CHARACTERIZATION OF THE HOMOGENEOUS POLYNOMIALS $P$ FOR WHICH $(P + Q)(D)$ ADMITS A CONTINUOUS LINEAR RIGHT INVERSE FOR ALL LOWER ORDER PERTURBATIONS $Q$

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Those homogeneous polynomials $P$ are characterized for which for arbitrary lower order polynomials $Q$ the partial differential operator $(P + Q)(D)$ admits a continuous linear right inverse if regarded as an operator from the space of all $C^\infty$-functions on $\mathbb{R}^n$ into itself. It is shown that $P$ has this property if and only if $P$ is of principal type and real up to a complex constant and has no elliptic factor.

1. Introduction.

The problem of L. Schwartz to characterize those linear partial differential operators $P(D)$ with constant coefficients that admit a continuous linear right inverse on $C^\infty(\Omega)$ or $\mathcal{D}'(\Omega)$, $\Omega$ an open set in $\mathbb{R}^n$, $n \geq 2$, was solved in Meise, Taylor, and Vogt [9]. They derived various equivalent conditions for this property. When $\Omega$ is convex, it is equivalent to a condition PL$(\Omega, \log)$ of Phragmén-Lindelöf type for plurisubharmonic functions on the algebraic variety

$$V(P) := \{z \in \mathbb{C}^n : P(-z) = 0\}.$$ 

Using this characterization they showed in [12], Theorem 4.1, that when $V(P)$ has PL$(\Omega, \log)$, then also $V(P_m)$ has PL$(\Omega, \log)$, where $P_m$ denotes the principal part of $P$, which is a homogeneous polynomial of degree $m$. In other words, if $P(D)$ admits a right inverse on $C^\infty(\Omega)$, so does $P_m(D)$. The converse implication fails in general, as the example $(\frac{\partial}{\partial x})^2 - (\frac{\partial}{\partial y})^2 + \frac{\partial}{\partial z}$ shows. Since the condition PL$(\Omega, \log)$ for $V(P_m)$ is easier to check than for $V(P)$, one would like to know additional conditions on $P_m$ which imply that for some or all lower degree perturbations $Q$ the operator $(P_m + Q)(D)$ admits a right inverse on $C^\infty(\Omega)$. A first result of this type is Corollary 5.8 of [12] which states the following: If $P_m$ is homogeneous of degree $m$, grad $P_m(z) \neq 0$ for all $z \in \mathbb{C}^n \setminus \{0\}$, and $V(P_m)$ satisfies PL$(\mathbb{R}^n, \log)$, then $V(P_m + Q)$ satisfies PL$(\mathbb{R}^n, \log)$ for each polynomial $Q$ of degree less than $m$.

In the present paper we prove the following extension of this result:
Theorem 1.1. For each polynomial $P_m \in \mathbb{C}[z_1, \ldots, z_n]$, homogeneous of degree $m \geq 2$, the following conditions are equivalent:

1. $(P_m + Q)(D)$ admits a continuous linear right inverse on $C^\infty(\mathbb{R}^n)$ and/or $\mathcal{D}'(\mathbb{R}^n)$ for each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ of degree less than $m$,
2. $\text{grad } P_m(x) \neq 0$ for each $x \in \mathbb{R}^n \setminus \{0\}$, $P_m$ is real up to a complex constant, and each irreducible factor of $P_m$ has a non-trivial real zero.

In particular, each operator $P(D)$ of principal type admits a right inverse on $C^\infty(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ whenever its principal part $P_m$ is real and no irreducible factor of $P_m$ is elliptic. Note that these operators $P(D)$ admit fundamental solutions with large lacunas, as the results of Meise, Taylor, and Vogt [8], [9] imply (see 4.8). Also, Theorem 1.1 proves finally what had been suggested by many examples (see [12], Example 4.9, [13], Lemma 4), namely that the existence of real non-zero singular points in $V(P_m)$ implies the existence of a perturbation $Q$ of degree less than $m$ for which $(P_m + Q)(D)$ does not admit a right inverse on $C^\infty(\mathbb{R}^n)$.

The proof of Theorem 1.1 in one direction is a modification of the result of Meise, Taylor, and Vogt [12] mentioned above. For the other direction we use the concept of quasihomogeneity of polynomials. We show that this notion together with [12], Lemma 4.7, provides a systematic method to find necessary conditions for $V(P)$ to satisfy PL($\mathbb{R}^n, \log$) which can be checked easily and directly on the given polynomial $P$.

2. Preliminaries.

In this section we introduce some of the definitions that are used in this paper. First we recall the definition of a weight function from [1], then we introduce conditions of Phragmén-Lindelöf type for algebraic varieties according to Meise, Taylor, and Vogt [9], [11], [12] and we explain the significance of these conditions.

Throughout the paper, $|\cdot|$ will denote the euclidean norm and $B_\epsilon(z) = \{w \in \mathbb{C}^n : |w - z| < \epsilon\}$ an open ball in that norm. Zero is not a natural number.

Definition 2.1. Let $\omega : [0, \infty[ \to ]0, \infty[$ be continuous and increasing and assume that it has the following properties:

- (a) $\omega(2t) = O(\omega(t))$
- (b) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$
- (c) $\log t = O(\omega(t))$, as $t$ tends to infinity
- (d) $x \mapsto \omega(e^x)$ is convex.

By abuse of notation, $\omega : z \mapsto |z|, z \in \mathbb{C}^n$, will be called a weight function. Throughout this paper we assume that $\omega(0) \geq 1$. It is easy to check that this can be assumed without loss of generality.
Note that each weight function satisfies $\omega(z) = o(|z|)$. Moreover, each weight function is plurisubharmonic in $\mathbb{C}^n$ in view of 2.1($\delta$).

**Definition 2.2.** Let $V$ be an algebraic variety of pure dimension $k$ in $\mathbb{C}^n$ and $\Omega$ an open subset of $V$. A function $u : \Omega \to [-\infty, \infty]$ will be called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on $\Omega_{reg}$, the set of all regular points of $V$ in $\Omega$, and satisfies

$$u(z) = \limsup_{\xi \to z_{reg}} u(\xi)$$

at the singular points of $V$ in $\Omega$. By $\text{PSH}(\Omega)$ we denote the set of all plurisubharmonic functions on $\Omega$.

**Definition 2.3.** Let $V \subset \mathbb{C}^n$ be an algebraic variety and let $\omega$ be a weight function. Then $V$ satisfies the condition PL($\mathbb{R}^n, \omega$) if the following holds:

There exists $A \geq 1$ such that for each $\rho > 1$ there exists $B > 0$ such that each $u \in \text{PSH}(V)$ satisfying ($\alpha$) and ($\beta$) also satisfies ($\gamma$), where:

($\alpha$) $u(z) \leq |\text{Im} z| + O(\omega(z))$, $z \in V$,
($\beta$) $u(z) \leq \rho |\text{Im} z|$, $z \in V$,
($\gamma$) $u(z) \leq A |\text{Im} z| + B \omega(z)$, $z \in V$.

2.4. Phragmén-Lindelöf conditions and continuous linear right inverses. To explain the significance of the condition PL($\mathbb{R}^n, \omega$), let $P(z) = \sum_{|\alpha| \leq m} a_{\alpha} z^\alpha$ be a complex polynomial of degree $m > 0$ and let

$$V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}$$

denote its zero variety. Then $V(P)$ satisfies PL($\mathbb{R}^n, \omega$) if and only if the linear partial differential operator

$$P(D) : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n), \quad P(D)f := \sum_{|\alpha| \leq m} a_{\alpha} i^{-|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$$

admits a continuous linear right inverse, where $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ is the Fréchet space of all $\omega$-ultradifferentiable functions of Beurling type (see [1]). This follows from the general characterization in Meise, Taylor, and Vogt [11]. Note that for $\omega(t) = \log(1+t)$, i.e., $\mathcal{E}_{(\omega)}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, this was obtained earlier in [9] and that Palamodov [15] proved that a differential complex of $C^\infty$-functions over $\mathbb{R}^n$ splits if and only if the associated varieties satisfy PL($\mathbb{R}^n, \log$).

From Meise, Taylor, and Vogt [12], 4.7, we recall the following lemma which for many examples was the only tool to show that they do not satisfy PL($\mathbb{R}^n, \omega$) for some weight function $\omega$.

**Lemma 2.5.** Let $V$ be an algebraic variety in $\mathbb{C}^n$ that satisfies PL($\mathbb{R}^n, \omega$) with constants $A > 0$ and $B_\rho$ for $\rho > 0$, according to 2.3. Assume that for some $M \geq 1$ and some $z_0 \in V$ we have $|\text{Im} z| \leq M |\text{Im} z_0|$ for all $z$ in the
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connected component \( \tilde{V}_{z_0} \) of \( z_0 \) in the set \( V \cap \{ z \in \mathbb{C}^n : |z - z_0| < t|\text{Im} \ z_0| \} \), where \( t \geq 2A + 4 \). Then \( z_0 \) satisfies \( |\text{Im} \ z_0| \leq B(A + 2)M + 1 \omega(z_0) \).

3. Quasihomogeneous Polynomials.

In this section we use the concept of quasihomogeneity together with the lemma of Meise, Taylor, and Vogt \([12]\) stated in 2.5 above to derive conditions on a given polynomial \( P \) which imply that \( V(P) \) fails \( \text{PL}(\mathbb{R}^n, \omega) \) for weight functions \( \omega \) which are growing not too fast. These conditions can be checked easily by looking at the powers of the monomials appearing in \( P \).

**Definition 3.1.** For \( d = (d_1, \ldots, d_n) \neq (0, \ldots, 0) \) with \( d_j \in \mathbb{N}_0, 1 \leq j \leq n \), a polynomial \( P \in \mathbb{C}[z_1, \ldots, z_n] \) is said to be \( d \)-quasihomogeneous of degree \( m \geq 0 \) if
\[
P(z) = \sum_{\langle d, \alpha \rangle = m} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,
\]
where \( \langle d, \alpha \rangle = \sum_{j=1}^n d_j \alpha_j \) and where not all \( a_\alpha \) vanish. The zero polynomial is considered to be \( d \)-quasihomogeneous of degree \(-\infty\).

**Remark.** The concept of quasihomogeneity is widely used in the theory of partial differential operators. We would like to mention, e.g., the theory of semi-elliptic operators (see Hörmander \([5]\)) and the recent books of Gindikin and Volevich \([2]\) and Laurent \([6]\).

**Lemma 3.2.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be \( d \)-quasihomogeneous of degree \( m > 0 \) and let \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) be a sum of \( d \)-quasihomogeneous polynomials of degrees less than \( m \). Assume further that the following conditions are fulfilled:

1. \( d_1 < d_j \) for \( 2 \leq j \leq n \),
2. there exists \( \zeta = (\zeta_1, \zeta'') \in V(P) \) with \( \zeta_1 \notin \mathbb{R}, \zeta'' \in \mathbb{R}^{n-1}, \) and \( \zeta'' \neq 0 \),
3. the polynomial \( \lambda \mapsto P(\lambda, \zeta'') \) does not vanish identically.

If \( V(P + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) for some weight function \( \omega \) and \( D = \max \{ d_j : \zeta_j \neq 0 \} \), then \( \omega \) satisfies \( t^{d_1/D} = O(\omega(t)) \) as \( t \) tends to infinity.

**Proof.** By (2) and (3) we can choose

\[
0 < \delta \leq \frac{1}{4} |\text{Im} \ \zeta_1|
\]

so that \( \zeta_1 \) is the only zero of \( \lambda \mapsto P(\lambda, \zeta_2, \ldots, \zeta_n) \) in the disk \( B_\delta(\zeta_1) \) and that

\[
\eta := \inf_{|\lambda| = \delta} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.
\]
By a compactness argument there exists \( \varepsilon_0 > 0 \) so that whenever \( |z_k - \zeta_k| \leq \varepsilon_0 \) for \( 2 \leq k \leq n \) and \( |\lambda| = \delta \) we have

\[
|P(\zeta_1 + \lambda, z_2, \ldots, z_n)| \geq \eta/2.
\]

Next fix \( R \geq 1 \) and let

\[
s(\lambda) := \frac{1}{R^n}(P + Q)(R^{d_1}(\zeta_1 + \lambda), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n), \quad \lambda \in \mathbb{C}.
\]

By hypothesis, we have \( Q = \sum_{k=0}^{m-1} Q_k \) where \( Q_k \) is zero or \( d \)-quasihomogeneous of degree \( k \). Since \( P \) is \( d \)-quasihomogeneous of degree \( m \), it follows that

\[
s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}}Q_k(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n).
\]

Hence there exists \( R_0 > 1 \) such that for \( R \geq R_0 \)

\[
|s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| \leq \eta/4 \quad \text{if } |\lambda| = \delta.
\]

Because of this and (3.1), Rouché’s theorem implies that for each \( R \geq R_0 \) there exists \( \lambda(R) \in \mathbb{C} \) satisfying \( |\lambda(R)| < \delta \) and \( s(\lambda(R)) = 0 \). Hence

\[
z(R) := (R^{d_1}(\zeta_1 + \lambda(R)), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n)
\]

belongs to \( V(P + Q) \). By (2) we have

\[
|\text{Im } z(R)| = R^{d_1}|\text{Im } (\zeta_1 + \lambda(R))|.
\]

Since \( |\lambda(R)| < \delta \leq \frac{1}{4}|\text{Im } \zeta_1| \), we have

\[
(3.2) \quad \frac{3}{4}R^{d_1}|\text{Im } \zeta_1| \leq |\text{Im } z(R)| \leq \frac{5}{4}R^{d_1}|\text{Im } \zeta_1|.
\]

Now assume that \( V(P + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) with constants \( A > 0 \) and \( B_\rho \) for \( \rho > 0 \), and let \( t := 2A + 4 \). We claim:

(*) There exist \( R_1 \geq R_0 \) and \( M > 0 \) such that for each \( R \geq R_1 \) and each \( z \) in the connected component \( V_z(r) \) containing \( z(R) \) of the set

\[
V(P + Q) \cap \{ z \in \mathbb{C}^n : |z - z(R)| < t|\text{Im } z(R)| \}
\]

we have \( |\text{Im } z| \geq \frac{1}{4t}|\text{Im } z(R)| \).

Assume for a moment that this claim is shown. Then it follows from Lemma 2.5 and (3.2) that for some constant \( C > 0 \) and all \( R \geq R_1 \) we have

\[
\frac{3}{4}R^{d_1}|\text{Im } \zeta_1| \leq |\text{Im } z(R)| \leq C\omega(z(R)).
\]

It is no restriction to assume \( \zeta_n \neq 0 \) and \( d_n = D \). Then there exists \( C_1 > 0 \) such that \( |z(R)| \leq C_1R^D \) for \( R \geq R_1 \) and hence

\[
R^{d_1} \leq C\omega(C_1R^D).
\]
By 2.1(α), this implies $R^{d_j/D} = O(\omega(R))$, as $R$ tends to infinity. Thus the proof of the lemma is complete once we have shown our claim ($\ast$). To do so, note that by (1) we can choose $\tilde{R}_1 \geq R_0$ so large that

$$2t|\text{Im} \zeta_1| \leq \varepsilon_0 \tilde{R}_1^{d_j-d_1} \quad \text{for} \quad 2 \leq j \leq n.$$ 

Then fix $R \geq \tilde{R}_1$ and define $\pi_{1,R} : \mathbb{C}^n \to \mathbb{C}$ by $\pi_{1,R}(z) := z_1/R^{d_1}$. Next note that for each $z \in \mathbb{C}^n$ with $|z - z(R)| \leq t|\text{Im} z(R)| \leq 2tR^{d_1}|\text{Im} \zeta_1|$ its coordinates $z_1, \ldots, z_n$ satisfy

$$(3.3) \quad \left| \frac{z_1}{R^{d_1}} - \zeta_1 \right| \leq \left| \frac{z_1}{R^{d_1}} - \pi_{1,R}(z(R)) \right| + |\lambda(R)| \leq 3t|\text{Im} \zeta_1|, \quad \left| \frac{z_j}{R^{d_j}} - \zeta_j \right| \leq \frac{2t|\text{Im} \zeta_1|}{R^{d_j-d_1}} \leq \varepsilon_0, \quad 2 \leq j \leq n.$$ 

Note further that

$$K := \{ w \in \mathbb{C}^n : |w_1 - \zeta_1| \leq 3t|\text{Im} \zeta_1|, \quad |w_j - \zeta_j| \leq \varepsilon_0, \quad 2 \leq j \leq n \}$$

is compact and hence

$$\max_{0 \leq k \leq m-1} \sup_{w \in K} |Q_k(w)| < \infty.$$ 

Therefore we can choose $R_1 \geq \tilde{R}_1$ so large that

$$(3.4) \quad \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(w) \leq \eta/4 \quad \text{for each} \quad R \geq R_1 \quad \text{and} \quad w \in K.$$ 

Next fix $R \geq R_1$ and assume that $z \in \mathbb{C}^n$ satisfies the inequalities in (3.3). Then the $d$-quasihomogeneity properties of $P$ and $Q$ imply

$$\frac{1}{R^m} (P + Q)(z) = P\left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) + \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k\left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right).$$

By (3.1) and (3.3) this implies

$$\left| \frac{1}{R^m} (P + Q)(z) \right| \geq \eta/4 \quad \text{if} \quad \left| \frac{z_1}{R^{d_1}} - \zeta_1 \right| = \delta.$$ 

This shows that

$$\pi_{1,R}(\tilde{V}_z(R)) \subset \mathbb{C} \setminus \{ \lambda \in \mathbb{C} : |\lambda - \zeta_1| = \delta \}.$$ 

Since $\pi_{1,R}$ is continuous and satisfies $|\pi_{1,R}(z(R)) - \zeta_1| = |\lambda(R)| < \delta$ and since $\tilde{V}_z(R)$ is connected, it follows that

$$\left| \frac{z_1}{R^{d_1}} - \zeta_1 \right| < \delta \leq \frac{1}{4}|\text{Im} \zeta_1| \quad \text{for each} \quad z \in \tilde{V}_z(R).$$
Hence we have for each \( z \in \widetilde{V}(R) \)
\[
|\text{Im } z| \geq |\text{Im } z_1| \geq |\text{Im } R^{d_1} \pi_1(R(z(R)))| - |\text{Im } R^{d_1} \pi_1(R(z(R))) - \text{Im } z_1| \\
\geq \frac{3}{4} R^{d_1} |\text{Im } \zeta_1| - \frac{1}{4} R^{d_1} |\text{Im } \zeta_1| = \frac{R^{d_1}}{2} |\text{Im } \zeta_1| \geq \frac{2}{5} |\text{Im } z(R)|.
\]
This shows that our claim holds with \( M = \frac{5}{2} \). \( \square \)

**Remark.** Note that the application of Meise, Taylor, and Vogt [12], Lemma 4.7, stated in Lemma 2.5, requires a good understanding of the given variety \( V \) in order to find the points \( z_0 \in V \) at which one can use this lemma. Lemma 3.2 and also Lemma 3.6 below show that there is a systematic way to find these points in \( V(P) \) if \( P \) has a non-trivial \( d \)-quasihomogeneous principal part with certain other properties. Therefore these lemmas are much easier to use than Lemma 2.5. We demonstrate this in the following examples.

**Examples 3.3.**

(a) Let \( P \in \mathbb{C}[z_1, z_2, z_3] \) be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_2 z_3.
\]
If \( V(P) \) satisfies \( \text{PL}(\mathbb{R}^3, \omega) \) for some weight function \( \omega \) then \( t^\frac{1}{2} = O(\omega(t)) \) as \( t \) tends to infinity. This is an immediate consequence of Lemma 3.2 and the following facts:
1. \( P \) is \( (1, 2, 3) \)-quasihomogeneous of degree 5
2. \( (\frac{1}{2}(-1 + i \sqrt{3}), 1, 1) \in V(P) \)
3. \( P(\lambda, 1, 1) = \lambda^2 + \lambda + 1 \).

(b) Let \( P \in \mathbb{C}[z_1, z_2, z_3] \) be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_2 z_3.
\]
If \( V(P) \) satisfies \( \text{PL}(\mathbb{R}^3, \omega) \) for some weight function \( \omega \) then \( t^\frac{1}{2} = O(\omega(t)) \) as \( t \) tends to infinity. This follows from Lemma 3.2 and the following facts:
1. \( P \) is \( (2, 3, 4) \)-quasihomogeneous of degree 8
2. \( (i, 0, 1) \in V(P) \)
3. \( P(\lambda, 0, 1) = \lambda^2 + 1 \).

(c) Let \( P \in \mathbb{C}[z_1, z_2, z_3] \) be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_2 - z_3^2.
\]
If \( V(P) \) satisfies \( \text{PL}(\mathbb{R}^3, \omega) \) for some weight function \( \omega \) then \( t^\frac{1}{2} = O(\omega(t)) \) as \( t \) tends to infinity. This follows immediately from Lemma 3.2 and the following facts:
1. \( P \) is \( (1, 2, 2) \)-quasihomogeneous of degree 4
2. \( (i, -1, 1) \in V(P) \)
(3) $P(\lambda, 1, -1) = -(\lambda^2 + 1)$.

To indicate that Lemma 3.2 can also be used to disprove conditions of Phragmén-Lindelöf type for homogeneous polynomials which have not been considered so far, we next recall the condition introduced by Hörmander [3] to characterize the differential operators $P(D)$ that are surjective on the space $\mathcal{A}(\Omega)$ of all real-analytic functions on a convex open set $\Omega \subset \mathbb{R}^n, n \geq 2$. We restrict our attention here to the case $\Omega = \mathbb{R}^n$.

**Definition 3.4.** Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m$.

(a) The variety $V(P_m)$ satisfies the condition HPL($\mathbb{R}^n$) if there exists $A \geq 1$ such that each $u \in \text{PSH}(V)$ satisfying $(\alpha)$ and $(\beta)$ also satisfies $(\gamma)$, where

\begin{align*}
(\alpha) & \quad u(z) \leq |z|, \quad z \in V(P_m), \\
(\beta) & \quad u(z) \leq 0, \quad z \in V(P_m) \cap \mathbb{R}^n, \\
(\gamma) & \quad u(z) \leq A|\text{Im} z|, \quad z \in V(P_m).
\end{align*}

(b) The variety $V(P_m)$ satisfies HPL($\mathbb{R}^n$, loc) at $\xi \in V(P_m) \cap \mathbb{R}^n$ if there exist $A \geq 0$ and $0 < r_2 < r_1$ such that each function $u$ which is plurisubharmonic on $V(P_m) \cap B_{r_1}(\xi)$ and satisfies $(\alpha)$ and $(\beta)$ also satisfies $(\gamma)$, where

\begin{align*}
(\alpha) & \quad 0 \leq u \leq 1 \text{ on } V(P_m) \cap B_{r_1}(\xi), \\
(\beta) & \quad u(z) \leq 0, \quad z \in V(P_m) \cap \mathbb{R}^n \cap B_{r_1}(\xi), \\
(\gamma) & \quad u(z) \leq A|\text{Im} z|, \quad z \in V(P_m) \cap B_{r_2}(\xi).
\end{align*}

**Remark.** For $P \in \mathbb{C}[z_1, \ldots, z_n]$ let $P_m$ denote the principal part of $P$. Hörmander has shown in [3] that the operator $P(D): \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective if and only if $V(P_m)$ satisfies HPL($\mathbb{R}^n$). The latter holds if and only if $V(P_m)$ satisfies HPL($\mathbb{R}^n$, loc) at each $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$.

**Example 3.5.** Let $P \in \mathbb{C}[z_1, \ldots, z_4]$ be defined as

$$P(z_1, \ldots, z_4) := z_1^2z_4 - z_2^2z_3.$$ 

Then $V(P)$ fails PL($\mathbb{R}^4, \omega$) for each weight function $\omega$ and $V(P)$ fails HPL($\mathbb{R}^4$). In particular $V(P)$ fails HPL($\mathbb{R}^4$, loc) at some $\xi \in V(P) \cap \mathbb{R}^n, |\xi| = 1$.

To show this, note first that $P$ is homogeneous. By Meise, Taylor, and Vogt [12], Theorem 4.1 and Corollary 2.9, this implies that $V(P)$ satisfies PL($\mathbb{R}^4, \log$) if and only if $V(P)$ satisfies PL($\mathbb{R}^4, \omega$) for each weight function $\omega$. Next note that:

1. $P$ is (2, 3, 4, 6)-homogeneous of degree 10
2. $(i, 1, -1, 1) \in V(P)$
3. $P(\lambda, 1, -1, 1) = \lambda^2 + 1$.

Therefore Lemma 3.2 implies that $V(P)$ fails PL($\mathbb{R}^4, t^{1/3}$). Hence it also fails PL($\mathbb{R}^4$, log). Since $P$ is irreducible and not elliptic, it follows from [12], Corollary 3.14, that $V(P)$ does not satisfy HPL($\mathbb{R}^4$). Since $V(P)$ satisfies the
Because of our assumptions on dimension condition, \( \dim(V(P) \cap \mathbb{R}^n) = n - 1 \), Theorem 3.13(4) of \cite{12} shows that \( V(P) \) fails HIP(\( \mathbb{R}^4, \text{loc} \)) at some \( \xi \in V(P) \cap \mathbb{R}^n, |\xi| = 1 \). Inspection of the proof of Lemma 3.2 shows that \( \xi = \lim_{R \to \infty} z(R)/|z(R)| \). So in the present example, \( \xi = (0, 0, 0, 1) \).

For our application we also need the following variant of Lemma 3.2, which for \( k = 1 \) is weaker than that lemma.

**Lemma 3.6.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be \( d \)-quasihomogeneous of degree \( m \) and let \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) be the sum of \( d \)-quasihomogeneous polynomials of degrees less than \( m \). Assume that for some \( k \), \( 1 \leq k < n \), the following conditions are fulfilled:

1. \( d_1 = \cdots = d_k < d_j \) for \( j > k \),
2. there exists \( \zeta = (\zeta', \zeta'') \in \mathbb{C}^k \times \mathbb{R}^{n-k} \) satisfying \( P(\zeta) = 0 \) and \( \zeta'' \neq 0 \),
3. if \( P(z', \zeta'') = 0 \) then \( \text{Im} z' \neq 0 \).

If \( V(P + Q) \) satisfies PL(\( \mathbb{R}^n, \omega \)) for some weight function \( \omega \) and \( D = \max\{d_j : \zeta_j \neq 0\} \), then \( \omega \) satisfies \( t^{d_1/D} = O(\omega(t)) \) as \( t \) tends to infinity.

**Proof.** From (2) and (3) it follows that the polynomial \( z' \mapsto P(z', \zeta'') \) is not constant. Since the hypotheses are invariant under a real linear change of coordinates in the \( z' \) variables, we may assume that \( z_1 \mapsto P(z_1, \zeta_2, \ldots, \zeta_n) \) is not constant. From this and (2) it follows that we can choose \( 0 < r < \frac{1}{4} |\text{Im} \zeta'| \) so that

\[
\delta := \inf_{|\lambda| = r} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.
\]

For each \( \tau \geq 1 \) we get from (3) that \( P \) does not vanish on the compact set

\[
L(\tau) := \{ x \in \mathbb{R}^n : |x_j| \leq \tau, 1 \leq j \leq k, x_j = \zeta_j, k + 1 \leq j \leq n \}.
\]

Hence there exists \( \varepsilon = \varepsilon(\tau) > 0 \) such that for \( B_{\varepsilon}(0) := \{ z \in \mathbb{C}^n : |z| \leq \varepsilon \} \) we have

\[
\eta(\tau) := \inf\{|P(z)| : z \in L(\tau) + B_{\varepsilon(\tau)}(0)\} > 0.
\]

Next note that by hypothesis we have \( Q = \sum_{k=0}^{m-1} Q_k \), where \( Q_k \) is either zero or \( d \)-quasihomogeneous of degree \( k \). Then fix \( R \geq 1 \) and consider the polynomial

\[
s(\lambda) := \frac{1}{R^m} (P + Q)(R^{d_1}(\zeta_1 + \lambda), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n).
\]

Because of our assumptions on \( d \)-quasihomogeneity, we have

\[
s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n).
\]
For \( \omega \) with constants \( 0 \leq R \leq 1 \) such that for each \( R \geq R_0 \),
\[
\sup_{|\lambda| = r} |s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| \leq \delta/2.
\]
Since \( P(\zeta) = 0 \), our choice of \( \delta \) shows that we can apply Rouché’s theorem to get the existence of a zero \( \lambda(R) \) of \( s \) satisfying
\[
|\lambda(R)| < r \leq \frac{1}{4} |\text{Im} \, \zeta'|.
\]

Now assume that \( V(P + Q) \) satisfies \( PL(\mathbb{R}^n, \omega) \) for some weight function \( \omega \) with constants \( A \geq 1 \) and \( B_\rho > 0 \) for \( \rho > 0 \). Then let \( t := 2A + 4 \) and define for \( R \geq R_0 \)
\[
z(R) := (R^{d_1}(\zeta_1 + \lambda(R)), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n).
\]
By \( V_R \) we denote the set \( V(P + Q) \cap B_{|\text{Im} \, z(R)|}(z(R)) \). We claim that the following holds:

\((*)\) There exist \( R_2 \geq R_0 \) and \( \sigma > 0 \) such that for \( R \geq R_2 \)
\[
|\text{Im} \, z| \geq \sigma |\text{Im} \, z(R)| \text{ for each } z \in V_R.
\]
To prove \((*)\) note that the choice of \( \lambda(R) \) and \( d_1 = \cdots = d_k \) imply \( z(R) \in V(P + Q) \) and \( |\text{Im} \, z(R)| = R^{d_1} |\text{Im} \, (\zeta_1 + \lambda(R), \zeta_2, \ldots, \zeta_k)| \). By the estimate for \( \lambda(R) \) this shows
\[
3 \frac{R^{d_1}}{4} |\text{Im} \, \zeta'| \leq |\text{Im} \, z(R)| \leq \frac{5}{4} R^{d_1} |\text{Im} \, \zeta'|.
\]
For \( z \in V_R \) this implies
\[
\left| \frac{z_1}{R^{d_1}} - \zeta_1 \right| \leq \frac{t |\text{Im} \, z(R)|}{R^{d_1}} + |\lambda(R)| \leq 2t |\text{Im} \, \zeta'|,
\]
\[
\left| \frac{z_j}{R^{d_j}} - \zeta_j \right| \leq 2t R^{d_1 - d_j} |\text{Im} \, \zeta'|, \quad 2 \leq j \leq n.
\]
Because of this and \((1)\) we can choose \( \tau \geq 1 \) and \( R_1 \geq R_0 \) so that
\[
\left| \frac{z_j}{R^{d_j}} \right| \leq \tau \text{ for } 1 \leq j \leq k \quad \text{and} \quad \left| \frac{z_j}{R^{d_j}} - \zeta_j \right| \leq \varepsilon(\tau) \text{ for } k + 1 \leq j \leq n
\]
whenever \( R \geq R_1 \) and \( z \in V_R \). Next note that
\[
0 = (P + Q)(z)
\]
\[
= R^m P \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) + R^m \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right)
\]
implies
\[
\left| P \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) \right| = \left| \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) \right| \leq \frac{C}{R}
\]
for some constant $C \geq 1$ and all $z \in V_R$, $R \geq R_1$. Hence we can choose $R_2 \geq R_1$, so that $\left| P \left( \frac{z_1}{R^d_1}, \ldots, \frac{z_n}{R^d_n} \right) \right| < \eta(\tau)$ whenever $R \geq R_2$, $z \in V_R$. By the definition of $\eta(\tau)$, this implies

$$R^{-d_1} |\text{Im} z'| = \left| \text{Im} \left( \frac{z_1}{R^d_1}, \ldots, \frac{z_n}{R^d_n} \right) \right| \geq \varepsilon(\tau)$$

and consequently, for $\sigma := \frac{4}{5} \frac{\varepsilon(\tau)}{|\text{Im} \zeta'|}$:

$$|\text{Im} z| \geq |\text{Im} z'| \geq \varepsilon(\tau) R^{d_1} = \sigma \frac{5}{4} R^{d_1} |\text{Im} \zeta'| \geq \sigma |\text{Im} z(R)|,$$

which proves $(*).$

From $(*)$ and Meise, Taylor, and Vogt [12], Lemma 4.7, it follows that there exists $B > 0$ such that

$$|\text{Im} z(R)| \leq B \omega(z(R))$$

for $R \geq R_2$. It is again no restriction to assume $\zeta_n \neq 0$ and $D = d_n$. From this it follows as in the proof of Lemma 3.2 that $R^{d_1/D} = O(\omega(R))$ as $R$ tends to infinity.

As an application, we give a short proof of a result of Meise and Taylor [7], 2.1.

**Corollary 3.7.** Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be of degree $m$ and assume that its principal part $P_m$ is real. Let $q \in \mathbb{C}[t]$ have degree $k < m$ and non-real leading coefficient. Set $Q(z, t) = P(z) + q(t)$ and let $\omega$ be a weight function with $\omega(t) = o(t^{k/m})$. Then $V(Q)$ does not satisfy $\text{PL}(\mathbb{R}^{n+1}, \omega)$.

**Proof.** We apply Lemma 3.6 in $n + 1$ variables with $d_1 = \cdots = d_n = k < m = d_{n+1}$. Let $b \in \mathbb{C} \setminus \mathbb{R}$ denote the leading coefficient of $q$. The $d$-quasihomogeneous principal part of $Q$ is $P_m(z) + bt^k$. Choose $\zeta' \in \mathbb{C}^n$ with $P_m(\zeta') = b$ and set $\zeta'' = -1$. Then (1), (2), and (3) of Lemma 3.6 are obviously satisfied. The claim follows from that lemma. \hfill $\square$

### 4. Main Results.

In this section we use the results of the previous one to characterize the homogeneous polynomials $P_m$ of degree $m$ in $n$ variables ($n \geq 2$) for which $V(P_m + Q)$ satisfies the condition $\text{PL}(\mathbb{R}^n, \log)$ for each perturbation $Q$ of degree less than $m$. This will also prove Theorem 1.1. For the proof we need the following lemma, which is a variation of Meise, Taylor, and Vogt [12], Lemma 5.2.

**Lemma 4.1.** For $P \in \mathbb{C}[z_1, \ldots, z_n]$ denote by $P_m$ its principal part and assume that $V(P_m)$ has $\text{PL}(\mathbb{R}^n, \log)$, that $\text{grad} P_m(x) \neq 0$ for $x \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$, and that for some weight function $\omega$ the following condition is fulfilled:
For each \( \xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1 \), there exist \( \delta_\xi, C_\xi, R_\xi > 0 \) such that
\[
\text{dist}(\zeta, V(P_m)) \leq C_\xi \omega(\zeta) \quad \text{whenever} \quad \zeta \in V(P) \text{ satisfies } |\zeta| \geq R_\xi \text{ and } |\frac{\zeta}{|\zeta|} - \xi| < \delta_\xi.
\]
Then \( V(P) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \).

The proof of Lemma 4.1 is quite analogous to that of [12], Lemma 5.2. Therefore, we will only sketch its main steps: Since \( V(P_m) \) has \( \text{PL}(\mathbb{R}^n, \log) \) by hypothesis, it follows from Meise, Taylor, and Vogt [12], Theorem 3.13, and [10], Theorem 5.1, that \( V(P) \) satisfies the condition (RPL) of [10], 2.2.

Hence there exists \( A_0 \geq 1 \) such that for each \( \rho > 1 \), there exists \( B_\rho > 0 \) such that each \( u \in \text{PSH}(V(P_m)) \) satisfying
\[
u(z) \leq |z| + o(|z|) \quad \text{and} \quad u(z) \leq \rho |\text{Im} z|, \quad z \in V(P)
\]
also satisfies
\[
u(z) \leq A_0 |z| + B_\rho, \quad z \in V(P).
\]
This a priori estimate and a compactness argument imply that it suffices to prove the desired Phragmén-Lindelöf estimate for each \( \xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1 \), in the intersection of \( V(P) \) with some small cone centered around \( \xi \) (for the precise argument we refer to the proof of Meise and Taylor [7], 4.5).

Using appropriate coordinates in such cones, these estimates are derived from (4.2) similarly as in the proof of [12], Lemma 5.2.

To state our main result, we recall the following definition from Hörmander [5], 10.4.11.

**Definition 4.2.** \( P \in \mathbb{C}[z_1, \ldots, z_n] \) is said to be of principal type if its principal part \( P_m \) satisfies
\[
\sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial z_j}(x) \right|^2 \neq 0 \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\}.
\]

Note that by Euler’s rule \( \langle x, \text{grad } P_m(x) \rangle = mP_m(x) \), so \( P \) is of principal type if and only if
\[
\text{grad } P_m(x) \neq 0 \quad \text{for each} \quad x \in \mathbb{R}^n \setminus \{0\} \quad \text{satisfying} \quad P_m(x) = 0.
\]

**Theorem 4.3.** Let \( n \geq 2 \) and let \( P_m \in \mathbb{C}[z_1, \ldots, z_n] \) be homogeneous of degree \( m \geq 2 \). Then the following conditions are equivalent:

1. For each \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) with \( \deg Q < m \), the variety \( V(P_m + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \log) \).

2. \( P_m \) is of principal type, \( P_m \) is real up to a complex constant, and each irreducible factor \( q \) of \( P_m \) has a real zero \( \xi \neq 0 \).

**Proof.** (1) \( \Rightarrow \) (2): By hypothesis, \( V(P_m) \) has \( \text{PL}(\mathbb{R}^n, \log) \). Hence it follows from Meise, Taylor, and Vogt [12], Theorem 3.13, that \( \dim V(q) \cap \mathbb{R}^n = n-1 \).
for each irreducible factor \( q \) of \( P_m \). Thus, the third condition in (2) is fulfilled.

To prove that \( P_m \) is of principal type, note first that by Meise, Taylor, and Vogt [13], Lemma 2, there exists \( \lambda \in \mathbb{C} \setminus \{0\} \) so that \( \lambda P_m \in \mathbb{R}[z_1, \ldots, z_n] \). Hence the second condition of (2) is fulfilled and it is no restriction to assume that \( P_m \) has real coefficients. To prove that \( \lambda P_m \) does not vanish on \( V(P_m) \cap (\mathbb{R}^n \setminus \{0\}) \) we argue by contradiction and assume that there exists \( \theta \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\}) \) satisfying \( \text{grad} P_m(\theta) = 0 \). After a real linear change of variables, we may assume \( \theta = e_n = (0, \ldots, 0, 1) \). Then we apply Taylor’s formula at \( \theta \) to get

\[
P_m(z', 1) = P_m(\theta + (z', 0)) = \sum_{k=\nu}^{m} q_k(z'),
\]

where \( q_k \in \mathbb{C}[z_1, \ldots, z_{n-1}] \) is zero or homogeneous of degree \( k \) and where \( q_{\nu} \neq 0 \). Then \( 2 \leq \nu \leq m \) since \( P_m \) and \( \text{grad} P_m \) vanish at \( \theta \). By the homogeneity of \( P_m \) it follows from (4.3) that for \( z_n \neq 0 \) we have

\[
P_m(z', z_n) = z_n^m P_m \left( \frac{z'}{z_n}, 1 \right) = z_n^m \sum_{k=\nu}^{m} q_k \left( \frac{z'}{z_n} \right) = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z').
\]

By continuity, this holds also when \( z_n = 0 \). Now let

\[
P(z) := P_m(z) + i z_n^{m-1} = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z') + i z_n^{m-1}
\]

and \( d := (\nu-1, \ldots, \nu-1, \nu) \). Then the monomial \( z_n^{m-1} \) has \( d \)-degree \( (m-1)\nu \) and the polynomials \( z_n^{m-k} q_k(z') \) have \( d \)-degree \( \nu m - k \), so they are decreasing in \( k \). Hence the \( d \)-quasihomogeneous principal part \( q \) of \( P \) equals

\[
q(z) = z_n^{m-\nu} q_{\nu}(z') + i z_n^{m-1}.
\]

To show that \( q \) satisfies the hypotheses of Lemma 3.6, we note that \( q_{\nu} \neq 0 \) implies the existence of \( \zeta' \in \mathbb{C}^{n-1} \) satisfying \( q_{\nu}(\zeta') = -i \). Then \( \zeta := (\zeta', 1) \) satisfies

\[
q(\zeta) = q_{\nu}(\zeta') + i = 0.
\]

Hence the conditions (1) and (2) of Lemma 3.6 are fulfilled. To show that also condition 3.6(3) holds, assume that for some \( z' \in \mathbb{C}^{n-1} \) we have

\[
0 = q(z', 1) = q_{\nu}(z') + i.
\]

Since \( q_{\nu} \) has real coefficients, this implies \( z' \notin \mathbb{R}^{n-1} \), which proves condition 3.6(3). Hence we can apply Lemma 3.6 to conclude that \( V(P) \) does not satisfy \( \text{PL}(\mathbb{R}^n, \log) \) in contradiction to the hypothesis (1).

(2) \( \Rightarrow \) (1): Since \( P_m \) is real up to a complex factor, it is no restriction to assume that \( P_m \) has real coefficients. By Meise and Taylor [7], Lemma 4.6, the hypothesis implies that \( P_m \) is a product of distinct, irreducible factors.
with real coefficients, each of which is of principal type. This implies that $P_m$ is locally hyperbolic at every real characteristic in the sense of Definition 6.4 of Hörmander [3]. Hence it follows from [3], Theorem 6.5, that $V(P_m)$ satisfies HPL($\mathbb{R}^n$). By hypothesis, no irreducible component of $V(P_m)$ is elliptic. Hence $V(P_m)$ satisfies PL($\mathbb{R}^n$, log) by Meise, Taylor, and Vogt [12], Corollary 3.14. Next fix $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with deg $Q < m$, $Q \neq 0$ and choose $C > 0$ such that $|Q(z)| \leq C(1 + |z|^{m-1})$, $z \in \mathbb{C}^n$. By the homogeneity of $P_m$, the function

$$x \mapsto \sum_{j=1}^n \left| \frac{\partial P_m}{\partial x_j}(x) \right|$$

is positively homogeneous of degree $m-1$ and does not vanish for $x \in \mathbb{R}^n \setminus \{0\}$ by hypothesis. This implies the existence of $\delta > 0$ and $A \geq 1$ such that

$$1 + |z|^{m-1} \leq A \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \{z \in \mathbb{C}^n : \text{Im} \ z < \delta |z|\} =: \Gamma.$$ 

Consequently, there exists $A'$ with

$$|Q(z)| \leq A' \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \Gamma.$$ 

Now fix $z \in \Gamma \cap (V(P_m + Q) \setminus V(P_m))$ and note that by Hörmander [4], Lemma 4.1.1, (which holds also for $\xi \in \mathbb{C}^n$) there exists $D > 0$ such that

$$\text{dist}(\zeta, V(P_m)) \sum_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(\zeta)}{P_m(\zeta)} \right|^{\frac{1}{\alpha}} \leq D, \quad \zeta \in \mathbb{C}^n \setminus V(P_m).$$

This and $P_m(z) = -Q(z)$ imply

$$\frac{1}{A'} \leq \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{Q(z)} \right| = \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{P_m(z)} \right| \leq \max_{\alpha \neq 0} \left( \frac{D}{\text{dist}(z, V(P_m))} \right)^{\alpha}$$

and hence the existence of $E > 0$ such that (by continuity)

$$\text{dist}(z, V(P_m)) \leq E, \quad z \in \Gamma \cap V(P_m + Q).$$

From this we get (1) by Lemma 4.1. 

\begin{remark}
Note that Theorem 4.3 and its Corollary 4.7 below extend Corollary 5.8 of Meise, Taylor, and Vogt [12]. Moreover, Theorem 4.3 shows that the characterizing condition is in fact weaker than the sufficient condition given there, since $P_m$ can be of principal type, while $V(P_m)$ has complex singularities. To see this, consider $P_4(x, y, z) := (x^2 + y^2 - z^2)(x^2 + z^2 - y^2/4)$ and note that $\{\lambda \cdot (i\sqrt{3/5}, 2\sqrt{2/5}, 1) : \lambda \in \mathbb{C}\}$ is a singular line for $V(P_4)$.

\begin{remark}
From Meise and Taylor [7], 4.8 and 3.4, it follows that each real homogeneous polynomial $P_m$ of principal type for which each irreducible
factor has a non-trivial real zero is also stable under certain real perturbations introducing an extra variable. More precisely, the variety 

$$\{ z \in \mathbb{C}^{n+1} : P_m(z_1, \ldots, z_n) = z_{n+1} \}$$

satisfies $\text{PL}(\mathbb{R}^{n+1}, \text{log})$, provided that $P_m$ is of real principal type and has no elliptic factor.

In the proof of Theorem 4.3 we used complex polynomials to show that $V(P_m + Q)$ fails $\text{PL}(\mathbb{R}^n, \text{log})$ if $V(P_m)$ satisfies $\text{PL}(\mathbb{R}^n, \text{log})$ while $P_m$ is not of principal type. In some cases, such as $P_2(x, y, z) = x^2 - y^2$, this is the only possible choice (see Meise, Taylor, and Vogt [12], Example 4.9). However, in other cases, real perturbations can also have the same effect, as the following example shows.

