $$F(\zeta) = \pm 2|\alpha_j| \log(\zeta - \zeta_j) + o(1)$$

as $\zeta \to \zeta_j$, for a fixed branch of the log. Suppose the fixed branch of (3.3) has minus sign, and let $\phi(z) = \text{Re} F \circ f^{-1}(z)$ be a corresponding branch in $\mathcal{P}$ for points near the corresponding side $l_j$. Now suppose $\mathcal{P}$ is convex and $F(\zeta)$ is analytically continued to an adjacent point, say $\zeta_{j+1}$, so that $\phi$ is then continued to a corresponding side $l_{j+1}$ having common endpoint $c_j$ with $l_j$. Since $\phi \to -\infty$ as $z \to l_j$, it remains to show that $\phi \to +\infty$ as $z \to l_{j+1}$. This effect has been noted for minimal surfaces [JS], and can be accomplished by a simple barrier argument. I thank Professor Finn for pointing this out.

Let $0 < \beta < \pi$ be the angle in $\mathcal{P}$ between $l_j$ and $l_{j+1}$. Suppose that $\phi \to -\infty$ on both open segments $l_j$ and $l_{j+1}$. Since $\phi$ satisfies the minimal surface equation, $\phi$ can only tend to $-\infty$ over line segments [O, p. 102]. Since we make no assumption at the common endpoint $c_j$, in order to get a contradiction we must show that $\phi \to -\infty$ at $c_j$ as well. We may assume that $c_j = (\pi/2, 0)$, and $l_j, l_{j+1}$ make the angle $\beta$ symmetrically with respect to the $x$ axis, opening toward the origin. Let $0 < \varepsilon < (\pi/2) \cot(\beta/2)$ be small enough so that the isosceles triangle $N$ formed by the sector and the line $x = \pi/2 - \varepsilon$ has the given branch of $F$ single valued. Then, two of
the sides of $N$ are contained in the segments $l_j$ and $l_{j+1}$, and the third is $x = \pi/2 - \varepsilon, -\delta < y < \delta$, where $\delta = \varepsilon \tan(\beta/2)$. If $\psi$ is the height function for Scherk's surface given by (3.1), then for any $M > 0$, clearly

$$\phi(x, y) < -\psi(x - \pi + \varepsilon, y) - M$$

on $\partial N \setminus \{c_j\}$. By the extended maximum principle [F1, pp. 342-343], it follows that (3.4) holds throughout $N$. Since $M > 0$ was arbitrary, it follows that $\phi \equiv -\infty$ on $N$, a contradiction. Thus $\phi = +\infty$ on $l_{j+1}$.

4. An application to the Riemann mapping theorem.

One of the most basic results in the theory of univalent harmonic mappings is the Riemann mapping theorem of Hengartner and Schober [HS1].

**Theorem A.** Let $D$ be a bounded simply connected domain whose boundary is locally connected. Fix $w_0 \in D$, and let $a(\zeta)$ be analytic in $U$, with $a(U) \subseteq U$. Then there exists a univalent harmonic mapping $f$ with the following properties.

a) $f$ maps $U$ into $D$ and $f(0) = w_0, f_z(0) > 0$.

b) $f$ satisfies the equation $(f_\zeta) = af_\zeta$.

c) Except for a countable set $E \subseteq \partial U$, the unrestricted limit $f^*(e^{it}) = \lim_{\zeta \to e^{it}} f(\zeta)$ exists and belongs to $\partial D$.

d) The one sided limits $\lim_{\tau \to t^+} f^*(e^{i\tau})$, $\lim_{\tau \to t^-} f^*(e^{i\tau})$ through values of $e^{i\tau} \not\in E$ exist and belong to $\partial D$; for $e^{it} \not\in E$ they are equal and for $e^{it} \in E$ they are different.

e) The cluster set of $f$ at $e^{it} \in E$ is the straight line segment joining the left and right limits in d).

If in Theorem A, the set $D$ is convex, and $a(\zeta)$ is a finite Blaschke product, one can say more [HS2, p. 203], [S-S, p. 473].

**Theorem B.** Let $f$ be as in Theorem A with $D$ bounded and convex, and $a(\zeta)$ a Blaschke product of order $n - 2$. Then $f(U)$ is a polygon with $n$ vertices all of which lie on $\partial D$.

We shall prove uniqueness in the case $a(\zeta) = \zeta^n$ and $D$ convex. The case of uniqueness when $D = U$ and $a(\zeta) = \zeta$ was done in [HS2, p. 204].