**Example 4.6.** Let $P(x, y, z) := x^2 z + y z^2 + y z$. The principal part $P_3(x, y, z) = x^2 z + y z^2 = (x^2 + y) z$ is hyperbolic with respect to $N = (0, -1, 1)$. Hence $V(P_3)$ satisfies $\text{PL}(\mathbb{R}^3, \text{log})$ by Meise, Taylor, and Vogt [9], 3.6, and 4.5 in connection with [12], 2.12. Obviously, $P_3$ is not of principal type. By Example 3.3(a), $V(P)$ does not satisfy $\text{PL}(\mathbb{R}^3, \omega)$ whenever $\omega(t) = o(t^{1/3})$.

**Corollary 4.7.** Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$ and of principal type. Then the following conditions are equivalent:

1. $V(P_m)$ satisfies $\text{PL}(\mathbb{R}^n, \text{log})$,
2. $P_m$ is real up to a complex constant and each irreducible factor $q$ of $P_m$ has a real zero $\xi \neq 0$,
3. there exist $k \in \mathbb{N}$, $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < km$ and a weight function $\omega$ so that $V(P_m^k + Q)$ satisfies $\text{PL}(\mathbb{R}^n, \omega)$,
4. for each $k \in \mathbb{N}$ and each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q = l < km$ we have:
   a) if $l \leq k(m - 1)$, then $V(P_m^k + Q)$ satisfies $\text{PL}(\mathbb{R}^n, \text{log})$,
   b) if $l > k(m - 1)$, then $V(P_m^k + Q)$ satisfies $\text{PL}(\mathbb{R}^n, 1 + t^3)$ for $\beta = 1 + \frac{l}{k} - m$.

**Proof.**

1. $\Rightarrow$ 2: This follows from Meise, Taylor, and Vogt [13], Lemma 2, and [12], Corollary 3.14.

2. $\Rightarrow$ 3: Since $P_m$ is of principal type, (2) implies that condition 4.3(2) is fulfilled. Hence $V(P_m)$ satisfies $\text{PL}(\mathbb{R}^n, \text{log})$ by Theorem 4.3. Thus (3) holds for $k = 1$ and $Q = 0$.

3. $\Rightarrow$ 1: If $V(P_m^k + Q)$ satisfies $\text{PL}(\mathbb{R}^n, \omega)$ then $V(P_m^k) = V(P_m)$ satisfies $\text{PL}(\mathbb{R}^n, \text{log})$ by Meise, Taylor, and Vogt [12], Theorem 4.1.

4. $\Rightarrow$ 3: This holds trivially.

2. $\Rightarrow$ 4: Since $P_m$ is of principal type, there exists $\eta > 0$ such that

$$|\text{grad } P_m(z)| > 0 \text{ for } z \in \mathbb{C}^n, |z| = 1 \text{ and } |\text{Im } z| \leq \eta.$$
Therefore standard arguments using homogeneity and compactness imply the existence of $\delta > 0$ such that

\[
(4.4) \quad \max_{0<|\alpha|\leq k} \left| (P_m^k)^{(\alpha)}(z) \right| \geq \delta |z|^{k(m-1)} \quad \text{for } z \in \mathbb{C}^n, |\text{Im} z| \leq \eta|z|, |z| \geq 1.
\]

Now fix $Q \in \mathbb{C}[z_1, \ldots, z_n]$ satisfying $\deg Q \leq k(m - 1)$. Then there exists $D \geq 1$ such that

\[
(4.5) \quad |Q(z)| \leq D \max_{|\alpha| > 0} \left| (P_m^k)^{(\alpha)}(z) \right| \quad \text{for } z \in \mathbb{C}^n, |z| \geq 1, |\text{Im} z| \leq \eta|z|.
\]

Now fix $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$, and let

\[
\Gamma(\xi, \eta, 1) := \left\{ z \in \mathbb{C}^n : \left| \frac{z}{|z|} - \xi \right| \leq \eta, |z| \geq 1 \right\}.
\]

For $z \in \Gamma(\xi, \eta, 1)$ we have $|\text{Im} z| \leq \eta|z|$ since $\xi$ is real. Now fix $\zeta \in V(P_m^k + Q) \cap \Gamma(\xi, \eta, 1)$ satisfying $P_m(\zeta) \neq 0$. Then $P_m(\zeta) = -Q(\zeta)$ and (4.5) imply the existence of $M \geq 1$ such that

\[
\frac{1}{M} \leq \sum_{|\alpha| > 0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{Q(\zeta)} \right|^{1/|\alpha|} = \sum_{|\alpha| > 0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{P_m^k(\zeta)} \right|^{1/|\alpha|}.
\]

Since by Hörmander [4], Lemma 4.1.1, there exists $C \geq 1$ such that

\[
\text{dist}(\zeta, V(P_m^k)) \sum_{|\alpha| > 0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{P_m^k(\zeta)} \right|^{1/|\alpha|} \leq C
\]

we conclude that

\[
\text{dist}(\zeta, V(P_m^k)) \leq CM \leq CM \log(2 + |\zeta|).
\]

Since $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$, was chosen arbitrarily, it follows from Lemma 4.1 that $V(P_m^k + Q)$ satisfies $\text{PL}(\mathbb{R}^n, \log)$ in this case.

If $k(m - 1) < l = \deg Q < km$ then $0 < \beta := 1 + \frac{l}{k} - m < 1$ and (4.4) implies the existence of $\delta > 0$ such that

\[
\max_{0<|\alpha|\leq k} \left| (P_m^k)^{(\alpha)}(z) \right| |z|^{|\alpha|} \geq \delta |z|^{k(m-1)-\beta k} = \delta |z|^l \quad \text{if } |z| \geq 1 \text{ and } |\text{Im} z| \leq \eta|z|.
\]

Hence there exists $D \geq 1$ such that

\[
|Q(z)| \leq D \max_{|\alpha| > 0} \left| (P_m^k)^{(\alpha)}(z) \right| (1 + |z|^\beta)^{|\alpha|} \quad \text{for } z \in \mathbb{C}^n, |\text{Im} z| \leq \eta|z|, |z| \geq 1.
\]

From this it follows as above that for each $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$, there exists $C_\xi > 0$ such that

\[
\text{dist}(\zeta, V(P_m^k)) \leq C_\xi (1 + |\xi|^\beta), \quad \zeta \in \Gamma(\xi, \eta, 1).
\]
By Lemma 4.1, this implies that \( V(P_k^m + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) for \( \omega(t) = 1 + t^3 \), as asserted.

Theorem 4.3 in connection with Meise, Taylor, and Vogt [8], Théorème, also implies the following result on the existence of fundamental solutions with large lacunas.

**Corollary 4.8.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be of principal type and of degree \( m \geq 2 \). If the principal part \( P_m \) of \( P \) is real up to a complex constant and if each irreducible factor \( q \) of \( P_m \) has a real zero \( \xi \neq 0 \) then the following holds:

For each \( r > 0 \) there exists \( R > 0 \) such that for each \( \xi \in \mathbb{R}^n, |\xi| > R \), there exists a fundamental solution \( E_\xi \in \mathcal{D}'(\mathbb{R}^n) \) of \( P(D) \) satisfying \( \text{Supp} E_\xi \subset \{ x \in \mathbb{R}^n : |x - \xi| \geq r \} \).

**References**


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CUBIC MODULAR EQUATIONS AND NEW Ramanujan-type SERIES FOR $1/\pi$

HENG HUAT CHAN AND WEN-CHIN LIAW

Dedicated to our advisor, Professor Bruce C. Berndt on his 60th birthday

In this paper, we derive new Ramanujan-type series for $1/\pi$ which belong to “Ramanujan’s theory of elliptic functions to alternative base 3” developed recently by B.C. Berndt, S. Bhargava, and F.G. Garvan.

1. Introduction.

Let $(a)_0 = 1$ and, for a positive integer $m$,

$$(a)_m := a(a + 1)(a + 2) \cdots (a + m - 1),$$

and

$$_2F_1(a, b; c; z) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m z^m}{(c)_m m!}, \quad |z| < 1.$$

In his famous paper “Modular equations and approximations to $\pi$” [10], S. Ramanujan offered 17 beautiful series representations for $1/\pi$. He then remarked that two of these series

$$(1.1) \quad \frac{27}{4\pi} = \sum_{m=0}^{\infty} (2 + 15m) \frac{(\frac{1}{2})_m (\frac{1}{3})_m (\frac{2}{3})_m}{(m!)^3} \left( \frac{2}{27} \right)^m$$

and

$$(1.2) \quad \frac{15\sqrt{3}}{2\pi} = \sum_{m=0}^{\infty} (4 + 33m) \frac{(\frac{1}{2})_m (\frac{1}{3})_m (\frac{2}{3})_m}{(m!)^3} \left( \frac{4}{125} \right)^m$$

“belong to the theory of $q_2$,” where

$$q_2 = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{\Gamma(\frac{1}{2}, \frac{2}{5}; 1; 1-k^2)}{2F_1(\frac{1}{2}, \frac{2}{5}; 1; k^2)} \right).$$

Ramanujan did not elaborate on his “theory of $q_2$,” neither did he provide details for his proofs of (1.1) and (1.2).
Ramanujan’s formulas (1.1) and (1.2) were first proved by J.M. Borwein and P.B. Borwein in 1987. Motivated by their study of Ramanujan’s series for $1/\pi$ associated with the classical theory of elliptic functions, they established the following result:

**Theorem 1.1 ([3, p. 186]).** Let

$$K(x) := 2 F_{1}(\frac{1}{3}, \frac{2}{3}; 1; x), \quad \text{and} \quad \dot{K}(x) := \frac{dK(x)}{dx}.$$\n
For $n \in \mathbb{Q}^{+}$, define the cubic singular modulus to be the unique number $\alpha_n$ satisfying

$$\frac{K(1 - \alpha_n)}{K(\alpha_n)} = \sqrt{n}. \quad (1.3)$$\n
Set

$$\epsilon(n) = \frac{3 \sqrt{3}}{8\pi} (K(\alpha_n))^{-2} - \sqrt{n} \left( \frac{3}{2} \alpha_n (1 - \alpha_n) \frac{\dot{K}(\alpha_n)}{K(\alpha_n)} - \alpha_n \right), \quad (1.4)$$\n
$$a_n := \frac{8 \sqrt{3}}{9} \left( \epsilon(n) - \sqrt{n} \alpha_n \right), \quad (1.5)$$\n
and

$$b_n := \frac{2 \sqrt{3n}}{3} \sqrt{1 - H_n}, \quad (1.6)$$\n
where

$$H_n := 4 \alpha_n (1 - \alpha_n). \quad (1.7)$$\n
Then

$$\frac{1}{\pi} = \sum_{m=0}^{\infty} (a_n + b_n m) \frac{(\frac{1}{2})_m (\frac{1}{3})_m (\frac{2}{3})_m m}{(m!)^3} H_n^m. \quad (1.8)$$\n
**Remark.** We state this theorem with a different definition of $\epsilon(n)$ than that given in [3]. We have avoided using elliptic integrals of the second kind and Legendre’s relation.

The Borweins’ theorem indicates that for each positive rational number $n$, we can easily derive a series for $1/\pi$ belonging to the “theory of $q_2$” if the values of $\alpha_n$ and $\epsilon(n)$ (the rest of the constants can be computed from these) are known. The computation of these constants for any given $n$, however, is far from trivial.
The Borweins’ method of evaluating $\alpha_n$ involves solving a quartic equation. More precisely, they show that when $n$ is an odd positive integer, $\alpha_n$ is the smaller of the two real solutions of the equation
\begin{equation}
\frac{(9 - 8\alpha_n)^3}{64\alpha_n^3(1 - \alpha_n)} = \frac{(4G_{3n}^{24} - 1)^3}{27G_{3n}^{24}},
\end{equation}
where $G_n$ is the classical Ramanujan-Weber class invariant defined by
\begin{equation}
G_n := 2^{-1/4}e^{\pi\sqrt{n}/24} \prod_{m=1}^{\infty} (1 + e^{-\pi\sqrt{n}(2m-1)}).
\end{equation}
Using known values for $G_{3n}$, they derive $\alpha_n$ for $n = 3$ and 5 from (1.9). For example, from (see [1, p. 190])
\begin{equation}
G_{15}^{12} = 8 \left(\frac{\sqrt{5} + 1}{2}\right)^4,
\end{equation}
they deduce that
\begin{equation}
\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{50}.
\end{equation}
When $n$ is an even positive integer, the corresponding formula between $\alpha_n$ and $g_{3n}$ is
\begin{equation}
\frac{(9 - 8\alpha_n)^3}{64\alpha_n^3(1 - \alpha_n)} = \frac{(4g_{3n}^{24} + 1)^3}{27g_{3n}^{24}},
\end{equation}
where $g_n$ is the other Ramanujan-Weber class invariant defined by
\begin{equation}
g_n := 2^{-1/4}e^{\pi\sqrt{n}/24} \prod_{m=1}^{\infty} (1 - e^{-\pi\sqrt{n}(2m-1)}).
\end{equation}
Using (1.10) and known values of $g_{3n}$, they compute $\alpha_n$ for $n = 2, 4, \text{ and } 6$. Together with the values of $\epsilon(n)$ for $n = 2, 3, 4, 5, \text{ and } 6$ [3, p. 190, Problem 20], they obtained five series for $1/\pi$. Ramanujan’s series (1.1) and (1.2) then correspond to $n = 4$ and 5, respectively. At the end of [3, Chapter 5, Section 5], the Borweins remark that their explanation of Ramanujan’s series (1.1) and (1.2) is “a bit disappointing” as they only have “well-concealed analogues of the original theory for $K$.”

In a recent paper, B. C. Berndt, S. Bhargava, and F. G. Garvan [2] succeeded in developing Ramanujan’s “corresponding theories” mentioned in [10]. One of these theories is Ramanujan’s “theory of $q_2$” and its discovery has motivated us to revisit Ramanujan’s series (1.1) and (1.2). This theory is now known as “Ramanujan’s theory of elliptic functions to alternative base 3” or “Ramanujan’s elliptic functions in the theory of signature 3.”

In this article, we derive some new formulas from the “theory of $q_2$” which will facilitate the computations of $\alpha_n$ and $\epsilon(n)$. With the aid of cubic
Russell-type modular equations (see [6]) and Kronecker’s Limit Formula, we discover new Ramanujan-type series for \(1 / \pi\) belonging to the “theory of \(q_2\).” An example of these series, which corresponds to \(n = 59\), is

\[
\frac{2153559\sqrt{3}}{\pi} = \sum_{m=0}^{\infty} \left( a + bm \right) \frac{\left( \frac{1}{3} \right)_m \left( \frac{2}{3} \right)_m \left( \frac{73 - 40\sqrt{3}}{2^{1/3} \cdot 23^2 (4 + 5\sqrt{3})} \right)^{3m}}{(m!)^3},
\]

where

\[ a := 1028358\sqrt{3} - 593849 \quad \text{and} \quad b := 19101285\sqrt{3} - 795. \]

Each term in this series gives approximately 10 decimal places of \(\pi\).

In Section 2, we recall some important results proved in [2] and establish new formulas satisfied by \(\epsilon(n)\) which lead to a new formula for \(a_n\). In Section 3, we describe our strategy for computing \(a_n\). In Section 4, we indicate that if \(3n\) is an Euler convenient number, then \(\alpha_n\), as well as other related cubic singular moduli, can be computed explicitly via Kronecker’s Limit Formula. These values are used to derive the constants \(a_n\), \(b_n\), and \(H_n\) listed in our final section.

2. Ramanujan’s elliptic functions in the theory of signature 3
(Ramanujan’s “theory of \(q_2\)”).

Define

\[
a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}
\]

and

\[
c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.
\]

**Theorem 2.1.** If

\[
q = \exp\left( -\frac{2\pi}{\sqrt{3}} \frac{K(1 - \alpha)}{K(\alpha)} \right),
\]

then

\[
\alpha = \frac{c^3(q)}{a^3(q)}.
\]

**Theorem 2.2** (Borweins’ Inversion Formula). We have

\[
a(q) = K\left( \frac{c^3(q)}{a^3(q)} \right) = K(\alpha),
\]

where \(K(\cdot)\) is defined in Theorem 1.1.
Theorem 2.1 and Theorem 2.2 are important results in Ramanujan’s theory of elliptic functions in the signature 3 which can be found in [2] as Lemma 2.9 and Lemma 2.6, respectively.

Let \( \alpha \) be given as in (2.2). Then it is known that (see [2, (4.4)] and [5, (4.7)])

\[
q \frac{d\alpha}{dq} = K^2(\alpha)\alpha(1 - \alpha). 
\]

The modulus \( \beta \) is said to have degree \( n \) over the modulus \( \alpha \) when there is a relation

\[
\frac{K(1 - \beta)}{K(\beta)} = n \frac{K(1 - \alpha)}{K(\alpha)}. 
\]

Hence, when \( q \) satisfies (2.1),

\[
q^n = \exp \left( -\frac{2\pi}{\sqrt{3}} \frac{K(1 - \beta)}{K(\beta)} \right),
\]

and applying (2.4) with \( q \) and \( \alpha \) replaced by \( q^n \) and \( \beta \), respectively, we deduce that

\[
q \frac{d\beta}{dq} = nK^2(\beta)\beta(1 - \beta). 
\]

Combining (2.6) and (2.4), we arrive at:

**Theorem 2.3.** If \( \beta \) has degree \( n \) over \( \alpha \), then

\[
m^2(\alpha, \beta) \frac{d\beta}{d\alpha} = n \frac{\beta(1 - \beta)}{\alpha(1 - \alpha)},
\]

where

\[
m(\alpha, \beta) := \frac{K(\alpha)}{K(\beta)}. 
\]

We call the quantity \( m(\alpha, \beta) \) the multiplier of degree \( n \) in the theory of signature 3. We are now ready to derive new formulas satisfied by \( \epsilon(n) \).

**Theorem 2.4.** Let \( \epsilon(r) \) be defined as in (1.4). Then

\[
\epsilon \left( \frac{1}{r} \right) = \frac{\sqrt{r} - \epsilon(r)}{r}.
\]

**Proof.** Set

\[
\tau = \frac{K(1 - \alpha)}{K(\alpha)}. 
\]

Then

\[
q \frac{d\tau}{d\alpha} K(\alpha) + \dot{K}(\alpha) \tau = -\ddot{K}(1 - \alpha).
\]
From (2.1) and (2.4), we deduce that
\[
\frac{d\alpha}{d\tau} = -\frac{2\pi}{\sqrt{3}} K^2(\alpha)\alpha(1 - \alpha).
\]

Hence,
\[
(2.9) \quad \dot{K}(1 - \alpha) = \frac{\sqrt{3}}{2\pi} \frac{1}{K(\alpha)\alpha(1 - \alpha)} - \dot{K}(\alpha)\tau.
\]

Next, note that from (1.3)
\[
\alpha = 1 \quad \text{and} \quad (2.10)
\]

Therefore, by (1.4) and (2.9) with \(\tau = \sqrt{r}\),
\[
\epsilon\left(\frac{1}{r}\right) = \frac{3\sqrt{3}}{8\pi} K^2(1 - \alpha) - \frac{\sqrt{r}}{3} \left(3\alpha(1 - \alpha)\dot{K}(1 - \alpha) - (1 - \alpha)\right)
\]
\[
= \frac{3\sqrt{3}}{8\pi} K^2(1 - \alpha) - \frac{3\sqrt{3}}{4\pi r K^2(\alpha)} + \frac{3\alpha(1 - \alpha)\dot{K}(\alpha)}{2\sqrt{r} K(\alpha)} + \frac{1}{\sqrt{r}} - \frac{\alpha}{\sqrt{r}}
\]
\[
= \sqrt{r} - \epsilon(r).
\]

\[\square\]

**Theorem 2.5.** Let
\[
m^* := m(\alpha_r, \alpha_{n^2r}) \quad \text{and} \quad \dot{m}^* := \frac{dm}{d\alpha}(\alpha_r, \alpha_{n^2r}).
\]

Then
\[
(2.11) \quad \epsilon(n^2r) = m^* - \sqrt{r} \left(\alpha - \frac{3}{2} m^{*-1}(1 - \alpha)\dot{m}^* - \frac{2\alpha m^*}{m^{*2}}\right).
\]

**Proof.** Suppose \(\beta\) has degree \(n\) over \(\alpha\). Then from (2.8), we deduce that
\[
(2.12) \quad \frac{m}{K(\beta)} \frac{dK(\beta)}{d\alpha} - \frac{K(\beta)}{d\alpha} \frac{dm}{d\alpha} = \frac{dK(\alpha)}{d\alpha}.
\]

Using (2.7), we may rewrite (2.12) as
\[
(2.13) \quad \frac{n\beta(1 - \beta) dK(\beta)}{K(\beta)} = \frac{m^2(1 - \alpha) dK(\alpha)}{K(\alpha)} - m\alpha(1 - \alpha) \frac{dm}{d\alpha}.
\]

Next, suppose \(\alpha = \alpha_r\). Then \(\beta = \alpha_{n^2r}\), and by (1.4), (2.8), and (2.13),
\[
\epsilon(n^2r) = \frac{3\sqrt{3}}{8\pi K^2(\alpha_{n^2r})} - n\sqrt{r} \left(\frac{3\alpha_{n^2r}(1 - \alpha_{n^2r})}{2K(\alpha_{n^2r})} \dot{K}(\alpha_{n^2r}) - \alpha_{n^2r}\right)
\]
\[
= \frac{3\sqrt{3} m^{*2}}{8\pi K^2(\alpha)}
\]
\[
- \sqrt{r} \left(\frac{3m^{*2}\alpha_r(1 - \alpha)}{2K(\alpha)} \dot{K}(\alpha_r) - \frac{3}{2} m^* \alpha_r(1 - \alpha)\dot{m}^* - \alpha_{n^2r}\right)
\]
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$$= m^2 \left( \epsilon(r) - \sqrt{r} \left( \alpha_r - \frac{3}{2} m^{-1} \alpha_r (1 - \alpha_r) \frac{m^*}{m^{*2}} \right) \right).$$

□

If we set $r = 1/n$ in (2.11) and use (2.10), we find that

$$\epsilon(n) = n \left( \epsilon \left( \frac{1}{n} \right) - \sqrt{\frac{1}{n}} \left( 1 - \alpha_n - \frac{3\alpha_n(1 - \alpha_n)}{2\sqrt{n}} \frac{dm}{d\alpha}(1 - \alpha_n, \alpha_n) \right) \right)$$

$$= -\epsilon(n) + 2\alpha_n\sqrt{n} + \frac{3\alpha_n(1 - \alpha_n)}{2} \frac{dm}{d\alpha}(1 - \alpha_n, \alpha_n).$$

Hence, we have:

**Theorem 2.6.**

$$\epsilon(n) = \sqrt{n\alpha_n} + \frac{3\alpha_n(1 - \alpha_n)}{4} \frac{dm}{d\alpha}(1 - \alpha_n, \alpha_n).$$

**Corollary 2.7.** With $a_n$ and $H_n$ defined in Theorem 1.1, we have

$$a_n = \frac{H_n}{2\sqrt{3}} \frac{dm}{d\alpha}(1 - \alpha_n, \alpha_n).$$

Theorems 2.4, 2.5, and 2.6 are the respective cubic analogues of [3, (5.1.5), Theorem 5.2, and (5.2.5)].

3. Computations of $a_n$.

It is clear from Corollary 2.7 that in order to compute $a_n$ it suffices to compute $\alpha_n$ and $dm/d\alpha$, where $m$ is the multiplier of degree $n$. We will discuss the computation of the latter in this section. Suppose there is a relation between $\alpha$ and $\beta$, where $\beta$ has degree $n$ over $\alpha$. Then we can determine $d\beta/d\alpha$ by implicitly differentiating the relation with respect to $\alpha$. Substituting $d\beta/d\alpha$ into (2.7), we conclude that $m$ can be expressed in terms of $\alpha$ and $\beta$. This implies that $dm/d\alpha$ is a function of $\alpha$ and $\beta$.

A relation between $\alpha$ and $\beta$ induced by (2.5) (i.e., when $\beta$ has degree $n$ over $\alpha$) is known as a modular equation of degree $n$ in the theory of signature 3. (We sometimes call these cubic modular equations.) Our discussion in the previous paragraph indicates that our computations of $dm/d\alpha$ depend on the existence of such modular equations.

The first few modular equations in the theory of signature 3 are given by Ramanujan in his notebooks. One of these is the following modular equation of degree 2:

$$\alpha \beta^{1/3} + \{(1 - \alpha)(1 - \beta)\}^{1/3} = 1.$$  (3.1)

Proofs of Ramanujan’s modular equations in the theory of signature 3 are now available in [2] and [6].
Recently, we showed that [6] if \( p \) is a prime, then there is a relation between 
\[
x := (\alpha \beta)^{s/6} \quad \text{and} \quad y := \{(1 - \alpha)(1 - \beta)\}^{s/6},
\]
when \((p + 1)/3 = N/s\) and \(\gcd(N, s) = 1\). Moreover, we proved that the degree of the polynomial 
satisfied by \( x \) and \( y \) is \( N \). This proves the existence of modular equations of prime degrees 
and we conclude that when \( m \) is a multiplier of degree \( p \),

\[
\frac{d m}{d \alpha} = F_p(\alpha, \beta),
\]

where \( F_p \) is a certain function in \( \alpha \) and \( \beta \). If we know the value of \( \alpha_p \), then 
the value of \( a_p \) follows by substituting \( \alpha = 1 - \alpha_p \) and \( \beta = \alpha_p \) into (3.2) and 
simplifying. Our simplification is done with the help of MAPLE V.

When \( n \) is not a prime, except for modular equations of degrees 4 and 9, it 
is difficult to derive a modular equation of degree \( n \). However, deriving such 
a modular equation is unnecessary. We illustrate our point with \( n = pq \). If 
\( \beta \) has degree \( q \) over \( \alpha \) then from a cubic modular equation of degree \( q \) 
and (2.7), we can write

\[
m_q = \frac{K(\alpha)}{K(\beta)} = G_q(\alpha, \beta),
\]

where \( m_q \) is the multiplier of degree \( q \) and \( G_q \) is a certain function of \( \alpha \) and 
\( \beta \). Similarly, we can deduce that if \( \gamma \) has degree \( p \) over \( \beta \), then from a cubic 
modular equation of degree \( p \), we may write

\[
m_p = \frac{K(\beta)}{K(\gamma)} = G_p(\beta, \gamma),
\]

where \( m_p \) is the multiplier of degree \( p \) and \( G_p \) is a certain function of \( \beta \) and 
\( \gamma \). It follows that \( \gamma \) has degree \( pq \) over \( \alpha \) and

\[
m_{pq}(\alpha, \gamma) = \frac{K(\alpha)}{K(\gamma)} = \frac{K(\alpha)}{K(\beta)} \cdot \frac{K(\beta)}{K(\gamma)} = m_q(\alpha, \beta) \cdot m_p(\beta, \gamma).
\]

Hence, differentiating with respect to \( \alpha \) and substituting \( \alpha = \alpha_1/(pq) \), we have

\[
\frac{d m_{pq}}{d \alpha}(1 - \alpha_{pq}, \alpha_{pq}) = m_p(\alpha_{q/p}, \alpha_{pq}) \frac{d m_q}{d \alpha}(1 - \alpha_{pq}, \alpha_{q/p})
+ m_q(1 - \alpha_{pq}, \alpha_{q/p}) \frac{d \beta}{d \alpha} \cdot \frac{d m_p}{d \beta}(\alpha_{q/p}, \alpha_{pq}).
\]

This allows us to compute \( a_{pq} \) provided we know modular equations of degrees \( p \) and \( q \) and the singular moduli \( \alpha_{pq} \) and \( \alpha_{q/p} \).

When \( n \) is a squarefree product of more than 2 primes, say \( n = p_1 p_2 \cdots p_l \), 
then the above idea can be extended with the computation of \( a_n \) reduced to 
that of finding modular equations of degrees \( p_1, p_2, \ldots, p_{l-1}, \) and \( p_l \), and 
constants \( \alpha_n/(p_{i_1}^2 \cdots p_{i_j}^2) \), where \( 1 \leq s \leq l - 1 \) and \( 1 \leq i_j \leq l \).
4. Euler’s convenient numbers, Kronecker’s Limit Formula, and cubic singular moduli.

An Euler convenient number is a number \( c \) satisfying the following criterion:

Let \( l > 1 \) be an odd number relatively prime to \( c \) which is properly represented by \( x^2 + cy^2 \). If the equation \( l = x^2 + cy^2 \) has only one solution with \( x, y \geq 0 \), then \( l \) is a prime number.

Euler was interested in these numbers because they helped him to generate large primes. The above criterion, however, is not very useful for finding these numbers.

Let \( d \) be squarefree, \( K = \mathbb{Q}(\sqrt{-d}) \), \( C_K \) denote the class group of \( K \) and \( C_K^0 \) be the subgroup of squares in \( C_K \). A genus group \( G_K \) is defined as the quotient group \( C_K/C_K^0 \). Gauss observed that \( G_K \simeq C_K \) if and only if \( d \) is a convenient number. (Some convenient numbers are not squarefree but Gauss’ criterion is also true for class groups of orders in \( K \).) Using this new criterion, Gauss determined 65 Euler convenient numbers \([8], [7, p. 60] \). We reproduce here those \( c \)'s \((\neq 3)\) which are squarefree and divisible by 3.

| \( h(-4c) := |C_{\mathbb{Q}(\sqrt{-4c})}| \) | Euler’s convenient number \( c \) |
|----------------|------------------|
| 2              | 6, 15            |
| 4              | 21, 30, 33, 42, 57, 78, 93, 102, 177 |
| 8              | 105, 165, 210, 273, 330, 345, 357, 462 |
| 16             | 1365             |

Table 1. Convenient numbers in Gauss’ table which are squarefree and divisible by 3 (except 3).

For each \( c \) in Table 1, we will deduce the corresponding values \( a_{c/3}, b_{c/3}, \) and \( H_{c/3} \), which in turn yield new series for \( 1/\pi \).

A group homomorphism \( \chi : G_K \longrightarrow \{\pm 1\} \) is known as a genus character. One can show that a genus character arises from a certain decomposition of \( D_K \), where \( D_K \) is the discriminant of \( K \). More precisely, if \( \chi \) is a genus character, then there exist \( d_1 \) and \( d_2 \) satisfying \( D_K = d_1d_2, \ d_1 > 0, \) and
\[ d_i \equiv 0 \text{ or } 1 \pmod{4}, \text{ such that for any prime ideal } p \text{ in } K, \]
\[
\chi([p]) = \begin{cases} 
\left( \frac{d_1}{N(p)} \right), & \text{if } N(p) \nmid d_1, \\
\left( \frac{d_2}{N(p)} \right), & \text{if } N(p) \mid d_1,
\end{cases}
\]

(4.1)

where \( N(p) \) is the norm of the ideal \( p \) and \( \left( \frac{\cdot}{\cdot} \right) \) denotes the Kronecker symbol.

If \([a]\) is an ideal class in \( C_K \) and \( a = \prod p^{\alpha_p} \), then we define
\[
\chi([a]) = \prod \chi([p])^{\alpha_p}.
\]

Theorem 4.1. Let \( \chi \) be a genus character arising from the decomposition \( D_K = d_1, \chi d_2, \chi \). Let \( h_{i,\chi} \) be the class number of the field \( \mathbb{Q}(\sqrt{d_{i,\chi}}) \), \( w_{2,\chi} \) be the number of roots of unity in \( \mathbb{Q}(\sqrt{d_{2,\chi}}) \), and \( \epsilon_{\chi} \) be the fundamental unit of \( \mathbb{Q}(\sqrt{d_{1,\chi}}) \). Let
\[
F([a]) = \sqrt{N([1, \tau])} |\eta(\tau)|^2,
\]
where \( N(\cdot) \) denotes the norm of a fractional ideal, \( \eta(z) \) denotes the Dedekind eta-function defined by
\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i z}), \quad \text{Im } z > 0,
\]
and
\[
\tau = \frac{\tau_2}{\tau_1}, \quad \text{Im } \tau > 0, \quad \text{where } \quad a = [\tau_1, \tau_2].
\]

Then
\[
\epsilon_{\chi}^{2h_{1,\chi}h_{2,\chi}/w_{2,\chi}} = \prod_{[a] \in C_K} F([a])^{-\chi([a])}.
\]

Theorem 4.1 follows from Kronecker’s First Limit Formula \([11, \text{p. 72, Theorem 6}]\). In \([9]\), K.G. Ramanathan applied Theorem 4.1 to compute products of the form
\[
t_n = \frac{1}{5\sqrt{5}} \left( \frac{\eta\left(\frac{1+\sqrt{-n/5}}{2}\right)}{\eta\left(\frac{1+\sqrt{-m}}{2}\right)} \right)^6
\]
when \( 5n \) is a convenient number. These products are then used to deduce special values of the Rogers-Ramanujan continued fraction. In the same article, he defined \([9, \text{Eq. (51)}]\)
\[
\mu_n = \frac{1}{3\sqrt{3}} \left( \frac{\eta(\sqrt{-n/3})}{\eta(\sqrt{-3n})} \right)^6
\]

(4.3)
and remarked that $\mu_n$ can be evaluated when $3n$ is one of the convenient numbers listed in Table 1 (15 is missing from his list). Ramanathan’s result can be stated as follows:

**Theorem 4.2 ([9, Theorem 4]).** Let $c$ be a convenient number listed in Table 1 and let $K = \mathbb{Q}(\sqrt{-c})$. Let $|t|$ be the ideal class containing $t$ such that $t^2 = (3)$. Then with the same notation as in Theorem 4.1

$$\mu_{c/3} = \prod_{\chi(|t|) = 1} \epsilon_{\chi}^{\frac{3e_{\chi}}{e_{\chi}}},$$

where the exponents are given by

$$e_{\chi} = \frac{2wh_{1,\chi}h_{2,\chi}}{w_{2,\chi}h},$$

with $h$ being the class number of $K$ and $w$ the number of roots of unity in $K$.

It turns out that Ramanathan’s $\mu_n$ is related to $\alpha_n$, namely $[5, (2.7)]$,

$$(4.4) \quad \frac{1}{\alpha_n} = \mu_n^2 + 1.$$ 

Hence, from Theorem 4.2, (4.1), and (4.4), we can determine $\alpha_n$ explicitly. Using the same technique as given in the proof of Theorem 4.2, one can compute $\alpha_n/(p_{i_j}^2 - p_{i_j})$, $1 \leq s \leq l - 1$ and $1 \leq i_j \leq l$, which will be needed in the evaluations of $a_n$.

We conclude this section with a list of singular moduli which will be needed in the evaluations of $a_n$, $b_n$, and $H_n$ with $n = c/3$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Cubic singular moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\alpha_2 = \frac{1}{2} - \frac{\sqrt{7}}{4}$</td>
</tr>
<tr>
<td>5</td>
<td>$\alpha_5 = \frac{1}{2} - \frac{11\sqrt{5}}{30}$</td>
</tr>
</tbody>
</table>

**Table 2.** Cubic singular moduli for $h(-12n) = 2$. 

Table 3. Cubic singular moduli for $h(-12n) = 4.$
Table 4. Cubic singular moduli for $h(-12n) = 8$.
<table>
<thead>
<tr>
<th>$n$</th>
<th>Cubic singular moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_{455} = \frac{1}{2}$</td>
<td>$\frac{52602592750172050462677}{24871948742554175611950} \sqrt{5} - \frac{5668214189343349857381}{49743589748510835122390} \sqrt{15}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{569659790275946071461}{49743589748510835122390} \sqrt{65} + \frac{538462633924678371678}{24871948742554175611950} \sqrt{1515}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{682503637304416627557}{994871949702167024478} \sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900} \sqrt{195}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{109593923135795012632}{994871949702167024478} \sqrt{1365}$</td>
</tr>
<tr>
<td>$\alpha_{91/5} = \frac{1}{2}$</td>
<td>$\frac{52602592750172050462677}{24871948742554175611950} \sqrt{5} + \frac{5668214189343349857381}{49743589748510835122390} \sqrt{15}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{569659790275946071461}{49743589748510835122390} \sqrt{65} - \frac{538462633924678371678}{24871948742554175611950} \sqrt{1515}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{682503637304416627557}{994871949702167024478} \sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900} \sqrt{195}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{109593923135795012632}{994871949702167024478} \sqrt{1365}$</td>
</tr>
<tr>
<td>$\alpha_{65/7} = \frac{1}{2}$</td>
<td>$\frac{52602592750172050462677}{24871948742554175611950} \sqrt{5} - \frac{5668214189343349857381}{49743589748510835122390} \sqrt{15}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{569659790275946071461}{49743589748510835122390} \sqrt{65} - \frac{538462633924678371678}{24871948742554175611950} \sqrt{1515}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{682503637304416627557}{994871949702167024478} \sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900} \sqrt{195}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{109593923135795012632}{994871949702167024478} \sqrt{1365}$</td>
</tr>
<tr>
<td>$\alpha_{35/13} = \frac{1}{2}$</td>
<td>$\frac{52602592750172050462677}{24871948742554175611950} \sqrt{5} + \frac{5668214189343349857381}{49743589748510835122390} \sqrt{15}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{569659790275946071461}{49743589748510835122390} \sqrt{65} - \frac{538462633924678371678}{24871948742554175611950} \sqrt{1515}$</td>
</tr>
<tr>
<td></td>
<td>$- \frac{682503637304416627557}{994871949702167024478} \sqrt{105} + \frac{25146509927196138320763}{497435897485108351223900} \sqrt{195}$</td>
</tr>
<tr>
<td></td>
<td>$+ \frac{109593923135795012632}{994871949702167024478} \sqrt{1365}$</td>
</tr>
</tbody>
</table>

Table 5. Cubic singular moduli for $h(-12n) = 16$. 
5. Values of $a_n$, $b_n$, and $H_n$. 

The values of $H_n$ follow immediately from the values of $\alpha_n$ by (1.7). From (1.6), it appears that we need to denest the expression $\sqrt{1-H_n}$ in order to determine $b_n$. The next simple lemma shows that this is not necessary.

**Lemma 5.1.** Let $\mu_n$ be defined as in (4.3). Then

$$b_n = \frac{2\sqrt{3n} \mu_n^2 - 1}{3 \mu_n^2 + 1}.$$

**Proof.** From (4.4), we deduce that

$$(5.1) \quad \frac{1}{1 - \alpha_n} = \frac{1}{\mu_n^2} + 1.$$

Hence, by (1.7), (4.4), and (5.1), we conclude that

$$\frac{4}{H_n} = \frac{\mu_n^2}{\mu_n^2 + 2} + 1.$$

Hence,

$$(5.2) \quad \sqrt{1 - H_n} = \sqrt{1 - \frac{4}{\mu_n^{-2} + \mu_n^{-2}} + 2} = \frac{\mu_n - \mu_n^{-1}}{\mu_n + \mu_n^{-1}}.$$  

Substituting (5.2) into (1.6) completes our proof of the lemma.  

Finally, to compute $a_n$, we use the method outlined in Section 3, together with the singular moduli given in Section 4. Our final results are shown in the following tables, grouped once again according to class numbers.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$</th>
<th>$b_n$</th>
<th>$H_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\frac{1}{3\sqrt{3}}$</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>5</td>
<td>$\frac{8}{15\sqrt{3}}$</td>
<td>$\frac{22}{5\sqrt{3}}$</td>
<td>$\frac{4}{125}$</td>
</tr>
</tbody>
</table>

**Table 6.** $a_n$, $b_n$, and $H_n$ for $h(-12n) = 2$. 

\[
\begin{array}{|c|c|c|c|}
\hline
n & a_n & b_n & H_n \\
\hline
7 & -\frac{10}{27} + \frac{7}{27}\sqrt{7} & \frac{13}{9}\sqrt{7} - \frac{7}{9} & -\frac{17}{27} + \frac{13}{54}\sqrt{7} \\
10 & \frac{25}{243}\sqrt[15]{15} - \frac{8}{243}\sqrt[6]{6} & \frac{70}{81}\sqrt[15]{15} + \frac{10}{81}\sqrt[6]{6} & \frac{223}{1458} - \frac{35}{729}\sqrt{10} \\
11 & \frac{6}{11} - \frac{13}{99}\sqrt{3} & \frac{45}{11}\sqrt{3} - \frac{5}{33}\sqrt{3} & -\frac{194}{1331} + \frac{225}{2662}\sqrt{3} \\
14 & \frac{21}{125}\sqrt{7} - \frac{82}{1125}\sqrt{3} & \frac{198}{125}\sqrt{7} + \frac{28}{375}\sqrt{3} & -\frac{1819}{31250} + \frac{198}{15625}\sqrt{21} \\
19 & \frac{1654}{3375} - \frac{133}{3375}\sqrt[19]{9} & \frac{5719}{1125} - \frac{13}{1125}\sqrt[19]{9} & \frac{8522}{421875} + \frac{3913}{843750}\sqrt[19]{9} \\
26 & \frac{1118}{44217}\sqrt[39]{39} - \frac{3967}{44217}\sqrt[3]{3} & \frac{4620}{4913}\sqrt[39]{39} + \frac{130}{4913}\sqrt[3]{3} & \frac{249913}{48275138} - \frac{34650}{24137569}\sqrt[3]{3} \\
31 & \frac{14662}{91125} + \frac{7843}{91125}\sqrt[31]{31} & \frac{217}{30375} + \frac{35113}{30375}\sqrt[31]{31} & -\frac{684197}{307546875} + \frac{245791}{615093750}\sqrt[31]{31} \\
34 & \frac{7157}{323433}\sqrt[33]{33} + \frac{62896}{323433}\sqrt[6]{6} & \frac{70}{107811}\sqrt[33]{33} + \frac{296140}{107811}\sqrt[6]{6} & \frac{3555313}{2582935938} - \frac{304850}{1291467969}\sqrt[33]{34} \\
59 & \frac{342786}{717853} - \frac{593849}{6490677}\sqrt{3} & \frac{6367095}{717853} - \frac{265}{2153559}\sqrt{3} & -\frac{1461224894}{30403462846931} + \frac{1687280175}{60806925693862}\sqrt{3} \\
\hline
\end{array}
\]

Table 7. $a_n$, $b_n$, and $H_n$ for $h(-12n) = 4$. 
<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$, $b_n$, and $H_n$</th>
</tr>
</thead>
</table>
| 35  | $a_{35} = \frac{558}{5} \sqrt{\frac{15}{15}} - \frac{364}{15} \sqrt{\frac{21}{21}} - \frac{577}{9} \sqrt{3}$  
$b_{35} = \frac{1701}{3} + \frac{581}{3} \sqrt{3} - \frac{126}{5} \sqrt{\frac{7}{7}} - \frac{366}{5} \sqrt{\frac{21}{21}}$  
$H_{35} = \frac{1210352}{125} - \frac{279351}{25} \sqrt{3} + \frac{91494}{25} \sqrt{7} + \frac{264132}{125} \sqrt{\frac{21}{21}}$ |
| 55  | $a_{55} = \frac{-1411054}{132651} + \frac{14375}{4913} \sqrt{\frac{15}{15}} + \frac{26884}{14739} \sqrt{\frac{33}{33}} - \frac{191450}{132651} \sqrt{\frac{55}{55}}$  
$b_{55} = \frac{-1423345}{44217} + \frac{50435}{4913} \sqrt{\frac{15}{15}} + \frac{27530}{4913} \sqrt{\frac{33}{33}} - \frac{185990}{44217} \sqrt{\frac{55}{55}}$  
$H_{55} = \frac{-40461636767}{651714365} + \frac{2329268305}{72412707} \sqrt{\frac{15}{15}} + \frac{782606510}{651714365} \sqrt{\frac{33}{33}} - \frac{5473886320}{651714365} \sqrt{\frac{55}{55}}$ |
| 70  | $a_{70} = \frac{57239}{1642545} \sqrt{7} + \frac{217912}{1642545} \sqrt{10} + \frac{18154}{182905} \sqrt{\frac{15}{15}} - \frac{5432}{60835} \sqrt{\frac{42}{42}}$  
$b_{70} = \frac{76766}{547515} \sqrt{7} + \frac{540694}{547515} \sqrt{10} + \frac{93548}{60835} \sqrt{\frac{15}{15}} - \frac{29314}{60835} \sqrt{\frac{42}{42}}$  
$H_{70} = \frac{-263701974157}{99924225075} - \frac{3413048639}{499621125375} \sqrt{6} + \frac{8992317139}{499621125375} \sqrt{\frac{105}{105}} - \frac{1429629212}{55513458375} \sqrt{\frac{55}{55}}$ |
| 91  | $a_{91} = \frac{-513055226}{17779581} + \frac{197125250}{17779581} \sqrt{7} - \frac{140862644}{17779581} \sqrt{\frac{15}{15}} + \frac{5294433}{17779581} \sqrt{\frac{91}{91}}$  
$b_{91} = \frac{-502667615}{5926527} + \frac{21464450}{5926527} \sqrt{7} - \frac{142555490}{5926527} \sqrt{\frac{15}{15}} + \frac{53880905}{5926527} \sqrt{\frac{91}{91}}$  
$H_{91} = \frac{302019804574232}{11707907427243} + \frac{1141527555432550}{11707907427243} \sqrt{7} - \frac{83858339971300}{11707907427243} \sqrt{\frac{133}{133}}$  
$+ \frac{633909424388075}{23415814854486} \sqrt{\frac{91}{91}}$ |
| 110 | $a_{110} = \frac{51466456301}{226152099801} \sqrt{3} - \frac{1605347400}{25128011089} \sqrt{10} + \frac{2302296150}{25128011089} \sqrt{\frac{55}{55}}$  
$b_{110} = \frac{-113011075870}{57584033267} \sqrt{3} + \frac{21911639310}{25128011089} \sqrt{10} + \frac{31690709820}{25128011089} \sqrt{\frac{55}{55}}$  
$+ \frac{8031352980}{25128011089} \sqrt{66}$  
$H_{110} = \frac{328032510163806603637}{126283382857813931842} + \frac{3496834326005152660}{631416941288906965921} \sqrt{22}$  
$- \frac{269221736736827282405}{631416941288906965921} \sqrt{\frac{35}{35}} + \frac{11479638881035691730}{631416941288906965921} \sqrt{\frac{110}{110}}$ |

Table 8. $a_n$, $b_n$, and $H_n$ for $h(-12n) = 8$. 
<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n$, $b_n$, and $H_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>115</td>
<td>$a_{115} = \frac{2453452114}{882792405} \sqrt{15} - \frac{731796}{32696015} \sqrt{69} + \frac{139937129}{882792405} \sqrt{115}$</td>
</tr>
<tr>
<td></td>
<td>$b_{115} = \frac{5026751821}{294264135} \sqrt{15} - \frac{19045012}{294264135} \sqrt{69} + \frac{132769793}{294264135} \sqrt{115}$</td>
</tr>
<tr>
<td></td>
<td>$H_{115} = \frac{19519366684794106}{14431896857830375} \sqrt{15} - \frac{5599864542570116}{16035440953203375} \sqrt{69} + \frac{3700660425371947}{28863793715760750} \sqrt{115}$</td>
</tr>
<tr>
<td>119</td>
<td>$a_{119} = \frac{103789302}{9720379} \sqrt{3} + \frac{43499625}{9720379} \sqrt{7} - \frac{72149336}{29161137} \sqrt{21}$</td>
</tr>
<tr>
<td></td>
<td>$b_{119} = \frac{318200715}{29161137} \sqrt{3} + \frac{162387225}{29161137} \sqrt{7} - \frac{60535540}{29161137} \sqrt{21}$</td>
</tr>
<tr>
<td></td>
<td>$H_{119} = \frac{49978710596750603}{1606258054361897} \sqrt{3} + \frac{28855221888962700}{1606258054361897} \sqrt{7} + \frac{10963595445145200}{1606258054361897} \sqrt{21}$</td>
</tr>
<tr>
<td>154</td>
<td>$a_{154} = \frac{965168}{4053225} \sqrt{6} - \frac{3910004}{182395125} \sqrt{7} - \frac{28870936}{182395125} \sqrt{22} + \frac{142457}{4053225} \sqrt{231}$</td>
</tr>
<tr>
<td></td>
<td>$b_{154} = \frac{2375828}{1351075} \sqrt{6} + \frac{11296216}{60798375} \sqrt{7} - \frac{5583106}{60798375} \sqrt{22} + \frac{836822}{1351075} \sqrt{231}$</td>
</tr>
<tr>
<td></td>
<td>$H_{154} = \frac{7319532242037247}{2710724428603125} \sqrt{33} + \frac{14157410807176}{301191603178125} \sqrt{31} + \frac{8770226416943}{301191603178125} \sqrt{115}$</td>
</tr>
</tbody>
</table>

Table 8 (continuation). $a_n$, $b_n$, and $H_n$ for $h(-12n) = 8$. 
NEW RAMANUJAN-TYPE SERIES FOR $1/\pi$

$a_{455} = -\frac{3595686812804845816546}{4974358974851083512239} - \frac{199639241839509967088008}{44769230773659751610151} \sqrt{3} + \frac{12114289251501127493868}{4974358974851083512239} \sqrt{13}$

$+ \frac{21170489873453104001440}{14923076924553250536717} \sqrt{21} + \frac{19045288924435485549578}{14923076924553250536717} \sqrt{39}$

$+ \frac{3501400086019335742242}{4974358974851083512239} \sqrt{91} + \frac{36482832707135043514012}{746153846227652683585} \sqrt{273}$

$b_{455} = -\frac{108868803097864065436089}{4974358974851083512239} - \frac{199460940107146922990240}{44769230773659751610151} \sqrt{3}$

$- \frac{32609426905353497981699991}{24871948748255147561195} \sqrt{7} + \frac{4777524611309163928990}{4974358974851083512239} \sqrt{13}$

$- \frac{9333522312361091775752}{4974358974851083512239} \sqrt{21} + \frac{2658123954621081666818}{4974358974851083512239} \sqrt{39}$

$+ \frac{1133642837868669714762}{4974358974851083512239} \sqrt{91} + \frac{35068395166781366975118}{2487179487455417561195} \sqrt{273}$

$H_{455} = -\frac{25593277575291678024530931497850444197001585383}{12372123605340761245091680055188536631396560500} \sqrt{3}$

$+ \frac{726431859849607816583487985232610666030207597}{618606180267038062254584002759426801569828025} \sqrt{7}$

$+ \frac{3839347276534358839899743258373110667699209791}{4948849442136304498036672022075414412558624200} \sqrt{13}$

$+ \frac{351649374516601338100434337317872485470333402}{618606180267038062254584002759426801569828025} \sqrt{21}$

$- \frac{56196149792473665089405401522944398076417222}{1237212360534076124509168005518853663139656050} \sqrt{39}$

$+ \frac{10324026218960132531682158178780641551536294}{393030901335190311272920013797134007849140125} \sqrt{273}$

$+ \frac{5358037718402253304261118862331927596113329}{2474424721068152249018336011037770720627931210} \sqrt{91}$

$+ \frac{1525258371640393899925036504420966961190929163}{1237212360534076124509168005518853663139656050} \sqrt{273}$

Table 9. $a_n$, $b_n$, and $H_n$ for $h(-12n) = 16$. 
Concluding remarks.

The common feature of all our series computed here is that they involve only simple quadratic numbers. The series corresponding to $n = 455$ gives us approximately 33 additional digits per term and it is the fastest convergent series belonging to the theory of $q_2$ known so far. It might also be the fastest convergent series for $1/\pi$ which involves only real quadratic numbers. One should compare this with the spectacular series discovered by the Borweins [4] which gives “25 digits per term” using only real quadratics. The fastest convergent series known so far is that given by the Borweins [4] which gives roughly 50 additional digits per term.

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References


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ON THE STABILITY OF CANONICAL FORMS OF SINGULAR LINEAR DIFFERENCE SYSTEMS

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Using the formal reduction by a method of deformation of orbits under the adjoint representation of GL(n, C), we have proved the existence and uniqueness (up to equivalence under GL(n, C)) of a formal canonical form of systems of singular linear difference equations. In this paper we study the stability of the irregular part of the canonical form under perturbation of the matrix coefficients.

1. Introduction and Notations.

The formal reduction of singular linear difference systems or of difference equations is studied in many ways: Formal classification, canonical forms or formal solutions (see [9], [4], [8], [2], [3]). One of the approaches is the reduction to canonical forms given in [3] by using the method of Babbitt and Varadarajan [1] for singular differential systems. We study in this paper the stability of the canonical forms of singular linear difference systems. Similar results for singular differential systems can be found in [1] or [7].

We shall use the following notations.

Let $K = \mathbb{C}((1/x))$ be the field of formal power series with coefficients in $\mathbb{C}$. $\phi$ is the $\mathbb{C}$-automorphism of $K$ defined by $\phi(x) = x+1$. For $q \in \mathbb{N}^*$, $x^{1/q}$ is a fixed root of $y^q = x$, $\mathcal{O}_q = \mathbb{C}[[x^{-1/q}]]$, $K_q = \mathbb{C}((x^{1/q}))$ and $\overline{K} = \bigcup_{q \in \mathbb{N}^*} K_q$ is the field of formal Puiseux power series over $\mathbb{C}$. $\phi$ can be extended to $\overline{K}$ by $\phi(x^{1/q}) = x^{1/q}(1 + x^{-1})^{1/q}$. Let $a \in K_q$ be nonzero, then it can be written in the form

$$a = a(x) = x^{-k/q} \sum_{j=0}^{+\infty} a_j x^{-j/q}, \quad a_0 \neq 0$$

where $k$ is an integer. We write $\text{ord}(a)$ for $k/q$, $(\text{ord}(0) = +\infty)$. For $A \in \text{gl}(n, K_q)$, $A \neq 0$, we define

$$\text{ord}(A) = \max \left\{ \frac{r}{q} \mid r \in \mathbb{Z}, A \in x^{-r/q} \text{gl}(n, \mathcal{O}_q) \right\}$$

and $\text{ord}(0) = +\infty$. 

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We consider systems of linear difference equations of the following type
\[ \phi(u) = Au \] (1)
where \( A \in \text{GL}(n, K_q) \), \( q \in \mathbb{N}^* \). One can write
\[ A = \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \text{GL}(n, K_q) \] (2)
where \( r \in \mathbb{Z}, A_{r+j} \in \text{gl}(n, \mathbb{C}) \) and \( A_r \neq 0 \).

Recall ([3]) that a matrix \( A \) or its associate system is said of level 0 if
\[ A = I + \sum_{m=q}^{\infty} \frac{A_m}{x^{m/q}}; \] of level \( \leq 1 \) if
\[ A = I + \sum_{m=0}^{\infty} \frac{A_{r+m}}{x^{(r+m)/q}, r \in \mathbb{N}^*, 1 \leq r < q, A_r \neq 0,} \] (3)
where \( I \) denotes the \( n \times n \) identity matrix.

Let \( T \in \text{GL}(n, K_q) \). The change \( \tilde{u} = Tu \) transforms the system (1) to \( \phi(\tilde{u}) = \tilde{A} \tilde{u} \) where
\[ \tilde{A} = T[A] \overset{\text{def}}{=} \phi(T)AT^{-1}. \]
We shall say that the matrices \( A, \tilde{A} \) (or the corresponding difference systems) are equivalent (under \( \text{GL}(n, K_q) \)).

We recall (cf. [3]) the definition of a canonical form for a matrix or its associate linear difference system.

**Definition 1.1.** Let \( p \in \mathbb{N}^* \). We shall say that a matrix \( B \in \text{GL}(n, K_p) \) is in canonical form if \( B = \frac{1}{x^{r'/p}} \bigoplus_{i=1}^{s} \frac{B_i}{x^{\ell_i}} \) with
\[ r \in \mathbb{Z}, \ \ell_i \in \frac{1}{p} \mathbb{N}, \ \ell_1 < \ell_2 < \cdots < \ell_s, \]
\[ B_i \in \text{GL}(n^{(i)}, \mathcal{O}_p), \ n^{(i)} \in \mathbb{N}^*, \]
\[ \sum_{i=1}^{s} n^{(i)} = n, \ B_i = \bigoplus_{\alpha=1}^{\ell_i} \lambda^{(i)}_{\alpha} \left( B^{(i)}_{\alpha} + \frac{C^{(i)}_{\alpha}}{x} \right) \]
where
\[ B^{(i)}_{\alpha} = f^{(i)}_{\alpha} + \frac{D^{(i)}_{\alpha,1}}{x^{r^{(i)}_{\alpha,1}}} + \cdots + \frac{D^{(i)}_{\alpha,j_{(i)_{\alpha}}}}{x^{r^{(i)}_{\alpha,j_{(i)_{\alpha}}}}} \]
and
- \( \lambda^{(i)}_{\alpha} \in \mathbb{C}^*, \lambda^{(i)}_{\alpha} \neq \lambda^{(i)}_{\beta} \) for \( \alpha \neq \beta \),
• $I^{(i)}_α$ is the $n^{(i)}_α \times n^{(i)}_α$ identity matrix, $n^{(i)}_α \in \mathbb{N}^*$, $\sum_{α=1}^{t_i} n^{(i)}_α = n^{(i)}$,
• $r^{(i)}_{α,j} \in \frac{1}{p} \mathbb{N}^*$, $r^{(i)}_{α,1} < r^{(i)}_{α,2} \cdots < r^{(i)}_{α,j_{α}^{(i)}} < 1$, $D^{(i)}_{α,j} \in \text{gl}(n^{(i)}_α, \mathbb{C})$ \(1 \leq j \leq j_{α}^{(i)}\) are nonzero diagonal matrices,
• $C^{(i)}_α \in \text{gl}(n^{(i)}_α, \mathbb{C})$ commutes with the $D^{(i)}_{α,j}$ for $1 \leq j \leq j_{α}^{(i)}$.

We make the convention that for $j_{α}^{(i)} = 0$, $B^{(i)}_α = I^{(i)}_α$.

We will call $\frac{1}{x^{r/p}} \bigoplus_{i=1}^{s} \frac{t^{(i)}_α}{x^{t_i}} \sum \frac{\lambda^{(i)}_α}{x^{r_i}} B^{(i)}_α$ the irregular part of the canonical form. The aim of this paper is to study the dependency of the irregular part in the canonical form of a singular linear difference system on the matrix coefficients $A_{r+j}$.

In [3] we have proved that for any matrix $A \in \text{GL}(n, K_q)$ there exist some $p \in q \mathbb{N}^*$ and $T \in \text{GL}(n, K_p)$ such that $T[A] \in \text{GL}(n, K_p)$ is in a canonical form and its irregular part is unique up to equivalence in $\text{GL}(n, \mathbb{C})$. It is based on the formal reduction using the method of Babbitt and Varadarajan [1], i.e., the method of deformation of orbits under the adjoint representation of $\text{GL}(n, \mathbb{C})$ in the nilpotent case of the leading matrix.

Recall that a canonical form for a matrix (or the associate difference system) of level $\leq 1$ is in the form:

$$I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C}{x}$$

where the $D_j(1 \leq j \leq k)$ are nonzero diagonal matrices, $0 < r_1 < \cdots < r_k$ are rational numbers and the matrix $C$ commutes with the matrices $D_j(1 \leq j \leq k)$. According to the convention of Definition 1.1, for $k = 0$ the canonical form is reduced to $I + C x^{-1}$. The canonical form of level $\leq 1$ is similar as in the differential case (see [1]). But for general difference systems the canonical form is more complicated.

We study in this paper the stability of the irregular part of the canonical form of a matrix or its associate linear difference system under perturbation of the matrix coefficients. A perturbed system of (1) is

$$\phi(u) = \left(A + \frac{P}{x^{(r+N)/q}}\right) u$$

with $N \in \mathbb{N}^*$ and $P \in \text{gl}(n, \mathbb{C})$, i.e., $\text{ord}(P) \geq 0$.

Note that in [2] the first author of this paper has studied similar problems for formal solutions. More precisely it is proved that the irregular part in a fundamental matrix of formal solutions of difference systems associated to a matrix of level $\leq 1$ (resp. of general systems) depends only on $A_r, A_{r+1}, \ldots, A_{r+(q-1)}$ (resp. $A_r, A_{r+1}, \ldots, A_{r+\nu+q-1}$) where $\nu$ denotes the integer such that $\nu \equiv \text{ord} (\text{det} x^{r/q} A)$. 

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We shall prove, by using the method of [1], similar results on canonical forms for these two difference systems. More precisely, we will prove in Section 3 that for systems of level \( \leq 1 \), if \( N \geq n(q - r) \), the two systems (1) and (4) have the same irregular part in their canonical forms. This result is similar to the differential case [1]. In the general case the situation is more complicated and is considered in Section 4. Basing on the method of [1] for differential systems, we use also frequently the formal reduction procedure of linear difference systems presented in [3]. We state some of the results of [1] and [3] in Section 2 for the use in the sequel.

2. Preliminaries.

We present now some preliminary results which will be used in the next sections (see also [1] and [3]).

For \( T \in \text{GL}(n, K_q) \) we define the lag (see also [1]) of \( T \) as

\[
\sigma_q(T) = \min \left\{ \frac{m}{q} \mid m \in \mathbb{N}, A \equiv 0 \pmod{x^{-m/q}} \Rightarrow TAT^{-1} \in \text{gl}(n, \mathcal{O}_q) \right\}.
\]

It is clear that if \( \sigma_q(T) \leq \frac{m}{q} \) then

\[
A \equiv B \pmod{x^{-m'/q}} \Rightarrow T[A] \equiv T[B] \pmod{x^{-(m'-m)/q}}.
\]

Therefore if one controls the lag of a transformation matrix \( T \), then one controls the first terms in the transformed system \( T[A] \).

One has immediately the following properties (see also [1], p. 10-11):

(i) If \( q' \) is a multiple of \( q \) then for \( T \in \text{GL}(n, K_q) \subset \text{GL}(n, K_{q'}) \), \( \sigma_{q'}(T) = \sigma_q(T) \). We will write \( \sigma_q(T) \) for \( \sigma_{q'}(T) \) in the sequel.

(ii) \( \sigma(T) = 0 \) for \( T \in \text{GL}(n, \mathcal{O}_q) \cdot \mathbb{Z}_q \) where \( \mathbb{Z}_q \) is the group of elements of the form \( x^{-k/q} \cdot 1 \) for \( k \in \mathbb{Z} \).

(iii) \( \sigma(T_1T_2) \leq \sigma(T_1) + \sigma(T_2) \) for \( T_1, T_2 \in \text{GL}(n, K_q) \).

(iv) \( \sigma(QTQ) = \sigma(T) \), \( Q, \bar{Q} \in \text{GL}(n, \mathcal{O}_q) \cdot \mathbb{Z}_q \) and \( T \in \text{GL}(n, K_q) \).

(v) If \( T = x^H \) for some semi-simple matrix \( H \) in \( \text{gl}(n, \mathbb{C}) \) with eigenvalues \( \lambda_i \in \frac{1}{q}\mathbb{Z} \) \((i = 1, \ldots, n)\) then

\[
\sigma(T) = \max_{1 \leq i,j \leq n} \{|\lambda_i - \lambda_j|\}.
\]

(vi) \( \sigma(T) = \sigma(T^{-1}) \) for \( T \in \text{GL}(n, K_q) \).

Let \( \mathcal{O}^\times \) be the group of units of \( \mathcal{O} = \bigcup_{q \in \mathbb{N}^*} \mathcal{O}_q \). We define

\[
\text{^\circ}\text{GL}(n, F) = \{ T \in \text{GL}(n, F) \mid \det T \in \mathcal{O}^\times \}
\]

where \( F \) may represent \( \overline{K}, K_q, \mathcal{O}_q \) etc. If \( H \) is semi-simple in \( \text{gl}(n, \mathbb{C}) \) with eigenvalues in \( \mathbb{Q} \), it is immediate that

\[
x^H \in \text{^\circ}\text{GL}(n, \overline{K}) \iff \text{tr}(H) = 0.
\]

We then have (cf. [1], Proposition 1.2).
(vii) Let \( T = \bigoplus_{i=1}^{m} T_i \) where \( T_i \in \text{oGL}(n_i, K_q) \) and \( n = \sum_{i=1}^{m} n_i \). Then \( T \in \text{oGL}(n, K_q) \) and \( \sigma(T) \leq \sigma(T_1) + \cdots + \sigma(T_m) \).

Let \( \mathcal{G} = \text{gl}(n, C) \). For \( M \in \mathcal{G} \), \( \mathcal{G}_M \) and \([\mathcal{G}, M]\) denote respectively the kernel and the image of the adjoint homomorphism \( \text{ad}(M) \). \( d(M) \) is the dimension of the \( \text{GL}(n, C) \)-orbit of \( M \) with respect to the adjoint representation of \( \mathcal{G} \).

**Proposition 2.1** ([6], [1]). Let \( Y \) be a nonzero nilpotent in \( \mathcal{G} \); then we can find \( H, X \in \text{sl}(n, C) \) such that \( H \) is semi-simple, \( X \) is nilpotent and

\[
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.
\]

\((Y, H, X)\) is called a standard triple.

**Proposition 2.2** ([1]). Let \( Y \) be a nonzero nilpotent and \((Y, H, X)\) a standard triple. Let \( Z \in \mathcal{G}_X, Z \neq 0 \). Suppose that \( Y + Z \) is nilpotent. Then \( d(Y + Z) > d(Y) \).

For a standard triple \((Y, H, X)\), we have \( \mathcal{G} = \mathcal{G}_X \oplus [\mathcal{G}, Y] \). Moreover there exists a basis \( \{Z_1, \ldots, Z_\ell\} \) of \( \mathcal{G}_X \) such that \( Z_1 = I, Z_j \in \text{sl}(n, C) \) for \( j \geq 2 \) (see [1], p. 15) and

\[
[H, Z_j] = \lambda_j Z_j, \quad \lambda_j \in \mathbb{N} \quad \text{for} \quad 1 \leq j \leq \ell.
\]

In particular \( \lambda_1 = 0 \). Define \( \Lambda = \max_{1 \leq j \leq \ell} \left( \frac{\lambda_j}{2} + 1 \right) \), then \( 1 \leq \Lambda \leq n \).

\( \{Z_1, \ldots, Z_\ell\} \) can be extended to a basis \( \{Z_1, \ldots, Z_\ell, Z_{\ell+1}, \ldots, Z_{n^2}\} \) of \( \mathcal{G} \) with the following properties:

For all \( j > \ell \), \( [H, Z_j] = \lambda_j Z_j, \quad \lambda_j \in \mathbb{Z}, \quad |\lambda_j| \leq \max_{1 \leq i \leq \ell} \lambda_i. \)

If \( M \in \text{gl}(n, C) \) is such that \([H, M] = cM\) for some \( c \in \mathbb{Z} \) then

\[
x^{\alpha H} M x^{-\alpha H} = x^{\alpha} M, \quad \text{for} \quad \alpha \in \mathbb{Q}.
\]

In particular

\[
(6) \quad x^{\alpha H} Y x^{-\alpha H} = x^{-2\alpha} Y; \quad x^{\alpha H} Z_j x^{-\alpha H} = x^{\lambda_j \alpha} Z_j.
\]

One has for \( \alpha \in \mathbb{Q} \)

\[
(7) \quad \sigma(x^{\alpha H}) \leq |\alpha| \max_{1 \leq j \leq n^2} \{|\lambda_j|\} \leq 2(\Lambda - 1)|\alpha|.
\]

We need also the following lemmas.

**Lemma 2.1** ([3]). Let a matrix \( A \in \text{GL}(n, K_q) \) be in one of the following forms,

\[
(I) \quad I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}}, \quad 1 \leq r < q.
\]
or

$$\quad (II) \quad x^{-r/q} \sum_{j=0}^{\infty} A_{r+j} x^{j/q}, \quad r \in \mathbb{Z};$$

where $A_{r+j} \in G$, $A_r \neq 0$. Let $\mathcal{L} \subset G$ be a linear subspace such that $G = \mathcal{L} + [G, A_r]$. Then there exist sequences $(T_j)_{j \geq 1}$ in $G$, $(A'_{r+j})_{j \geq 1}$ in $L$,

$$T = \prod_{j=\infty}^{1} \left( I + \frac{T_j}{x^{j/q}} \right) = \lim_{J \to \infty} \prod_{j=J}^{1} \left( I + \frac{T_j}{x^{j/q}} \right)$$

such that $A'_r = A_r$ and

$$T[A] = I + \sum_{j=0}^{\infty} \frac{A'_{r+j}}{x^{(r+j)/q}}, \quad \text{in the case (I)},$$

or

$$T[A] = x^{-r/q} \sum_{j=0}^{\infty} \frac{A'_{r+j}}{x^{j/q}}, \quad \text{in the case (II)}.$$

Moreover $A'_{r+j}$ only depends on $A_r, A_{r+1}, \ldots, A_{r+j}$.

**Corollary 2.1** ([3]; Splitting lemma). Let notations be as in the above lemma. Let $\Sigma$ be the set of eigenvalues of $A_r$, $P_{\lambda}$ be the matrix of the projection of $\mathbb{C}^n$ on the eigenspace corresponding to $\lambda$ in $\Sigma$. Let $S$ be the semi-simple part of $A_r$. Choose $L = G_S$. Then $A'_{r+j}$ commutes with $P_{\lambda}$ for $j \geq 1$; moreover $T[A] = \bigoplus_{\lambda \in \Sigma} A'_{\lambda}$ where

$$A'_{\lambda} = I + \sum_{j=0}^{\infty} \frac{P_{\lambda} A'_{r+j}}{x^{(r+j)/q}}, \quad \text{in the case (I)},$$

$$A'_{\lambda} = x^{-r/q} \sum_{j=0}^{\infty} \frac{P_{\lambda} A'_{r+j}}{x^{j/q}}, \quad \text{in the case (II)}.$$

**Corollary 2.2** ([3]). Let notations be as in the above lemma. Assume that $A_r$ is nilpotent and $(A_r, H, X)$ a standard triple. Let $\mathcal{L} = G_X$ and let $m \geq 2$ be an integer. There exists

$$T = \left( I + \frac{T_{m-1}}{x^{(m-1)/q}} \right) \cdots \left( I + \frac{T_1}{x^{1/q}} \right) \in \text{GL}(n, K_q)$$

such that

$$T[A] = I + \sum_{j=0}^{\infty} \frac{A'_{r+j}}{x^{(r+j)/q}}, \quad \text{in the case (I)}$$

$$T[A] = x^{-r/q} \sum_{j=0}^{\infty} \frac{A'_{r+j}}{x^{j/q}}, \quad \text{in the case (II)}.$$


with $A'_{r} = A_{r}$, $A'_{r+j} \in G_{X}$ for $1 \leq j < m$. Furthermore for $j \in \mathbb{N}^{*}$, $A'_{r+j}$ only depends on $A_{r}, \ldots, A_{r+j}$.

**Lemma 2.2 ([3]).** Let a matrix $B \in \text{GL}(n, K_{p})$ be in the form

$$B = I + \frac{D_{1}}{x^{1/p}} + \cdots + \frac{D_{p-1}}{x^{(p-1)/p}} + \frac{C}{x} + R_{B}$$

where the $D_j \in \text{gl}(n, \mathbb{C})$ are diagonal matrices, $C \in \text{gl}(n, \mathbb{C})$, $\text{ord}(R_{B}) > 1$. Then $B$ is equivalent to a canonical matrix of the form $I + \frac{D_{1}}{x^{1/p}} + \cdots + \frac{D_{p-1}}{x^{(p-1)/p}} + \frac{C'}{x}$ for some $C' \in \text{gl}(n, \mathbb{C})$.

### 3. Difference Systems of Level $\leq 1$

We consider at first, as in [3], difference systems of level $\leq 1$, i.e., systems (1) with matrix $A$ in the special form (3):

$$A = I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \text{GL}(n, K_{q})$$

with $A_{r} \neq 0, 1 \leq r < q$.

We will prove, using the method of [1], that the irregular part of a canonical form of difference systems of level $\leq 1$ is determined by the matrices $A_{r}, A_{r+1}, \ldots, A_{r+n(q-r)-1}$. Similar result for formal solutions has been proved in [2] by a different method.

Recall (cf. [3]) that a canonical form for matrices of level $\leq 1$ is in the form:

$$(8) \quad A_{\text{cano}} = I + \frac{D_{1}}{x^{r_1}} + \cdots + \frac{D_{k}}{x^{r_k}} + \frac{C_{A}}{x} \in \text{GL}(n, K_{p})$$

for some $p \in \mathbb{N}^{*}$ and the irregular part of this canonical form is $I + \frac{D_{1}}{x^{r_1}} + \cdots + \frac{D_{k}}{x^{r_k}}$. If $A$ is of level 0, the irregular part in its canonical form is reduced to $I$.

Since the irregular part is the first terms of a canonical form, from (5) one needs to make normalizations by matrices with convenient lags. The following proposition shows that, by a transformation matrix with a lag not exceeding a certain number, a difference system of level $\leq 1$ with nilpotent leading matrix can be converted to a new one with non nilpotent leading matrix.

**Proposition 3.1.** Let a matrix $A = I + \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \text{GL}(n, K_{q})$ with $A_{r} \neq 0$ be of level $\leq 1$, i.e., $1 \leq r < q$. Then we can find a matrix $U \in \text{GL}(n, K_{p})$ for some $p \in q\mathbb{N}^{*}$ and $1 \leq s \leq p$, such that:

1. $U[A] = I + \sum_{j=0}^{\infty} \tilde{A}_{s+j}x^{-(s+j)/p} \in \text{GL}(n, K_{p})$ where either $U[A]$ is of level 0 in which case $s = p$ or $\tilde{A}_{s}$ is not nilpotent.
\( (2) \) \( \sigma(U) \leq (n - 1)(\frac{a}{p} - \frac{c}{q}) \).

**Proof.** If \( A_r \) is not nilpotent then the proposition is true with \( s = r, \ p = q \) and \( U = I, \ \sigma(U) = 0. \)

If \( A_r \) is nilpotent we prove it by downward induction on \( d(A) \), the dimension of the GL\((n, \mathbb{C})\)-orbit of \( A_r \).

Let \( (Y, H, X) \) be a standard triple with \( Y = A_r \). We apply at first the Corollary 2.2 (for the case (I)) with \( m = \Lambda(q - r) \). Recall that

\[
T = \left( I + \frac{T_{\Lambda(q-r)-1}}{x^{(r+\Lambda(q-r)-1)/q}} \right) \cdots \left( I + \frac{T_1}{x^{r/q}} \right)
\]

and

\[
A' = T[A] = I + \frac{A_r}{x^{r/q}} + \frac{A'_{r+1}}{x^{(r+1)/q}} + \cdots
\]

where \( A'_{r+j} \in G_X \) for \( 1 \leq j < \Lambda(q - r) \). Then \( T \in \mathfrak{o}GL(n, K_q) \) and \( \sigma(T) = 0 \) according to the property (ii) of Section 2.

We can write

\[
A'_{r+j} = \sum_{i=1}^{\ell} a_{r+j,i} Z_i, \quad 1 \leq j < \Lambda(q - r),
\]

\[
A'_{r+j} = \sum_{i=1}^{n^2} a_{r+j,i} Z_i, \quad j \geq \Lambda(q - r).
\]

Define

\[
E = \left\{ \frac{j}{\ell} + 1 \middle| 1 \leq j < \Lambda(q - r), 1 \leq i \leq \ell, a_{r+j,i} \neq 0 \right\}.
\]

Let

\[
\beta = \begin{cases} \inf E & \text{if } E \neq \emptyset \\ \infty & \text{otherwise} \end{cases}
\]

Define \( \alpha = \min\{q - r, \beta\} \) and \( S = x^{\alpha H/(2q)} \). It is clear that \( \beta > 0, \ \alpha > 0. \)

According to (7), \( \sigma(S) \leq (\Lambda - 1)\alpha/q \leq (n - 1)\alpha/q. \) Since \( \text{tr}(H) = 0, \ S \in \mathfrak{o}GL(n, K_p). \) According to (6) we have,

\[
S[Y] = (1 + x^{-1})^{\alpha H/(2q)} x^{-\alpha/q} Y,
\]

\[
S[Z_j] = (1 + x^{-1})^{\alpha H/(2q)} x^{\alpha \lambda_j/(2q)} Z_j, \quad 1 \leq j \leq n^2.
\]
Therefore

\[ A'' \overset{\text{def}}{=} S[A'] = (1 + x^{-1})^{\alpha H/(2q)} \left[ I + x^{-(r+\alpha)/q} \left( Y 
\right. \right.
\]

\[ + \sum_{1 \leq j < \Lambda(q-r)} \frac{a_{r+j,i}Z_i}{x^{\frac{j}{q}(\frac{1}{2}+1)}a} + \sum_{\Lambda(q-r) \leq j \leq n^2} \frac{a_{r+j,i}Z_i}{x^{\frac{j}{q}(\frac{1}{2}+1)a}} \right]. \]

If \( \alpha = q - r \) then \( s = p = q \) and \( \sigma(S) \leq (n-1) \left( \frac{s}{p} - \frac{r}{q} \right) \). \( A'' \) is of level 0.

If \( 0 < \alpha < q - r \), write \( \alpha = r'/q' \). Then

\[ A'' = S[A'] = ST[A] = I + Y' x^{-\tilde{r}/\tilde{q}} + \cdots \in GL(n, \mathbb{O}_q) \]

where \( \tilde{r} = 2(q'r' + r') \), \( \tilde{q} = 2q'q \) and \( Y' = Y + \sum_{(j,i) \in \Omega} a_{r+j,i}Z_i \) with

\[ \Omega = \left\{ (j,i) \mid 1 \leq j < \Lambda(q-r), 1 \leq i \leq \ell, a_{r+j,i} \neq 0, \alpha = \frac{j}{\tilde{q} + 1} \right\}. \]

Moreover \( Y' = Y + Z \neq Y \) with \( Z \in \mathcal{G}_X \). We have \( \sigma(S) \leq (n-1) \frac{s}{\tilde{q}} = (n-1) \left( \frac{s}{p} - \frac{r}{q} \right) \). There are two distinct cases.

(a) If \( Y' \) is not nilpotent (this case occurs when \( d(Y) \) has the maximal dimension, i.e., when \( Y \) is a principal nilpotent) then take \( p = \tilde{q}, s = \tilde{r} \), and

\[ \sigma(S) \leq (n-1) \left( \frac{s}{p} - \frac{r}{q} \right). \]

Hence \( U = ST \in \mathcal{O}_Q \) has the claimed properties.

(b) If \( Y' \) is nilpotent, then \( d(Y') > d(Y) \) according to Proposition 2.2. The induction hypothesis is applicable to \( A'' \). One deduces the existence of a \( S' \in GL(n, K_p) \) for some \( p \in \tilde{q}\mathbb{N}^* \) and \( 1 \leq s \leq p \) such that

\[ \sigma(S') \leq (n-1) \left( \frac{s}{p} - \frac{\tilde{r}}{\tilde{q}} \right) \] and \( S'[A'] \) has the property (1). Let \( U = S'ST \) then

\[ \mathcal{O}(U) \leq \sigma(S') + \sigma(S) \leq (n-1) \left( \frac{s}{p} - \frac{\tilde{r}}{\tilde{q}} \right) + (n-1) \left( \frac{\tilde{r}}{\tilde{q}} - \frac{r}{q} \right) \]

\[ = (n-1) \left( \frac{s}{p} - \frac{r}{q} \right). \]

The next proposition proves that for a system of level \( \leq 1 \) one can obtain the irregular part of a canonical form with a transformation matrix whose lag is not greater than the number \( (n-1) \left( 1 - \frac{s}{p} \right) \).
Proposition 3.2. Let $A$ be as in the above proposition. Then we can find a matrix $U \in \mathfrak{gl}(n, K_p)$ for some $p \in qN^*$, such that

1. there exists a canonical form

$$A_{\text{cano}} = I + \frac{D_1}{x} + \cdots + \frac{D_k}{x^k} + \frac{C_A}{x} \in \mathfrak{gl}(n, K_p)$$

such that $U[A] = A_{\text{cano}} + R_A \in \mathfrak{gl}(n, K_p)$ with $\text{ord}(R_A) > 1$.

2. $\sigma(U) \leq (n-1)(1-\frac{r}{q})$.

Remark. With the convention of Definition 1.1, if $k > 0$ then $D_j(1 \leq j \leq k)$ are nonzero diagonal matrices and for $k = 0$, $A_{\text{cano}} = I + C_Ax^{-1}$ is of level 0.

Proof. We prove the proposition by induction on $n$. For $n = 1$ one can take $U = 1$. Suppose $n > 1$. We assume the assertion in dimension $< n$.

Assume at first that $A_r$ has at least two distinct eigenvalues. By applying the Corollary 2.1 we obtain a matrix $T = \prod_{j=1}^{1}(I + T_jx^{-j/q})$. Take

$$T_A = \prod_{j=n(q-r)-1}^{1} \left( I + \frac{T_j}{x^{j/q}} \right) \quad \text{and} \quad A' = T_A[A].$$

Let $A''$ be the matrix obtained from $A'$ by omitting all terms of $x^{-j/q}$ with $j \geq r + n(q-r)$. Then $A' = A'' + E$ where $\text{ord}(E) \geq (r + n(q-r))/q$ and $A''$ commutes with the spectral projections of $A_r$.

If $A_r = \bigoplus \lambda A^{(r)}_{\lambda}$ with $n_{\lambda} = \dim \left( A^{(r)}_{\lambda} \right)$ then $A'' = \bigoplus \lambda A''_{\lambda}$. By induction we can find matrices $U_{\lambda} \in \mathfrak{gl}(n_{\lambda}, K_p)$ verifying the condition (1) and $\sigma(U_{\lambda}) \leq (n_{\lambda} - 1)\left(1 - \frac{r}{q}\right)$. We now use the property (vii) (cf. Section 2) to conclude that if $U' = \bigoplus \lambda U_{\lambda}$, then $U' \in \mathfrak{gl}(n, K_p)$ and

$$\sigma(U') \leq \sum_{\lambda} \sigma(U_{\lambda}) \leq \sum_{\lambda} (n_{\lambda} - 1) \left(1 - \frac{r}{q}\right) \leq (n - 2) \left(1 - \frac{r}{q}\right)$$

and $U'[A'']$ verifies the condition (1). We now check that $\text{ord}(U'[E]) > 1$ by using (5):

$$\text{ord}(U'[E]) \geq \frac{r + n(q-r)}{q} - \frac{(n-2)(q-r)}{q} > 1.$$ 

Then $U = U'T_A$ has the claimed properties.

We now consider the case where $A_r$ has a unique eigenvalue, $A_r = \omega I + Y$ where $Y$ is nilpotent. We proceed by induction on the number $k = k(A)$ in the canonical form. If this number is 0 then by the above proposition one can find a matrix $T \in \mathfrak{gl}(n, K_p)$ for some $p \in qN^*$ such that $\sigma(T) \leq (n - 1)\left(1 - \frac{r}{q}\right)$ and $T[A]$ is of level 0.
We may thus suppose that $k \geq 1$. If $\omega \neq 0$ then let $A = (1 + \omega x^{-r/q})$. We have $k(\tilde{A}) < k(A)$. Therefore the induction hypothesis is applicable to $\tilde{A}$ and proves the proposition for $A$.

Suppose now that $\omega = 0$ so that $A_r = Y$ is a nonzero nilpotent matrix. Let $U_1$ be chosen to satisfy the conditions of Proposition 3.1 for some $p^* \in qN^*$. Then

$$A^* = U_1[A] = I + A_s^*x^{-s/p^*} + \cdots$$

and $\sigma(U_1) \leq (n - 1) \left( \frac{s}{p^*} - \frac{r}{q} \right)$. Either $A^*$ is of level 0 in which case $s = p$ the proof is thus finished or $A_s^*$ is not nilpotent which we consider in the following.

If $A_s^*$ has at least two distinct eigenvalues, the earlier result allows us to find a matrix $U^* \in ^*GL(n, K)$ for some $p \in p^*N$ such that $\sigma(U^*) \leq (n - 1) \left( 1 - \frac{s}{p^*} \right)$ and $U^*[A^*]$ has the property (1). If $U = U^*U_1$, then one has immediately the second assertion:

$$\sigma(U) \leq (n - 1) \left( 1 - \frac{s}{p^*} \right) + (n - 1) \left( \frac{s}{p^*} - \frac{r}{q} \right) = (n - 1) \left( 1 - \frac{r}{q} \right).$$

If $A_s^*$ has a single eigenvalue $\omega^*$ (which should be nonzero), one can write

$$A^* = (1 + \omega^*x^{-s/p^*})A^*$$

where $k(A^*) < k(A^*) = k(A)$. The induction hypothesis applied to the matrix

$$A^* = I + A_s^*x^{-s/p^*} + \cdots$$

(with $s' \geq s$)

gives a matrix $U^*$ having properties (1) and

$$\sigma(U^*) \leq (n - 1) \left( 1 - \frac{s'}{p^*} \right) \leq (n - 1) \left( 1 - \frac{s}{p^*} \right).$$

As before we take $U = U^*U_1$ and note that $\sigma(U) \leq (n - 1) \left( 1 - \frac{r}{q} \right)$. The proof is thus complete. 

Let $A \in GL(n, K_q)$ be a matrix of level $\leq 1$ as in the above propositions. We denote by $\Omega(A, m)$ the set of matrices $B \in GL(n, K_q)$ of the same form as $A$ with $B_{r+j} = A_{r+j}$ for all $0 \leq j < m$, i.e., $B \equiv A(\mod x^{-(r+m)/q})$.

**Corollary 3.1.** Let notations be as in the proposition, $m = n(q - r)$. If $B \in \Omega(A, m)$ then

$$U[B] = I + \frac{D_1}{x^{r_1}} + \cdots + \frac{D_k}{x^{r_k}} + \frac{C_B}{x} + R_B \in GL(n, K_q)$$

where $\text{ord}(R_B) > 1$. If further $B \equiv A(\mod x^{-(r+m')/q})$ for some $m' > m$ then $C_B = C_A$. 

Proof. \( \sigma(U) \leq (n - 1) \left( 1 - \frac{r'}{q} \right) \). From \( B \equiv A (\text{mod } x^{-(r+m)/q} \) and (5) we have

\[
U[A] \equiv U[B] \left( \text{mod } x^{-\left[ \frac{r+m}{q} - (n-1)\left(1-\frac{r}{q}\right) \right]} \right)
\]

and

\[
\frac{r+m}{q} - (n-1) \left( 1 - \frac{r}{q} \right) = 1,
\]

proving the first assertion.

If \( B \equiv A \left( \text{mod } x^{-\frac{r+m'}{q}} \right) \) then

\[
U[A] \equiv U[B] \left( \text{mod } x^{-\left[ 1+\frac{m'+m}{q} \right]} \right),
\]

proving the second statement. \( \square \)

According to Lemma 2.2, the canonical form of \( A_{\text{cano}} + R_A \) is

\[
I + D_1 x^{r_1} + \cdots + D_k x^{r_k} + C' x
\]

where only the matrix \( C' \) may be different from \( C_A \) of \( A_{\text{cano}} \).

The following theorem is now immediate.

**Theorem 3.1.** Let \( A \) be a matrix as in the above propositions. Let \( m = n(q-r) \). If \( B = I + B_r x^{-r/q} + \cdots + B_r x^{-r/q} + C' x \) where only the matrix \( C' \) may be different from \( C_A \) of \( A_{\text{cano}} \), then \( A \) and \( B \) are either both of level 0 or both not, and have canonical forms with the same irregular part.

As a consequence of this theorem, for systems of level \( \leq 1 \), the irregular part in a fundamental matrix of formal solutions depends only on the matrix coefficients \( A_{r+j}, 0 \leq j < n(q-r) \) (see also [2]).

**4. General Difference Systems.**

We now consider general difference systems of the form (1). We study at first as in the preceding section the nilpotent case, i.e., the case where the leading matrix \( A_r \) is nilpotent. We prove that for \( N \geq 2\nu + nq \) the two difference systems (1) and (4) have the same irregular part in their canonical forms.

**Definition 4.1.** Let notations be as in Definition 1.1. We shall say that a matrix \( B' \) is in quasi-canonical form if

\[
B' = \frac{1}{x^{r/p}} \bigoplus_{i=1}^{s} B'_i x^{r_i}
\]

with

\[
B'_i = \bigoplus_{\alpha=1}^{t_i} \lambda^{(i)}_{\alpha} \left( B^{(i)}_{\alpha} + C^{(i)}_{\alpha} x + R^{(i)}_{\alpha} \right)
\]

where \( \text{ord}(R^{(i)}_{\alpha}) > 1 \).

**Remark.** A quasi-canonical form of a matrix of level \( \leq 1 \) is simply a matrix of the form \( A_{\text{cano}} + R_A \) where \( A_{\text{cano}} \) is a canonical matrix of the form (8) and \( \text{ord}(R_A) > 1 \). It is clear that the matrices \( B \) and \( B' \) have the same irregular part according to Lemma 2.2.

At first we prove in the following proposition that one can always reach a non nilpotent leading matrix by a transformation with a convenient lag.
Proposition 4.1. Let \( A = \sum_{j=0}^{\infty} \frac{A_{r+j}}{x^{(r+j)/q}} \in \text{GL}(n,K_q) \) with \( r \in \mathbb{Z}, A_r \neq 0 \).

Let \( \nu \) be the integer such that \( \frac{\nu}{q} = \text{ord}(\det x^{r/q}A) \). Then we can find a matrix \( U \in \overset{\circ}{\text{GL}}(n,K_p) \) for some \( p \in qN^* \) so that

1. \( \tilde{A} = U[A] = \tilde{A}_s x^{-s/p} + \cdots \in \text{GL}(n,K_p) \), where \( \tilde{A}_s \) is not nilpotent.
2. \( \sigma(U) \leq (1 - \frac{1}{n}) \left( \frac{\nu}{q} - \frac{\tilde{\nu}}{p} \right) \) where \( \frac{\tilde{\nu}}{p} = \text{ord}(\det x^{s/p} \tilde{A}) \).

Proof. If \( A_r \) is not nilpotent then \( s = r, p = q \) and \( U = I, \sigma(U) = 0 \).

If \( A_r \) is nilpotent we prove it by induction on \( d(A_r) \), the dimension of the \( \text{GL}(n,C) \)-orbit of \( A_r \). Let \( Y = A_r \) and \((Y,H,X)\) a standard triple. We apply at first Corollary 2.2 (for the case II) with \( m = \nu \Lambda + 1 \). We have

\[
T = \prod_{j=\nu \Lambda}^{1} \left( I + \frac{T_j}{x^{j/q}} \right) \in \overset{\circ}{\text{GL}}(n,K_q)
\]

and \( \sigma(T) = 0 \) according to the property (ii) of the Section 2. Let \( A' = T[A] \). Then

\[
A' = x^{-r/q} \left( Y + \frac{A'_{r+1}}{x^{1/q}} + \cdots + \frac{A'_{r+\nu \Lambda}}{x^{(\nu \Lambda)/q}} + \cdots \right)
\]

with \( A'_{r+j} \in G_X \) for \( j = 1, \ldots, \nu \Lambda \). Furthermore for \( j \in N^*, A'_{r+j} \) only depends on \( A_r, \ldots, A_{r+j} \).