The proof involves a combinatorial argument with the level sets of the height function. Such arguments are often useful in the theory of partial differential equation, and in particular the minimal surface equation [F1], [FO], [JS], [Se].

**Theorem 3.** The solution $f(\zeta)$ to the Riemann mapping theorem above with $D$ convex and

$$a(\zeta) = \zeta^{n-2}$$
is unique for each \( n = 3, 4, \ldots \)

**Proof.** Let \( f_1 \) and \( f_2 \) be Riemann mappings corresponding to \( D \). We may assume \( f_1(0) = f_2(0) = 0 \). Let \( \Delta \) be a disk centered at 0, and contained in \( f_1(U) \cap f_2(U) \).

If \( n \) is even, then \( \hat{U} = U \) and if \( n \) is odd \( \hat{U} \) is a two sheeted cover of \( U \) with branch point over 0. Similarly, if \( \hat{U}_1 \) corresponds to \( f_1(U) \) and \( \hat{U}_2 \) to \( f(U_2) \), then \( \hat{U}_1 \) and \( \hat{U}_2 \) are one or two sheeted according as \( n \) is even or odd.

We consider the case where \( n \) is odd. The even case goes the same way, but is simpler since one can bypass discussion of Riemann surfaces.

Let \( \varphi_j, \psi_j, \tilde{\varphi}_j, \tilde{\psi}_j, \tilde{F}_j, \tilde{U}_j, \pi_j, \ j = 1, 2 \) be the quantities of §2 defined for \( f_1 \) and \( f_2 \) respectively. We may assume that \( \tilde{F}_1(0) = \tilde{F}_2(0) = 0 \). If \( \hat{\Delta} \) represents the Riemann surface of \( \sqrt{z} \) over \( \Delta \), then we may consider \( \hat{\Delta} \subseteq \tilde{U}_1 \) and \( \hat{\Delta} \subseteq \tilde{U}_2 \), so that \( \tilde{F}_1 \) and \( \tilde{F}_2 \) may both be considered as defined for all \( \hat{z} \in \hat{\Delta} \). For brevity of notation, we shall write \( \tilde{F} \) for \( \tilde{F} \circ \pi^{-1} \).

Since the analytic dilatation for \( f_1(\zeta) \) and \( f_2(\zeta) \) is 0 when \( \zeta = 0 \), it follows from (1.2), (4.1), and a) of Theorem A, that

\[
(4.2) \quad f_j(\zeta) = c_j\zeta(1 + o(1)) \quad (\zeta \to 0, \ c_j > 0, \ j = 1, 2).
\]

Then, from (2.1), (4.1), (4.2), and Theorem 1 we may take determinations of \( \tilde{F}_1 \) and \( \tilde{F}_2 \) in \( \hat{\Delta} \) so that

\[
(4.3) \quad \tilde{\varphi}_j(z) + i\tilde{\psi}_j(z) = \tilde{F}_j(z) = d_jz^{n/2}(1 + o(1)) \quad (j = 1, 2 \ z \to 0)
\]

with \( d_1, d_2 > 0 \) and \( z^{n/2} \) is some fixed branch.

Having thus fixed branches in (4.3) we may then take a constant \( \lambda > 0 \) such that

\[
(4.4) \quad \tilde{F}_1(z) - \lambda \tilde{F}_2(z/\lambda) = Cz^{n/2}(1 + o(1)) \quad (z \to 0)
\]

for some constant \( C \) and integer \( p \geq n \). We suppose \( \lambda \geq 1 \); otherwise we interchange \( \tilde{F}_1 \) and \( \tilde{F}_2 \). Now, the change from \( F(z) \) to \( \lambda F(z/\lambda) \) corresponds to replacing \( f \) by \( \lambda f \). Then the analytic dilatation is unchanged, and following the change in (2.1) it gives the parametrization \( \zeta \to (\lambda f(\zeta), \Re \lambda F(\zeta)) \).