Write

\[
A'_{r+j} = \sum_{i=1}^{\ell} a_{r+j,i} Z_i, \quad 1 \leq j \leq \nu \Lambda,
\]

\[
A'_{r+j} = \sum_{i=1}^{n^2} a_{r+j,i} Z_i, \quad j > \nu \Lambda.
\]

Define

\[
E = \left\{ \left. \frac{j}{\Lambda + 1} \right| 1 \leq j \leq \nu \Lambda, 1 \leq i \leq \ell, a_{r+j,i} \neq 0 \right\}.
\]

We claim that \( E \neq \emptyset \) and \( \inf E \leq \nu \) since \( \det(\sum_{j=0}^{\nu} A_{r+j} x^{-j/q}) \neq 0 \). Let \( \beta = \inf E > 0 \) and \( S = x^{\beta H/(2q)} \). By (7), \( \sigma(S) \leq (\Lambda - 1) \beta / q \). According to (6),

\[
S[Y] = (1 + x^{-1})^{\beta H/(2q)} x^{-\beta/q} Y,
\]

\[
S[Z_i] = (1 + x^{-1})^{\beta H/(2q)} x^{\beta A_i/(2q)} Z_i, \quad 1 \leq i \leq n^2.
\]
Hence

\[
S[A'] = (1 + x^{-1})^{\beta H/(2q) x^{-(r+\beta)/q}} \left[ Y + \sum_{1 \leq j \leq \nu A, 1 \leq i \leq \ell} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q} \left[ j - \left( \frac{\lambda_i}{2} + 1 \right) \beta \right]}} + \sum_{j > \nu A, 1 \leq i \leq n^2} \frac{a_{r+j,i} Z_i}{x^{\frac{1}{q} \left[ j - \left( \frac{\lambda_i}{2} + 1 \right) \beta \right]}} \right].
\]

Write \( \beta = \frac{r'}{q'} \) with \( r', q' \in \mathbb{N}^* \). Recall that \( 0 \leq \beta \leq \nu \). For all \( j > \nu A \) and \( 1 \leq i \leq n^2 \), \( j > (\lambda_i^2 + 1) \beta > 0 \). Then \( S[A'] \in \text{GL}(n, \mathcal{O}_{2qq'}) \). More precisely,

\[
A'' = S[A'] = x^{-r''/q''} \left[ Y' + O(x^{-1/q''}) \right]
\]

where

\[
Y' = Y + \sum_{(j,i) \in \Omega} a_{r+j,i} Z_i \neq Y.
\]

The summation is over the (nonempty) set

\[
\Omega = \left\{ (j,i) \big| 1 \leq j \leq \nu A, 1 \leq i \leq \ell, a_{r+j,i} \neq 0, \beta = \frac{j}{\frac{\lambda_i}{2} + 1} \right\}.
\]

Let \( \frac{\nu''}{q''} = \text{ord} (\text{det} x^{-r''/q''} A'') \). Since \( H \) is semi-simple and \( \text{tr}(H) = 0 \) then \( S \in \text{GL}(n, K_p) \) and we have also

\[
\sigma(S) \leq (\Lambda - 1) \frac{\beta}{q} \leq (n - 1) \left( \frac{r''}{q''} - \frac{r}{q} \right) = \left( 1 - \frac{1}{n} \right) \left( \frac{\nu}{q} - \frac{\nu''}{q''} \right).
\]

We distinguish two cases.

(a) \( Y' \) is not nilpotent (we have this case if \( d(Y) \) is of the maximal dimension, i.e., if \( Y \) is a principal nilpotent). We take \( s = r'', p = q'' \), \( U = ST \). Then \( U \in \text{GL}(n, K_p) \), \( \tilde{A} = U[A] = A'' \) verifies the assertion (1). With \( \tilde{\nu} = \nu'' \) the second assertion follows from

\[
\sigma(U) \leq \sigma(S) \leq \left( 1 - \frac{1}{n} \right) \left( \frac{\nu}{q} - \frac{\tilde{\nu}'}{p} \right).
\]

(b) \( Y' \) is nilpotent. We have \( d(Y') > d(Y) \) by the Proposition 2.2. The induction hypothesis is applicable to \( A'' \). One deduces the existence of \( U_1 \in \text{GL}(n, K_p) \) for some \( p \in q'' \mathbb{N}^* \) such that \( \sigma(U_1) \leq (1 - \frac{1}{n}) \left( \frac{\nu''}{q''} - \frac{\tilde{\nu}'}{p} \right) \) and

\[
\tilde{A} = U_1[A''] = \tilde{Y} x^{-s/p} + \cdots \in \text{GL}(n, K_p)
\]
with $\tilde{Y}$ non nilpotent. Let $U = U_1 ST$. Then $U \in \mathcal{O}GL(n, K_p)$. $U[A]$ has the property (1) and $\sigma(U) \leq \sigma(U_1) + \sigma(S) \leq (1 - \frac{1}{n}) \left( \frac{\nu}{q} - \frac{\tilde{\nu}}{q} \right)$.

The next proposition shows that one can obtain the irregular part of a canonical form by a transformation matrix with a lag not exceeding a certain number that depends only on $n, q$ and $\nu$.

**Proposition 4.2.** Let $A$ and $\nu$ be as in the above proposition and $m$ an integer $\geq \nu$. We can find a matrix $T \in \mathcal{O}GL(n, K_p)$ for some $p \in q\mathbb{N}^*$ such that

1. $T[A] = A_{q-\text{cano}} + R_A$ where $A_{q-\text{cano}}$ is a quasi-canonical matrix as in the Definition 4.1 and $\text{ord}(R_A) > \frac{r + m}{q} + 1$.
2. $\sigma(T) \leq n - 1 + \frac{\nu}{q}$.

**Proof.** If $r \neq 0$ one considers $x^{r/q}A$ in the place of $A$. Then we can assume that $r = 0$. We prove the theorem by induction on $n$. It is trivial for $n = 1$.

Suppose $n > 1$. We assume the assertion in dimension $< n$.

Assume that the leading matrix $A_0$ has at least two distinct eigenvalues. Then according to Corollary 2.1, there exists $\tilde{T} = \prod_{j=1}^{\infty} (I + T_j x^{j/q}) \in \mathcal{O}GL(n, K_q)$ such that

$$\tilde{T}[A] = \sum_{j=0}^{\infty} A_j x^{-j/q}$$

where $A_0' = A_0$ and $A_j$ commutes with $A_0$ for all $j \geq 1$. Take $N = \max\{\nu \Lambda + 1, m + \nu + nq\}$ and

$$T_A = \prod_{j=N-1}^{1} \left( I + \frac{T_j}{x^{j/q}} \right) \in \mathcal{O}GL(n, K_q).$$

Then $A' = T_A[A] = A'' + R$ where $A'' = \sum_{j=0}^{N-1} A_j' x^{-j/q}$ and $\text{ord}(R) \geq N/q$. $A''$ commutes with the spectral projections of $A_0$.

If $A_0 = \bigoplus_{\lambda} A^{(0)}_{\lambda}$ with $n_{\lambda} = \dim(A^{(0)}_{\lambda})$ then $A'' = \bigoplus_{\lambda} A''_{\lambda}$. Let

$$\frac{\nu_{\lambda}}{q} = \text{ord}(\det x^{r_{\lambda}/q} A''_{\lambda}) \geq 0$$

with $r_{\lambda}/q = \text{ord}(A''_{\lambda}) \geq 0$. We can find matrices $U_{\lambda} \in \mathcal{O}GL(n_{\lambda}, K_p)$ such that $U_{\lambda}[A''_{\lambda}] = A_{q-\text{cano}}^{(\lambda)} + R_{\lambda}$ with $A_{q-\text{cano}}^{(\lambda)}$ in quasi-canonical form of dimension $n_{\lambda}$, $\text{ord}(R_{\lambda}) > \frac{r_{\lambda} + m}{q} + 1 \geq \frac{m}{q} + 1$ and

$$\sigma(U_{\lambda}) \leq n_{\lambda} - 1 + \frac{\nu_{\lambda}}{q}.$$
Since \( \nu \geq \sum \lambda \nu_\lambda, n = \sum \lambda n_\lambda, \) we now use the property (vii) (cf. Section 2) to conclude that if \( U' = \bigoplus \lambda U_\lambda, \) then \( U' \in \mathfrak{o} \text{GL}(n, K_p) \) and

\[
\sigma(U') \leq \sum \lambda \sigma(U_\lambda) \leq n - 2 + \frac{\nu}{q}.
\]

And \( U'[A']' \) has the property (1). We now check that

\[
\text{ord}(U'[R]) \geq \frac{N}{q} - (n - 2) - \frac{\nu}{q} > \frac{m}{q} + 1.
\]

Then \( T = U'T_A \) has the claimed properties.

We now consider the case where \( A_0 \) has a unique eigenvalue, \( A_0 = \omega I + Y \) with \( Y \) nilpotent. Then either \( \omega = 0 \) or not.

**Case 1.** \( \omega = 0 \), then \( A_0 \) is nilpotent. According to Proposition 4.1, one can find a matrix \( U \in \mathfrak{o} \text{GL}(n, K_p), \tilde{p} \in q\mathbb{N}^* \) with

\[
\sigma(U) \leq \left(1 - \frac{1}{n}\right) \left(\frac{\nu}{q} - \frac{\tilde{\nu}}{\tilde{p}}\right) \leq \frac{\nu}{q} - \frac{\tilde{\nu}}{\tilde{p}}
\]

such that

\[
\tilde{A} = U[A] = \tilde{A}_s x^{-s/\tilde{p}} + \cdots \in \text{GL}(n, K_{\tilde{p}})
\]

where \( \tilde{A}_s \) is not nilpotent and \( \tilde{p} = \text{ord} (\det x^{s/\tilde{p}} \tilde{A}) \). Two cases may occur:

(a) \( \tilde{A}_s \) has at least two distinct eigenvalues.

According to the above result one can find a matrix \( T' \in \mathfrak{o} \text{GL}(n, K_p) \) with \( p \in \tilde{p} \mathbb{N} \) such that \( \sigma(T') \leq n - 1 + \frac{\tilde{\nu}}{\tilde{p}} \) and \( T'[\tilde{A}] \) is in the desired form (1). Let \( T = T'U \) then \( T \in \mathfrak{o} \text{GL}(n, K_p) \) and

\[
\sigma(T) \leq \sigma(T') + \sigma(U) \leq n - 1 + \frac{\nu}{q}.
\]

(b) \( \tilde{A}_s = wI + Y \) has only one nonzero eigenvalue \( w \) so that \( \tilde{\nu} = 0 \) and \( Y \) is nilpotent.

- If \( Y = 0 \) then \( \tilde{A} = x^{-s/\tilde{p}} w A' \) where \( A' = I + \sum_{j=0}^{\infty} A'_{r'+j} x^{-(r'+j)/\nu'} \in \text{GL}(n, O_{\nu'}) \) is a matrix of level \( \leq 1 \) with \( r' \geq 2 \) and \( p' = 2\tilde{p} \).
- If \( Y \neq 0 \) then let \( (Y, H, X) \) be a standard triple. Let \( p' = 2n\tilde{p} \) and \( S = x^{H/(2n\tilde{p})} \). Then \( S \in \mathfrak{o} \text{GL}(n, K_{p'}) \) and \( \sigma(S) \leq \frac{n-1}{n\tilde{p}} \). Write \( \tilde{A} = x^{-s/\tilde{p}} (wI + B) \) with \( B = Y + \sum_{j=1}^{\infty} \tilde{A}_{s+j} x^{-j/\tilde{p}} \).

Write \( \tilde{A}_{s+j} = \sum_{k=1}^{n^2} \tilde{a}_{s+j,k} Z_k \). According to (6), one has

\[
S[wI] = (1 + x^{-1})^{H/(2n\tilde{p})} w I, \quad S[Y] = (1 + x^{-1})^{H/(2n\tilde{p})} x^{-1/(n\tilde{p})} Y
\]

and

\[
S[Z_k] = (1 + x^{-1})^{H/(2n\tilde{p})} x^{\lambda_k/(2n\tilde{p})} Z_k \quad \text{for} \quad 1 \leq k \leq n^2.
\]
Hence
\[
S[B] = \left(1 + x^{-1}\right)^{\nu} x^{-1/n} \left(Y + \sum_{j=1}^{n^2} \frac{\tilde{a}_{j+k} Z_k}{x^{j + \left(\frac{j}{2} + 1\right) \frac{j}{n}}} \right).
\]

For all \(j \geq 1\), since \(\frac{j}{2} + 1 \leq n\), \(j - \left(\frac{j}{2} + 1\right) \frac{1}{n} \geq 0\). Hence \(\text{ord}(S[B]) \geq \frac{1}{np} = \frac{2}{p'}\). One has therefore\( S[\tilde{A}] = x^{-s/p} w A' \) where
\[
A' = I + \sum_{j=0}^{\infty} A'_{r' j} x^{(r' j)/p'} \in \text{GL}(n, \mathcal{O}_{p'}) \text{ with } r' \geq 2
\]
is a matrix of level \(\leq 1\).

If \(r' \geq p'\) then the matrix \(A'\) is of level 0. We are through.

If \(r' < p'\) we apply Proposition 3.2 to the matrix \(A'\) to obtain an integer \(p' \in p' \mathbb{N}^*\) and a matrix \(U' \in \sigma \text{GL}(n, K_p)\) such that \(\sigma(U') \leq (n - 1) \left(1 - \frac{r'}{p'}\right)\) and
\[
U'[A'] = A'_{\text{cano}} + R_{A'} = A'_{q-\text{cano}} \in \text{GL}(n, K_p)
\]
where \(A'_{\text{cano}}\) is a canonical matrix of the form (8) of level \(\leq 1\) and \(\text{ord}(R_{A'}) > 1\). Let \(T = U'SU\) then we have the assertion (1) and
\[
\sigma(T) \leq \sigma(U') + \sigma(S) + \sigma(U) \\
\leq (n - 1) \left(1 - \frac{r'}{p'} + 2 + \frac{\nu}{p'}\right) + \frac{\nu}{q} \leq n - 1 + \frac{\nu}{q}.
\]

Case 2. If \(\omega \neq 0\) then \(\nu = 0\) and the treatment is the same as in the case (b). \(\square\)

**Theorem 4.1.** Let notations be as above. Take \(m = \nu\) and \(N = 2\nu + nq\). Let \(B\) be a matrix in the same form as \(A\) such that \(B_{r+j} = A_{r+j}\) for all \(0 \leq j < N\). Let \(T\) be as in the above proposition. Then \(T[B] = B_{q-\text{cano}} + R_B\) where \(B_{q-\text{cano}}\) is a quasi-canonical matrix and \(\text{ord}(R_B) \geq \frac{r+N}{q} + 1\). The two matrices \(A\) and \(B\) have the same irregular part in their canonical forms.

**Proof.** Since \(\frac{\nu}{q} = \sum_{i=1}^{s} n^{(i)} \ell_i\) and \(\ell_i \geq 0\), for all \(i \in \{1, \ldots, s\}\), one has \(\ell_i \leq \frac{\nu}{q}\), where \(n^{(i)}\) and \(\ell_i\) are as in \(A_{q-\text{cano}}\) (see Definition 4.1 and Definition 1.1).

From \(\sigma(T) \leq n - 1 + \frac{\nu}{q}\), \(B \equiv A(\text{mod } x^{-(r+N)/q})\) and (5) we obtain
\[
T[A] \equiv T[B] \left(\text{mod } x^{-\left[\frac{r+N}{q} - (n-1) - \frac{\nu}{q}\right]}\right).
\]

Since \(\frac{r+N}{q} - (n-1) - \frac{\nu}{q} = \frac{r+\nu}{q} + 1\) and \(T[A] = A_{q-\text{cano}} + R_A\) with \(\text{ord}(R_A) > \frac{r+\nu}{q} + 1\) the first assertion follows since \(\ell_i \leq \frac{\nu}{q}\) for all \(1 \leq i \leq s\). The second one follows from Lemma 2.2. \(\square\)
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WEIGHTED GRAPH LAPLACIANS AND ISOPERIMETRIC INEQUALITIES

F.R.K. Chung and Kevin Oden

We consider a weighted Cheeger’s constant for a graph and we examine the gap between the first two eigenvalues of Laplacian. We establish several isoperimetric inequalities concerning the unweighted Cheeger’s constant, weighted Cheeger’s constants and eigenvalues for Neumann and Dirichlet conditions.

1. Introduction.

The study of eigenvalue ratios and gaps has a long and prolific history. The motivation stems not only from their physical relevance but also from the significance of their geometric content. The early seminal work of Polyá and S. Szegö [29] lay the foundation for the geometric study of eigenvalues. One of the main techniques involves deriving isoperimetric inequalities, in one form or another, to associate geometric constraints with analytic invariants of a given manifold. The isoperimetric methods have been developed by Cheeger [13], among others, to bound the first eigenvalue of a compact manifold by the isoperimetric constant. Further generalizations of Cheeger’s constant can be attributed to Yau [32], Croke [20], Brooks [6], and others. The question of the extent to which the eigenvalues of the Laplace operator characterize a compact manifold has been investigated by Yau [30], Sunada [31], Brooks [7, 8, 9], Gordon, Webb and Wolpert [24], just to name a few. Numerous related results can be found in [3, 5, 11, 14, 22, 23, 27].

The ideas developed in this paper have their roots in results of the continuous setting which have been contributed by numerous people. For example, the early work of Payne, Polya and Weinberger [28] used geometric arguments to develop quite general bounds on eigenvalue gaps. Hile and Protter [25] and later Ashbaugh and Benguria [1] have obtained sharp upper bounds on the ratio of the first two Dirichlet eigenvalues of a compact manifold.

Davies [21] first transformed the problem to a weighted $L^2$ space with weighted operator in the continuous setting and he considered eigenvalue gaps for the weighted cases. In this paper, we introduce the weighted Cheeger constant of a graph which is a discrete analogue of the results of Cheng and Oden [15].
For an induced subgraph S of a graph G, the weighted Cheeger constant arises quite naturally by considering a weighted Laplacian (using the first Dirichlet eigenfunction u). The following study parallels in many respects the study in [15] for the continuous cases. We establish a weighted Cheeger’s inequality for the first eigenvalue $\lambda_u$ of the weighted Laplacian of a graph. We derive several inequalities involving the unweighted eigenvalue $\lambda$ and weighted eigenvalues $\lambda_u$ as well as the Dirichlet eigenvalues $\lambda_{D,i}$ and Neumann eigenvalues $\lambda_{N,i}$. (The detailed definition will be given in Section 2.) For example, we show that the following relation between the weighted eigenvalue $\lambda_u$, and the spectral gap of the Dirichlet eigenvalues:

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{2 - \lambda_{D,1} - \lambda_{D,2}} h_u^2.$$  

We prove the following eigenvalue inequality involving the unweighted and weighted eigenvalues, the Neumann eigenvalues and the Dirichlet eigenvalues.

$$\lambda_u - \lambda_{D,1} \geq \lambda_{N,1}.$$  

We also prove the following inequality involving the Dirichlet eigenvalues, the unweighted Cheeger’s constant $h$ and the weighted constant $h_u$.

$$h \leq 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1} \cdot h_u}.$$  

For a strongly convex subgraph S of an abelian homogeneous graph $\Gamma$, we show that

$$\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{8kD^2}$$  

where $D$ denotes the diameter of S and $k$ is the degree of $\Gamma$ (which is regular). For undefined terminology in graph theory and spectral geometry, the reader is referred to [4, 16] and [12, 30], respectively.

The organization of this paper is as follows: In §2 we give basic definitions and describe basic properties for the Laplacian of graphs. In §3 we define a weighted graph Laplacian and its associated first eigenvalue and the weighted Cheeger’s constant. In §4 we prove the weighted Cheeger’s inequalities. In §5, we establish several isoperimetric inequalities concerning Neumann and Dirichlet eigenvalues.

2. Preliminaries.

We consider a graph $G = (V, E)$ with vertex set $V(G)$ and edge set $E(G)$. The value of a function $f : V(G) \to \mathbb{R}$ at a vertex $y$ is denoted by $f_y$. For $y \in V(G)$, we let $d_y$ denote the degree of $y$ (which is the number of vertices adjacent to $y$). We define the normalized Laplacian of $G$ to be the following
matrix:
\[
L(x, y) = \begin{cases} 
1 & \text{if } x = y, \text{ and } d_x \neq 0, \\
-\frac{1}{\sqrt{d_x d_y}} & \text{if } x \text{ and } y \text{ are adjacent}, \\
0 & \text{otherwise}.
\end{cases}
\]

The eigenvalues of \( L \) are denoted by \( 0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1} \) and when \( G \) is \( k \)-regular, it is easy to see that
\[
L = I - \frac{1}{k} A,
\]
where \( A \) is the adjacency matrix of \( G \). It is often convenient to write \( L \) as a product of simpler matrices for a connected graph.
\[
L = T^{-1/2} LT^{-1/2},
\]
where \( L \) denotes the combinatorial Laplacian defined as follows:
\[
L(x, y) = \begin{cases} 
d_x & \text{if } x = y, \\
-1 & \text{if } x \text{ and } y \text{ are adjacent}, \\
0 & \text{otherwise},
\end{cases}
\]
and \( T \) is the diagonal matrix with the \((x, x)\)-th entry having value \( d_x \).

For a regular graph of degree \( k \), \( L \) is just a multiple \( k \) of \( \hat{L} \). For a general graph, our definition of the normalized Laplacian leads to a clean version of the Cheeger inequality for graphs [17] (also see (4)), while the Cheeger inequality using the combinatorial Laplacian involves additional complications concerning scaling. The advantages of the normalized Laplacian are perhaps due to the fact that it is consistent with the formulation in spectral geometry and in stochastic processes. In the rest of the paper, we will call \( \hat{L} \) the Laplacian, for short.

Associated with \( L \) is the positive definite quadratic form \( Q(f) = \langle f, Lf \rangle \).

For any real-valued function \( f \) we have
\[
\frac{Q(g)}{\langle g, g \rangle} = \frac{\langle g, T^{-1/2} LT^{-1/2} g \rangle}{\langle g, g \rangle} = \frac{\langle f, Lf \rangle}{\langle T^{1/2} f, T^{1/2} f \rangle} = \frac{\sum_{x \sim y} (f_x - f_y)^2}{\sum_x f_x^2 d_x}
\]
where \( f = T^{-1/2} g, x \sim y \) denotes \( x \) is adjacent to \( y \), and the sum \( \sum_{x \sim y} \) ranges over all unordered adjacent pairs \( x \) and \( y \).
Let $\lambda$ denote the least nontrivial eigenvalue of $L$ of a graph $G$. The eigenvalue $\lambda$ is closely related to the isoperimetric invariant, so called the Cheeger constant, defined as follows:

In a graph $G$, the volume of a subset $X$ of the vertex set $V$, denoted by $\text{vol}(X)$, is defined to be $\sum_{x \in X} d_x$.

**Definition.** The Cheeger constant of a graph $G$ with vertex set $V$ is defined to be

$$h = \min_{X \subseteq V(G)} \frac{e(X, V \setminus X)}{\min\{\text{vol}(X), \text{vol}(V \setminus X)\}}$$

where $e(X, V \setminus X)$ denotes the number of edges between $X$ and $V \setminus X$.

The Cheeger inequality for a graph $G$ states [17] that

$$2h \geq \lambda \geq \frac{h^2}{2}.$$  

(4)

We will establish several variations of the Cheeger inequality by considering eigenvalues of induced subgraphs of a graph.

In a graph $G$ with vertex set $V$, an induced subgraph on a subset $S$ of $V$ has vertex set $S$ and edge set consisting of all edges with both endpoints in $S$. We denote the induced subgraph determined by $S$ also by $S$ if there is no confusion. The extension $S'$ of $S$ consists of all the edges $\{x, y\}$ with at least one endpoint in $S$. The boundary of $S$, denoted by $\delta S$, is defined to be $\{x \in V(G) \setminus S : x$ is adjacent to some $y \in S\}$. We now define various eigenvalues associated with the induced subgraph $S$ that we shall study:

**Definition.** The Neumann Eigenvalue $\lambda_{N,1}$ of the induced subgraph $S$ is defined to be

$$\lambda_{N,1} = \inf_{f \neq 0} \frac{\sum_{(x,y) \in S'} (f_x - f_y)^2}{\sum_{x \in S} f_x^2 d_x}$$

(5)

where the infimum is taken over all nontrivial functions $f : S \cup \delta S \to \mathbb{R}$ satisfying, for each $x \in \delta S$,

$$\sum_{y \in S, y \sim x} (f_x - f_y) = 0.$$  

(6)

We remark that (6) is called the Neumann boundary condition for a function $f : V \to \mathbb{R}$. It corresponds to the Neumann boundary condition in the continuous setting. That is,  

$$\frac{\partial f}{\partial \nu}(x) = 0$$
for \( x \in \delta S \) where \( \nu \) is the normal direction orthogonal to the tangent hyper-plane at \( x \).

**Definition.** The Dirichlet Eigenvalues of \( S \) is defined as follows:

\[
\lambda_{D,1} = \inf_{f|\delta S = 0} \sum_{\{x,y\} \in S'} \frac{(f_x - f_y)^2}{\sum_{x \in S} (f_x - f'_x)^2 d_x}
\]

where \( f|\delta S = 0 \) means \( f(x) = 0 \) for \( x \in \delta S \). In general, we define

\[
\lambda_{D,i} = \inf_{f|\delta S = 0} \sup_{f' \in C_{i-1}} \sum_{\{x,y\} \in S'} \frac{(f_x - f_y)^2}{\sum_{x \in S} (f_x - f'_x)^2 d_x}
\]

where \( C_i \) is the subspace spanned by the \( j \)-th eigenfunctions \( \phi_j \) with eigenvalue \( \lambda_{D,j} \) for \( j \leq i \).

### 3. Weighted Graph Laplacian.

Let \( u \) be the first eigenfunction for Dirichlet conditions of the induced sub-graph \( S \) achieving \( \lambda_{D,1} \) in (7). Here are some useful facts about \( u \) and \( \lambda_{D,1} \) which follows from the definitions (see [16]):

**Fact 1:** \( u \geq 0 \) on \( S \) and \( u = 0 \) on \( \delta S \).

**Fact 2:** \( \lambda_{D,1} < 1 \).

**Fact 3:** For \( x \in S \), we have

\[
\sum_{\{x,y\} \in S'} (u_x - u_y) = \lambda_{D,1} u_x d_x.
\]

**Fact 4:**

\[
\sum_{x} \sum_{y \{x,y\} \in S'} (u_x - u_y) = \sum_{\{x,y\} \in S'} (u_x - u_y)^2.
\]

We will use \( u \) to define a weighted Laplacian \( L_u \) as follows:

\[
L_u(x, y) = \begin{cases} 
\frac{u_x^2}{d_x^2} & \text{if } x = y \text{ and } d_x \neq 0, \\
-\frac{u_x u_y}{\sqrt{d_x d_y}} & \text{if } x \text{ and } y \text{ are adjacent}, \\
0 & \text{otherwise},
\end{cases}
\]

where \( d_x \) is the degree of \( x \) in \( G \).
The first eigenvalue $\lambda_u$ of $L_u$ satisfies

$$\lambda_u = \inf_{f \neq 0} \frac{\sum_{x \in S'} (f_x - f_y)^2 u_x u_y}{\sum_{x \in S} f_x^2 u_x^2 d_x}.$$  

We define the weighted Cheeger’s constant $h_u$ to be

$$h_u = \min \frac{\sum_{x \in X, y \in S \setminus X} u_x u_y}{\sum_{x \in X} u_x^2 d_x}$$

where the minimum ranges over all $X \subseteq S$ and $\sum_{x \in X} u_x^2 d_x \leq \sum_{x \in S \setminus X} u_x^2 d_x$.

4. Weighted Cheeger’s inequalities.

We will first give a functional formulation of the weighted Cheeger’s constant which will be used later. As in the continuous setting [30], this shows the connection between the functional properties of a graph and its spectral properties.

**Theorem 1.** Suppose $u$ is a nonnegative vector in $\mathbb{R}^n$ (i.e. $u_i \geq 0$ for all $i$) where $n = V(S)$. Then

$$h_u = \inf_{f \neq 0} \sup_{C \in \mathbb{R}} \int_{-\infty}^{\infty} g(\sigma) d\sigma$$

**Proof.** We choose $C$ to satisfy:

1) For $\sigma < 0$ we have $\sum_{f_x - C < \sigma} u_x^2 d_x \leq \sum_{f_x - C \geq \sigma} u_x^2 d_x.$

2) For $\sigma > 0$ we have $\sum_{f_x - C < \sigma} u_x^2 d_x \geq \sum_{f_x - C \geq \sigma} u_x^2 d_x.$

Let $g(\sigma) = \sum_{\{x,y\} \in E} u_x u_y$. Then we have

$$\sum_{x \sim y} |f_x - f_y| u_x u_y = \int_{-\infty}^{\infty} g(\sigma) d\sigma$$
\[= \int_{-\infty}^{0} \frac{g(\sigma)}{f_x - C \leq \sigma} \sum_{f_x - C \leq \sigma} u_x^2 \, dx \, d\sigma + \int_{0}^{\infty} \frac{g(\sigma)}{f_x - C \geq \sigma} \sum_{f_x - C \geq \sigma} u_x^2 \, dx \, d\sigma\]

\[\geq h_u \int_{-\infty}^{0} \sum_{f_x - C \leq \sigma} u_x^2 \, dx \, d\sigma + h_u \int_{0}^{\infty} \sum_{f_x - C \geq \sigma} u_x^2 \, dx \, d\sigma\]

\[= h_u \sum_{x \in S} |f_x - C| u_x^2 \, dx.\]

Conversely, suppose \(X_0 \subset S\) is a subset such that

\[h_u = \frac{\sum_{x \in X_0, y \in S \setminus X_0} u_x u_y}{\sum_{x \in X_0} u_x^2 \, dx}.

Define \(f\) as follows:

\[f_x = \begin{cases} 1 & x \in X_0 \\ -1 & x \in S \setminus X_0 \end{cases} \]

Then

\[\inf_{f} \sup_{C \in \mathbb{R}} \left( \sum_{x \sim y} |f_x - f_y| u_x u_y \right) \leq \sup_{C} \sum_{x \in X_0, y \in S \setminus X_0} u_x u_y + \sum_{x \sim y} |1 - C| u_x^2 \, dx + \sum_{x \sim y} |1 + C| u_x^2 \, dx.\]

We consider \(\inf_{C} \left( \sum_{x \in X_0} |1 - C| u_x^2 + \sum_{x \in S \setminus X_0} |1 + C| u_x^2 \right),\) for \(-1 \leq C \leq 1\).

Define

\[f(c) = \left( \sum_{x \in S \setminus X_0} u_x^2 \, dx - \sum_{x \in X_0} u_x^2 \, dx \right) \cdot C + \left( \sum_{x \in S \setminus X_0} u_x^2 \, dx + \sum_{x \in X_0} u_x^2 \, dx \right)\]

on the interval \(-1 \leq C \leq 1\). Since \(\sum_{x \in S \setminus X_0} u_x^2 \, dx - \sum_{x \in X_0} u_x^2 \, dx \geq 0\), \(f\) has a minimum at \(C = -1\) by elementary calculation. Therefore,

\[\inf_{f} \sup_{C \in \mathbb{R}} \sum_{x \sim y} |f_x - C| u_x^2 \, dx \leq \sum_{x \sim y} 2u_x u_y \leq h_u \sum_{x \in X_0} 2u_x^2 \, dx \leq h_u^{\sum_{x \in X_0} 2u_x^2 \, dx} \leq h_u\]
which completes the proof of Theorem 1.

The above theorem leads to several Cheeger-type inequalities concerning eigenvalue gaps. We will show that the eigenvalue gap $\lambda_{D,2} - \lambda_{D,1}$ is, in fact, the first eigenvalue of the weighted Laplacian defined in §3.

**Proposition 1.1.** Suppose $u$ is the first Dirichlet eigenfunction of the Laplacian on the induced subgraph $S$ of $G$. Let $\lambda_u$ be the first eigenvalue of the $u$-weighted Laplacian, $\mathcal{L}_u$. Then,

$$\lambda_u = \lambda_{D,2} - \lambda_{D,1}.$$  

**Proof.** For any function $f : S \cup \delta S \to \mathbb{R}^+$, by using Fact 3 in Section 3 we have

$$\lambda_{D,1} \sum_x f_x^2 u_x^2 dx$$

$$= \sum_x f_x^2 u_x \sum_{y \sim x} (u_x - u_y)$$

$$= \sum_{x \sim y} (u_x - u_y)(f_x^2 u_x - f_y^2 u_y)$$

$$= \sum_{x \sim y} (f_x^2 u_x - f_y^2 u_y)^2 - \left( f_y^2 u_x u_y + f_y^2 u_x u_y - f_x f_y u_x u_y - 2 f_x f_y u_x u_y \right)$$

$$= \sum_{x \sim y} (f_x u_x - f_y u_y)^2 - \sum_{x \sim y} (f_x - f_y)^2 u_x u_y.$$  

Therefore,

$$\lambda_u = \inf_{f \neq 0} \frac{\sum_{x \sim y} (f_x - f_y)^2 u_x u_y}{\sum_{x} f_x^2 u_x^2 dx}$$

$$= \inf_{f \neq 0} \frac{\sum_{x \sim y} (f_x u_x - f_y u_y)^2}{\sum_{x} f_x^2 u_x^2 dx} - \lambda_{D,1}$$

$$= \inf_{g \neq 0} \frac{\sum_{x \sim y} (g_x - g_y)^2}{\sum_{x} g_x^2 dx} - \lambda_{D,1}$$

$$= \lambda_{D,2} - \lambda_{D,1}.$$  

In the preceding proof of the proposition we have also shown the following:

**Corollary 1.1.**
\[
\inf_{f \neq 0} \frac{\sum_{x \sim y} (f_x u_x - f_y u_y)^2}{\sum_x f_x^2 u_x^2 d_x} = \frac{\sum_{x \sim y} (g_x - g_y)^2}{\sum_x g_x^2 d_x} = \lambda_{D,2}.
\]

**Proposition 1.2.**
\[
2(1 - \lambda_{D,1}) \geq \lambda_u.
\]

*Proof.* Let \( f \) denote the eigenfunction achieving the Dirichlet eigenvalue \( \lambda_{D,2} \). We consider
\[
2 \sum_x f_x^2 u_x \sum_{y \sim x} u_y \geq \sum_{x \sim y} 2(f_x^2 + f_y^2) u_x u_y \\
\geq \sum_{x \sim y} (f_x - f_y)^2 u_x u_y.
\]
Using the fact that
\[
\sum_{y \sim x} u_y = (1 - \lambda_{D,1}) u_x d_x
\]
we then have
\[
2(1 - \lambda_{D,1}) \sum_x f_x^2 u_x^2 d_x \geq \sum_{x \sim y} (f_x - f_y)^2 u_x u_y.
\]
This implies
\[
2(1 - \lambda_{D,1}) \geq \frac{\sum_{x \sim y} (f_x - f_y)^2 u_x u_y}{\sum_x f_x^2 u_x^2 d_x} = \lambda_{D,2}.
\]

We will use the above facts to prove several versions of weighted Cheeger’s inequalities. The following proof is somewhat similar to the unweighted case in [15].

**Theorem 2.**
\[
\lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{2 - \lambda_{D,1} - \lambda_{D,2}} h_u^2.
\]
Proof. Let $f$ denote the function achieving $\lambda_u$ in (8). We label the vertices of $G$, so that $f_i \equiv f_{v_i} \leq f_{i+1}$ and let $p$ denote the least integer such that $f_p \geq 0$. For each $i$, $1 \leq i \leq |S|$ we consider the cut $C_i = \{v_j, v_k\} \in E(S) : 1 \leq j \leq i \leq k \leq n$. We define $\alpha$ to be

$$\alpha = \min_{1 \leq i \leq |S|} \frac{\sum_{\{j,k\} \in C_i} 2u_ju_k}{\min \left( \sum_{j \leq i} u_j^2d_j, \sum_{j > i} u_j^2d_j \right)}.$$ 

It is clear that $\alpha \geq h_u$. Without loss of generality, we may assume

$$\sum_{j \leq p} u_j^2d_j \leq \sum_{j > p} u_j^2d_j.$$ 

We define

$$V_+ = \{x : f_x \geq 0\}$$

$$E_+ = \{\{x, y\} : x \in V_+ \text{ or } y \in V_+\}.$$ 

We define

$$g_x = \begin{cases} f_x & \text{if } x \in V_+ \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$\lambda_u f_x u_x d_x = \sum_{y \sim x} (f_x - f_y)u_xu_y$$

we have

$$\lambda_u = \sum_{x \in V_+} f_x \sum_{\{x, y\} \in E_+} (f_x - f_y)u_xu_y$$

$$\geq \sum_{\{x, y\} \in E_+} (g_x - g_y)^2u_xu_y$$

$$= \sum_{x \in V_+} f_x^2u_x^2d_x$$

$$= W.$$
Then we have

\[
W = \frac{\left( \sum_{(x,y) \in E_+} (g_x - g_y)^2 u_x u_y \right) \left( \sum_{(x,y) \in E_+} (g_x + g_y)^2 u_x u_y \right)}{ \left( \sum_{x \in V_+} f_x^2 u_x^2 d_x \right) \cdot \left( \sum_{(x,y) \in E_+} (g_x + g_y)^2 u_x u_y \right) \left( \sum_{\{x,y\} \in E_+} \left| g_x^2 - g_y^2 \right| u_x u_y \right)^2}.
\]

Since

\[
\sum_{y \sim x} u_y = u_x d_x - \lambda_D u_x d_x
\]

we have

\[
W \geq \frac{\left( \sum_{\{x,y\} \in E_+} \left| g_x^2 - g_y^2 \right| u_x u_y \right)^2}{\left( \sum_{x \in V_+} f_x^2 u_x^2 d_x \right) \cdot \left( \sum_{x \in V_+} f_x^2 u_x^2 d_x \right) \cdot \left( 2 - 2\lambda_D - W \right) \left( \sum_{i \leq p} |f_{i+1}^2 - f_i^2| \sum |u_x u_y| \right)^2}.
\]

\[
\geq \frac{\left( \sum_{\{x,y\} \in E_+} \left| g_x^2 - g_y^2 \right| u_x u_y \right)^2}{\left( \sum_{x \in V_+} f_x^2 u_x^2 d_x \right) \cdot \left( \sum_{x \in V_+} f_x^2 u_x^2 d_x \right) \cdot (2 - 2\lambda_D - \lambda_u)}.
\]
\[
\begin{align*}
\geq & \left( \sum_{i \leq p} |f_i^2 - f_{i+1}^2| \sum_{j \leq i} u_j^2 \right)^2 \\
& \left( \sum_{x \in V_+} f_x^2 u_x^2 dx \right) (2 - 2\lambda_{D,1} - \lambda_u) \\
& \alpha^2 \left( \sum_{x \in V_+} f_x^2 u_x^2 dx \right)^2 \\
& (2 - 2\lambda_{D,1} - \lambda_u) \left( \sum_{x \in V_+} f_x^2 u_x^2 dx \right) \\
\geq & \frac{\alpha^2}{2 - 2\lambda_{D,1} - \lambda_u} \\
\geq & \frac{h_u^2}{2 - \lambda_{D,1} - \lambda_{D,2}}
\end{align*}
\]

by using Proposition 1.1.

Here is another analogue of the Cheeger inequality relating the spectral gaps of Dirichlet eigenvalues to the weighted Cheeger’s constant. Its proof follows immediately from Theorem 2 and Fact 2.

**Corollary 2.1.**

\[
\lambda_{D,2} - \lambda_{D,1} \geq \frac{h_u^2}{2(1 - \lambda_{D,1})} \geq \frac{h_u^2}{2}.
\]

A theorem of Payne, Polya and Weinberger [28] gives

\[
\lambda_{D,k+1} - \lambda_{D,k} \leq \frac{4}{n \cdot k} \sum_{i=1}^{k} \lambda_{D,i}
\]

for Dirichlet eigenvalues of a bounded domain \(\Omega \subset \mathbb{R}^n\). It would be of interest to prove a similar inequality for graphs.

### 5. Several isoperimetric inequalities.

It was shown in [15] that the continuous analogue of \(h_u\) was bounded below by \(c \cdot h\), where \(c\) is a constant depending on the dimension of the manifold and its rolling sphere radius and \(h\) is the unweighted Neumann Cheeger’s constant. In the discrete setting, similar relationships can be found between the various unweighted and weighted Cheeger’s constants as well as the unweighted and weighted eigenvalues. The following results have their origins in the work of
Payne, Polya and Weinberger [28] as well as the subsequent developments by Ashbaugh and Benguria [1], Hile and Protter [25], and Hile and Xu [26].

**Theorem 3.**

$$\lambda_u - \lambda_{D,1} \geq \lambda_{N,1}.$$ 

**Proof.** From the definition, we have

$$\lambda_{N,1} = \inf_{f \neq 0} \frac{\sum_{(x,y) \in S'} (f_x - f_y)^2}{\sum_{x \in S} f_x^2 d_x},$$

subject to the Neumann boundary condition

$$\sum_{y \sim x} (f_x - f_y) = 0 \text{ for any } x \in \partial S.$$ 

Using the Neumann boundary condition, we have

$$\sum_{(x,y) \in S'} (f_x - f_y)^2 = \sum_{x \in S} f_x \sum_{y \sim x} (f_x - f_y).$$

Let $h$ be the eigenfunction for the weighted Laplacian and set $f = h \cdot u$. Then we have

$$\lambda_{N,1} \leq \frac{\sum_{x \in S} f_x \cdot \sum_{y \sim x} (f_x - f_y)}{\sum_{x \in S} f_x^2 d_x}$$

$$= \frac{\sum_x h_x u_x \cdot \sum_{y \sim x} (h_x u_x - h_y u_y)}{\sum_x h_x^2 u_x^2 d_x}$$

$$= \frac{\sum_{(x,y) \in S'} (h_x - h_y)^2 u_x u_y - \sum_{x \in S} \sum_{y \sim x} h_x^2 u_x (u_x - u_y)}{\sum_x h_x^2 u_x^2 d_x}$$

$$= \frac{\lambda_{D,1} \sum_x h_x^2 u_x^2 d_x}{\sum_x h_x^2 u_x^2 d_x}$$

$$= \lambda_u - \frac{\lambda_{D,1}}{\sum_x h_x^2 u_x^2 d_x}$$

$$= \lambda_u - \lambda_{D,1}.$$

□

Now by using Proposition 1.1, we have the following.
Corollary 3.1.
\[ \lambda_{D,2} \geq 2\lambda_{D,1} + \lambda_{N,1}. \]

In particular the theorem implies that \( \lambda_{D,2} - \lambda_{D,1} \geq \lambda_{N,1} \). One of the authors and S.T. Yau [18] studied \( \lambda_{N,1} \) on subgraphs of homogeneous graphs. When the induced subgraph \( S \) is strongly convex (i.e., all shortest paths in the host graph joining two vertices in \( S \) are contained in \( S \), see [19] for more details) it was proved in [18] that
\[ \lambda_{N,1} \geq \frac{1}{8kD^2}, \]
where \( k \) is the degree of the \( S \) and \( D \) is the diameter. This immediately gives:

**Corollary 3.2.** Suppose \( S \) is a strongly convex subgraph of an invariant abelian homogeneous graph \( \Gamma \) with edge generating set \( K \) consisting of \( k \) generators. Then
\[ \lambda_{D,2} - \lambda_{D,1} \geq \frac{1}{8kD^2} \]
where \( D \) is the diameter of \( S \).

The preceding results can be related to several bounds of \( \lambda_{N,1} \) in terms of the heat kernel, as examined in [19].

The weighted Cheeger’s constant incorporates more information about the graph in its definition. So in principle one would expect estimates involving the weighted Cheeger to be better than ones involving only the unweighted Cheeger’s constants. In this respect, the following isoperimetric inequality is of interest, and can be contrasted with the upper bounds developed by Buser [10] in the continuous setting.

**Theorem 4.** Suppose \( h \) is the Cheeger’s constant of the induced subgraph \( S \) and \( h_u \) the weighted Cheeger’s constant. Then,
\[ h \leq 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1} \cdot h_u}. \]

**Proof.** From Theorem 1, we have
\[ h = \inf_{f \neq 0} \sup_{|C|} \sum_{x \sim y} \frac{|f_x - f_y|}{C \sum_x |f_x - C| d_x}. \]

Since \( V(S) \) and \( E(S) \) are finite sets, there is some \( X \subseteq S \) which achieves \( h \). Using the first Dirichlet eigenfunction \( u \), we define
\[ f_x - C = \begin{cases} u_x^2 & \text{if } x \in X, \\ -u_x^2 & \text{if } x \in S \setminus X. \end{cases} \]
where $C$ is as defined in the proof of Theorem 1. Therefore we have

$$h \leq \frac{\sum_{x \sim y} (u_x^2 + u_y^2) + \sum_{x \sim y} |u_x^2 - u_y^2|}{\sum_x u_x^2 d_x} \sum_{x \sim y} (|u_x^2 - u_y^2| - (u_x - u_y)^2) \leq \frac{\sum_{x \sim y} (|u_x^2 - u_y^2| - (u_x - u_y)^2) \sum_{x \sim y} u_x^2 d_x}{\sum_x u_x^2 d_x} = \lambda_{D,1} + 2h_u$$

We note that

$$\sum_{x \sim y \in \{x,y\in X\} \text{ or } \{x,y\in S\setminus X\}} (|u_x^2 - u_y^2| - (u_x - u_y)^2) \leq \sum_{x \sim y} |u_x - u_y| 2 \min\{u_x, u_y\} \leq \sum_{x \sim y} |u_x - u_y| \sqrt{u_x u_y} \leq 2 \left( \sum_{x \sim y} (u_x - u_y)^2 \right)^{1/2} \left( \sum_{x \sim y} u_x u_y \right)^{1/2} \leq 2 \left( \sum_x u_x^2 d_x \right)^{1/2} \sqrt{\lambda_{D,1} h_u},$$

by using the definition of $\lambda_{D,1}$ and $h_u$. Therefore, we have

$$h \leq 2h_u + \lambda_{D,1} + 2\sqrt{\lambda_{D,1} h_u}$$

as claimed.

□

References


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COTILTING MODULES AND BIMODULES

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Cotilting modules and bimodules over arbitrary associative rings are studied. On the one hand we find a connection between reflexive modules with respect to a cotilting (bi)module $U$ and a notion of $U$-torsionless linear compactness. On the other hand we provide concrete examples of cotilting bimodules over linearly compact noetherian serial rings.

Cotilting theory is a generalization of Morita duality in a sense that is analogous to that in which tilting theory is a generalization of Morita equivalence. Indeed, cotilting modules first appeared as vector space duals of tilting modules over finite dimensional algebras (see, e.g., [H, IV, 7.8]), just as injective cogenerators are such duals of progenerators. Later, R.R. Colby [Cb1] studied finitely generated cotilting bimodules over noetherian rings, proving that they induce finitistic generalized Morita dualities, similar to the finite dimensional algebra case. More recently, in [Cb2] he investigated a more general class of representable dualities, namely (nonfinitistic) generalized Morita dualities. He proved that the existence of such a duality implies the existence of a second pair of functors between classes that complement the reflexive ones, obtaining a result which is close to a dual form of the celebrated Tilting Theorem [BrBu], [HaRi].

For arbitrary rings $R$ and $S$, a Morita duality between left $S$-modules and right $R$-modules is given by the contravariant Hom functors induced by a so called Morita bimodule $SW_R$, namely, one such that (i) the classes of $W$-reflexive modules contain $R_R$, $W_R$, $SS$ and $SW$, and are closed under submodules, factor modules and extensions; or, equivalently, (ii) $SW_R$ is balanced, and $W_R$ and $SW$ are injective cogenerators. Colby’s generalized Morita dualities in [Cb2] are those induced by a bimodule $SU_R$ such that a natural weakening of (i) holds (just closure under factor modules is left out). Generalizing the notion of injective cogenerator, the authors of [CpDeTo] and [CpToTr] defined a cotilting module $U_R$ over a ring $R$ as one such that $\text{Cogen}(U_R) = \text{Ker Ext}^1_R(-, U_R)$. In [CpDeTo, Proposition 1.7] it is shown that this notion is dual to that of tilting module by means of the following

**Proposition.** A module $U_R$ is a cotilting module if and only if it satisfies the conditions

(1) $\text{inj dim}(U_R) \leq 1$, }
\( \text{(2) } \text{Ext}^1_R(U_R^\alpha, U_R) = 0 \text{ for any cardinal } \alpha, \)
\( \text{(3) } \text{Ker Hom}_R(-, U_R) \cap \text{Ker Ext}^1_R(-, U_R) = 0. \)

To obtain a homological generalization of (ii), as in [Cp], we say that a balanced bimodule \( sU_R \) in which both \( U_R \) and \( sU \) are cotilting modules is a \textit{cotilting bimodule}.

In this paper we continue the study of cotilting (bi)modules over arbitrary rings that was begun in [Cp]. There it was shown that any cotilting bimodule \( sU_R \) induces a pair of dualities between quite large subcategories of torsion-free and torsion modules in Mod-\( R \) and \( S \)-Mod, respectively. This result naturally generalizes Morita dualities to torsion theories, and it is still dual to the Tilting Theorem.

A third major component of Morita duality theory is B. Müller’s theorem [X, Corollary 4.2] that the reflexive modules relative to a Morita bimodule are precisely the linearly compact modules. In Section 1 we investigate the related notion of torsionless linear compactness and its connection to the reflexivity of modules. This allows us to find a bridge between Colby’s generalized Morita duality and cotilting bimodules by showing that a cotilting bimodule \( U \) induces a generalized Morita duality if and only if the classes of the \( U \)-reflexive modules coincide with those of the \( U \)-torsionless linearly compact modules. This is accomplished, in part, by answering a question posed in [Cp].

Perhaps the most accessible collection of examples of tilting modules over non-artinian rings are those over hereditary noetherian serial rings. They and their endomorphism rings were classified in [CbFu]. In Section 2 we show that the Morita dual of a tilting module possesses most of the properties of a cotilting bimodule. Then in Section 3 we employ these results and Warfield’s theorems on noetherian serial rings in [Wa] to show that the dual of any tilting module over a noetherian serial ring with selfduality is a cotilting bimodule. Thus we obtain a class of concrete examples of cotilting bimodules that are not, in general, finitely generated.

We denote by \( R \) and \( S \) two arbitrary associative rings with unit, and by Mod-\( R \) and \( S \)-Mod the category of all unitary right \( R \)- and left \( S \)-modules, respectively. All the classes of modules that we introduce are to be considered as full subcategories of modules closed under isomorphisms. Given a module \( U \), we denote by add\((U)\) the class of all direct summands of any finite direct sum of copies of \( U \), and by Cogen\((U)\) the class of all modules \textit{cogenerated} by \( U \), that is all the modules \( M \) such that there exists an exact sequence \( 0 \to M \to U^\alpha \), for some cardinal \( \alpha \). We denote by Rej\(_U\)(\( -\)) the \textit{reject} radical, defined by Rej\(_U\)(\( M \)) = \( \cap \{ \text{Ker}(f) \mid f \in \text{Hom}_R(M, U) \} \), i.e., the least submodule \( M_0 \) of \( M \) such that \( M/M_0 \) belongs to Cogen\((U)\). Given a bimodule \( sU_R \), we denote by \( \Delta \) both the functors Hom\(_R\)(\( -\), \( U \)) and Hom\(_S\)(\( -\), \( U \)), and by \( \Gamma \) both the functors Ext\(_1^R\)(\( -\), \( U \)) and Ext\(_1^S\)(\( -\), \( U \)). For
any module $M$, $\delta_M : M \to \Delta^2(M)$ denotes the evaluation morphism. $M$ is called $\Delta$-reflexive if $\delta_M$ is an isomorphism. Note that if $U_R$ is a cotilting module, then $(\text{Ker} \Delta, \text{Ker} \Gamma)$ is a torsion theory in $\text{Mod-}R$, associated to the idempotent radical $\text{Rej}_{U_R}(-) = \text{Ker}(\delta_-)$. For further notation, we refer to [AF], [S] and [CE].

1. Reflexivity and torsionless linear compactness.

We start this section pointing out some facts on $\Delta$-reflexivity of modules, with respect to a cotilting module $U_R$, which generalize part of [Cp, Lemma 4 and Proposition 5]:

**Lemma 1.1.** Let $U_R$ be a cotilting module, and let $S = \text{End}(U_R)$. Then:

(a) $U_R$ and $SS$ are $\Delta$-reflexive.

(b) If $SL \in S\text{-Mod}$ is a factor of any $\Delta$-reflexive module (in particular, if $SL$ is finitely generated), then $\delta_L$ is an epimorphism.

(c) If $SL \in \text{Cogen}(SU)$ is a factor of any $\Delta$-reflexive module, then $L$ is $\Delta$-reflexive.

(d) For any $M_R \in \text{Mod-}R$, we have $\text{Coker}(\delta_M) \subseteq \text{Ker} \Gamma$.

(e) Let $M_R \in \text{Mod-}R$. Then $M_R$ is $\Delta$-reflexive if and only if $M_R \in \text{Ker} \Gamma$ and $\Delta(M)$ is $\Delta$-reflexive.

(f) If $M_R \in \text{Ker} \Gamma$ and $\Delta(M_R)$ is a factor of any $\Delta$-reflexive module (in particular, if $\Delta(M_R)$ is finitely generated), then $M_R$ is $\Delta$-reflexive.

(g) If $L_R \subseteq M_R$, $M_R$ is $\Delta$-reflexive and $M/L \in \text{Ker} \Gamma$, then $L_R$ is $\Delta$-reflexive.

**Proof.** (a) $\Delta^2(U_R) \cong \Delta(SS) \cong U_R$ and $\Delta^2(SS) \cong \Delta(U_R) \cong SS$ canonically.

(b) Let $K \rightarrow M \rightarrow L \rightarrow 0$ be an exact sequence in $S\text{-Mod}$, with $M$ $\Delta$-reflexive. Then we have the exact sequence $0 \rightarrow \Delta(L) \rightarrow \Delta(M) \rightarrow I \rightarrow 0$, where $I$ embeds into $\Delta(K)$, so that $\Gamma(I) = 0$. Therefore we get the commutative exact diagram

$$
\begin{array}{ccc}
M & \longrightarrow & L \\
\downarrow \delta_M & & \downarrow \delta_L \\
\Delta^2(M) & \longrightarrow & \Delta^2(L) \\
\end{array}
$$

which shows that $\delta_L$ is epic.

(c) Clearly $SL \in \text{Cogen}(SU)$ if and only if $\delta_L$ is monic. We can conclude by (b).

(d) By adjunction, we get $\Delta(\delta_M) \circ \delta_{\Delta(M)} = \text{id}_{\Delta(M)}$, so that $\Delta(\delta_M)$ is epic. Therefore, from the exact sequence $0 \rightarrow M/\text{Rej}_{U}(M) \rightarrow \Delta^2(M) \rightarrow \text{Coker}(\delta_M) \rightarrow 0$ we see that $\Gamma(\text{Coker}(\delta_M)) \rightarrow \Gamma(\Delta^2(M) = 0$.

(e) Again from the identity $\Delta(\delta_M) \circ \delta_{\Delta(M)} = \text{id}_{\Delta(M)}$, we see that if $\delta_M$ is an isomorphism, then $\delta_{\Delta(M)}$ is too, and of course $M \in \text{Cogen}(U_R) = \text{Ker} \Gamma$. 

Conversely, if $\delta_M$ is an isomorphism, then $\Delta(\delta_M)$ must be monic, i.e., Coker($\delta_M$) $\in$ Ker $\Delta$. Moreover Coker($\delta_M$) $\in$ Ker $\Gamma$ because of (d). Since (Ker $\Delta$, Ker $\Gamma$) is a torsion theory, we conclude that Coker($\delta_M$) = 0, i.e., $\delta_M$ is epic. Under the further assumption that $M \in$ Ker $\Gamma$, we conclude that $\delta_M$ is an isomorphism.

(f) Since $\Delta(M) \in$ Cogen($sU$), (c) applies, giving $\Delta(M)$ reflexive. We conclude by (e).

(g) From the exact sequence $0 \rightarrow L \rightarrow M \rightarrow M/L \rightarrow 0$ in $\text{Mod-}R$, by assumption we get the exact sequence $0 \rightarrow \Delta(M/L) \rightarrow \Delta(M) \rightarrow \Delta(L) \rightarrow 0$. We conclude using (e) and (f).

□

It is well known that linear compactness plays a fundamental role in the study of duality. Here we introduce a concept of linear compactness with respect to a torsion theory, drawing inspiration from [GpGaWi, §3]:

**Definition 1.2.** Let $(T, F)$ be a torsion theory in $\text{Mod-}R$. Then a right $R$-module $M$ is called $F$-linearly compact if $M \in F$ and for any inverse system of morphisms $\{p_\lambda : M \rightarrow M_\lambda\}$ with $M_\lambda \in F$ and Coker($p_\lambda$) $\in$ $T$, for all $\lambda$'s, it happens that Coker($\lim\limits_{\leftarrow} p_\lambda$) $\in$ $T$.

If $U_R$ is a cotilting module, a module $M \in$ $\text{Mod-}R$ is called $U$-torsionless linearly compact ($U$-tl.l.c., for short) if $M$ is Ker $\Gamma$-linearly compact.

Note that $M \in$ $\text{Mod-}R$ is linearly compact iff $M$ is $\text{Mod-}R$-linearly compact, i.e., it is linearly compact with respect to the trivial torsion theory ($\{0\}$, $\text{Mod-}R$). In particular if $U_R$ is a cotilting module, then the $U$-torsionless linear compactness coincides with the usual linear compactness iff $U_R$ is an injective cogenerator.

Torsionfree linear compactness is inherited by any inverse limit of the type in Definition 1.2, as the following result due to A. Tonolo shows:

**Proposition 1.3.** Let $(T, F)$ be a torsion theory in $\text{Mod-}R$, and let $M \in$ $\text{Mod-}R$ be $F$-linearly compact. Then for any inverse system $\{p_\lambda : M \rightarrow M_\lambda\}$ with $M_\lambda \in F$ and Coker($p_\lambda$) $\in$ $T$, the module $\lim\limits_{\leftarrow} M_\lambda$ is $F$-linearly compact too.

Proof. First of all, let us note that $\lim\limits_{\leftarrow} M_\lambda \in F$, because $F$ is closed under inverse limits. Next, let $\{p'_\mu : \lim\limits_{\leftarrow} M_\lambda \rightarrow M'_\mu : \mu \in \Lambda'\}$ be any inverse system with $M'_\mu \in F$ and Coker($p'_\mu$) $\in$ $T$ for all $\mu$'s. Let us prove that Coker($\lim\limits_{\leftarrow} p'_\mu$) $\in$ $T$. Note that the cokernel of each map $p'_\mu \circ \lim\limits_{\leftarrow} p_\lambda$, for all $\mu$'s, is in $T$, because it is an extension of a factor of Coker($\lim\limits_{\leftarrow} p_\lambda$), which is in $T$, by the torsion module Coker($p'_\mu$). Hence, by assumption, we get that the morphism $\lim\limits_{\leftarrow} (p'_\mu \circ \lim\limits_{\leftarrow} p_\lambda) \cong \lim\limits_{\leftarrow} p'_\mu \circ \lim\limits_{\leftarrow} p_\lambda$ has a torsion cokernel. This implies that Coker($\lim\limits_{\leftarrow} p'_\mu$) $\in$ $T$. □

In [Cp, Proposition 10] it was proved that if $sU_R$ is a cotilting bimodule, then any $U$-tl.l.c. module is $\Delta$-reflexive; and the question of whether the
converse is true was posed. To give a partial answer, we start with a theorem which generalizes a well known result, substantially due to Müller [Mu] (see also [X, Theorem 4.1]):

**Theorem 1.4.** Let $U_R$ be a cotilting module, and let $S = \text{End}(U_R)$. Then the following are equivalent for any $M \in \text{Mod}-R$:

1. $M$ is $U$-tl.l.c.
2. $M$ is $\Delta$-reflexive, and for all $L \hookrightarrow \Delta(M)$ we have $\text{Coker} \Delta(i) \in \text{Ker} \Delta$.

**Proof.** (1) $\Rightarrow$ (2). Let $M_R$ be $U$-tl.l.c., let $L$ be a submodule of $\Delta(M)$ and let $\{L_\lambda : \lambda \in \Lambda\}$ be the upward directed family of the finitely generated submodules of $L$. Thus, if we denote by $i_\lambda : L_\lambda \hookrightarrow \Delta(M)$ the canonical inclusions, we get $\lim L_\lambda = L$ and $\lim i_\lambda = i$. Let now $p_\lambda = \Delta(i_\lambda) \circ \delta_M : M \to \Delta(L_\lambda)$. Then $\{p_\lambda : \lambda \in \Lambda\}$ is an inverse system of morphisms in $\text{Ker} \Gamma$. In order to show that $\text{Coker} p_\lambda \in \text{Ker} \Delta$ for any $\lambda$, let us consider the commutative diagram in $\text{Mod}-R$

\[
\begin{array}{cccccc}
L_\lambda & \xrightarrow{i_\lambda} & \Delta(M) & \xrightarrow{} & \Delta(M) \\
\downarrow & & \downarrow & & \downarrow \\
\Delta^2(L_\lambda) & \xrightarrow{\Delta^2(i_\lambda)} & \Delta^3(M) & \xrightarrow{\Delta(\delta_M)} & \Delta(M)
\end{array}
\]

where $\delta_{L_\lambda}$ is an isomorphism because of Lemma 1.1(c), which proves that $\Delta(p_\lambda) = \Delta(\delta_M) \circ \Delta^2(i_\lambda)$ is monic, i.e., $\text{Coker}(p_\lambda) \in \text{Ker} \Delta$.

Thus the hypothesis (1) applies, giving $\text{Coker}(\lim p_\lambda) \in \text{Ker} \Delta$. Moreover

\[\lim p_\lambda = \lim \Delta(i_\lambda) \circ \delta_M \cong \Delta(\lim i_\lambda) \circ \delta_M = \Delta(i) \circ \delta_M.\]

First, if we choose $L = \Delta(M)$ we clearly get $\text{Coker}(\delta_M) \cong \text{Coker}(\lim p_\lambda) \in \text{Ker} \Delta$. On the other hand, since $M \in \text{Ker} \Gamma$, $\delta_M$ is injective and $\text{Coker}(\delta_M) \in \text{Ker} \Gamma$ because of Lemma 1.1(d). Therefore $\text{Coker}(\delta_M) = 0$, i.e., $M$ is $\Delta$-reflexive.

Finally, in the case $L$ is arbitrary, since $\delta_M$ is an isomorphism, from (*) we get $\text{Coker}(\Delta(i)) \cong \text{Coker}(\lim p_\lambda) \in \text{Ker} \Delta$.

(2) $\Rightarrow$ (1). Let $\{p_\lambda : M \to M_\lambda\}$ be an inverse system of morphisms in $\text{Mod}-R$, with $M, M_\lambda \in \text{Ker} \Gamma$ and $\text{Coker}(p_\lambda) \in \text{Ker} \Delta$ for all $\lambda$'s.

In the sequel, we will refer to the following exact sequences

\[
0 \longrightarrow K_\lambda \longrightarrow M \xrightarrow{\alpha_\lambda} I_\lambda \longrightarrow 0 \quad (\text{ex1})
\]

\[
0 \longrightarrow I_\lambda \xrightarrow{\beta_\lambda} M_\lambda \longrightarrow C_\lambda \longrightarrow 0 \quad (\text{ex2})
\]

with $K_\lambda = \text{Ker}(p_\lambda)$, $I_\lambda = \text{Im}(p_\lambda)$, $C_\lambda = \text{Coker}(p_\lambda)$ and $\beta_\lambda \circ \alpha_\lambda = p_\lambda$.

First, let us prove that all the $K_\lambda$, $I_\lambda$, $M_\lambda$ are $\Delta$-reflexive. Note that the sequence (ex1) is in $\text{Ker} \Gamma$, and $M$ is $\Delta$-reflexive by assumption, so that from Lemma 1.1(g) we obtain that $K_\lambda$ is $\Delta$-reflexive too. Moreover, looking
at the embedding $\Delta(\alpha) : \Delta(I) \hookrightarrow \Delta(M)$, by hypothesis we have that $\text{Coker}(\Delta^2(\alpha)) \in \text{Ker} \Delta$. Thus we obtain the commutative exact diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & M & \longrightarrow & I & \longrightarrow & 0 \\
\text{inf} & & \text{inf} & & \text{inf} & & \text{inf} & & \\
0 & \longrightarrow & \Delta^2(K) & \longrightarrow & \Delta^2(M) & \longrightarrow & \Delta^2(I) & \longrightarrow & \text{Coker}(\Delta^2(\alpha)) & \longrightarrow & 0 \\
\end{array}
\]

from which we get (thanks to Lemma 1.1(d)) $\text{Coker}(\Delta^2(I)) \cong \text{Coker}(\delta) \in \text{Ker} \Gamma$. Thus $\text{Coker}(\delta(I)) = 0$, i.e., $I$ is $\Delta$-reflexive. Next, from (ex2) we get the commutative exact diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & \beta & \longrightarrow & M & \longrightarrow & C & \longrightarrow & 0 \\
\text{inf} & & \text{inf} & & \text{inf} & & \text{inf} & & \text{inf} & & \\
0 & \longrightarrow & \Delta^2(I) & \longrightarrow & \Delta^2(M) & \longrightarrow & \Delta^2(C) & \longrightarrow & \text{Coker}(\Delta^2(I)) & \longrightarrow & 0 \\
\end{array}
\]

where $C \in \text{Ker} \Delta$ by assumption. Let us prove that $C' \in \text{Ker} \Delta$ too. From the embedding

\[
\begin{array}{ccccccccc}
0 & = & \Delta(C) & \longrightarrow & \Delta(M) & \longrightarrow & \Delta(p) & \longrightarrow & \Delta(M) \\
\end{array}
\]

we get, by hypothesis, that $\text{Coker}(\Delta^2(p)) \in \text{Ker} \Delta$. From $\Delta^2(p) = \Delta^2(\beta) \circ \Delta^2(\alpha)$ we see that $C' = \text{Coker}(\Delta^2(\beta)) \in \text{Ker} \Delta$ too. Therefore, applying the functor $\Delta$ to the previous diagram we obtain the commutative exact diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Delta(M) & \longrightarrow & \Delta(p) & \longrightarrow & \Delta(M) & \longrightarrow & \Delta(I) & \longrightarrow & 0 \\
\text{inf} & & \text{inf} & & \text{inf} & & \text{inf} & & \text{inf} & & \\
0 & \longrightarrow & \Delta^3(M) & \longrightarrow & \Delta^3(M) & \longrightarrow & \Delta^3(I) & \longrightarrow & \text{Coker}(\Delta^3(p)) & \longrightarrow & 0 \\
\end{array}
\]

which shows that $\Delta(\delta_M)$ is monic. Since $\Delta(\delta_M) \circ \Delta(\delta_M) = \text{id}_{\Delta(M)}$, we conclude that $\Delta(\delta_M)$ is an isomorphism, so that $M$ is $\Delta$-reflexive, because of Lemma 1.1(e).

Finally, from (ex3), we derive the embedding $\lim \Delta(p) : \lim \Delta(M) \hookrightarrow \Delta(M)$, so that $\text{Coker}(\Delta(\lim \Delta(p))) \in \text{Ker} \Delta$ by assumption. Therefore we get the commutative exact diagram

\[
\begin{array}{ccccccccc}
\Delta^2(M) & \longrightarrow & \Delta(\lim \Delta(p)) & \longrightarrow & \Delta^2(M) & \longrightarrow & \text{Coker}(\Delta(\lim \Delta(p))) & \longrightarrow & 0 \\
\text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \text{inf} & \\
M & \longrightarrow & \lim M & \longrightarrow & \lim M & \longrightarrow & \text{Coker}(\lim p) & \longrightarrow & 0 \\
\end{array}
\]

which shows that $\text{Coker}(\lim p) \cong \text{Coker}(\Delta(\lim \Delta(p))) \in \text{Ker} \Delta$. \qed

The next result points out some good properties of $U$-tl.l.c. modules.

**Corollary 1.5.** Let $U_R$ be a cotilting module.
(a) If $M \to M' \to T \to 0$ is exact in $\text{Mod-}R$, and $M$ is $U$-tl.l.c., $M' \in \text{Ker } \Gamma$ and $T \in \text{Ker } \Delta$, then $M'$ is $U$-tl.l.c. too.

(b) If $M \in \text{Mod-}R$ is a factor of any $U$-tl.l.c. module, then $\delta_M$ is surjective and $M/\text{Rej}_U(M)$ is $U$-tl.l.c. too.

Proof. (a) is an immediate consequence of Proposition 1.3. In order to prove (b), let us consider an epimorphism $L \xrightarrow{\varphi} M \to 0$, with $L$ $U$-tl.l.c. From Theorem 1.4 we get that $L$ is $\Delta$-reflexive and, considering the embedding $0 \to \Delta(M) \xrightarrow{\Delta(\varphi)} \Delta(L)$, also that $\text{Coker}(\Delta^2(\varphi)) \in \text{Ker } \Delta$. On the other hand, from the commutative exact diagram

\[
\begin{array}{cccc}
L & \xrightarrow{\varphi} & M & \to 0 \\
\downarrow{\delta_L} & & \downarrow{\delta_M} & \\
\Delta^2(L) & \xrightarrow{\Delta^2(\varphi)} & \Delta^2(M) & \to \text{Coker}(\Delta^2(\varphi)) & \to 0
\end{array}
\]

we see that $\text{Coker}(\Delta^2(\varphi)) \cong \text{Coker}(\delta_M) \in \text{Ker } \Gamma$, because of Lemma 1.1(d). Hence $\text{Coker}(\Delta^2(\varphi)) \cong \Gamma(S/I)$, so that $\delta_M$ is surjective and $M/\text{Rej}_U(M) \cong \Delta^2(M)$ is $U$-tl.l.c. because of (a).

Proposition 1.6. Let $U_R$ be a cotilting module and let $S = \text{End}(U_R)$. Then $U_R$ is $U$-tl.l.c. if and only if $\Delta \Gamma(S/I) = 0$ for every left ideal $I$ of $S$.

Proof. The module $U_R$ is $\Delta$-reflexive because of Lemma 1.1(a). Therefore, by Theorem 1.4, $U_R$ is $U$-tl.l.c. if and only if for any exact sequence of the form $0 \to I \to \Delta(U_R) \cong SS \to S/I \to 0$ it happens that $\text{Coker}(\Delta(i)) \in \text{Ker } \Delta$. Finally, from the previous sequence we get the exact sequence $0 \to \Delta(S/I) \to \Delta(S) \xrightarrow{\Delta(i)} \Delta(I) \to \Gamma(S/I) \to 0$, which shows that $\text{Coker}(\Delta(i)) \cong \Gamma(S/I)$. □

We switch now to the case of a cotilting bimodule.

Corollary 1.7. Let $S U_R$ be a cotilting bimodule and let $S S (R_R, \text{respectively})$ be noetherian. Then $U_R$ ($SU$, respectively) is $U$-tl.l.c.

Proof. By assumption, for any left ideal $I$ of $S$ the cyclic module $S/I$ is finitely presented, and so it belongs to the class $\mathcal{C}$, as proved in [Cr, Proposition 5 d)]. Moreover, from [Cr, Theorem 6 a)], we get $\Gamma(\mathcal{C}) \subseteq \text{Ker } \Delta$, so that $\Delta \Gamma(S/I) = 0$. We finish the proof applying Proposition 1.6. □

We are now ready to answer the question posed in [Cr, Remark 11].

Theorem 1.8. Let $S U_R$ be a cotilting bimodule. The following conditions are equivalent for any module $M_R \in \text{Ker } \Gamma$:

1. $M_R$ is $U$-tl.l.c.
2. $M_R$ is $\Delta$-reflexive and for all $S L \leq \Delta(M)$ we have $\Delta \Gamma(\Delta(M)/L) = 0$. 


(3) Any $S$-submodule of $\Delta(M)$ is $\Delta$-reflexive.

Proof. (1) $\iff$ (2). Since $\Gamma\Delta(M) = 0$, for any embedding $i : SL \rightarrow \Delta(M)$ we get $\text{Coker} (\Delta(i)) = \Gamma(\Delta(M)/L)$. Now apply Theorem 1.4.

(2) $\Rightarrow$ (3). For any $SL \leq \Delta(M)$ we get the exact sequence $\Delta^2(M) \rightarrow \Delta(L) \rightarrow \Gamma(\Delta(M)/L) \rightarrow 0$ where, by assumption, since (1) $\iff$ (2), $\Delta^2(M)$ is $U$-tl.l.c. and $\Gamma(\Delta(M)/L) \in \text{Ker} \Delta$. So Corollary 1.5(a) applies, giving $\Delta(L)$ reflexive. Since $L$ is clearly in $\text{Ker} \Gamma$, from Lemma 1.1(e) we obtain that $L$ is $\Delta$-reflexive.

(3) $\Rightarrow$ (2). By assumption $\Delta(M)$ is $\Delta$-reflexive, and so $M$ is $\Delta$-reflexive too, because of Lemma 1.1(e). Next, for any $SL \leq \Delta(M)$ we get the canonical exact sequence $0 \rightarrow L \rightarrow \Delta(M) \rightarrow \Delta(M)/L \rightarrow 0$, with both $L$ and $\Delta(M)$ $\Delta$-reflexive. Then $\Delta\Gamma(\Delta(M)/L) = 0$ because of [Cp, Lemma 4 d)]. □

Corollary 1.9. Let $SU_R$ be a cotilting bimodule. The following conditions are equivalent:

(1) every $\Delta$-reflexive right $R$-module is $U$-tl.l.c.,

(2) the class of all the $\Delta$-reflexive left $S$-modules is closed under submodules.

Proof. Apply (1) $\iff$ (3) of Theorem 1.8. □

We now have the following connection between cotilting bimodules and those bimodules $SU_R$ that induce Colby’s generalized Morita dualities [Cb2] in the sense that the classes of $\Delta$-reflexive modules are closed under extensions and submodules, and contain $SS$ and $RR$, respectively.

Corollary 1.10. Let $SU_R$ be a cotilting bimodule. Then $SU_R$ induces a generalized Morita duality if and only if the class of the $\Delta$-reflexive modules coincides with the class of the $U$-torsionless linearly compact modules, both in $S$-Mod and in $\text{Mod-R}$.

Proof. For any cotilting bimodule $SU_R$, the regular modules $SS$ and $RR$ are $\Delta$-reflexive, because of Lemma 1.1(a), and, similarly, any extension of two $\Delta$-reflexive modules is $\Delta$-reflexive too, because of [Cp, Proposition 5 a)]. Now apply Corollary 1.9. □

2. Morita duals of tilting bimodules.

Originally cotilting bimodules arose as $k$-duals of tilting bimodules. Namely, consider two finite dimensional $k$-algebras $R$ and $S$, and denote by $D(-)$ the vector space $k$-duality. In this context a cotilting bimodule is just the dual $D(RVS)$ of a finite dimensional tilting bimodule $RVS$, so cotilting theory for finite dimensional algebras is just a perfect dual of tilting theory. Moreover, since $D(RR)$ is an injective cogenerator in $R$-Mod and adjunction induces a natural isomorphism of left $S$-modules $D(V_S) \cong \text{Hom}_R(RVS, D(RR))$, it
follows that \( D(V_S) \) is a cotilting left \( S \)-module in our sense (see the proof of 2.4 below). Arguing in the same way for \( D(RV) \), we obtain that \( SU_R = D(RV_S) \) is a cotilting bimodule in our sense.

Nevertheless, if we do not restrict our attention to finitely generated modules, cotilting theory is as far from tilting theory as Morita duality is from Morita equivalence. Even in this classical case, the theory seems to be quite hidden: We do not know, for instance, if the equivalent conditions of Corollary 1.9 hold true.

Obviously, a natural way to generalize this construction is to consider Morita duals of tilting bimodules. In this pursuit we are fortunate that standard methods yield the following extensions of the adjointness of the functors \( \text{Hom}_A(V,-) \) and \( V \otimes_S - \) and of the contravariant functors \( \text{Hom}_A(-,W) \) and \( \text{Hom}_R(-,W) \) induced by bimodules \( A V_S \) and \( A W_R \) (see [AF, §20]):

**Lemma 2.1.** Let \( S N \) and \( M_R \) be modules and \( A V_S \) and \( A W_R \) be bimodules.

(a) If \( AW \) is injective, then there are natural isomorphisms

\[
\text{Hom}_A(\text{Tor}_n^S(V,N),W) \cong \text{Ext}_S^n(N,\text{Hom}_A(V,W))
\]

for \( n = 1, 2, \ldots \).

(b) If \( AW \) and \( W_R \) are both injective, then there are natural isomorphisms

\[
\text{Ext}_A^n(V,\text{Hom}_R(M,W)) \cong \text{Ext}_R^n(M,\text{Hom}_A(V,W))
\]

for \( n = 1, 2, \ldots \).

**Proof.** (a) This is [CE, page 120, Proposition 5.1].

(b) Being unable to find a reference for this part, we shall sketch a proof. Let

\[
\cdots \to P_2 \to P_1 \to P_0 \to AV \to 0
\]

be a projective resolution of \( AV \), and note that the conditions on \( W \) yield an injective resolution

\[
0 \to \text{Hom}_A(V,W) \to \text{Hom}_A(P_0,W) \to \text{Hom}_A(P_1,W) \to \cdots
\]

of \( \text{Hom}_A(V,W)_R \). Then (see [R, Chapter 7]) one obtains the desired isomorphisms from the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_A(P_0,\text{Hom}_R(M,W)) & \to & \text{Hom}_A(P_1,\text{Hom}_R(M,W)) \\
\cong & & \cong \\
\text{Hom}_R(M,\text{Hom}_A(P_0,W)) & \to & \text{Hom}_R(M,\text{Hom}_A(P_1,W)) \\
\end{array}
\]

\[\square\]

For the remainder of this section \( A \) and \( R \) are supposed to be Morita dual rings via faithfully balanced bimodule \( AW_R \) that is a (linearly compact) injective cogenerator on both sides. Moreover we assume that \( AV \) is
a (linearly compact) tilting module with endomorphism ring $S = \text{End}(A V)$, and we let

$$SU_R = \text{Hom}_A(V, W).$$

We further assume that $A V$ is not projective (equivalently, not a (pro)generator), so that the bimodule $SU_R$ is not just another Morita bimodule.

For convenience sake, given any bimodule $AM_B$ we shall denote by $\Delta_M$ the two contravariant functors $\text{Hom}_?(-, AM_B)$ and by $\Gamma_M$ their first derived functors $\text{Ext}^1_A(-, AM_B)$, where $? = A$ or $B$. Also we put $H_M = \text{Hom}_A(M, -)$, $T_M = M \otimes_B -$,

$$H'_M = \text{Ext}^1_A(M, -) \quad \text{and} \quad T'_M = \text{Tor}^B_1(M, -).$$

Thus by adjointness we have

$$\Delta_U \cong H_V \Delta_W : \text{Mod-}R \rightarrow S\text{-Mod}$$

and

$$\Delta_U \cong \Delta_W T_V : S\text{-Mod} \rightarrow \text{Mod-}R$$

and by Lemma 2.1

$$\Gamma_U \cong H'_V \Delta_W : \text{Mod-}R \rightarrow S\text{-Mod}$$

and

$$\Gamma_U \cong \Delta_W T'_V : S\text{-Mod} \rightarrow \text{Mod-}R.$$ 

Also there are natural transformations

$$\delta : \text{id}_{\text{Mod-}R} \rightarrow \Delta_U \Delta_U \quad \text{and} \quad \delta : \text{id}_{S\text{-Mod}} \rightarrow \Delta_U \Delta_U$$

and

$$\gamma : \Gamma_U \Gamma_U \rightarrow \text{id}_{\text{Mod-}R} \quad \text{and} \quad \gamma : \Gamma_U \Gamma_U \rightarrow \text{id}_{S\text{-Mod}}$$

with the $\delta$’s via the usual evaluation maps, and the $\gamma$’s derived from the natural transformations of the Tilting Theorem [CbFu, 1.4] and the $\Delta_W$’s. Thus we obtain

**Duality 2.2.** There are dualities

$$\Delta_U : \mathcal{Y}_R \rightleftharpoons SU : \Delta_U$$

$$\Gamma_U : \mathcal{X}_R \rightleftharpoons SU : \Gamma_U$$

where the $\mathcal{Y}$’s and $\mathcal{X}$’s are the full subcategories on whose objects the $\delta$’s and the $\gamma$’s, respectively, are isomorphisms.

Let us denote by $A \mathcal{C}$ and $\mathcal{C}_R$ the classes of all linearly compact left $A$- and right $R$-modules, respectively. Moreover, $(A \mathcal{T}, A \mathcal{F})$ denotes the torsion theory generated by the tilting module $A V$, and $(S \mathcal{T}, S \mathcal{F})$ the torsion theory cogenerated by the cotilting module $SU = \text{Hom}_A(A V, A W)$ (see the proof of 2.4 below).

By assumption, the bimodule $A W_R$ induces a duality of the form

$$\Delta_W : \mathcal{C}_R \rightleftharpoons A \mathcal{C} : \Delta_W$$
and the tilting bimodule \( A V \) induces the two equivalences
\[
H_V : A T \rightleftharpoons s F : T_V \quad \text{and} \quad H'_V : A F \rightleftharpoons s T : T'_V.
\]

Therefore, letting
\[
A \mathcal{X} = A \mathcal{C} \cap A T \quad \text{and} \quad A \mathcal{Y} = A \mathcal{C} \cap A F
\]
we see that
\[
X_R \supseteq \Delta W(A \mathcal{Y}), \quad \mathcal{Y}_R \supseteq \Delta W(A \mathcal{X}), \quad s \mathcal{X} \supseteq H'_V(A \mathcal{Y}), \quad s \mathcal{Y} \supseteq H_V(A \mathcal{X}).
\]

Since \( A V \) is a tilting module, \( A V \) and \( A W \) belong to \( A \mathcal{X} \). Thus
\[
U_R = \Delta W(A V) \in \mathcal{Y}_R, \quad R_R = \Delta W(A W) \in \mathcal{Y}_R, \quad sU = H_V(A W) \in s \mathcal{Y}, \quad sS = H_V(A V) \in s \mathcal{Y},
\]
so, in particular, we have:

2.3. Balance. The bimodule \( SU_R \) is faithfully balanced.

Since \( A V \) is a \( * \)-module, \( SU = \text{Hom}_A(A V, A W) \) and \( A W \) is an injective cogenerator, as in [CpToTr, 2.3 3)], we obtain:

2.4. Properties of \( SU \). \( SU \) is a cotilting module.

One would hope that \( U_R \) is one too. Perhaps not in general, but we do have the following:

2.5. Properties of \( U_R \).

(a) There is an exact sequence \( 0 \to U_R \to W' \to W'' \to 0 \), where \( W', W'' \in \text{add}(W_R) \). In particular \( U_R \) is finitely cogenerated and \( \text{inj dim}(U_R) \leq 1 \).

(b) There is an exact sequence \( 0 \to U' \to U'' \to W_R \to 0 \), where \( U', U'' \in \text{add}(U_R) \). In particular \( \text{Ker} \Delta_U \cap \text{Ker} \Gamma_U = 0 \).

(c) \( \Delta_W(A \mathcal{X}) \subseteq \text{Ker} \Gamma_U \). In particular \( \text{Ext}^1_R(M, U) = 0 \) for all \( M_R \to U^n_R \) (\( n \) finite).

Proof. (a) Since \( A V \) is a tilting module, there is an exact sequence of the form \( 0 \to A' \to A'' \to A V \to 0 \), with \( A', A'' \in \text{add}(A A) \). Now apply \( \Delta_W \).

(b) Similarly to the previous case, applying \( \Delta_W \) to the exact sequence \( 0 \to A A \to V' \to V'' \to 0 \), where \( V', V'' \in \text{add}(A V) \), we obtain the required exact sequence. Finally, applying \( \text{Hom}_R(M, -) \) to that, we see that \( \text{Hom}_R(M, U) = 0 = \text{Ext}^1_R(M, U) \) implies \( \text{Hom}_R(M, W) = 0 \), and so \( M = 0 \).

(c) For any \( M \in \Delta_W(A \mathcal{X}) \) we clearly have \( \Delta_W(M) \in A \mathcal{X} \subseteq A T = \text{Ker} \text{Ext}^1_A(V, -) \). Therefore, we see by Lemma 2.1(b) that \( \text{Ext}^1_R(M, U) \cong \text{Ext}^1_A(V, \Delta_W(M)) = 0 \).

From 2.3, 2.4 and 2.5 we immediately have:

Proposition 2.6. The bimodule \( SU_R \) is a cotilting bimodule if and only if \( \text{Ext}^1_R(U^\alpha, U) = 0 \) for any cardinal \( \alpha \).
3. Cotilting bimodules over noetherian serial rings.

In [CbFu] Colby and Fuller determined all the tilting bimodules \( R V S \) over a noetherian serial ring \( R \). In this concluding section we shall see that if \( R \) has self-duality induced by \( R W R \) then \( SU R = \text{Hom}_R(V, W) \) is a cotilting bimodule. Thus we obtain a large collection of cotilting bimodules (that are not even finitely generated) in addition to the classical ones over finite dimensional algebras.

According to [Wa, Theorem 5.11], a noetherian serial ring is a finite direct sum of indecomposable artinian serial rings and prime noetherian serial rings. Warfield proved that every finitely generated module and every injective module over such a ring is a direct sum of uniserial modules. The structure of artinian serial rings is well known (see [AF, §32]).

Let \( R \) be a prime noetherian serial ring with right Kupisch series \( e_1 R, \ldots, e_n R \) so that, setting \( J = J(R) \)
\[
e_1 J \cong e_2 R, \ldots, e_{n-1} J \cong e_n R \quad \text{and} \quad e_n J \cong e_1 R
\]
(see [CbFu, §3]). According to Warfield [Wa]
\[
e_i R > e_i J > e_i J^2 > \ldots \quad \text{and} \quad Re_i > Je_i > J^2 e_i > \ldots
\]
are complete lists of the submodules of \( e_i R \) and \( Re_i \), for \( i = 1, \ldots, n \). Thus, setting \( S_i = e_i R/e_i J \), the composition factors of \( e_i R \) are, from the top down,
\[
S_i, S_{i+1}, \ldots, S_n, S_1, S_2, \ldots, S_n, \ldots.
\]
On the other hand, as Warfield showed, every finitely generated indecomposable \( R \)-module is uniserial. It follows that the indecomposable injective \( R \)-modules are also uniserial. There are just \( n + 1 \) indecomposable injective right \( R \)-modules
\[
E_1 = E(S_1), \ldots, E_n = E(S_n) \quad \text{and} \quad E_0
\]
with \( \text{Soc}(E_0) = 0 \), each \( E_i \) is artinian, and for any \( i = 1, \ldots, n \) the submodules of \( E_i \) are
\[
0 < \text{Soc}(E_i) < \text{Soc}^2(E_i) < \ldots
\]
where \( \text{Soc}^k(M) = \text{Ann}_M(J^k) \). And the composition factors of \( E_i \), from the bottom up, are
\[
S_i, S_{i-1}, \ldots, S_1, S_n, S_{n-1}, \ldots, S_1, \ldots
\]
while the composition factor of \( E_0 \) are
\[
\ldots, S_n, \ldots, S_i, S_{i-1}, \ldots, S_1, S_n, S_{n-1}, \ldots, S_i, S_{i-1}, \ldots, S_1, \ldots
\]
In particular any proper factor of an indecomposable injective module is the injective envelope of its socle, and every proper submodule of \( E_0 \) is isomorphic to an indecomposable projective module.
Lemma 3.1. Let $R$ be a noetherian serial ring. If $X_R$ is an indecomposable $R$-module of finite length, then for any cardinal $\alpha$ there is a cardinal $\gamma$ such that $X^\alpha \cong X^{(\gamma)}$.

Proof. Let $Q = R/\text{Ann}_R(X)$. Then $X_Q$ is a faithful indecomposable module over the artinian QF-3 ring $Q$. Thus $X$ is the unique indecomposable injective projective right $Q$-module (see [AF, §31 and §32]). But $X^\alpha$ is both injective and, since $Q$ is artinian, projective. Moreover $X_Q^\alpha$ is a direct sum of indecomposable modules, since $Q$ is artinian.

Lemma 3.2. Let $R$ be a prime noetherian serial ring with indecomposable injective modules $E_1, \ldots, E_n$ and $E_0$ as above. Then for any cardinal $\alpha$ there are cardinals $\beta, \gamma$ such that $E_i^\alpha \cong E_i^{(\beta)} \oplus E_0^{(\gamma)}$.

Proof. Since $R$ is semiperfect and $J$ is finitely generated, we see that
\[
\text{Soc}(E_i^\alpha) = \text{Ann}_E^\alpha(J) = \text{Ann}_E(J)^\alpha = \text{Soc}(E_i)^\alpha.
\]
But if $i \neq j$ then $\text{Soc}(E_i)e_j = 0$. Thus $\text{Soc}(E_i^\alpha) = S_i^{(\beta)}$. So we see that $E_i^\alpha \cong E_i^{(\beta)} \oplus E$ with $\text{Soc}(E) = 0$. But the only indecomposable injective module with zero socle is $E_0$, so $E \cong E_0^{(\gamma)}$.

Proposition 3.3. If $U$ is a finitely cogenerated module over a noetherian serial ring $R$ such that $\text{Ext}_R^1(U, U) = 0$, then $\text{Ext}_R^1(U^\alpha, U) = 0$ for any cardinal $\alpha$.