Let \( \varphi_3, \psi_3, \tilde{\varphi}_3, \tilde{\psi}_3 \) correspond to \( f_3 = \lambda f_2 \) so that \( f_3(U) \), is nothing more than \( f_1(U) \) dilated by the constant \( \lambda \geq 1 \), and (4.5) becomes

\[
(4.5) \quad \tilde{F}_1(\hat{z}) - \tilde{F}_3(\hat{z}) = Cz^{n/2}(1 + o(1)) \quad (z \to 0).
\]

**Case 1.** \( C = 0 \) for every \( p \). Since \( \tilde{F}_1(z^2) - \tilde{F}_3(z^2) \) is real analytic, then \( \tilde{F}_1 \equiv \tilde{F}_3 \). Thus, in particular \( \lambda = 1 \) and \( f_1(U) = f_3(U) = \mathcal{P} \). In order to show that \( f_1 \equiv f_3 \) we use the subordination principle of [BHH, p. 170]. Briefly, since \( \mathcal{P} \) is a convex polygon by Theorem B, and \((f_1)_z(0), (f_3)_z(0) > 0\), we may apply the argument principle in [BHH, p. 170] to

\[
G(z) = (f_3)_z(0)f_1(z) - (f_1)_z(0)f_3(z)
\]
to deduce that \((f_1)_2(0) = (f_3)_2(0)\). Then, another application of the argument principle as in [BHH] to \(G_2(z) = (1 + \varepsilon)f_1(z) - f_3(z) (\varepsilon \to 0)\) shows that \(f_1 \equiv f_3\).

**Case 2.** \(C \neq 0\) for some \(p \geq n\). In this case, near the origin on \(\Delta\), by (4.5) there are \(2p + 4\) level curves \(\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0\) emanating from \(\tilde{0}\). Between the level curves, \(\tilde{\varphi}_1 - \tilde{\varphi}_3\) alternates in sign. In order to analyze the component sets between the level sets, we must modify \(f_3\).

Let \(\eta_1, \eta_2, \ldots\) be homeomorphisms of \(|\zeta| = 1\) onto the boundary of \(\lambda D\), which converge to the (step function) boundary values of \(f_3\), and let \(f_3^{(n)}\), \(n = 1, 2, \ldots\) their corresponding Poisson integrals so that \(f_3^{(n)} \to f_3\) uniformly on compact subsets of \(U\).

The level sets of \(\tilde{\varphi}_1 - \tilde{\varphi}_3 = 0\) create \(2p + 4\) disjoint component open sets \(O_1, O_2, \ldots, O_{2p+4}\) where \(\tilde{\varphi}_1 - \tilde{\varphi}_3 > 0\) in \(O_{2j-1}\) and \(\tilde{\varphi}_1 - \tilde{\varphi}_3 < 0\) in \(O_{2j}\) for \(j = 1, \ldots, p + 2\). These components alternate in position around the origin.

For \(\varepsilon > 0\) we can find nonempty components at \(O_1(\varepsilon), O_2(\varepsilon), \ldots, O_{2p+4}(\varepsilon)\) where \(\tilde{\varphi}_1 - \tilde{\varphi}_3^{(n)} > \varepsilon\) in \(O_{2j-1}(\varepsilon)\), \(\tilde{\varphi} - \tilde{\varphi}_3^{(n)} = \varepsilon\) on \(\partial O_{2j-1}(\varepsilon)\), \(\Delta \cap O_{2j-1}(\varepsilon) \subseteq O_{2j-1}, j = 1, \ldots, 2p\), and analogous statements hold for \(O_{2j}(\varepsilon), j = 1, \ldots, p + 2\).

Now, \(f_3^{(j)}(U) = \lambda D\), so by the maximum principle for solutions to the minimal surface equation, the level sets forming the boundaries of the \(O_j(\varepsilon)\)'s must extend to points over the boundary of \(P = f_1(U)\). As in [FO, pp. 357-358], we observe that since \(\tilde{F}_1\) is \(\pm \infty\) over the sides of \(P\) by Theorem 2, if a component \(O_j(\varepsilon)\) has a boundary point over an interior point of a side of \(P\), then the boundary must contain that side. Since, by Theorem B, \(P\) has \(n\) sides, then \(\tilde{P} = \pi_1^{-1}(P)\) has \(2n\) sides. This implies that there are at most \(2n\) sets \(O_j(\varepsilon)\) whose boundaries have interior points over \(\partial P\). If \(O_j(\varepsilon)\) were a component whose boundary contained no points over \(\partial P\), then its boundary could only be interior points over \(P\), or vertices. As pointed out in [FO, p. 358], this is impossible by a theorem of Finn [F1, pp. 342-343]. Thus, \(2p + 4 \leq 2n\). Since \(p \geq n\), we obtain a contradiction and the theorem is proved. □

**References**


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