Proof. Since $U$ is finitely cogenerated, we have
\[
U = E_{i_1} \oplus \cdots \oplus E_{i_k} \oplus X_1 \oplus \cdots \oplus X_l
\]
where $E_{i_j} = E(S_{i_j}), j = 1, \ldots, k$, and $X_i, i = 1, \ldots, l$, are uniserial modules of finite length. Thus by Lemmas 3.1 and 3.2 we have
\[
U^\alpha = E_{i_1}^{(\beta_1)} \oplus \cdots \oplus E_{i_k}^{(\beta_k)} \oplus E_0^{(\gamma)} \oplus X_1^{(\delta_1)} \oplus \cdots \oplus X_l^{(\delta_l)}.
\]
Now, since $\text{Ext}_R^1(-, X_i)$ converts direct sums to direct products, we need only check that $\text{Ext}_R^1(E_0, X_i) = 0$ for all $i = 1, \ldots, l$. To this end, consider the minimal injective resolution
\[
0 \longrightarrow X_i \longrightarrow E_i \longrightarrow E_j \longrightarrow 0.
\]
Here we need to show that
\[
\text{Hom}_R(E_0, E_i) \xrightarrow{\text{Hom}_R(E_0, \nu)} \text{Hom}_R(E_0, E_j) \longrightarrow 0
\]
is exact. So let $0 \neq \beta \in \text{Hom}_R(E_0, E_j)$ with $K = \text{Ker}(\beta)$. Then there is $m \in \mathbb{N}$ such that $E_0/KJ^m \cong E_{i_1}$. But $\text{Ext}_R^1(E_{i_1}, X_i) = 0$, being a direct summand of $\text{Ext}_R^1(U, U)$, so
\[
\text{Hom}_R(E_0/KJ^m, E_i) \xrightarrow{\text{Hom}_R(E_0/KJ^m, \nu)} \text{Hom}_R(E_0/KJ^m, E_j) \longrightarrow 0
\]
is exact. Thus, setting \( \eta : E_0 \to E_0/KJ^m \), we have a commutative diagram

\[
\begin{array}{ccc}
E_0 & \xrightarrow{\beta} & E_j \\
\downarrow{\eta} & & \downarrow{}
\end{array}
\]

\[
\begin{array}{ccc}
E_0/KJ^m & \xrightarrow{\overline{\beta}} & E_j \\
\downarrow{\gamma} & & \downarrow{}
\end{array}
\]

which shows that \( \text{Hom}_R(E_0, \nu)(\gamma \circ \eta) = \nu \circ \gamma \circ \eta = \overline{\beta} \circ \eta = \beta \).

\[\Box\]

**Theorem 3.4.** Let \( R \) be a noetherian serial ring with self-duality induced by a bimodule \( RW_R \). If \( R \) is a tilting module and \( S = \text{End}(R \vee) \), then \( SU_R = \text{Hom}_R(V, W) \) is a cotilting bimodule.

**Proof.** According to Proposition 2.6, it only remains to observe that \( \text{Ext}^1_R(U^\alpha, U) = 0 \) for any \( \alpha \). And this is true thanks to Proposition 3.3, since \( \text{Ext}^1_R(U, U) = 0 \) and \( U_R \) is finitely cogenerated because of 2.5(c) and (a).

Let us pause to point out a couple of facts about self-duality for noetherian serial rings.

**Proposition 3.5.** If \( R \) is a left linearly compact indecomposable prime noetherian serial ring, then \( R \) has a self-duality.

**Proof.** Assume, as we may, that \( R \) is basic. Let \( E = E_1 \oplus \cdots \oplus E_n \) be the minimal cogenerator. Then \( E \) is artinian, hence linearly compact. Thus, setting \( S = \text{End}(R E) \), \( S_S \) is linearly compact and the bimodule \( R E_S \) defines a Morita duality. Now it is easy to see that \( A_k = \text{Ann}_E(J^k) \) is the minimal cogenerator over \( R/J^k \), and that the bimodule \( R/J^k A_k S/\text{Ann}_S(A_k) \) defines a Morita duality. But \( R/J^k \) is a basic QF-ring (see [AF, §32.6]) and hence \( R/J^k A_k \cong R/J^k R/J^k \). But then

\[
S/\text{Ann}_S(A_k) \cong \text{End}(R/J^k A_k) \cong R/J^k
\]

as rings. Now both \( \{ J^k \mid k \geq 1 \} \) and \( \{ \text{Ann}_S(A_k) \mid k \geq 1 \} \) are downward directed sets of ideals with \( \cap_k J^k = 0 \) [Wa, Theorem 5.11] and so \( \cap_k \text{Ann}_S(A_k) = 0 \). Therefore, since \( R R \) and \( S_S \) are both linearly compact, we have

\[
R \cong \lim \limits_{\longrightarrow} R/J^k \cong \lim \limits_{\longrightarrow} S/\text{Ann}_S(A_k) \cong S.
\]

\[\Box\]

As Warfield [Wa] showed, a prime noetherian serial ring \( R \) is isomorphic to the \( n \times n \ (D : M) \)-upper triangular matrix ring \( \text{UTM}_n(D : M) \), consisting of those matrices over a local noetherian serial ring \( D \) whose entries below
the main diagonal all come from the unique maximal ideal $M$ of $D$. It follows from Proposition 3.5 and $X$, Theorem 4.3, Lemma 4.9 and Proposition 3.3 that $R$ has self-duality if and only if $D$ is linearly compact. According to $Wb$ and $DiMl$, any artinian serial ring has self-duality. Thus from Proposition 3.5 and $Mu$ (see again $X$, Theorem 4.3) we have:

**Proposition 3.6.** A noetherian serial ring has a self-duality if and only if it is left (equivalently right) linearly compact.

Finally, we note that any tilting module $_R V$ over a hereditary noetherian ring (which was shown to be a finitistic cotilting module in $CbFu$) satisfies at least two of the three conditions needed to be a cotilting module in our sense whenever $R$ has selfduality.

**Proposition 3.7.** Let $R$ be a hereditary linearly compact noetherian serial ring and let $_R V$ be a tilting module. Then $_R V$ is a finitistic cotilting module with $\text{Ext}^1_R(V^\alpha, V) = 0$ for all cardinal numbers $\alpha$.

**Proof.** According to $CbFu$, Proposition 2.1, $R V$ is a finitistic cotilting module, and since it is finitely generated

$$RV = P \oplus T,$$

with $P$ finitely generated projective and $T = T_1 \oplus \cdots \oplus T_l$, with all the $T_i$’s uniserial modules of finite length. Since $\text{Ext}^1_R(V, V) = 0$, and since, by Lemma 3.1,

$$V^\alpha = P^\alpha \oplus T_1^{(\delta_1)} \oplus \cdots \oplus T_l^{(\delta_l)},$$

it only remains to show that $\text{Ext}^1_R(P^\alpha, P) = 0$ and $\text{Ext}^1_R(P^\alpha, T_i) = 0$ for $i = 1, \ldots, l$.

Let $_R W_R$ induce a self-duality and observe that the canonical right $R$-isomorphism

$$R \longrightarrow \text{Hom}_R(\text{Hom}_R(R_R, W), W) \longrightarrow \text{Hom}_R(R W_R, R W_R)$$

is also a left $R$-map. Now $P^\alpha$ is flat by Chase’s Theorem $[AF$, 19.20], since $R$ is noetherian, and so by Lemma 2.1(a)

$$\text{Ext}^1_R(P^\alpha, R R) \cong \text{Ext}^1_R(P^\alpha, \text{Hom}_R(R W_R, R W))$$

$$\cong \text{Hom}_R(\text{Tor}^1_R(W, P^\alpha), W) = 0.$$

Thus, assuming, as we may, that $P$ is a direct summand of $R R$, we do have

$$\text{Ext}^1_R(P^\alpha, P) = 0.$$

On the other hand, if $T_i$ has length $m$, and $A = R/J^m$, then $_A T_i$ is injective $[AF$, Theorem 32.6] and

$$R T_i \cong \text{Hom}_A(A A_R, _A T_i),$$
so that
\[
\text{Ext}^1_R(P^\alpha, R T_i) \cong \text{Ext}^1_R(P^\alpha, \text{Hom}_A(A A_R, A T_i)) \\
\cong \text{Hom}_R(\text{Tor}^R_1(A_R, P^\alpha), A T_i) = 0.
\]

Remark 3.8. (1) Krause and Saorín [KrSa, Proposition 3.8] have recently shown that if \(M_R\) is a finitely generated module, then every \(M^\alpha\) is isomorphic to a direct summand of some \(M(5)\) if and only if \(S = \text{End}(M_R)\) is left coherent and right perfect and \(s M\) is finitely presented. Thus we see that if \(R\) is right artinian and (hence) \(S\) is left artinian in a cotilting triple \((S, U, R)\) in the sense of [Cb1, §2], then \(\text{Ext}^1_R(U^\alpha, U) = 0 = \text{Ext}^2_S(U^\alpha, U)\) for any cardinal \(\alpha\).

(2) Over rings of finite representation type, cotilting triples yield more examples of cotilting modules. Indeed, in a cotilting triple \((S, U, R)\), if it happens that \(R\) is a ring of finite representation type (so that every \(R\)-module is a direct sum of finitely generated modules), then since \(U_R\) is a finitisitic cotilting module [Cb1, Theorem 3.3], we also have \(\text{Ker} \hom_R(-, U_R) \cap \text{Ker} \text{Ext}^1_R(-, U_R) = 0\), so that \(U_R\) is a cotilting module in the present sense. If in addition \(S\) has finite representation type (in particular, if \(R\) is hereditary [CbFu, Proposition 2.2]), then \(S U_R\) is a cotilting bimodule.

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References


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HARDY'S UNCERTAINTY PRINCIPLE
ON SEMISIMPLE GROUPS

M. COWLING, A. SITARAM, AND M. SUNDARI

A theorem of Hardy states that, if $f$ is a function on $\mathbb{R}$ such that $|f(x)| \leq C e^{-\alpha |x|^2}$ for all $x$ in $\mathbb{R}$ and $|\hat{f}(\xi)| \leq C e^{-\beta |\xi|^2}$ for all $\xi$ in $\mathbb{R}$, where $\alpha > 0$, $\beta > 0$, and $\alpha \beta > 1/4$, then $f = 0$. Sitaram and Sundari generalised this theorem to semisimple groups with one conjugacy class of Cartan subgroups and to the $K$-invariant case for general semisimple groups. We extend the theorem to all semisimple groups.

1. Introduction.

The Uncertainty Principle states, roughly speaking, that a nonzero function $f$ and its Fourier transform $\hat{f}$ cannot both be sharply localised. This fact may be manifested in different ways. The version of this phenomenon described in the abstract is due to Hardy [3]; we call it Hardy's Uncertainty Principle. Considerable attention has been devoted recently to discovering new forms of and new contexts for the Uncertainty Principle (see [2] for a recent comprehensive survey). In particular, Sitaram and Sundari [4] generalised Hardy's Uncertainty Principle to connected semisimple Lie groups with one conjugacy class of Cartan subgroups and to the $K$-invariant case for general connected semisimple Lie groups. We extend the theorem of Sitaram and Sundari [4], and establish a form of Hardy's Uncertainty Principle for all connected semisimple Lie groups with finite centre.

2. The theorem.

Let $G$ be a connected real semisimple Lie group with finite centre. Let $KAN$ be an Iwasawa decomposition of $G$, and let $MAN$ be the associated minimal parabolic subgroup of $G$. The Lie algebras of $G$ and $A$ are denoted by $\mathfrak{g}$ and $\mathfrak{a}$. The Killing form of $\mathfrak{g}$ induces an inner product on $\mathfrak{a}$ and hence on the dual $\mathfrak{a}^*$; in both cases the corresponding norms are denoted by $|\cdot|$. Haar measures on $K$ and $G$ are fixed; that on $K$ is normalised so that the total mass of $K$ is 1. Integrals over $G$ and $K$ are relative to these Haar measures.

Any irreducible unitary representation $\mu$ of $M$ may be realised as the left-translation representation on a finite-dimensional subspace $\mathcal{H}_\mu$ of $C(M)$, the space of continuous complex-valued functions on $M$. For such a $\mu$, and $\lambda$ in
the complexification $a^\ast_\mathbb{C}$ of $a^\ast$, we define the space $\mathcal{H}^0_{\mu,\lambda}$ to be the subspace of $C(G)$ of all functions $\xi$ with the properties that

$$\xi(gan) = \xi(g) \exp((i\lambda - \rho) \log a) \quad \forall g \in G \quad \forall a \in A \quad \forall n \in N$$

and

$$m \mapsto \xi(gm) \in \mathcal{H}_\mu \quad \forall g \in G.$$ 

Note that such functions are determined by their restrictions to $K$, i.e., effectively we are dealing with a subspace of $C(K)$. The representation $\pi^0_{\mu,\lambda}$ of $G$ is the left-translation representation of $G$ on this space. We define the inner product $\langle \xi, \eta \rangle$ of $\xi$ and $\eta$ in $\mathcal{H}^0_{\mu,\lambda}$ to be

$$\int_K \xi(k) \overline{\eta(k)} \, dk;$$

$\| \cdot \|$ denotes the associated norm.

Denote by $\mathcal{H}_{\mu,\lambda}$ the completion of $\mathcal{H}^0_{\mu,\lambda}$ with this norm, and by $\pi_{\mu,\lambda}$ the extension of $\pi^0_{\mu,\lambda}$ to $\mathcal{H}_{\mu,\lambda}$. The space $\mathcal{H}_{\mu,\lambda}$ may be identified with a subspace of $L^2(K)$, and $\mathcal{H}^0_{\mu,\lambda}$ with the space of continuous functions in $\mathcal{H}_{\mu,\lambda}$.

For $\mu$ in $\widehat{M}$ and $\lambda$ in $a^\ast$, the representation $\pi_{\mu,\lambda}$ is unitary. This representation lifts to a representation of $L^1(G)$ by integration, as follows. First, for $f$ in $L^1(G)$ and $\xi$ and $\eta$ in $\mathcal{H}^0_{\mu,\lambda}$, the integral

$$\int_G f(g) \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle \, dg$$

converges, to $B_f(\xi, \eta)$ say. Next, $B_f$ is a sesquilinear form on $\mathcal{H}_{\mu,\lambda}$. Thus there exists a unique bounded operator, denoted $\pi_{\mu,\lambda}(f)$, such that

$$\langle \pi_{\mu,\lambda}(f)\xi, \eta \rangle = \int_G f(g) \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle \, dg \quad \forall \xi, \eta \in \mathcal{H}_{\mu,\lambda}.$$

We denote by $\| \cdot \|$ the operator norm of such operators, relative to the given norm on $\mathcal{H}_{\mu,\lambda}$. If $\lambda \in a^*_C \setminus a^\ast$, then the matrix coefficients $g \mapsto \langle \pi_{\mu,\lambda}(g)\xi, \eta \rangle$ need not be bounded, and for general $f$ in $L^1(G)$ it may not be possible to define $\pi_{\mu,\lambda}(f)$. However, for $f$ which decays sufficiently rapidly at infinity in $G$, in particular for $f$ in the theorem below, $\pi_{\mu,\lambda}(f)$ may still be defined by the procedure above.

**Theorem.** Suppose that $C$, $\alpha$, $C_\mu$, $\beta_\mu$ are positive constants and $\alpha \beta_\mu > 1/4$ for all $\mu$ in $\widehat{M}$, and that $f$ is a measurable function on $G$ such that

$$|f(kak')| \leq C \exp(-\alpha|\log a|^2) \quad \forall k, k' \in K \quad \forall a \in A$$

and

$$\|\pi_{\mu,\lambda}(f)\| \leq C_\mu \exp(-\beta_\mu|\lambda|^2) \quad \forall \mu \in \widehat{M} \quad \forall \lambda \in a^\ast.$$ 

Then $f = 0$. 
Proof. Let $\sigma$ and $\tau$ be irreducible representations of $K$, with characters $\chi_{\sigma}$ and $\chi_{\tau}$. Define $f_{\sigma,\tau}$ by the formula

$$f_{\sigma,\tau}(g) = \dim \sigma \dim \tau \int_K \int_K \chi_{\sigma}(k) \chi_{\tau}(k') f(kgk') \, dk \, dk'.$$

By a straightforward estimate,

$$|f_{\sigma,\tau}(kak')| \leq C (\dim \sigma \dim \tau)^2 \exp(-\alpha |\log a|^2) \quad \forall k, k' \in K \quad \forall a \in A.$$

Further, $\pi_{\mu,\lambda}(f_{\sigma,\tau})$ is the composition $P_{\sigma} \pi_{\mu,\lambda}(f) P_{\tau}$, where $P_{\sigma}$ and $P_{\tau}$ are the projections of $L^2(K)$ onto the $\sigma$-isotypic and $\tau$-isotypic subspaces, so that

$$\|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| \leq C_{\mu} \exp(-\beta_{\mu}|\lambda|^2) \quad \forall \mu \in \hat{M} \quad \forall \lambda \in a^*.$$ 

Now the arguments of Section 3 of [4] show that, if $\alpha_{\mu}$ is chosen such that $0 < \alpha_{\mu} < \alpha$ and $\alpha_{\mu}\beta_{\mu} > 1/4$, then

$$\|\pi_{\mu,\lambda}(f_{\sigma,\tau})\| \leq C_{\sigma,\tau,\mu} \int_G \Phi_{1,\text{Re}(\lambda)}(x) |f(x)| \, dx \leq C'_{\sigma,\tau,\mu} \exp\left(\frac{|\lambda|^2}{4\alpha_{\mu}}\right) \quad \forall \mu \in \hat{M} \quad \forall \lambda \in a^*_C,$$

where $\Phi_{1,\text{Re}(\lambda)}$ denotes the usual elementary spherical function, and hence that

$$\pi_{\mu,\lambda}(f_{\sigma,\tau}) = 0 \quad \forall \mu \in \hat{M} \quad \forall \lambda \in a^*_C.$$ 

By Harish-Chandra’s subquotient theorem (see G. Warner [5, p. 452]), if $\pi$ is any irreducible unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$, then there exist $\mu$ in $\hat{M}$ and $\lambda$ in $a^*_C$ and closed subspaces $S_0$ and $S_1$ of $\mathcal{H}_{\mu,\lambda}$ such that $\pi$ is Naimark equivalent to the quotient representation $\pi_{\mu,\lambda}$ of $\pi_{\mu,\lambda}$ on $S_1/S_0$. This means that there is an intertwining operator $A_{\pi}$ with dense domain and range between $(\pi, \mathcal{H}_\pi)$ and $(\pi_{\mu,\lambda}, S_1/S_0)$. Consequently $\pi(f_{\sigma,\tau}) = 0$, first on the domain of $A_{\pi}$ by the intertwining property, and then on all $\mathcal{H}_\pi$ by continuity. In summary,

$$\langle \pi(f_{\sigma,\tau})\xi, \eta \rangle = 0 \quad \forall \xi, \eta \in \mathcal{H}_\pi,$$

and therefore, summing over $\sigma$ and $\tau$, we see that

$$\langle \pi(f)\xi, \eta \rangle = 0 \quad \forall \xi, \eta \in \mathcal{H}_\pi.$$

It follows that $\pi(f) = 0$ for all $\pi$ in $\widehat{G}$, the unitary dual of $G$, whence $f = 0$ by the Plancherel theorem. \qed

The argument of this paper may also be applied in other contexts. For instance, we may show the following: if $f$ is a measurable function on $G$, rapidly decreasing in the sense that for any $B$ in $\mathbb{R}^+$ there exists $A$ in $\mathbb{R}^+$ such that

$$|f(kak')| \leq A \exp(-\alpha B |\log a|) \quad \forall k, k' \in K \quad \forall a \in A,$$
and if on each principal series induced from the minimal parabolic subgroup, the group-theoretic Fourier transform vanishes on a set of positive Plancherel measure, then $f$ is zero. This is a qualitative uncertainty principle related to [1].

References


RESTRICTIONS OF $\Omega_m(q)$-MODULES TO ALTERNATING GROUPS

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We consider the restriction of an irreducible $F\Omega_m(q)$-module $M$ to a subgroup $H$ where $F^*(H) \cong A_n$ and where $F$ is algebraically closed with $(\text{char}(F), q) \neq 1$. Given certain restrictions on the highest weight of $M$, we show that if $m > n^6$, then $M\downarrow H$ is reducible.

1. Introduction.

In the study of the maximal subgroups of classical groups, the following question arises: Given an absolutely irreducible module $M$ for $K$ and a subgroup $H$, when does $M\downarrow H$ remain absolutely irreducible? In this article $K \cong \Omega_m(q)$ is the commutator subgroup of an $m$-dimensional orthogonal group over $F_q$, and $F^*(H) \cong A_n$ is the alternating group of degree $n$. We treat the case that the field of definition of $M$ has characteristic dividing $q$.

Let $F$ be an algebraically closed field containing $F_q$, the field with $q$ elements, such that $\text{char}(F) > 3$. Then $K < \overline{K}$ where $\overline{K} \cong \Omega_m(F)$ and we may assume that $M$ is a $FK$-module. By [6, Theorem 43], every absolutely irreducible $FK$-module is the restriction of an irreducible $FK$-module of the same weight. So we may assume that $M = M(\lambda)$ is an irreducible $FK$-module with highest weight $\lambda$. Let $\ell = [m/2]$ be the Lie rank of $\overline{K}$ and let $\{\lambda_i\}$ be the fundamental dominant weights of $\overline{K}$. The labeling of these weights corresponds to the labeling of the Dynkin diagrams for $\overline{K}$ as given in [3].

Hypothesis 1.1. Assume the following are true:

1. If $m$ is even, then $\lambda = \left( \sum_{i=1}^{\ell-2} a_i \lambda_i \right) + a_{\ell-1}(\lambda_{\ell-1} + \lambda_{\ell}); a_i \in \mathbb{Z}, a_i \geq 0$.

2. If $m$ is odd, then $\lambda = \left( \sum_{i=1}^{\ell-1} a_i \lambda_i \right) + 2a_\ell \lambda_\ell; a_i \in \mathbb{Z}, a_i \geq 0$.

3. If $\mu_i = \sum_{j=i}^{\ell} a_j$, $m$ even or if $\mu_i = \sum_{j=i}^{\ell} a_j$, $m$ odd then
   
   (a) $\mu_1 < p = \text{char}(F_q)$;
   
   (b) $1 < \sum \mu_i = k < \ell$.

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Conditions (1) and (2) imply that $M$ is not a faithful module for any proper covering group of $K$. We now state our main result:

**Theorem 1.2.** Assume that $H, K$ and $M = M(\lambda)$ are as above with $n, m \geq 10$ and $(q, 6) = 1$. Suppose further that $\lambda$ satisfies Hypothesis 1.1. If $m > n^6$, then $M\downarrow H$ is reducible.

Our strategy is to produce a small subspace in $M$ with a large stabilizer in $H$ and then, using Frobenius reciprocity, produce an upper bound for $\dim(M)$. We produce a lower bound for $\dim(M)$ as an $FK$-module using the length of the Weyl group orbit of a subdominant weight in $M$. The result then follows by comparing these two bounds.

2. A construction of $W(\lambda)$.

In this section we construct the Weyl module $W(\lambda)$ of $K$ with highest weight $\lambda$. Then $M$ is a homomorphic image of this module. Our construction proceeds by first constructing the Weyl module $W(\lambda)$ for a complex Lie group $G$ of the same type and rank as $K$, then we use Kostant's $\mathbb{Z}$-form to produce $W(\lambda)$. For notational convenience we assume that $\{\lambda_i\}$ are the fundamental dominant weights for $G$ as well as for $K$, and accordingly, assume that $\lambda$ is a dominant weight of $G$.

Let $V$ be a complex, $m$-dimensional vector space possessing a non-degenerate orthogonal form $f(\ ,\ )$ and let $B$ be a basis for $V$ so that

$$B = \begin{cases} \{e_i, f_i | 1 \leq i \leq \ell\} & \text{if } m \text{ is even} \\
\{e_i, f_i, x | 1 \leq i \leq \ell\} & \text{if } m \text{ is odd}
\end{cases}$$

with $f(e_i, e_j) = f(f_i, f_j) = f(x, e_i) = f(x, f_j) = 0$, $f(e_i, f_j) = \delta_{i,j}$ and $f(x, x) = 2$. We then define $G = \Omega(V)$ and let $T$ be the maximal torus of $G$ with respect to $B$. Set $V_e = \langle e_i | 1 \leq i \leq \ell \rangle$ and $V_f = \langle f_i | 1 \leq i \leq \ell \rangle$.

Suppose that $\lambda$ satisfies hypothesis 1.1 and $d = \max\{i | \mu_i \neq 0\}$ so that $\mu = (\mu_1, \ldots, \mu_d)$ is a proper partition of $k$. Let $T$ be the tableau of shape $\mu$ with entries $t_{i,j} = j + \sum_{s<i} \mu_s$. Define the following subgroups of the symmetric group $\mathcal{S}_k$:

$$\mathcal{R}_\mu = \{ \sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same row as } t_{i,j} \text{ for all } i, j \}$$

$$\mathcal{C}_\mu = \{ \sigma \in \mathcal{S}_k \mid \sigma(t_{i,j}) \text{ lies in the same column as } t_{i,j} \text{ for all } i, j \}$$

and elements of $\mathcal{C}_k$:

$$r_\mu = \sum_{\sigma \in \mathcal{R}_\mu} \sigma \quad \text{ and } \quad c_\mu = \sum_{\sigma \in \mathcal{C}_\mu} \text{sgn}(\sigma)\sigma$$
Define $\kappa_{i,j} : V^\otimes k \to V^\otimes (k-2)$ by $\kappa_{i,j}(v_i \otimes \cdots \otimes v_k) = f(v_i, v_j)(v_1 \otimes \cdots \otimes \hat{v_i} \otimes \cdots \otimes \hat{v_j} \otimes \cdots \otimes v_k)$ for $1 \leq i < j \leq k$ and set

$$K = \bigcap_{i,j} \ker(\kappa_{i,j}).$$

$S_k$ acts on $V^\otimes k$ by place permutation, specifically:

$$\sigma(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}.$$ 

This action commutes with the diagonal action of $G$ on $V^\otimes k$.

Given $v \in V^\otimes k$, we define one additional element $r^\mu_k$ of the group algebra $CS_k$ as follows: Let $R^\mu_\mu = \{ \sigma \in R^\mu \mid \sigma(v) = v \}$ and let $\{ s_i \}$ be a left transversal for $R^\mu_\mu$ in $R^\mu$. Define $r^\mu_k = \sum s_i$. Notice that $r^\mu_k(v) = |R^\mu_\mu| r^\mu_k(v)$.

By [2, Theorem 19.22], $W(\lambda) = c_\mu r^\mu_\mu(V^\otimes k) \cap K$ is the Weyl module for $G$ with highest weight $\lambda$. Since $V$ is a complex vector space, $c_\mu r^\mu_\mu(V^\otimes k) = \langle c_\mu r^\mu_\mu(v) \mid v \in V^\otimes k \rangle$.

Define $V_Z = Z[\delta]$ and let $V = V_Z \otimes F$. Then $\tilde{f}(\ , ) = f(\ , ) \otimes 1_F$ is a non-degenerate orthogonal form on $V$. Without loss of generality, we may assume that $K = \Omega(V)$. Moreover if $\tilde{e}_i = e_i \otimes 1_F$, $\tilde{f}_i = f_i \otimes 1_F$ and $\tilde{x} = x \otimes 1_F$, then

$$\mathcal{B} = \begin{cases} 
\{ \tilde{e}_i, \tilde{f}_i \mid 1 \leq i \leq \ell \} & \text{if } m \text{ is even} \\
\{ \tilde{e}_i, \tilde{f}_i, \tilde{x} \mid 1 \leq i \leq \ell \} & \text{if } m \text{ is odd}
\end{cases}$$

is a standard basis for $V$ with respect to $\mathcal{B}(\ , )$. We identify $r^\mu_\mu$ and $c_\mu$ with the elements $r^\mu_\mu \otimes 1_F$ and $c_\mu \otimes 1_F$ of $FS_k$.

Suppose that $L \subset \text{End}(V)$ is the adjoint module for $G$ so that $L$ is a complex Lie algebra of type $D_\ell$ or $B_\ell$. Let $\Delta = \{ r_1, \ldots, r_\ell \}$ be the set of simple roots corresponding to the torus $T$ and let $\Phi$ be the root system generated by $\Delta$. Set $\Delta_0 = \{ r_1, \ldots, r_{\ell-1} \}$ and let $\Phi_0 \subset \Phi$ be the subset generated by $\Delta_0$. Using the setup of [1, §11.2], $\{ \epsilon_r, h_r \mid r \in \Phi, 1 \leq i \leq \ell \}$ is a Chevalley basis for $L$ and $\{ \epsilon_r, h_r \mid r \in \Phi_0, 1 \leq i \leq \ell - 1 \}$ is a Chevalley basis for $L_0 \subset L$ where $L_0$ is a Lie algebra of type $A_{\ell-1}$. Let $G_0 < N_G(V_e \oplus V_f)$ such that $G_0 \cong SL_{\ell}(C)$. Then, by [1, Theorem 11.3.2], $G = \langle \exp(\zeta_r) \mid r \in \Phi, \zeta \in C \rangle$ and $G_0 = \langle \exp(\zeta_r) \mid r \in \Phi_0, \zeta \in C \rangle$. Note that neither $G$ nor $G_0$ is the adjoint group for $L$ or $L_0$, respectively. We may consider $V_e$ to be the natural module for $G_0$. Under this identification, $V_f$ is the dual of $V_e$.

Assume that $U(L)$ is the universal enveloping algebra of $L$. From [3, §26], Kostant’s $Z$-form $U_Z(L)$ is the $Z$-span of $\{ \epsilon^*_m/m! \mid r \in \Phi, m \in Z^+ \}$. Given any vector $v$ of weight $\lambda$ in $W(\lambda)$, $U_Z(L)v \otimes_Z F = W(\lambda)$ where $W(\lambda)$ is the Weyl module for $K$ with highest weight $\lambda$. By the previous remarks, $U_Z(L_0) \subset U_Z(L)$, which implies that $U_Z(L_0)v \otimes_Z F \subset W(\lambda)$.

Define $v_{\mu} = \otimes_{j=1}^{\mu_i} e_i$ and $v_{\mu} = \otimes_{i=1}^{d} v_{\mu_i}$.
Lemma 2.1. We have:

1. $c_\mu(v_\mu)$ is a vector of weight $\lambda$ in $W(\lambda)$;
2. $\mathcal{U}_\mathbb{Z}(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V^\otimes k) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k}$.

Proof. First note that $\mathcal{R}_\mu \subset \mathcal{R}_\mu$ so that $r_\mu v_\mu = v_\mu$ and that $c_\mu(v_\mu) \neq 0$. This implies that $c_\mu(v_\mu) \in c_\mu r_\mu(V^\otimes k)$. It is clear that $c_\mu(v_\mu) \in \mathcal{K}$ so we have $c_\mu(v_\mu) \in W(\lambda)$. Let $t \in T$ and write $t = \text{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1})$ or $t = \text{diag}(t_1, \ldots, t_\ell, t_1^{-1}, \ldots, t_\ell^{-1}, t')$ depending on the parity of $m$. Then

$$tv = c_\mu(tc_\mu(v_\mu)) = c_\mu \left( \bigotimes_{i=1}^d t_i^{\mu_i} v_{\mu_i} \right) = \left( \prod_{i=1}^d t_i^{\mu_i} \right) c_\mu(v_\mu).$$

From the definition of $\mu$ it follows that $c_\mu(v_\mu)$ is a vector of weight $\lambda$ and so (1) follows. With the identification of $V_\epsilon$ with the natural module of $G_0$, we see by [2, Theorem 15.15] that $c_\mu r_\mu(V^\otimes k)$ is the Weyl module for $G_0$ corresponding to the partition $\mu$ of $k$ via the Schur functor. The argument above restricted to $t \in T \cap G_0$ shows that $c_\mu(v_\mu)$ is a highest weight vector in $c_\mu r_\mu(V^\otimes k)$. In particular $\mathcal{U}(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V^\otimes k)$. Using the proof of [4, Theorem 8.3.1], we have

$$\mathcal{U}_\mathbb{Z}(L_0)c_\mu(v_\mu) = c_\mu r_\mu(V^\otimes k) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k}$$

which completes our proof. \hfill \Box

Lemma 2.2. Suppose $\varpi = \varpi_{i_1} \otimes \cdots \otimes \varpi_{i_k}$ where \{\varpi_i\} is a collection of mutually orthogonal, linearly independent singular vectors. Then:

1. If $\text{sgn}(\sigma_c)\sigma_c r_\mu(\varpi) \neq -\varpi$ for all $\sigma_c \neq 1 \in \mathcal{C}_\mu$, $\sigma_r \in \mathcal{R}_\mu$, then $c_\mu r_\mu(\varpi) \neq 0$;
2. $c_\mu r_\mu(\varpi) \in \mathcal{W}(\lambda)$.

Proof. Since $\varpi$ is a summand of $c_\mu r_\mu(\varpi)$ and all other summands of $c_\mu r_\mu(\varpi)$ have the form $\text{sgn}(\sigma_c)\sigma_c r_\mu(\varpi)$, part (1) must hold. There is $g \in \overline{K}$ such that $g(\varpi_{i_j}) = \alpha_{i_j} \varpi_{i_j}$ such that $\alpha_{i_j} \neq 0$ for all $1 \leq i \leq k$. If $w = e_{i_1} \otimes \cdots \otimes e_{i_k}$, then $r_\mu = r_\mu^w$. As

$$c_\mu r_\mu(w) \in c_\mu r_\mu(V^\otimes k) \cap \mathbb{Z}[e_1, \ldots, e_d]^{\otimes k},$$

Lemma 2.1 implies that $c_\mu r_\mu(w) \in \mathcal{U}_\mathbb{Z}(L)v$. Writing $\varpi = \alpha_{i_1} \varpi_{i_1} \otimes \cdots \otimes \alpha_{i_k} \varpi_{i_k}$, we then have

$$c_\mu r_\mu^w(\varpi) \in \mathcal{U}_\mathbb{Z}(L)v \otimes_{\mathbb{Z}} F = \mathcal{W}(\lambda).$$

Finally, as $\mathcal{W}(\lambda)$ is a $\mathbb{F}\overline{K}$-module, $g^{-1}c_\mu r_\mu^w(\varpi) = c_\mu r_\mu^w(\varpi) \in \mathcal{W}(\lambda)$. \hfill \Box

Though $\mathcal{W}(\lambda)$ is an irreducible module for $G$, $\mathcal{W}(\lambda)$ may not be an irreducible module for $\overline{K}$; however, it does possess a unique maximal submodule by [6, Lemma 80] which we denote by $\text{Rad}(\mathcal{W}(\lambda))$. Moreover, $M \cong \overline{\mathcal{W}(\lambda)}/\text{Rad}(\mathcal{W}(\lambda))$. 

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We now discuss the orthogonal forms on $V^\otimes k$ and $W(\lambda)$. Suppose $v, w \in V^\otimes k$ where $v = v_1 \otimes \cdots \otimes v_k$ and $w = w_1 \otimes \cdots \otimes w_k$. We define $f^k(\ ,\ )$ by

$$f^k(v, w) = \prod_{i=1}^{k} f(v_i, w_i).$$

$f^k(\ ,\ )$ is a non-degenerate, $G$-invariant orthogonal form on $V^\otimes k$. This form is also invariant under the action of $S_k$. Note that

$$f^k[c_\mu(v), c_\mu(w)] = \sum_{\sigma \in C_\mu} \text{sgn}(\sigma) f^k[\sigma(v), c_\mu(w)]$$

$$= \sum_{\sigma \in C_\mu} \text{sgn}(\sigma) f^k[v, \sigma^{-1} c_\mu(w)]$$

$$= \sum_{\sigma \in C_\mu} f^k[v, c_\mu(w)]$$

$$= |C_\mu| f^k[v, c_\mu(w)].$$

We define $f^k_\mu(\ ,\ )$ on $c_\mu(V^\otimes k)$ by

$$f^k_\mu[c_\mu(v), c_\mu(w)] = f^k[v, c_\mu(w)].$$

By a similar argument as above, we see that $f^k[v, c_\mu(w)] = f^k[w, c_\mu(v)]$, so this form is symmetric. Since $f^k(\ ,\ )$ is bilinear and $G$-invariant, $f^k_\mu(\ ,\ )$ is also bilinear and $G$-invariant. Therefore $f^k_\mu(\ ,\ )$ is a $G$-invariant orthogonal form on $W(\lambda) \subset c_\mu(V^\otimes k)$. As before, $f^k(\ ,\ ) = f^k(\ ,\ ) \otimes 1_F$ is a $K$-invariant orthogonal form on $V^\otimes k$ and $f^k_\mu(\ ,\ ) = f^k_\mu(\ ,\ ) \otimes 1_F$ is a $K$-invariant orthogonal form on $W(\lambda)$. This form is possibly degenerate. We denote the radical of this form as $\overline{W}(\lambda)^\perp$. The following lemma is generally known, although we present a proof:

**Lemma 2.3.** $\text{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$.

**Proof.** Define $\overline{\nu}_{-\mu} = \bigotimes_{j=1}^{\mu} j_i$ and $\overline{\mu} = \bigotimes_{i=1}^{d} \overline{\nu}_{-\mu}$. Noting that $r_{\overline{\mu}}^{\overline{\nu}_{-\mu}} = 1$, $c_\mu(\nu_{-\mu}) \neq 0 \in W(\lambda)$ by Lemma 2.2. A similar argument as in the proof of Lemma 2.1 shows that $c_\mu(\nu_{-\mu})$ is a vector of weight $-\lambda$. Hypothesis 1.1 implies that $d < \ell$. In particular, there is an element $\omega_0$ of the Weyl group of $K$ such that $\omega_0[c_\mu(\nu_{-\mu})] = c_\mu(\nu_{\mu})$. This means that $M = M(\lambda)$ must be self-dual. Clearly we have that $\overline{W}(\lambda)^\perp \subset \text{Rad}(\overline{W}(\lambda))$ and that $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ is non-degenerate, so this latter module is also self-dual. Since $M$ is self-dual and is a homomorphic image of $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$, $\overline{W}(\lambda)/\overline{W}(\lambda)^\perp$ must possess a submodule isomorphic to $M$. Since $M \cong \overline{W}(\lambda)/\text{Rad}(\overline{W}(\lambda))$ and $\overline{W}(\lambda)/\text{Rad}(\overline{W}(\lambda))$ does not possess a constituent which is isomorphic to $M$, we must have $\text{Rad}(\overline{W}(\lambda)) = \overline{W}(\lambda)^\perp$ and our result follows. \qed
Lemma 2.4. Let \( \{ \nu_i, \omega_i \mid 1 \leq i \leq k \} \) be a hyperbolic basis for some \( 2k \)-dimensional subspace of \( \overline{V} \). Set \( \nu = \nu_1 \otimes \cdots \otimes \nu_k \) and \( \omega = \omega_1 \otimes \cdots \otimes \omega_k \).

Then:

1. \( c_{\mu} r_{\mu}(\nu) \neq 0, c_{\mu} r_{\mu}(\omega) \neq 0; \)
2. \( c_{\mu} r_{\mu}(\nu), c_{\mu} r_{\mu}(\omega) \in \overline{W}(\lambda); \)
3. \( f_{\mu} c_{\mu} r_{\mu}(\nu), c_{\mu} r_{\mu}(\omega) \neq 0. \)

Proof. Parts (1) and (2) follow from Lemma 2.2 since \( r_{\nu} \mu = r_{\omega} \mu = r_{\mu} \) and the \( \nu_i \) are distinct, similarly for \( \omega_i \). If \( \sigma_1, \sigma_2 \in S_k \), then

\[
\prod_{i=1}^{k} f_{i} \left[ \sigma_1(\nu), \sigma_2(\omega) \right] = \prod_{i=1}^{k} \overline{f}_{\sigma_1^{-1}(i), \sigma_2^{-1}(i)}(\nu, \omega) = \begin{cases} 1 & \text{if } \sigma_1 = \sigma_2 \\ 0 & \text{otherwise}. \end{cases}
\]

Recall that \( \mathcal{R}_{\mu} \cap \mathcal{C}_{\mu} = 1 \). Then we have

\[
f_{\mu} c_{\mu} r_{\mu}(\nu), c_{\mu} r_{\mu}(\omega) = \prod_{\sigma \in \mathcal{R}_{\mu}} \overline{f}_{\sigma(\nu), \sigma(\omega)}(\nu, \omega) = \prod_{\sigma \in \mathcal{R}_{\mu}} \overline{f}_{\sigma(\nu), \sigma(\omega)}(\nu, \omega) = |\mathcal{R}_{\mu}|.
\]

Part (3) then follows as \( |\mathcal{R}_{\mu}| = \prod_{i=1}^{d} \mu_i! \) and \( \mu_i < \text{char}(F_q) \) for all \( i \). \( \square \)

Lemma 2.5. \( M \) possesses a vector of weight \( \lambda_k \).

Proof. Let \( \{ \nu_i, \omega_i \mid 1 \leq i \leq k \} \) be a subset of our standard basis \( \overline{B} \) for \( \overline{V} \). By part (2) of Lemma 2.4, \( c_{\mu} r_{\mu}(\nu_1 \otimes \cdots \otimes \nu_k) \in \overline{W}(\lambda) \). An argument similar to that used in Lemma 2.1 shows that \( c_{\mu} r_{\mu}(\nu_1 \otimes \cdots \otimes \nu_k) \) is a vector of weight \( \lambda_k \). Hence \( \lambda_k \) is a subdominant weight of \( \lambda \). Condition (3) of Hypothesis 1.1 insures that \( \lambda \) is \( p \)-restricted. Therefore using the results of \([5]\), \( M \) possesses a vector of weight \( \lambda_k \). \( \square \)

3. Elementary abelian 3-subgroup \( E_k \).

Assume that \( k \leq n/3 - 2 \) and recall that \( F^*(H) \) possesses a subgroup \( H_0 \) isomorphic to \( S_{n-2} \). Let

\[
E_k \cong \langle (123), (456), \ldots, (3k - 2, 3k - 1, 3k) \rangle < A_n
\]

be a subgroup of \( H_0 \) generated by commuting 3-cycles in \( F^*(H) \) so that \( E_k \) is an elementary abelian 3-group of rank \( k \). Then

\[
N_k = N_{H_0}(E_k) \cong S_3 \wr S_k \times S_{n-3k-2}
\]

\[
C_k = C_{H_0}(E_k) \cong E_k \times S_{n-3k-2}
\]
and let $H_k < C_k$ so that $H_k \cong S_{n-3k-2}$. Note that $C_{N_k}(H_k) \cong S_3 \wr S_k$ and this subgroup controls fusion in $E_k$. Let $\sigma \neq 1 \in E_k$ and assume that $\sigma$ is the product of $k_1$ disjoint 3-cycles. Then $C_{N_k}(\sigma) \cong \mathbb{Z}_3 \wr S_{k_1} \times S_3 \wr S_{k-k_1} \times S_{n-3k-2}$ which implies $|\sigma^N| = 2^{k_1}(k_1)$.

Let $\varphi \in E_k^* = \text{Hom}(E_k, F^*)$. The group $N_k$ acts on this group by $\varphi^g : \sigma \mapsto \varphi(g^{-1}\sigma g)$ for $g \in N_k$, $\sigma \in E_k$. We abuse notation slightly and define $\varphi^{-1}$ by $\varphi^{-1} : \sigma \mapsto \varphi(\sigma^{-1})$ for all $\sigma \in E_k$. Recall that $\text{In}_{N_k}(\varphi) = \{g \in N_k | \varphi^g = \varphi\}$ is the inertia group of $\varphi$ in $N_k$ and note that $H_k \in \text{In}_{N_k}(\varphi)$.

If $\varphi \in E_k^*$ is non-trivial, then the previous remarks concerning the action of $N_k$ on $E_k$ imply that $[N_k : \text{In}_{N_k}(\varphi)] = 2^{k_1}(k_1)$ for some $k_1$, $1 \leq k_1 \leq k$ and that $\varphi^{-1} \in \varphi^{N_k}$. Since $(k_1) \geq k$ unless $k = k_1$, in which case $2^{k_1} \geq 2k$, we have $[N_k : \text{In}_{N_k}(\varphi)] \geq 2k$.

4. Decomposition of $\nabla \downarrow E_k$ and $C_k$-invariant subspace of $\nabla(\lambda)$.

We continue to assume that $k \leq n/3 - 2$ and we now consider the restriction of $\nabla$ to $E_k$. Since $\text{char}(F) \neq 3$, we have $\nabla \downarrow E_k \cong \bigoplus_{\varphi \in E_k^*} \nabla_\varphi$ where $\nabla_\varphi$ is the homogeneous component of $\varphi$. Let $\bar{\varphi}_1 \in \nabla_{\varphi_1}$ and $\bar{\varphi}_2 \in \nabla_{\varphi_2}$. Then $\varphi(g\bar{\varphi}_1, g\bar{\varphi}_2) = \varphi(g\varphi_{12})(\bar{\varphi}_1, \bar{\varphi}_2)$ for all $g \in E_k$. If $\varphi_1^{-1} \neq \varphi_2$ then $(\bar{\varphi}_1, \bar{\varphi}_2) = 0$ which implies $\nabla_{\varphi_1} \perp \nabla_{\varphi_2}$ when $\varphi_1^{-1} \neq \varphi_2$. Since $\nabla$ is non-degenerate, $\dim(\nabla_{\varphi_i}) = \dim(\nabla) - \dim(\nabla_{\varphi_i})$ and it follows that $\nabla_{\varphi} \oplus \nabla_{\varphi^{-1}}$ must be non-degenerate and therefore possesses a hyperbolic basis.

Pick $\varphi \neq 1$ so that $\nabla_{\varphi} \neq 0$. Since $g\nabla_{\varphi} = \nabla_{\varphi^g}$ for $g \in N_k$, we may consider $\nabla_{\varphi}$ to be an $F\text{In}_{N_k}(\varphi)$-module. Let $E_{k-1}^*$ be a maximal subgroup of $E_k^*$ which does not contain $\varphi$. Define $O_+ = \varphi E_{k-1}^* \cap \varphi^{N_k}$ and $O_- = \varphi^{-1} E_{k-1}^* \cap \varphi^{N_k}$ so that $O_+ \cup O_- = \varphi N_k$ and $|O_+| = |O_-| \geq k$. Moreover $\varphi_i \in O_+$ if and only if $\varphi_i^{-1} \in O_-$. We assume that $O_+ = \{\varphi_i\}$ and that $O_- = \{\varphi_i^{-1}\}$. Then $\left( \bigoplus_{\varphi_i \in O_+} \nabla_{\varphi_i} \right) \oplus \left( \bigoplus_{\varphi_i^{-1} \in O_-} \nabla_{\varphi_i^{-1}} \right)$ is an $F\text{In}_{N_k}$-submodule of $\nabla_{\varphi_i}$ if $\varphi_i \in \varphi^{N_k}$ then, as $C_{N_k}(H_k)$ also controls fusion in $E_k^*$, there is a $g \in C_{N_k}(H_k)$ such that $g\nabla_{\varphi_i} = \nabla_{\varphi'_{i}}$. In particular $\nabla_{\varphi} \cong \nabla_{\varphi'}$ as $F H_k$-modules. Define $D = \dim(\nabla_{\varphi})$ so that $D = \dim(\nabla_{\varphi_i})$ for all $i$.

Given the above decomposition, we form the following:

$$\nabla_+ = \bigotimes_{i=1}^{k} \nabla_{\varphi_i} \quad \text{and} \quad \nabla_- = \bigotimes_{i=1}^{k} \nabla_{\varphi_i^{-1}}.$$

Recall that $D = \dim(\nabla_{\varphi_i})$ and assume that $\{\bar{\varphi}_{ij}, \bar{\varphi}_{ij}^{-1} | 1 \leq j \leq D\}$ is a hyperbolic basis for $\nabla_{\varphi_i} \oplus \nabla_{\varphi_i^{-1}}$. Define $\varphi^{j_1 \ldots j_k} = \bigotimes_{i=1}^{k} \varphi_{ij_i}$, and $\varphi^{j_1 \ldots j_k} = \bigotimes_{i=1}^{k} \bar{\varphi}_{ij_i}$. Then $\{\varphi^{j_1 \ldots j_k}, \bar{\varphi}^{j_1 \ldots j_k} | 1 \leq j \leq D\}$ forms a hyperbolic basis for $\nabla_+ \oplus \nabla_-$. If $\sigma \in S_k$, then $\sigma(\varphi^{j_1 \ldots j_k}) = \varphi^{j_1 \ldots j_k}$ if and only if $\sigma = 1$.
since the $V_{\varphi_i}$ are distinct. Moreover, $r_{\mu}^{\pi_1, \ldots, j_k} = r_{\mu}$ for all $\pi^{j_1, \ldots, j_k} \in \nabla_+$. Similarly for $\overline{\pi^{j_1, \ldots, j_k}} \in \nabla_-$. 

By parts (1) and (2) of Lemma 2.4, and as $\nabla_{\pm}$ are both totally singular, $c_{\mu} r_{\mu}(\nabla_{\pm}) \subset \overline{W}(\lambda)$. By part (3) of Lemma 2.4, $\overline{F}_{\mu}[c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k}), c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k})] \neq 0$. Whenever $(j_1, \ldots, j_k) \neq (j'_1, \ldots, j'_k)$, we have that $\overline{F}_{\mu}[c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k}), c_{\mu} r_{\mu}(\overline{\pi}^{j'_1, \ldots, j'_k})] = 0$. Therefore $\{ c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k}), c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k}) | 1 \leq j_i \leq D \}$ is a hyperbolic basis for 

$$c_{\mu} r_{\mu}(\nabla_{+}) \bigoplus c_{\mu} r_{\mu}(\nabla_{-}).$$

**Lemma 4.1.** We have:

1. $\nabla_{\pm} \cong c_{\mu} r_{\mu}(\nabla_{\pm})$ as $\mathbb{F}C_k$-modules;
2. If $k$ is even, then $C_k$ stabilizes a 1-dimensional subspace of $M$;
3. If $k$ is odd, then $C_k$ stabilizes a $D$-dimensional subspace of $M$.

**Proof.** Given the hyperbolic basis $\{\overline{\pi}^{j_1, \ldots, j_k}, \overline{\pi}^{j_1, \ldots, j_k} | 1 \leq j_i \leq D \}$ for $\nabla_{+} \oplus \nabla_{-}$, it is clear that the map $\overline{\pi}^{j_1, \ldots, j_k} \mapsto c_{\mu} r_{\mu}(\overline{\pi}^{j_1, \ldots, j_k})$ is a $C_k$-invariant bijection. Therefore $\nabla_{+} \cong c_{\mu} r_{\mu}(\nabla_{+})$ as $\mathbb{F}C_k$-modules. The case for $\nabla_{-}$ follows by a similar argument, proving part (1). Suppose that $k$ is even and recall that $\overline{\nabla}_{\varphi_i} \cong \overline{\varphi}_{\varphi_j}$ and $\overline{\nabla}_{\varphi_{i^{-1}}} \cong \overline{\varphi}_{\varphi_{j^{-1}}}$ as $\mathbb{F}H_k$-modules. As $H_k \cong S_{\overline{n}^{-2k-2}}$ and all irreducible $\mathbb{F}S_{\overline{n}^{-2k-2}}$ are self-dual, $H_k$ stabilizes a 1-dimensional subspace of $\overline{\nabla}_{\varphi_i} \otimes \overline{\nabla}_{\varphi_j}$. It follows by induction that $H_k$ stabilizes a 1-dimensional subspace of $\overline{\nabla}_{+}$. If $k$ is odd, then the same argument leads to a $D$-dimensional subspace being stabilized by $H_k$. As $E_k$ acts as scalars on $\nabla_{\pm}$, these spaces are, in fact, stabilized by $C_k$. Using part (1), $C_k$ stabilizes a subspace $\overline{W}_0$ of one of these dimensions in $\overline{W}(\lambda)$. Since $c_{\mu} r_{\mu}(\nabla_{+}) \bigoplus c_{\mu} r_{\mu}(\nabla_{-})$ possesses a hyperbolic basis, $\overline{W}_0 \cap \overline{W}(\lambda)^{\perp} = 0$. If we let 

$$M_0 = \left( \overline{W}_0 + \overline{W}(\lambda)^{\perp} \right) / \overline{W}(\lambda)^{\perp}$$

then Lemma 2.3 implies that $M_0 \subset \overline{W}(\lambda)/\overline{W}(\lambda)^{\perp} \cong M$, hence (2) and (3).

**5. Proof of Theorem 1.2.**

We are now in a position to prove Theorem 1.2:

Since $M$ possesses a vector $\overline{\pi}_{\lambda_k}$ of weight $\lambda_k$ by Lemma 2.5, we can produce a lower bound for $\dim(M)$ as follows: Let $\text{Weyl}(\overline{K})$ be the Weyl group of $\overline{K}$ and recall that $\ell$ is the Lie rank of $\overline{K}$. We compute $C_{\text{Weyl}(\overline{K})}(\lambda_k)$ using [3, §13.1], and compute $|\lambda_k^{\text{Weyl}(\overline{K})}|$, whence 

$$(1) \quad \dim(M) \geq |\lambda_k^{\text{Weyl}(\overline{K})}| = 2^{k} \binom{\ell}{k}.$$
Case 1. First suppose that \( k \geq n/3 - 1 \). We assume that \( \dim(V) \geq 2n^4 \), so \( \ell \geq n^4 \). Since \( \dim(M) \leq \sqrt{|H|} \leq \sqrt{n!} \), Eq. (1) implies that \( 2^k \binom{\ell}{k} \leq \sqrt{n!} \). Trivially, \( 2^{n^4/2} > \sqrt{n!} \) for all \( n \geq 1 \), so that \( k < n^{4}/2 < \ell/2 \). Using the fact that \( \binom{\ell}{k_1} < \binom{\ell}{k_2} \) if \( k_1 < k_2 < \ell/2 \), \( \binom{\ell}{k} \) will be minimal when \( k = n/3 - 1 \) and \( \ell = n^4 \). Note also that \( \binom{\ell}{k} = \prod_{i=1}^{k} \binom{\ell - i + 1}{k - i + 1} \geq \frac{(\ell - k + 1)^k}{k^k} \). We have:

\[
2^{n/3-1} \left( \frac{n^4}{n/3 - 1} \right) < \sqrt{n!},
\]

\[
2^{n/3-1} \frac{(n^4 - n/3 + 2)^{n/3-1}}{(n/3)^{n/3-1}} < (n^{1/2})^{n-1},
\]

\[
2^{n/3-1} (n^3 - 1)^{n/3-1} < n^{(n-1)/2},
\]

\[
n^{n-3} < n^{(n-1)/2},
\]

\[
n - 3 < (n - 1)/2,
\]

\[
n < 5.
\]

This contradicts our assumption that \( n \geq 10 \), so that \( \dim(V) \leq 2n^4 \) or \( k < n/3 - 1 \).

Case 2. We assume that \( k < n/3 - 1 \) and that \( k \) is odd. Lemma 4.1 and Frobenius reciprocity imply \( \dim(M) \leq D[H : C_k] \). Since \( D \geq \frac{\ell}{2k} \) and \([H : C_k] = \frac{n!}{2(k(3^k)(n-3k-2))!} \), we have \( \dim(M) \leq \frac{\ell}{2k} \frac{n!}{3^{k}(n-3k-2)!} \). Combining with (1) we get:

\[
2^k \binom{\ell}{k} \leq \frac{\ell}{2k} \frac{n!}{2(k(3^k)(n-3k-2))!},
\]

\[
2^k \binom{\ell - 1}{k - 1} \leq \frac{n^{3k+2}}{3^{k-1}},
\]

\[
2^k \frac{(\ell - k + 1)^{k-1}}{(k - 1)^{k-1}} \leq \frac{n^{3(k-1)n^5}}{3^{k-1}},
\]

\[
2^k \frac{\ell - k}{k - 1} \leq \frac{n^3}{3} n^{5/(k-1)}.
\]

Observing that \((k - 1)n^{5/(k-1)} < n^3\) when \( k \geq 3 \) and \( n \geq 10 \), we have

\[
2\ell < \frac{n^0 + 2n}{3} < n^6.
\]

Case 3. Finally we assume that \( k < n/3 - 1 \) and that \( k \) is even. Again Lemma 4.1 and Frobenius reciprocity imply \( \dim(M) \leq [H : C_k] \leq \frac{n!}{2(k(3^k)(n-3k-2))!} \). Combining with (1) we get:

\[
2^k \binom{\ell}{k} \leq \frac{n!}{3^{k}(n-3k-2)!},
\]
\[ 2^k \left( \frac{\ell - k + 1}{k} \right)^k < \frac{n^{3k+2}}{3^k} = \frac{n^{3k}}{3^k} n^2, \]
\[ 2 \frac{\ell - k}{k} < \frac{n^3}{3} n^{2/k}, \]
\[ 2 \ell < \frac{n^5 + 3n}{9}. \]

In all cases, \( 2\ell < n^6 \), which implies that \( \dim(\mathbb{V}) \leq n^6 \). This completes the proof of Theorem 1.2. \qed

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References

HOW LARGE ARE THE SPECTRAL GAPS?

ALEX IOSEVICH AND STEEN PEDERSEN

Let $D$ be a bounded domain in $\mathbb{R}^n$ whose boundary has a Minkowski dimension $\alpha < n$. Suppose that $E_\Lambda = \{e^{2\pi i x \cdot \lambda}\}_{\lambda \in \Lambda}$, $\Lambda$ an infinite discrete subset of $\mathbb{R}^n$, is a frame of exponentials for $L^2(D)$, with frame constants $A, B$, $A \leq B$. Then if

$$R \geq C \left( \frac{B |\partial D|_\alpha}{A |D|} \right)^{\frac{1}{n-\alpha}},$$

where $C$ depends only on the ambient dimension $n$ and $|\partial D|_\alpha$ denotes the Minkowski content, then every cube of sidelength $R$ contains at least one element of $\Lambda$. We give examples that illustrate the extent to which our estimates are sharp.

Let $D$ be a domain of finite Lebesgue measure in $\mathbb{R}^n$. Suppose that $L^2(D)$ has a frame of exponentials of the form $E_\Lambda = \{e^{2\pi i x \cdot \lambda}\}, \lambda \in \Lambda$, a discrete infinite subset of $\mathbb{R}^n$, with frame constants $A, B$, $A \leq B$, in the sense that

$$(*) \quad A||f||^2_{L^2(D)} \leq \sum_{\lambda} |\hat{f}(\lambda)|^2 \leq B||f||^2_{L^2(D)},$$

where $f : D \to \mathbb{C}$, and $\hat{f}$ denotes the standard Fourier transform. In this paper we will work with frames rather than exponential basis because $L^2$ of every bounded domain has frames, whereas exponential basis are generally hard to come by. (See [Fug].) The following quantities were introduced by Beurling. See [Br].

(1) $D_R^+ = \max_{x \in \mathbb{R}^n} \# \{\Lambda \cap Q_R(x)\}$, where $Q_R(x)$ is a cube of sidelength $2R$ centered at $x$, and let

(2) $D_R^- = \min_{x \in \mathbb{R}^n} \# \{\Lambda \cap Q_R(x)\}$.

It follows from results proved by Landau ([Lan], see also [GR]) that if $D$ is a bounded domain then

(3) $\limsup_{R \to \infty} \frac{D_R^-}{(2R)^n} \geq |D|.$

If the set $E_\Lambda$ is actually an orthogonal basis for $L^2(D)$ then the inequality (3) is actually an equality for both $D_R^-$ and $D_R^+$. These results show that,
asymptotically, a sufficiently large cube centered at any point contains the number of elements of Λ at least equal to its volume multiplied by the Lebesgue measure of D. In this paper we will show that if the Minkowski dimension, α, of the boundary ∂D is smaller than the ambient dimension n, then there exists

\[ R = C \left( \frac{B|\partial D|_α}{A|D|} \right)^{\frac{1}{n-α}}, \]

where C only depends on n and \( |\partial D|_α = \lim_{\epsilon \to 0} |e^{\alpha n}| \{ x : d(x, \partial D) < \epsilon \} \)
denotes the α-dimensional upper Minkowski content of ∂D, such that a cube of sidelength 2R centered at any point contains at least one element of Λ. Note that if ∂D is, say, piecewise smooth, then \( \alpha = n - 1 \) and \( R = C \frac{B|\partial D|}{A|D|} \).

A note on notation. The letter C below shall denote a generic constant which may change from line to line. We shall give more precise information about the constants when appropriate.

Our main result is the following.

**Theorem 1.** Let D denote a domain in \( \mathbb{R}^n \) with finite non-zero Lebesgue measure whose boundary ∂D has Minkowski dimension α < n in the sense that

\[ |\{ x \in \mathbb{R}^n : d(x, \partial D) < \epsilon \}| \leq C \epsilon^{n-α}. \]

Then there exists C depending only on n, such that if

\[ R \geq C \left( \frac{B|\partial D|_α}{A|D|} \right)^{\frac{1}{n-α}}, \]

then

\[ \Lambda \cap Q_R(\mu) \neq \emptyset \]

for every \( \mu \in \mathbb{R}^n \), and any set Λ such that \( E_\Lambda \) is an exponential frame for \( L^2(D) \), with frame constants \( A, B \), \( A \leq B \) where \( Q_R(\mu) \) denotes the cube of sidelength 2R centered at \( \mu \).

In other words, our result shows, at least if \( A = B \), that if D has a fixed volume, then the maximum gap between the elements of Λ is no larger than a fixed constant times the the Minkowski content of the boundary. Moreover, the rate of increase depends on the Minkowski dimension of ∂D. This idea is illustrated by the following simple example.

**Example 2.** Let \( D = [0, a_1] \times [0, a_2] \times \cdots \times [0, a_n] \), \( a_1 \geq a_2 \geq \cdots \geq a_n > 0 \), \( \Pi_{j=1}^n a_j = 1 \). We can take \( \Lambda = \Pi_{j=1}^n \frac{1}{a_j} \mathbb{Z} \). It is not hard to see that the largest cube that does not intersect Λ has sidelength \( 2R = \frac{1}{a_n} \). The measure of ∂D
is $2 \sum_{j=1}^{n} \frac{1}{a_{j}}$. It follows that

$$\frac{1}{4n} \leq \frac{R}{|\partial D|} \leq \frac{1}{4},$$

so $R$ grows linearly with $|\partial D|$.

**Example 3.** We now spice up the above example to illustrate the fractal phenomenon. Let $D$ be a domain constructed by taking a square $[0,1]^2$ and replacing the upper and lower segments by identical fractal curves of Minkowski dimension $1 < \alpha < 2$. It is not hard to see that $\Lambda$ may be taken to be $\mathbb{Z}^2$. (See [Fug].) We now blow up the domain by the factor of $t > 1$ (i.e., we apply the matrix $tI$, where $I$ is the identity matrix). Let $tD$ denote the image of $D$ under that mapping. The set $\Lambda$ must now be taken to be $(\frac{1}{t}\mathbb{Z})^2$, which tells us that $R$ in Theorem 1 should be $\approx \frac{1}{t}$. On the other hand, $|\partial tD|_\alpha \approx t\alpha$, and $|tD| = t^2$, so Theorem 1 gives us $R \approx (\frac{t^\alpha}{t^{2\alpha}})^{\frac{1}{2\alpha}} = \frac{1}{t}$.

The following example shows that if the Lebesgue measure $|D| = 0$ the conclusion of Theorem 1 no longer holds.

**Example 4.** Let $D \subset [0,1]$ denote the Cantor type set consisting of numbers that do not have 1 or 3 in their base 4 expansion. Let $m$ denote the unique probability measure supported on $D$ (see [Fal]) given by the equation

$$\int f(t)dm(t) = \frac{1}{2} \int f\left(\frac{t}{4}\right)dm(t) + \frac{1}{2} \int f\left(\frac{t+2}{4}\right)dm(t).$$

One can check that

$$\hat{m}(t) = e^{\pi i \frac{3}{2} t} \prod_{j=0}^{\infty} \cos\left(\frac{\pi t}{2 \cdot 4^n}\right).$$

If $\Lambda$ is the set of non-negative integers whose base 4 expansion does not contain 2 or 3, then $E_\Lambda$ is an orthonormal basis of $L^2(m)$. (See [JP].)

In particular this shows that the conclusion of Theorem 1 fails miserably in this case.

**Example 5.** In this example we shall see that there exist families of domains with piecewise smooth boundaries such that the volume of each domain is 1, the length of the boundary tends to infinity, but $R$, in the sense of Theorem 1, may always be taken to be $\frac{1}{2} + \epsilon$, for any $\epsilon > 0$.

Let $D_k$ denote the unit square in $\mathbb{R}^2$ where the upper and lower edges are replaced by a sawtooth function with $k$ teeth where the height of each tooth is $\frac{1}{k}$. The length $|\partial D_k|$ goes to infinity as $k \to \infty$. The set $\Lambda$ for each $D_k$ is $\mathbb{Z}^2$, so $R$, in the sense of Theorem 1, may always be taken to be $\frac{1}{2} + \epsilon$, for any $\epsilon > 0$. This says that the inequality (6) does not sharply describe the behavior of $R$ in this case. However, the proof of Theorem 1 (see the discussion at the end of the proof of Theorem 1 below) shows that in some
cases \( R \) may be taken to be \( C \frac{\text{diameter}(D)}{|D|} \), where \( C \) depends only on \( n \). We shall see that the example given in this paragraph falls into that category.

In all the previous examples we used frames which were actually orthogonal bases. However, this phenomenon persists in the cases when orthogonal exponential basis do not exist and we have to make do with frames.

**Example 6.** Let \( B_r \) denote the disc of radius \( r \) in \( \mathbb{R}^2 \) centered at the origin. It was shown in [JP2] that \( \Lambda = \frac{1}{2} \mathbb{Z}^2 \) is frame for \( L^2(B_r) \) with constants \( A = B = 4r^2 \). Note that we do not have orthogonal basis because, in particular, that would imply that \( A = B = |B_r| = \pi r^2 \). It is well known that \( B_r \) does not have orthogonal basis of exponentials. See [Fug].

It is clear that \( R \), in the sense of Theorem 1 must be taken to be greater than \( \frac{1}{4}r \), which is exactly what Theorem 1 predicts.

The key estimate (see Lemma 9 below) involved in the proof of Theorem 1 is

\[
\sum_{\lambda \notin Q_R(\mu)} |\hat{\chi}_D(\lambda - \mu)|^2 \leq C \frac{|\partial D|_n}{R^{n-\alpha}},
\]

for any \( \mu \in \mathbb{R}^n \), where \( C \) depends only on the dimension and on the frame constant \( B \).

This estimate is similar to the estimate that comes up in the theory of irregularities of distributions, (see [Mgr, p. 110]), namely that for any domain \( S \) whose boundary is a piecewise \( C^1 \) curve \( C \)

\[
\int_{|t| \geq R} |\hat{\chi}_S(t)|^2 dt \leq \frac{|C|}{2\pi^2 R}.
\]

In fact, our proof of the estimate (11) given in Lemma 9 below uses an idea from the proof of the estimate (12) given by Brandolini, Colzani, and Travaglini in [BCT].

The proof of Theorem 1 is based on the following sequence of lemmata.

**Lemma 7.** For any \( f \in L^2(D) \) define

\[
F_D f(\xi) = \int_D e^{-2\pi i x \cdot \xi} f(x) dx,
\]

and let \( \hat{f} \) denote the standard Fourier transform

\[
\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x) dx.
\]

Let \( t_h f(x) = f(x + h) \), and let \( \chi_D \) denote the characteristic function of \( D \). Then

\[
F_D t_h \chi_D(\lambda) = e^{2\pi i \lambda \cdot h} \hat{\chi}_{D \cap D+h}(\lambda),
\]

\[
F_D t_{-h} \chi_D(\lambda) = \hat{\chi}_{D \cap D+h}(\lambda),
\]
and

(17) \[ F_D \chi_D(\lambda) = \hat{\chi}_D(\lambda). \]

The proof is straightforward.

**Lemma 8.** Let \( D \) be as above. Then

(18) \[ \int_D |\chi_D(x + h) - \chi_D(x - h)|^2 dx \leq C|h|^{n-\alpha}, \]

and

(19) \[ \int_D |\chi_D(x) - \chi_D(x - h)|^2 dx \leq C|h|^{n-\alpha}, \]

with \( C \leq C'|\partial D| \alpha \), where \( C' \) depends only on \( n \).

**Remark.** We note again that even though the estimate \( C \leq C'|\partial D| \alpha \) is best possible over all \( h \)'s, for special choices of \( h \), the estimate is frequently much better. (See Example 5 above.)

To prove (19) note that the left hand side equals \(|\{D - (D + h)\} \cup \{(D + h) - D\}| \leq \{|x \in \mathbb{R}^n : d(x, \partial D) < h\}| \leq C|\partial D| |h|^{n-\alpha} \). The proof of (18) is similar.

The key lemma is the following. (See [BCT] for a similar argument.)

**Lemma 9.** Let \( D \) be as above and let \( \Lambda \) be such that \( E_\Lambda \) is a frame of \( L^2(D) \) with frame constants \( A \) and \( B \), \( A \leq B \). Then

(20) \[ \sum_{\{\lambda \in Q_{2k+1} - Q_{2k}\}} |\hat{\chi}_D(\lambda)|^2 \leq CB2^{-k(n-\alpha)}, \]

where \( Q_R = Q_R(0, \ldots, 0) \), and \( C \) as in Lemma 8.

To prove Lemma 9 chose \( N \) boxes \( A^j_k \) and \( N \) vectors \( h_j \) such that \( 2^{-k} \leq |h_j| \leq 2^{-k+1}, \bigcup_{j=1}^N A^j_k = Q_{2k+1} - Q_{2k}, \) and

(21) \[ |e^{2\pi i \lambda \cdot h_j} - 1| \geq 1, \quad \lambda \in A^j_k. \]

Clearly this can be done in any dimension \( n \), for a sufficiently large \( N = N(n) \).

Now, by triangle inequality

(22) \[ \left( \sum_{A_k^j} |\hat{\chi}_D(\lambda)|^2 \right)^{\frac{1}{2}} \]

\[ \leq \left( \sum_{A_k^j} |\hat{\chi}_D \cap D + h_j(\lambda)|^2 \right)^{\frac{1}{2}} + \left( \sum_{A_k^j} |\hat{\chi}_D(\lambda) - \hat{\chi}_D \cap D + h_j(\lambda)|^2 \right)^{\frac{1}{2}} \]
= I + II.

By Lemma 7, the frame property, and Lemma 8 we get

\[ I^2 \leq \sum_{\Lambda} |\hat{\chi}_D(\lambda) - \hat{\chi}_{D \cap D + h_j}(\lambda)|^2 \]

\[ = \sum_{\Lambda} |F_D \chi_D(\lambda) - F_D t_{-h_j} \chi_D(\lambda)|^2 \]

\[ \leq B \int_D |\chi_D(x) - \chi_D(x - h_j)|^2 dx \]

\[ \leq CB|h_j|^{n-\alpha} \leq CB2^{-k(n-\alpha)}. \]

On the other hand, by (21), Lemma 7, the frame property, and Lemma 8 we get

\[ I^2 \leq \sum_{A_k} |\hat{\chi}_{D \cap D + h_j}(\lambda)|^2 |e^{2\pi i \lambda h} - 1|^2 \]

\[ \leq \sum_{\Lambda} |\hat{\chi}_{D \cap D + h_j}(\lambda)|^2 |e^{2\pi i \lambda h} - 1|^2 \]

\[ = \sum_{\Lambda} |F_D t_{h_j} \chi_D(\lambda) - F_D t_{-h_j} \chi_D(\lambda)|^2 \]

\[ \leq B \int_D |\chi_D(x + h_j) - \chi_D(x - h_j)|^2 dx \]

\[ \leq CB|h_j|^{n-\alpha} \leq CB2^{-k(n-\alpha)}. \]

**Proof of Theorem 1.**

Since \(E_\Lambda\) is a frame for \(L^2(D)\) if and only if \(E_{\Lambda - \mu}\) is also a frame for \(L^2(D)\) (with the same frame constants) for any \(\mu \in \mathbb{R}^n\), and our estimates do not depend on the choice of \(\Lambda\), it is sufficient to consider the case \(\mu = (0, \ldots, 0)\).

By the frame property and Lemma 7 we get

\[ A|D| \leq \sum_{\Lambda} |F_D \chi_D(\lambda)|^2 = \sum_{\Lambda} |\hat{\chi}_D(\lambda)|^2 \]

\[ = \sum_{Q_R} |\hat{\chi}_D(\lambda)|^2 + \sum_{\lambda \notin Q_R} |\hat{\chi}_D(\lambda)|^2. \]

Using Lemma 9 we see that if \(R = 2^{k_0}\),

\[ \sum_{\lambda \notin Q_R} |\hat{\chi}_D(\lambda)|^2 = \sum_{k=k_0}^{\infty} \sum_{Q_{2k+1-Q_{2k}}} |\hat{\chi}_D(\lambda)|^2 \leq CB2^{-k_0(n-\alpha)} = \frac{BC}{R^{n-\alpha}}. \]
So by (25) and (26)

$$
\sum_{Q_R} |\hat{\chi}_D(\lambda)|^2 \geq A|D| - \frac{BC}{R^{n-\alpha}}
$$

which proves that if \( R > \left( \frac{BC}{A|D|} \right)^{\frac{1}{n-\alpha}} \), then

$$
\Lambda \cap Q_R \neq \emptyset.
$$

Moreover, the above proof shows that \( C \leq C' |\partial D|_\alpha \) where \( C' \) depends only on \( n \).

Remark. In the proof above the key estimate is \( |\{D \cap D - h\}| \leq C|h|^{n-\alpha} |\partial D|_\alpha \). While this is the best possible estimate uniform in \( h \), in the proof we are have a wide choice of \( h \)'s as long as \( |h| = 2^{-k} \) and the estimates (18), (19), and (21) are satisfied.

This observation can be used to handle the family of examples given by Example 5 above. For convenience we take \( \Lambda = (\frac{1}{2}, 0) + \mathbb{Z}^2 \). We can now take all \( h \)'s in the proof of Theorem 1 of the form \( h = (h_1, 0) \) and for this choice of \( h \)'s it is easy to check that \( |\{D_k \cap D_k - h\}| \leq C|h| |\text{diameter}(D_k)| \), where \( C \) is a uniform constant, since the “teeth” of \( D_k \)'s point in the \( y \)-direction. Since \( |\text{diameter}(D_k)| \) is uniformly bounded above and below, the lack of sharpness of Theorem 1 exposed in Example 5 is resolved for this family of examples.

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FINITE RELATIVE DETERMINATION AND RELATIVE STABILITY

León Kushner and Brasil Terra Leme

This paper is divided into three parts. The first part deals with the equivalence of finite determination on the right and finite relative determination (with respect to $S$) under some conditions on $S$. The second part deals with infinite determinacy (with respect to $S$, a germ of a closed set of $\mathbb{R}^n$). Both generalize results of P. Porto [P] for a big family of closed subsets $S$ of $\mathbb{R}^n$. The third part is a special case which is quite interesting, when $S$ coincides with the closure of its interior.

Introduction.

This paper continues the work done in [K]. In that paper there were proven results of finite relative determination for particular algebraic subsets of $\mathbb{R}^n$. Here we continue in this direction. In the first part we prove the equivalence of finite determination on the right and finite relative determination for a big family of algebraic subsets, generalizing the results of [P-L]. In the second part we continue with the concept of infinite determinacy and remarking the importance of quasihomogeneous polynomials. In the third part we generalize the results on relative stability in [P-L] and [P] for a broader family of closed subsets of $\mathbb{R}^n$, such as good semianalytic subsets.

Notation.

We shall work in $\mathcal{E}(n)$, the local algebra of $C^\infty$ function germs of $\mathbb{R}^n$ to $\mathbb{R}$ around the origin with maximal ideal $m(n)$. The powers of $m(n)$ will be denoted by $m(n)^k$ and $m(n)^\infty = \cap_{k=1}^\infty m(n)^k$. For $I = (i_1, \ldots, i_n)$ a multi-index of natural numbers and $x = (x_1, \ldots, x_n)$ we shall write $x^I = x_1^{i_1} \cdots x_n^{i_n}$ and $|I| = i_1 + \cdots + i_n$, also for a germ $f$, $\frac{\partial^{|I|} f}{\partial x^I} = \frac{\partial^{i_1} f}{\partial x_1^{i_1}} \cdots \frac{\partial^{i_n} f}{\partial x_n^{i_n}}$.

For $S$ a subset of $\mathbb{R}^n$, $\overline{0} \in S$, $\text{cl}(S)$ and $\text{int}(S)$ will denote the closure and the interior of $S$ respectively and $G_S$ will be the group of germs of diffeomorphisms $\phi$ at $\overline{0}$, such that $\phi(x) = x \forall x \in S$. Also $d(x, S)$ will denote the usual distance from the point $x$ to the subset $S$.

Finally if $f$ is a germ, $\partial f / \partial x_i$ will be the partial derivatives of $f$ and $\left\langle \frac{\partial f}{\partial x_i} \right\rangle$ will be the ideal of $\mathcal{E}(n)$ generated by them. Also for a germ $f$,
If \( j^k f(x) \) will be the Taylor expansion of \( f \) at the point \( x \) up to degree \( k \) and it is called the \( k \)-jet of \( f \) at \( x \). We will denote by \( J^k(n, 1) \) the \( \mathbb{R} \)-vector space of all polynomials in \( n \)-coordinates up to degree \( k \). In the case \( k = \infty \) we understand \( j^\infty f(x) \) the Taylor series of \( f \) at \( x \). Also \( J^\infty(n, 1) \), the set of all these jets will be identified with the formal power series ring \( \mathbb{R}[[x_1, \ldots, x_n]] \).

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1. Finite Relative Determination and Finite Determination on the Right.

**Definition 1.**

(a) Let \( S \) be a germ of a subset of \( \mathbb{R}^n \), \( f \) be a germ with \( f(\overline{0}) = 0 \) and let \( k \leq \infty \). We shall say that \( f \) is \( k \)-determined relative to \( G_S \) if whenever \( g \) is a germ such that \( j^k g(\overline{0}) = j^k f(\overline{0}) \) and \( g - f \) vanishes at \( S \), there exists \( h \) in \( G_S \) with \( g = f \circ h \). In the case \( S = \{ \overline{0} \} \) we say that \( f \) is finitely or infinitely determined on the right according to \( k \) is finite or not. In general if \( k \) is finite, then we say that \( f \) is finitely determined relative to \( G_S \).

(b) Let \( I \) be an ideal of \( \mathcal{E}(n) \) and \( S = z(I) \) the germ of the common zeroes of \( I \) (we suppose \( \overline{0} \in S \)). We denote by \( \text{rad} I \) the ideal of \( \mathcal{E}(n) \) consisting of all germs vanishing at \( S \), and we say that \( I \) is radical if \( I = \text{rad} I \).

**Lemma 2** (Artin-Rees). If \( I \) is an ideal of \( \mathbb{R}[[x]] = \mathbb{R}[[x_1, \ldots, x_n]] \), there exists \( k \) such that \( I \cap M^m = M^{m-k}(I \cap M^k) \) (\( \forall m \geq k \)).

We shall denote \( \mathcal{A}(I) \) the minimum \( k \) satisfying the equality of Lemma 2. Consider \( \mathbb{R}[[x]] \) the algebra of formal power series, \( M \) its maximal ideal, and the canonical projection \( \pi : \mathcal{E}(n) \rightarrow \mathbb{R}[[x]] \) which sends a germ to its Taylor infinite series and \( J \) an ideal of \( \mathcal{E}(n) \), we will get by Artin-Rees lemma for \( l = \mathcal{A}(\pi(J)) \), \( M^m \cap \pi(J) = M^{m-l}(M^l \cap \pi(J)), \forall m \geq l \). Hence applying \( \pi^{-1} \) to the above equality and intersecting each member with \( J \) we get

\[
(*) \quad J \cap m(n)^m = m(n)^{m-l}(J \cap m(n)^l) + J \cap m(n)^{\infty}, \quad (\forall m \geq l).
\]

We shall denote \( \mathcal{A}(J) \) the minimum \( l \) satisfying this equality, therefore \( \mathcal{A}(J) \leq \mathcal{A}(\pi(J)) \).

Since \( m(n)^k \supseteq \text{ker} \pi \), then \( \pi(J \cap m(n)^k) = \pi(J) \cap \pi(m(n)^k) = \pi(J) \cap M^k \). If we apply the epimorphism \( \pi \) to the equality \((*)\), we get \( M^m \cap \pi(J) = M^{m-l}(M^l \cap \pi(J)), \forall m \geq l \). Therefore \( \mathcal{A}(\pi(J)) \leq \mathcal{A}(J) \) and hence \( \mathcal{A}(J) = \mathcal{A}(\pi(J)) \).

In case \( I \) is a radical ideal of \( \mathcal{E}(n) \), we get in fact \( I \cap m(n)^m = m(n)^{m-k}(I \cap m(n)^k) \forall m \geq k \).
Theorem 3. Consider \( I \) a finitely generated ideal of \( \mathcal{E}(n) \). Then for any \( k < \infty \), \( I \cap m(n)^k \) is also finitely generated.

Proof. Consider \( g_1, \ldots, g_s \) generators of \( I \) and let \( f = \sum_{i=1}^{s} h_i g_i \). Then we have

\[
f = \sum_{i=1}^{s} h_i^{(k)} g_i + \sum_{i=1}^{s} h_i^{[k]} g_i,
\]

where \( h_i^{(k)} \) is the \((k-1)\)-jet of \( h_i \) and \( h_i^{[k]} = h_i - h_i^{(k)} \).

Hence as vector spaces \( I = V + m(n)^k I \), where \( V \) is the vector space generated by \( \{ x^l g_i \} \) with \( |l| \leq k - 1 \). Therefore \( I \cap m(n)^k = V \cap m(n)^k + m(n)^k I \). It is clear that a basis of the subspace \( V \cap m(n)^k \) of \( V \) and the generators of \( m(n)^k I \) generate \( I \cap m(n)^k \) as an ideal of \( \mathcal{E}(n) \).

\[\square\]

Theorem 4. Suppose \( I \) is a radical ideal of \( \mathcal{E}(n) \), \( I \cap m(n)^k \) a finitely generated ideal and \( I \cap m(n)^k \subseteq \text{Im}(n) \left\langle \frac{\partial f}{\partial x_i} \right\rangle \) with \( k \geq A(I) \). Then \( f \) is \( k \)-determined relative to \( G_S \), where \( S = z(I) \).

Theorem 5. Let \( f \) be a \( k \)-determined germ relative to \( G_S \), \( S = z(I) \) and \( I \) a radical finitely generated ideal. Then \( I \cap m(n)^{k+1} \subseteq I \left\langle \frac{\partial f}{\partial x_i} \right\rangle + m(n)^{k+2} \cap I \).

Joining Theorems 4 and 5 we get for \( I \) a finitely generated ideal, the following:

Theorem 6. Let \( f \) be a germ, \( I \) a finitely generated radical ideal, \( S = z(I) \), and \( k \geq A(I) \). Then \( f \) is finitely determined relative to \( G_S \) if and only if there exists a number \( l \) greater or equal than \( k \) such that \( m(n)^l \cap I \subseteq I \left\langle \frac{\partial f}{\partial x_i} \right\rangle \).

The proofs of the above theorems can be found in [K], Theorems 11 and 15.

We can change Theorem 4 in the following way.

Theorem 7. Let \( I \) be a radical ideal, \( k = A(I) \) and suppose that \( I \cap m(n)^k \) is finitely generated. Let \( l \) be a natural number such that \( m(n)^l I \subseteq m(n) I \left\langle \frac{\partial f}{\partial x_i} \right\rangle \). Then \( f \) is \((k + l - 1)\) determined relative to \( G_S \), where \( S = z(I) \).

Proof. Let \( g \) be a germ with \( g \equiv f \) on \( S \) and \( j^{k+l-1} g(0) = j^{k+l-1} f(0) \).

If we define the trivial homotopy \( F(x, t) = (1 - t) f(x) + t g(x) \) we get \( \frac{\partial F}{\partial t} = g - f \in m(n)^{k+l} \cap I \) and \( \frac{\partial F}{\partial x_i} = \frac{\partial F}{\partial x_i} + t \left( \frac{\partial f}{\partial x_i} - \frac{\partial g}{\partial x_i} \right) \).
Since $m(n)^{k+l} \cap I \subseteq m(n)^l I$ and $Im(n)^{p-1} \subseteq I \cap m(n)^p \forall p$, we get
\[
\left( m(n)^{k+l} \cap I \right) \mathcal{E} (n + 1) \subseteq m(n) I \left\langle \frac{\partial F}{\partial x_i} \right\rangle \mathcal{E} (n + 1) + m(n) \left( m(n)^{k+l} \cap I \right) \mathcal{E} (n + 1).
\]

By Nakayama’s lemma we arrive to the inclusion:
\[
\left( m(n)^{k+l} \cap I \right) \mathcal{E} (n + 1) \subseteq m(n) I \left\langle \frac{\partial F}{\partial x_i} \right\rangle \mathcal{E} (n + 1).
\]

Hence $\frac{\partial F}{\partial t} = \sum h_i (x, t) \frac{\partial F}{\partial x_i}$ with $h_i (x, t) \equiv 0$ for $x \in S$, $t$ near $t_0$ ($t_0$ fixed). We now proceed in the usual way. □

**Remark 1.**

(a) If $I = m(n)$ then $k = 1$ and we get that $m(n)^{l+1} \subseteq m(n)^{2} \left\langle \frac{\partial F}{\partial x_i} \right\rangle$ implies $f$ is $l-$determined on the right ([M]).

(b) If $I = \langle x_1, \ldots, x_s \rangle$ then $k = 1$ and we get that $m(n) I \subseteq m(n) I \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ implies $f$ is $l-$determined relative to $G_S$, $S = \{0\} \times \mathbb{R}^{n-s}$ ([P-L]).

**Corollary 8.** Let $f$ be a germ, $I$ a radical ideal, $k = \mathcal{A} (I)$ and $I \cap m(n)^k$ be finitely generated. Suppose that $m(n)^l \subseteq m(n) \left\langle \frac{\partial f}{\partial x_i} \right\rangle$. Then $f$ is $(k+l-1)$-determined relative to $G_S$. Hence finite determination on the right implies finite relative determination.

**Proof.** Since $m(n)^l \subseteq m(n) \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ then $m(n)^l I \subseteq m(n) I \left\langle \frac{\partial f}{\partial x_i} \right\rangle$. We now use Theorem 7. □

We are now interested in determining for which ideals $I$ we have the converse of Corollary 8. For this purpose we need the following:

**Theorem 9.** Let $A$ be a commutative ring, $I, J, K$ ideals of $A$ with $I = \langle g_1, \ldots, g_k \rangle$. Suppose that $a g_i = 0$ for all $i$ and $a \in J^k + K$ implies $a = 0$. Then if $IJ \subset IK$ hence $J^k \subseteq K$.

**Proof.** Let $m_1, \ldots, m_k$ be arbitrary elements of $J$, then $g_i m_i = \sum_{j=1}^k g_j d_{ij}$ \forall $i$ with $1 \leq i \leq k$ ($d_{ij} \in K$). In matricial notation we can write
\[
(*) \quad C \begin{pmatrix} g_1 \\ \vdots \\ g_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where} \quad C = (\delta_{ij} m_i - d_{ij}).
\]

If we multiply $(*)$ by the adjoint of $C$ we get $(\det C) g_i = 0 \forall i$, but $\det C = m_1 \cdots m_k + b$ with $b \in K$. Hence by hypothesis $\det C = 0$ and then $m_1 \cdots m_k \in K$, therefore $J^k \subseteq K$. □
Corollary 10. Let $A = E(n)$, $J = m(n)^1$, (or $J = m(n)\infty$), $K = \langle \frac{\partial f}{\partial x_i} \rangle$ and $I$ ideal with $I = \langle g_1, \ldots, g_k \rangle$. Suppose that $hg_i = 0$ for all $i$ and $h \in m(n)^{1k} + \langle \frac{\partial f}{\partial x_i} \rangle$ (or $h \in m(n)\infty + \langle \frac{\partial f}{\partial x_i} \rangle$) implies $h \equiv 0$. Then if $I m(n)^1 \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$ (or $I m(n)\infty \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$) hence $m(n)^{1k} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$ (or $m(n)\infty \subseteq \langle \frac{\partial f}{\partial x_i} \rangle$) and $f$ is $(1k + 1)$-determined on the right ($\infty$-determined on the right).

This result motivates us to find examples where $I$ is a finitely generated radical ideal satisfying the hypothesis of the above corollary.

Example 11. Let $I$ be a radical ideal generated by a non trivial analytic germ $g$. If $hg = 0$ then $h \equiv 0$ and we will have finite relative determination implies determination on the right ($hg \equiv 0 \implies h^{-1}(0) \cup g^{-1}(0) = \mathbb{R}^n$ but $g^{-1}(0)$ is a closed set with empty interior, therefore $h^{-1}(0) = \mathbb{R}^n$).

Example 12. Consider in $E(3)$ the ideal $I$ generated by $\{x_1x_2, x_1x_3, x_2x_3\}$. It is easy to see that $I$ is radical and $A(I) = 2$. Moreover if we denote $P_1 = x_1x_2$, $P_2 = x_1x_3$ and $P_3 = x_2x_3$, we get for $i \neq j$, $i \neq k$, $j \neq k$ that the closure of $z(P_i) \cap z(P_j) = z(P_k)$ is a plane and does not contain $z(I)$, which is the union of the three axes, hence it does not satisfy the hypothesis of Theorem 20 [K], but the conclusion is still true. We give a proof since it is important for the converse of Corollary 8.

Proposition 13. With the above notation, if $f$ is $m$-determined relative to $G_S$, where $S = z(I)$ are the coordinate axes, then $f$ is $(2m - 2)$-determined on the right.

Proof. By Theorem 15 ([K]) we know that $m(3)^{m+1} \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$ which in this case is equivalent to $Im(3)^{m-1} \subseteq I \langle \frac{\partial f}{\partial x_i} \rangle$. We shall show that $m(3)^{2m-1} \subseteq \langle \frac{\partial f}{\partial x_i} \rangle m(3)^2$ and hence $f$ is $(2m - 2)$-determined on the right.

Any mixed monomial of $m(3)^{2m-1}$ has a factor of $I$ times a monomial of degree $2m - 3$, hence for $m \geq 2$ it is contained in the Jacobian ideal. We now give the proof for $x_1^{2m-1}$, the other two are similar,

\[ x_1^{2m-1}(x_1x_2) = x_1x_2 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{2j} + x_2x_3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{3j}. \]

If we denote $\phi = 3 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j}$ we get that the zeroes of $\phi$ contain $\{x_3 = 0\}$ and the zeroes of $x_1\phi$ contain $\{x_3 = 0\} \cup \{x_1 = 0\}$, hence $x_1\phi \in I = I$. From (*) and the definition of $\phi$, $x_1^m = x_1 \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} h_{1j} + x_1\phi$, therefore $x_1^{2m-1} \in m(3)^2 \langle \frac{\partial f}{\partial x_i} \rangle$ and $f$ is $(2m - 2)$-determined. □
Remark 2. By Corollary 10, since $I = \hat{I} = \langle x_1 x_2, x_1 x_3, x_2 x_3 \rangle$ then $\text{Im}(3)^{m-1} \subseteq I \left( \frac{\partial f}{\partial z_i} \right)$ implies $m(3)^{m-1} \subseteq \langle \frac{\partial f}{\partial z_i} \rangle$ and $f$ is $(3m-2)$-determined on the right.

Definition 14. Let $f_1, \ldots, f_r$ be germs in $m(n)$. We say that they are linearly independent if their gradients denoted by $\nabla f_1, \ldots, \nabla f_r$ are linearly independent at the origin.

Lemma 15. Let $f_1, \ldots, f_r$ be linearly independent germs. Then the ideal $I$ generated by them is radical.

Proof. Let $H$ be the germ of the common zeroes of $I$ and $P_{r+1}, \ldots, P_n$ linear polynomials such that $\nabla f_1(\tilde{0}), \ldots, \nabla f_r(\tilde{0}), \nabla P_{r+1}(\tilde{0}), \ldots, \nabla P_n(\tilde{0})$ is a basis of $\mathbb{R}^n$. Thus $\phi = (f_1, \ldots, f_r, P_{r+1}, \ldots, P_n)$ is a germ of diffeomorphism. Let $f \in \hat{I}$ hence $f \equiv 0$ on $H$ if and only if $f \circ \phi^{-1} \equiv 0$ on $\{0\} \times \mathbb{R}^{n-1}$. By Hadamard’s lemma we get $f \circ \phi^{-1}(x_1, \ldots, x_n) = \sum f_i x_i$, therefore $f = \sum f_i g_i$, and hence $f$ belongs to the ideal $I$.

Lemma 16. Let $I_1, \ldots, I_r$ be radical ideals in $\mathcal{E}(n)$. Then their intersection is also a radical ideal.

Proof. In general $\text{rad}(\bigcap_{i=1}^r I_i) \subseteq \cap_{i=1}^r \text{rad} I_i$, hence we get

$$\cap_{i=1}^r I_i \subseteq \text{rad} \cap_{i=1}^r I_i \subseteq \cap_{i=1}^r \text{rad} I_i = \cap_{i=1}^r I_i.$$  

Therefore the equality $\cap_{i=1}^r I_i = \text{rad} \cap_{i=1}^r I_i$. □

Lemmas 15 and 16 generate a special collection of algebraic sets. They are called bouquets of subspaces.

Example 17. Consider $I \subseteq \mathcal{E}(3)$ the ideal generated by $x$ and $yz$, hence $z(I)$ is the union of the $y$-axis and $z$-axis, they are not in general position (in $\mathbb{R}^3$). By Lemma 16, $I$ is clearly a radical ideal since $I = I_1 \cap I_2$ where $I_1 = \langle x, y \rangle$ and $I_2 = \langle x, z \rangle$.

Definition 18. Let $I$ be a finitely generated ideal of $\mathcal{E}(n)$, $I = \langle g_1, \ldots, g_k \rangle$. We say that $I$ is integral if $S = z(I)$ is nowhere dense.

We now arrive at the main theorem of this section.

Theorem 19. Let $I$ be a finitely generated ideal of $\mathcal{E}(n)$ which is radical. Then if $f$ is finitely determined relative to $G_S$, $S = z(I)$, hence $f$ is finitely determined on the right.

Proof. Suppose $I = \langle g_1, \ldots, g_k \rangle$ and that $hg_i \equiv 0 \forall i$. Therefore $z(h) \cup z(g_i) = \mathbb{R}^n \forall i$ and hence $z(h) \cup z(I) = \mathbb{R}^n$. Since $I$ is an integral ideal, see $[\mathbb{R}]$, $z(h) = \mathbb{R}^n$ and hence $h \equiv 0$. On the other side there exists a natural number $p$ such that $m(n)^p I \subset \langle \frac{\partial f}{\partial z_i} \rangle I$. By Corollary 10, we get $m(n)^{pk} I \subset \langle \frac{\partial f}{\partial z_i} \rangle$ and therefore $f$ is $(pk + 1)$-determined on the right. □
Corollary 20. Consider $I_1, \ldots, I_r$ ideals each of them generated by linearly independent linear polynomials and $S$ the union of their common zeroes (bouquet of subspaces). Then a germ $f$ is finitely determined on the right if and only if $f$ is finitely determined relative to $G_S$.

We finish this section with an observation about homogeneous polynomials.

Proposition 21. Let $h_1, \ldots, h_k$ be homogeneous polynomials of degree $s_1, \ldots, s_k$ respectively and let $s$ be the maximum of these degrees. Hence if $I$ is the ideal generated by $h_1, \ldots, h_k$ we get $\mathcal{A}(I) \leq s$.

Proof. We have to show that $(I \cap m(n)^s)m(n)^r = I \cap m(n)^{s+r} \forall r \geq 0$. Let $f \in I \cap m(n)^{s+r}$, hence we have the following equalities.

$$(*) \quad f = h_1 g_1 + \ldots + h_k g_k$$

$$(**) \quad 0 = j^{s+r-1} f(0) = h_1 j^{s+s_1-1-s_1} g_1(0) + \ldots + h_k j^{s+s_k-1-s_k} g_k(0).$$

Subtracting $(**)$ from $(*)$ we get $f = h_1 \tilde{g}_1 + \cdots + h_k \tilde{g}_k$, where $\tilde{g}_i \in m(n)^{r+s-s_i}$.

Hence each $\tilde{g}_i$ is a sum of elements of the form $\frac{h_i^j}{h_i^j}$, with $h_i^j \in m(n)^r$ and $h_i^j$ is a homogeneous monomial of degree $s - s_i$.

Therefore $f$ is a sum of elements of the form $(h_i^j h_i^j) h_i^j$, with $(h_i^j h_i^j) \in I \cap m(n)^s$, so $f \in (I \cap m(n)^s)m(n)^r$. We have shown that $I \cap m(n)^{s+r} \subseteq (I \cap m(n)^s)m(n)^r \forall r \geq 0$. The other inclusion is obvious.

2. Infinite determinacy on germs of closed subsets of $\mathbb{R}^n$.

In this section we will assume that $S$ is a germ of a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of $S$.

Definition 22. Let $S \subseteq \mathbb{R}^n$ be a germ of a closed set such that $\tilde{0}$ is an accumulation point of $S$. We say that a germ $f$ in $\mathcal{E}(n)$ is $S$–infinitely determined if given a germ $g$ such that $j^\infty g(x) = j^\infty f(x) \forall x \in S$ there exists a germ of a diffeomorphism $\phi$ such that $g = f \circ \phi$.

We denote by $\mathcal{E}(S,n)$ the ideal of $\mathcal{E}(n)$ consisting of the germs $f$ such that $j^\infty f(x) = 0$ for all $x \in S$. If $f$ is a germ in this ideal, we can write $f = gh$ where $(g,h) \subseteq m(n)^\infty$ and $h(x) > 0$ for $x \neq 0$. Then $j^\infty g(x) = 0 \forall x \neq 0, x \in S$ and therefore $\mathcal{E}(S,n) \subseteq \mathcal{E}(S,n)m(n)^\infty$. We get in fact the equality.

Remark 3. If $f \in \mathcal{E}(S,n)$ then for all multi-index $I$, $\frac{\partial^{\mu(I)} f}{\partial x^I} \in \mathcal{E}(S,n)$. 
Definition 23. A germ $f$ is $S$-infinitesimally stable if $\mathcal{E}(S, n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S, n)$.

Theorem 24. Let $S$ be a germ of a closed subset of $\mathbb{R}^n$ such that the origin is an accumulation point of $S$. If $f$ is $S$-infinitesimally stable then $f$ is $S$-infinitely determined.

Proof. Let $g(x)$ be a germ such that $j^\infty g(x) = j^\infty f(x) \forall x \in S$. We define the homotopy $F(x, t) = tg(x) + (1 - t) f(x)$ . Consider the following $\mathcal{E}(n + 1)$-modules $N = \mathcal{E}(n + 1)\langle \frac{\partial f}{\partial x_i} \rangle$ and $K = \mathcal{E}(n + 1)\langle \frac{\partial F}{\partial x_i} \rangle$. If $h \in N$, we have $h(x, t) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(x) h_i(x, t) = \sum_{i=1}^{n} \frac{\partial E}{\partial x_i}(x, t) h_i(x, t) + t \sum_{i=1}^{n} \frac{\partial (f-g)}{\partial x_i}(x) h_i(x, t)$. Since $\frac{\partial (f-g)}{\partial x_i}(x) \in \mathcal{E}(S, n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S, n)$, we get $N \subseteq K + \mathcal{E}(S, n)N$, and by Nakayama’s lemma, $N \subseteq K$ and hence $\mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial F}{\partial x_i} \rangle \subset \mathcal{E}(n + 1) \subseteq \mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial F}{\partial x_i} \rangle \mathcal{E}(n + 1)$. Since $g - f \in \mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial F}{\partial x_i} \rangle \mathcal{E}(n + 1)$, hence $\frac{\partial F}{\partial x_i} \in \mathcal{E}(S \times \mathbb{R}, n + 1) \langle \frac{\partial F}{\partial x_i} \rangle \mathcal{E}(n + 1)$. We now proceed in the usual way. \(\Box\)

Proposition 25. If $f$ is a germ, finitely (infinitely) determined on the right, it is $S$-infinitesimally stable and therefore $S$-infinitely determined.

Proof. By our hypothesis we have

$$m(n)^\infty \subseteq \left\langle \frac{\partial f}{\partial x_i} \right\rangle \text{ and } \mathcal{E}(S, n) \subseteq \mathcal{E}(S, n)m(n)^\infty \subseteq \mathcal{E}(S, n) \left\langle \frac{\partial f}{\partial x_i} \right\rangle.$$

\(\Box\)

Example 26.

(a) Let $f$ be a germ and $k$ a natural number, denote by $I_k$ the ideal generated by $f^k$. Suppose $\mathcal{E}(S, n) \subseteq I_k$, $f^k \in \left\langle \frac{\partial f}{\partial x_i} \right\rangle$ and $j^\infty f(x) \neq 0 \forall x \in T$, where the closure of $T$ is $S$. If $h \in \mathcal{E}(S, n)$, $h = f^k g$ where $g \in \mathcal{E}(S, n)$. Therefore $\mathcal{E}(S, n) \subseteq \mathcal{E}(S, n) \left\langle f^k \right\rangle \subseteq \mathcal{E}(S, n) \left\langle \frac{\partial f}{\partial x_i} \right\rangle$, and thus $f$ is $S$-infinitely determined.

(b) In particular let $S = \{(x_1, \ldots, x_n)|x_1 \leq 0\}$. Then for the germs $f_1(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$, and $f_2(x_1, \ldots, x_n) = x_1$, we get $j^\infty f_1(x) \neq 0$ and $j^\infty f_2(x) \neq 0 \forall x \in \text{int } S$. Since $S = \text{cl}(\text{int } S)$ and $\mathcal{E}(S, n) \subseteq \left\langle f_i^k \right\rangle$ for $i = 1, 2$ (Proposition 5.4 of Chapter V, [T]), and clearly $f_i^k \in \left\langle \frac{\partial f}{\partial x_j} \right\rangle$ for $i = 1, 2$, we get that $f_1^k, f_2^k$ are $S$-infinitesimally determined. ([P-L]).

Definition 27.

(a) Let $S$ be a closed subset of $\mathbb{R}^n$ (containing the origin) and $f$ a germ with $f(\hat{0}) = 0$. We say that $f$ satisfies a Lojasiewicz inequality for $S$ if for any $K$, a germ of a compact set with $\hat{0} \in K$, there exist constants $c > 0$ and $\alpha \geq 0$ such that $|f(x)| \geq cd(x, S)^\alpha$ for all $x \in K$. 
(b) Let \( I \) be a finitely generated ideal of \( \mathcal{E}(n) \) and \( S \) the germ of its common zeroes. We say that \( I \) is a Lojasiewicz ideal if there exists \( f \) in \( I \) satisfying a Lojasiewicz inequality for \( S \).

(c) Let \( f \in m(n) \) and \( S \) a closed subset of \( \mathbb{R}^n \), we say that \( f \) satisfies a Jacobi-Lojasiewicz condition for \( S \) if \( |\nabla f| \) satisfies a Lojasiewicz inequality for \( S \).

**Remark 4.** If \( \{f_1, \ldots, f_s\} \) is a set of generators of a Lojasiewicz ideal \( I \), then \( \sum_{i=1}^s f_i^2, \sum_{i=1}^s |f_i| \) and \( \max \{f_1^2, \ldots, f_s^2\} \) also satisfy a Lojasiewicz inequality for \( S \).

**Definition 28.** Let \( (b_i) \) be a sequence of positive real numbers converging to zero. We say that a sequence of real numbers \( (a_i) \) is flat along \( (b_i) \) if given \( r > 0 \) there exists a natural number \( N = N(r) \) such that \( |a_i| \leq b_i^r \) for \( i \geq N \). Sequences of vectors, matrices, jets are flat along a sequence \( (b_i) \) if each entry is flat along \( (b_i) \). A sequence is flat along a sequence \( (x_i) \) of nonzero vectors in \( \mathbb{R}^n \) converging to the vector \( \vec{0} \) if it is flat along the sequence \( (|x_i|) \). In the case of \( \infty-jets \), we ask for a uniform \( N = N(r) \) for all entries. Here we are identifying \( \sum_{\alpha} a_{\alpha} \frac{(x-x_0)^{\alpha}}{\alpha!} \) with \( (a_{\alpha}) \).

**Remark 5.** We can change \( r > 0 \) for \( r = n, n \) a natural number since for \( n > r \), we get \( b_i^n \leq b_i^r (0 \leq b_i \leq 1) \).

We state an interesting equivalence.

**Lemma 29.** A germ \( g \) does not satisfy a Lojasiewicz inequality for a closed subset \( S \) if and only if there exists a sequence of vectors \( x_i \in \mathbb{R}^n - S \) converging to the vector \( \vec{0} \) such that \( (g(x_i)) \) is flat along \( (d(x_i,S)) \).

**Remark 6.** For a germ \( g \) not identically zero we can choose \( g(x_i) \neq 0 \forall i \).

**Definition 30.** Let \( S \) be a closed subset of \( \mathbb{R}^n \). Then \( M(S, n) \) is the set of maps \( \phi : \mathbb{R}^n - S \longrightarrow \mathbb{R} \) such that if \( K \) is a germ of a compact set and \( I \) a multi-index of natural numbers, there exist constants \( c > 0 \) and \( \alpha > 0 \) such that \( \left| \frac{\partial^{d+1} \phi}{\partial x^d} (x) \right| \leq c d(x, S)^{-\alpha} \) for all \( x \in K - S \).

We state the following proposition (Chapter IV, Proposition 4.2 of [T]).

**Proposition 31.** Let \( \phi \in M(S, n) \) and \( f \in \mathcal{E}(S, n) \). Then we can extend \( \phi f \) in a unique way to a germ in \( \mathcal{E}(S, n) \), denoted also by \( \phi f \).

**Theorem 32.** Let \( f \) be a germ, \( S \) a germ of a closed subset of \( \mathbb{R}^n \) such that \( \vec{0} \) is an accumulation point of \( S \). Suppose that \( f \) satisfies a Jacobi-Lojasiewicz condition for \( S \). Then \( f \) is \( S \)-infinitesimally stable and therefore \( S \)-infinitely determined.

**Proof.** Consider \( g = |\nabla f|^2 \), we shall show that \( \mathcal{E}(S, n) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{E}(S, n) \).

Let \( K \) be a germ of a compact subset and \( g_1 \) be a representative of \( g \); for
each \( I \) multi-index there exists \( C_I \) constant such that \( \left| \frac{\partial^{\|I\|}(\frac{1}{g_1})}{\partial x^I} \right| \leq \frac{C_I}{c^{\|I\|+1}d(x,S)\alpha^{\|I\|+1}} \) \( \forall x \in K \). Since \( g_1 \) satisfies a Lojasiewicz inequality for \( S \), there exist \( c > 0 \) and \( \alpha \geq 0 \) such that \( |g_1(x)| \geq cd(x,S)^\alpha \forall x \in K-S \) and therefore \( \left| \frac{\partial^{\|I\|}(\frac{1}{g_1})}{\partial x^I} \right| \leq \frac{C_I}{c^{\|I\|+1}d(x,S)\alpha^{\|I\|+1}} \) \( \forall x \in K-S \), hence \( \frac{1}{g_1} \in M(S,n) \). Now for \( h \in \mathcal{E}(S,n) \) and \( x \notin S \) we have \( h(x) = \frac{h(x)}{g_1(x)}g_1(x) \), extend \( \frac{h(x)}{g_1(x)} \) to a germ \( H \) in \( \mathcal{E}(S,n) \) and \( h = Hg_1 \) in \( \mathcal{E}(S,n) \left( \frac{\partial f}{\partial x^i} \right) \). Therefore we get \( \mathcal{E}(S,n) \subseteq \mathcal{E}(S,n) \left( \frac{\partial f}{\partial x^i} \right) \) and \( f \) is \( S \)-infinitesimally stable.

**Lemma 33** ([W, Lemma 3.3]). Suppose there exist a sequence \((w_i)\) in \( J^k(n,1), k \leq \infty \), a sequence \((x_i)\) in \( \mathbb{R}^n - \{0\} \) converging to the origin and a germ \( f \) such that \( q_i = w_i - j^k f(x_i) \) is flat along \( (x_i) \). Then there exists a germ \( g \) such that \( j^k g(x_i) = w_i \) holds for \((x_i)\) subsequence of \((x_i)\), and \( j^\infty g(0) = j^\infty f(0) \).

**Lemma 34.** Suppose there exist a sequence \((w_i)\) in \( J^k(n,1), k \leq \infty \), a sequence \((x_i)\) in \( \mathbb{R}^n - S \) converging to zero and a germ \( f \in \mathcal{E}(n) \) such that \( (q_i) = (w_i - j^k f(x_i)) \) is flat along \((d(x_i,S))\), where \( S \) is a closed subset of \( \mathbb{R}^n(0 \in S) \). Then there exists a germ \( g \in \mathcal{E}(n) \) such that \( j^\infty g(x) = j^\infty f(x) \forall x \in S \) and \( j^k g(x_i) = w_i \) holds for a subsequence of \((x_i)\).

**Proof.** If \( k \) is finite, then we transform \( q_i \) into an \( \infty \)-jet in such a way that all the terms of order greater than \( k \) of \( q_i \) are zero. Thus we will assume \( k = \infty \).

We define \( Q \), a Taylor field, by \( q_i \) at \( x_i \) and by the zero series on \( S \). We want to show that \( Q \) is a \( C^\infty \) Whitney field. It is enough to show (Proposition 1.5 of Chapter IV, [T]) for each \( m \) and each multi-index \( I \) with \( |I| \leq m \), that \( (R^m_y Q)^I(x) = o(|x-y|^{m-|I|}) \), where \( (R^m_y Q)^I(x) = Q^I(x) - \sum_{|L|\leq m-|I|}Q^{I+L}(y)(x-y)^L \).

If \( \{x, y\} \subseteq S \) then the proof is obvious. In the case \( \{x, y\} \subseteq \{x_i\} \cup \{0\} \) we proceed as in the proof of Lemma 3.3 of [W]. If \( \{x, y\} = \{x_j, s\}, s \in S \), we use the flatness of \( (q_i) \) along \((d(x_i,S))\) to obtain for each natural number \( l \) another \( N(l) \) such that \( |(R^m_s)^I(x_j)| = |q^I_{x_j}| \leq d(x_j, S)^l \leq d(x_j, s)^l \) and \( |(R^m_s)^I(s)| \leq \sum_{|L|\leq m-|I|}q^I_{x_j}^{I+L} \langle x-x_j \rangle^L \leq C d(x_j, s)^l \) for \( j \geq N(l) \), where \( C \) is a positive real number depending only on \( m \) and \( I \). Let \( l = m + 1 \).

Hence, using Whitney Extension Theorem (Theorem 3.1 of Chapter IV, [T]), there exists a smooth germ \( q \) such that \( j^\infty q(x) = 0 \forall x \in S \) and \( j^\infty q(x_i) = q_i \). If \( g = f + q \), we see that \( g \) has the desired properties.

**Theorem 35.** Let \( f \) be a germ, \( S \) a closed subset of \( \mathbb{R}^n \) and \( 0 \) an accumulation point of \( S \). Hence if \( f \) is \( S \)-infinitesimally determined, then \( f \) satisfies a Jacobi-Lojasiewicz condition for \( S \).
Proof. We shall prove the theorem by contradiction. Then there is a sequence \((x_j)\) in \(\mathbb{R}^n - S\) converging to the origin such that \((|\nabla f(x_j)|)\) is flat along \((d(x_j), S))\). Choose \((y_j)\) a sequence of regular values of \(f\) converging to zero and such that \((f(x_j) - y_j)\) is flat along \((d(x_j), S))\). It clearly follows that \((y_j, 0) - (f(x_j), \nabla f(x_j))\) is flat along \((d(x_j), S))\).

If we denote \(q_j = (y_j, 0) - (f(x_j), \nabla f(x_j))\) and setting \(k = 1\) in the previous lemma, there exists a germ \(g\) such that \(j^1 g(x_j) = (y_j, 0)\) and \(g - f \in \mathcal{E}(S, n)\). Now since \(f\) is \(S\)-infinitely determined, \(f\) and \(g\) must have the same critical and regular values, which is not the case, since the points \(y_j\) are regular values for \(f\) but critical values for \(g\). □

As a consequence of Theorems 24, 32 and 35 we get the main theorem of part II:

**Theorem 36.** Let \(f \in \mathcal{E}(n)\). The concepts of \(S\)-infinitesimally stability, \(S\)-infinite determinacy and the Jacobi-Lojasiewicz condition at \(S\) are equivalent for the germ \(f\) and \(S\) a germ of a closed subset of \(\mathbb{R}^n\) with \(\bar{0}\) an accumulation point of \(S\).

### 3. A special case.

**Definition 37.** Let \(S\) be a germ of a closed subset of \(\mathbb{R}^n\) such that \(\bar{0} \in \text{cl}(\text{int})\). We say that a germ \(f\) is \(S\)-stable, if given a germ \(g\) such that \(g(x) = f(x)\) \(\forall x \in S\), there exists a germ of a diffeomorphism \(\phi \in G_S\) such that \(g = f \circ \phi\).

Note that if \(\text{cl}(\text{int}) = S\), the previous definition is apparently much stronger than Definition 23. In this case \(f(x) = g(x)\) \(\forall x \in S\) and \(j^\infty g(x) = j^\infty f(x)\) \(\forall x \in S\) are equivalent but now we restrict ourselves to the group \(G_S\), hence the diffeomorphism must be the identity on \(S\).

**Example 38.**

(a) Let \(S = \{(x, y) \in \mathbb{R}^2 | x \leq 0 \text{ and } y = 0\}\), then \(S\) is closed but \(\bar{0} \notin \text{cl}(\text{int} S)\).

(b) Let \(S = \{(x, y) \in \mathbb{R}^2 | x^4 - x^3 - xy^2 \geq 0\}\), in this case \(S = \text{cl}(\text{int} S)\).

(c) Let \(S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - x^3 \leq 0\}\), in this case \(\bar{0} \in \text{cl}(\text{int} S)\) but clearly \(\text{cl}(\text{int} S) \neq S\).

For \(S\) any germ of subset of \(\mathbb{R}^n\)containing the origin, we let \(C_S(\mathbb{R}^n)\) be the \(\mathbb{R}\)-algebra of germs constant at \(S\). It is a local algebra with maximal ideal \(m(S)\) consisting of germs of \(C_S(\mathbb{R}^n)\) vanishing at \(S\). In fact \(m(S)\) is an ideal of \(\mathcal{E}(n)\).

**Remark 7.** If \(f \in m(S)\) and \(S = \text{cl}(\text{int} S)\) we have \(f \in m(n)^\infty\) and \(\frac{\partial^{|I|} f}{\partial x^I} \in m(S)\) for all multi-index \(I\). We also get in this case the equality \(m(S) = m(n)^\infty m(S)\).
Lemma 39. Let $S$ a subset of $\mathbb{R}^n$. Suppose $S_0$ is a nonempty open subset of $S$. Then $\operatorname{cl}(S_0) = \operatorname{cl}(\operatorname{int} S)$ if and only if $\operatorname{int} (S - S_0) \subseteq \operatorname{cl}(S_0)$.

Proof. We decompose $\operatorname{int} S$ in the following way: $\operatorname{int} S = S_0 \cup \operatorname{int} (S - S_0) \cup T$, where $\operatorname{int} T = \emptyset$. Then $\operatorname{cl}(\operatorname{int} S) = \operatorname{cl}(S_0) \cup \operatorname{cl}(\operatorname{int} (S - S_0))$, since $\operatorname{cl}(T) \subseteq \operatorname{cl}(S_0 \cup \operatorname{int}(S - S_0))$. Hence $\operatorname{cl}(\operatorname{int} S) = \operatorname{cl}(S_0)$ if and only if $\operatorname{cl}(\operatorname{int} (S - S_0)) \subseteq \operatorname{cl}(S_0)$ and this is equivalent to $\operatorname{int} (S - S_0) \subseteq \operatorname{cl}(S_0)$. □

Definition 40. Let $A$ be a closed subset of $\mathbb{R}^n$. We say that $A$ is good if there exists a locally finite partition $\mathcal{P}$ of $A$ into $C^0$-submanifolds of $\mathbb{R}^n$, called strata, such that if $X \in \mathcal{P}$ and $\dim X < n$, then there exists a nonempty open stratum $Y \in \mathcal{P}$ such that $X \subset \operatorname{cl}(Y)$.

We clearly have the next:

Proposition 41. Suppose that $S$ is a good subset of $\mathbb{R}^n$. Then $\operatorname{cl}(S) = \operatorname{cl}(\operatorname{int} S)$.

Joining Lemma 39 and Proposition 41 we get the following:

Proposition 42. Let $P_1, \ldots, P_s$ be real continous functions on $\mathbb{R}^n$ such that $S = \{x | P_i(x) \leq 0 \ \forall \ i\}$ is good and define $S_0 = \{x | P_i(x) < 0 \ \forall \ i\}$. Suppose $\operatorname{int} (S - S_0) \subseteq \operatorname{cl}(S_0)$. Then $\operatorname{cl}(S_0) = S$.

Remark 8. If $P_1, \ldots, P_s$ are real analytic functions on $\mathbb{R}^n$ then $S = \{x | P_i(x) \leq 0 \ \forall \ i\}$ will be good if we have for a decomposition of $S$, that whenever $T$ is a stratum of lower dimension, then there exists a nonempty open stratum $T'$ such that $T \subset \operatorname{cl}(T')$. Obviously there are more examples of good sets than the semianalytical ones. For this purpose see for instance Sections 1 and 2 of [V-M].

We remind here the following:

Definition 43. Suppose that $S$ is a closed subset of $\mathbb{R}^n$ containing the origin and such that $S = \operatorname{cl}(\operatorname{int} S)$. We say that $f$ is $S$-infinitesimally stable if $\mathcal{m}(S) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{m}(S)$.

Theorem 44. Suppose $S$ is a closed subset of $\mathbb{R}^n$ such that $\bar{0} \in S$ and $S = \operatorname{cl}(\operatorname{int} S)$. If $f$ is $S$-infinitesimally stable then $f$ is $S$-stable.

Proof. Following the proof of Theorem 24 we start with $g$ a germ such that $g(x) = f(x) \ \forall \ x \in S$, therefore $\frac{\partial f}{\partial t}(x) = \frac{\partial g}{\partial t}(x) \ \forall \ x \in S$, and we arrive to the inclusion $\mathcal{m}(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{C}_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R}) \subseteq \mathcal{m}(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{C}_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$.

Since $\frac{\partial f}{\partial t} = g - f \in \mathcal{m}(S \times \mathbb{R}) \subseteq \mathcal{m}(S \times \mathbb{R}) \langle \frac{\partial f}{\partial x_i} \rangle \mathcal{C}_{S \times \mathbb{R}}(\mathbb{R}^n \times \mathbb{R})$, then $\frac{\partial f}{\partial t}(x, t) = \sum_{i=1}^{n} h_i(x, t) \frac{\partial f}{\partial x_i}(x, t)$, with $h_i(x, t) \in \mathcal{m}(S \times \mathbb{R})$, hence $h_i(x, t) = 0 \ \forall \ (x, t) \in S \times \mathbb{R}$. When we integrate, the required diffeomorphism will belong to $G_S$. □
Proposition 45. If \( f \in \mathcal{E}(n) \) is a finitely (infinitely) determined on the right, then \( f \) is \( S \)-infinitesimally stable and therefore \( S \)-stable for \( S = \text{cl}(\text{int} S) \).

Proof. Since \( m(S) = m(n)\infty m(S) \) and \( m(n)^k \subseteq \langle \frac{\partial f}{\partial x_i} \rangle \) for some \( k \leq \infty \), we get that \( m(S) \subseteq \langle \frac{\partial f}{\partial x_i} \rangle m(S) \). We now use Theorem 44.

Definition 46. Let \( P \) be a polynomial in variables \( x_1, \ldots, x_n \). We say that \( P \) is quasihomogeneous of degree \( l \) and weights \( k_1, \ldots, k_n \) if \( P(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^l P(x_1, \ldots, x_n) \).

For \( P \) quasihomogeneous we get \( \frac{\partial P}{\partial x_j}(t^{k_1}x_1, \ldots, t^{k_n}x_n) = t^{l-k_j} \frac{\partial P}{\partial x_j}(x_1, \ldots, x_n) \).

Also if we write \( P = \sum a_I x^I \), for a quasihomogeneous polynomial we obtain for any multi-index \( I = (i_1, \ldots, i_n) \), \( i_1 k_1 + \ldots + i_n k_n = l \) \( (a_I \neq 0) \).

Theorem 47. Let \( P(x) \) be a quasihomogeneous polynomial and \( S \) a closed subset of \( \mathbb{R}^n \) containing the origin and such that \( S = \text{cl}(\text{int} S) \). Suppose that \( m(S) \subseteq \langle P \rangle \) and that \( z(P) \cap \text{int} S = \phi \). Then \( P \) is \( S \)-infinitesimally stable. In the case \( S = \{ x | P(x) \leq 0 \} \) is a good semialgebraic set, we can skip the equality \( z(P) \cap \text{int} S = \phi \).

Proof. By hypothesis we get \( m(S) \subseteq \langle P \rangle \) and \( P \in \langle \frac{\partial P}{\partial x_i} \rangle \), this together with \( z(P) \cap \text{int} S = \phi \) give the result using Example 26. For the second part it is obvious that \( z(P) \cap \text{int} S = \phi \) since \( S \) is a good semialgebraic set.

As in the previous section, we get the following:

Theorem 48. Let \( f \in \mathcal{E}(n) \), \( S \) be a closed subset of \( \mathbb{R}^n \) such that the origin is an accumulation point of \( S \) and \( S = \text{cl}(\text{int} S) \). Then the concepts for \( f \) of \( S \)-infinitesimally stability, \( S \)-stability and the Jacobi-Lojasiewicz condition for \( S \) are equivalent.

Proof. Our Theorem 44 shows that \( S \)-infinitesimally stability implies \( S \)-stability. Now as in Theorem 32, we show that the Jacobi-Lojasiewicz condition at \( S \) implies \( S \)-infinitesimally stability. Since Lemma 34 is true for any closed subset of \( \mathbb{R}^n \), the proof of Theorem 35 will be true in the case \( S = \text{cl}(\text{int} S) \), and hence \( S \)-stability implies the Jacobi-Lojasiewicz condition of \( f \) for \( S \).

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BRAIDED-LIE BIALGEBRAS

Shahn Majid

We introduce braided Lie bialgebras as the infinitesimal version of braided groups. They are Lie algebras and Lie coalgebras with the coboundary of the Lie cobracket an infinitesimal braiding. We provide theorems of transmutation, Lie biproduct, bosonisation and double-bosonisation relating braided Lie bialgebras to usual Lie bialgebras. Among the results, the kernel of any split projection of Lie bialgebras is a braided-Lie bialgebra. The Kirillov-Kostant Lie cobracket provides a natural braided-Lie bialgebra on any complex simple Lie algebra, as the transmutation of the Drinfeld-Sklyanin Lie cobracket. Other nontrivial braided-Lie bialgebras are associated to the inductive construction of simple Lie bialgebras along the $C$ and exceptional series.

1. Introduction.

Braided geometry has been developed in recent years as a natural generalisation of super-geometry with the role of $\mathbb{Z}/2\mathbb{Z}$ grading played by braid statistics. It is also the kind of noncommutative geometry appropriate to quantum group symmetry because the modules over a strict quantum group (a quasitriangular Hopf algebra [3]) form a braided category, hence any object covariant under the quantum group is naturally braided. In particular, one has braided groups [6] as generalisations of super-groups or super-Hopf algebras. The famous quantum-braided plane with relations $yx = qxy$ is a braided group with additive coproduct [7]. We refer to [8], [10] for introductions to the 50-60 papers in which the theory of braided groups is developed.

In a different direction, Drinfeld [2] has introduced Lie bialgebras as an infinitesimalisation of the theory quantum groups. This concept has led (on exponentiation) to an extensive theory of Poisson-Lie groups, as well as to a Yang-Baxter theoretic approach to classical results of Lie theory, such as a new proof of the Iwasawa decomposition and the structure of Bruhat cells; see for example [9], [5]. For an introduction to quantum groups and Lie bialgebras, see [10].

We now combine these ideas for the first time by introducing the infinitesimal theory of braided groups. All computations and results will be in
the setting of Lie algebras, although motivated from the theory of braided
groups. In fact, there are several different concepts of precisely what one
may mean by the infinitesimal theory of braided groups. Firstly, one may
keep the braided category in which one works fixed and look at algebras
which depart infinitesimally from being commutative. In the category of
vector spaces this leads to Drinfeld’s notion of Poisson-Lie group. Then one
can consider the coalgebra also in an infinitesimal form, which leads in the
category of vector spaces to Drinfeld’s notion of Lie bialgebra. In the case
of a braided category one already has the notion of braided-Lie algebra [11]
and, adding to this, one could similarly consider a Lie bialgebra in a braided
category. By contrast, we now go further and let the braiding also depart
infinitesimally from the usual vector space transposition. In principle, the
degree of braiding is independent of the degree of algebra commutativity
or coalgebra cocommutativity. Thus one could have infinitesimally braided
algebras, coalgebras and Hopf algebras as well. However, the case which ap-
ppears to be of most interest, on which we concentrate, is the case in which all
three aspects are made infinitesimal simultaneously, which we call a braided-
Lie bialgebra. The formal definition appears in Section 2. It consists of a
Lie algebra \( b \) equipped with further structure.

In Section 3 we provide the Lie version of the basic theorems from the
theory of braided groups. These basic theorems connect braided groups and
quantum groups by transmutation [12], [6] and bosonisation [13], [14] pro-
cedures, thereby establishing (for example) the existence of braided groups
associated to all simple Lie algebras. The theorems in Section 3 likewise
connect braided-Lie bialgebras with quasitriangular Lie bialgebras and es-
tablish the existence of the former. The Lie versions of biproducts [20] and
of the more recent double-bosonisation theorem [15] are covered as well. For
example, the Lie version of the theory of biproducts states that the kernel
of any split Lie bialgebra projection \( g \to f \) is a braided-Lie bialgebra \( b \), and
\( g = b \bowtie f \).

In Section 4 we study some concrete examples of braided-Lie bialgebras,
including ones not obtained by transmutation. The simplest are ones with
zero braided-Lie cobracket as the infinitesimal versions of the \( q \)-affine plane
braided groups in [7]. As an application of braided-Lie bialgebras, their
bosonisations provide maximal parabolic or inhomogeneous Lie bialgebras.
Meanwhile, double-bosonisation allows the formulation in a basis-free way
of the notion of adjoining a node to a Dynkin diagram. For every simple Lie
bialgebra \( g \) and braided-Lie bialgebra \( b \) in its category of modules we obtain
a new simple Lie bialgebra \( b \bowtie g \bowtie b^{\text{op}} \) as its double-bosonisation. This pro-
vides the inductive construction of all complex simple Lie algebras, complete
with their Drinfeld-Sklyanin quasitriangular Lie bialgebra structure (which
is built up inductively at the same time). Some concrete examples are given
in detail.
These results have been briefly announced in [16, Sec. 3], of which the present paper is the extended text. We work over a general ground field $k$ of characteristic not 2.

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2. Braided-Lie bialgebras.

We will be concerned throughout with the Lie version of braided categories obtained as module categories over quantum groups. In principle one could also formulate an abstract notion of ‘infinitesimal braiding’ as a Lie version of a general braided category, but since no examples other than the ones related to quantum groups are known we limit ourselves essentially to this concrete setting. Some slight extensions (such as to Lie crossed modules) will be considered as well, later on.

As a Lie version of a strict quantum group we use Drinfeld’s notion of a quasitriangular Lie bialgebra [2], [3]. We recall that a Lie bialgebra is a Lie algebra $g$ equipped with linear map $\delta : g \to g \otimes g$ forming a Lie coalgebra (in the finite dimensional case this is equivalent to a Lie bracket on $g^*$) and being a 1-cocycle with values in $g \otimes g$ as a $g$-module by the natural extension of ad. It is quasitriangular if there exists $r \in g \otimes g$ obeying $dr = \delta$ in the Lie algebra complex, obeying the Classical Yang-Baxter Equation (CYBE)

$$[r^{(1)}, r'^{(1)}] \otimes r^{(2)} \otimes r'^{(2)} + r^{(1)} \otimes [r^{(2)}, r'^{(1)}] \otimes r'^{(2)} + r^{(1)} \otimes r'^{(1)} \otimes [r^{(2)}, r'^{(2)}] = 0$$

and having ad-invariant symmetric part $2r_+ = r + \tau(r)$, where $\tau$ is transposition. We use the conventions and notation similar to [10, Ch. 8], with $r = r^{(1)} \otimes r^{(2)}$ denoting an element of $g \otimes g$ (summation understood) and $r'$ denoting another distinct copy of $r$. We also use $d\xi = \xi^{(1)} \otimes \xi^{(2)}$ to denote the output in $g \otimes g$ for $\xi \in g$ (summation understood). A quasitriangular Lie bialgebra is called factorisable if $2r_+$ is surjective when viewed as a map $g^* \to g$.

In view of the discussion above, we are interested in Lie-algebraic objects living in the category $g\mathcal{M}$ of modules over a quasitriangular Lie bialgebra $g$. If $V$ is a $g$-module, we define its infinitesimal braiding to be the operator

$$\psi : V \otimes V \to V \otimes V, \quad \psi(v \otimes w) = 2r_+ \triangleright (v \otimes w - w \otimes v)$$

where $\triangleright$ denotes the left action of $g$.

Lemma 2.1. Let $b \in g\mathcal{M}$ be a $g$-covariant Lie algebra. Then the associated $\psi : b \otimes b \to b \otimes b$ is a 2-cocycle $\psi \in Z^2_{ad}(b, b \otimes b)$. 

Proof. The proof that $d\psi = 0$ is a straightforward computation in Lie algebra cohomology. We use covariance of $b$ in the form: $\xi \triangleright [x, y] = [\xi \triangleright x, y] + [x, \xi \triangleright y]$ for all $\xi \in g$. Then,

$$(d\psi)(x, y, z) = -\psi([x, y], z) + \psi([x, z], y) - \psi([y, z], x) + \text{ad}_x \psi(y, z) - \text{ad}_y \psi(x, z) + \text{ad}_z \psi(x, y)$$

$$= 2r_+ \triangleright ([x, y] \otimes z + z \otimes [x, y]) + [x, 2r_+^{(1)} \triangleright y] \otimes r_+^{(2)} \triangleright z$$

$$+ 2r_+^{(1)} \triangleright y \otimes [x, r_+^{(2)} \triangleright z] - [x, 2r_+^{(1)} \triangleright z] \otimes r_+^{(2)} \triangleright y - 2r_+^{(1)} \triangleright y \otimes [x, r_+^{(2)} \triangleright y] + \text{cyclic}$$

$$= -2r_+^{(1)} \triangleright x, y \otimes r_+^{(2)} \triangleright z + 2r_+^{(1)} \triangleright y \otimes [x, r_+^{(2)} \triangleright z] - [x, 2r_+^{(1)} \triangleright z] \otimes r_+^{(2)} \triangleright y$$

$$+ 2r_+^{(1)} \triangleright z \otimes [r_+^{(2)} \triangleright x, y] + \text{cyclic} = 0$$

on using the cyclic invariance in $x, y, z$ and antisymmetry of the Lie bracket.

Note that this works for any element $2r_+ \in g \otimes g$ in the definition of $\psi$. □

Definition 2.2. A braided-Lie bialgebra $b \in gM$ is a $g$-covariant Lie algebra and $g$-covariant Lie coalgebra with cobracket $\delta : b \rightarrow b \otimes b$ obeying $\forall x, y \in b$,

$$\delta([x, y]) = \text{ad}_x \delta y - \text{ad}_y \delta x - \psi(x \otimes y); \quad \psi = 2r_+ \triangleright (\text{id} - \tau),$$

i.e., $\delta$ obeys the coJacobi identity and $d\delta = \psi$.

The definition is motivated from that of a braided group, where the coproduct fails to be multiplicative up to a braiding $\Psi$ [6]. The results in the next section serve to justify it further.

3. Lie versions of braided group theorems.

The existence of nontrivial quasitriangular Lie bialgebra structures is known [3] for all simple $g$ at least over $\mathbb{C}$. Our first theorem ensures likewise the existence of braided-Lie bialgebras.

Theorem 3.1. Let $i : g \rightarrow f$ be a map of Lie bialgebras, with $g$ quasitriangular. There is a braided-Lie bialgebra $b(g, f)$, the transmutation of $f$, living in $gM$. It has the Lie algebra of $f$ and for all $x \in f$, $\xi \in g$,

$$\delta x = \delta x + r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x, \quad \xi \triangleright x = [i(\xi), x].$$

Proof. We first verify that $\delta$ as stated is indeed a $g$-module map. Thus

$$\delta(\xi x) = \delta(i(\xi), x) + r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x$$

$$= \xi \triangleright \delta x - [x, i(\xi_{(1)})] \otimes i(\xi_{(2)}) - i(\xi_{(1)}) \otimes [x, i(\xi_{(2)})]$$

$$+ r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x$$

$$= \xi \triangleright \delta x - [x, i(\xi), i(r^{(1)})] \otimes i(r^{(2)}) - [x, i(r^{(1)})] \otimes [i(\xi), i(r^{(2)})]$$

$$+ r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x$$
\[ -[i(\xi), i(r^{(1)})] \otimes [x, i(r^{(2)})] - i(r^{(1)}) \otimes [x, [i(\xi), i(r^{(2)})]] \\
+ r^{(1)} \xi \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \xi \otimes x \\
= \xi \triangleright \delta x + [\xi, r^{(1)}] \triangleright x \otimes i(r^{(2)}) - [x, i(r^{(1)})] \otimes [i(\xi), i(r^{(2)})] \\
- [i(\xi), i(r^{(1)})] \otimes [x, i(r^{(2)})] + i(r^{(1)}) \otimes [\xi, r^{(2)}] \triangleright x \\
+ r^{(1)} \xi \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \xi \otimes x \\
= \xi \triangleright \delta x + r^{(1)} \triangleright x \otimes [i(\xi), i(r^{(2)})] + [i(\xi), i(r^{(1)})] \otimes r^{(2)} \triangleright x \\
+ \xi r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes \xi r^{(1)} \triangleright x \\
= \xi \triangleright \delta x \\
\]

where we used the definitions of \( \triangleright \) and \( \delta \) and the fact that \( g \) is quasitriangular, so that \( \delta \xi = [\xi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\xi, r^{(2)}] \).

Antisymmetry of the output of \( \delta \hat{\phi} \) is clear. Next we verify the coJacobi identity,

\[(id \otimes \delta \hat{\phi}) \delta x \\
+ \text{cyclic} = (id \otimes \delta \hat{\phi}) \delta x + r^{(1)} \triangleright x \otimes \delta \hat{\phi}(r^{(2)}) - i(r^{(2)}) \otimes \delta \hat{\phi}(r^{(1)} \triangleright x) + \text{cyclic} \\
= x^{(1)} \otimes r^{(1)} \triangleright x^{(2)} \otimes i(r^{(2)}) - x^{(1)} \otimes i(r^{(2)}) \otimes r^{(1)} \triangleright x^{(2)} + r^{(1)} \triangleright x \otimes \delta \hat{\phi}(r^{(2)}) \\
+ r^{(1)} \triangleright x \otimes i([r^{(1)}, r^{(2)}]) \otimes i(r^{(2)}) - r^{(1)} \triangleright x \otimes i(r^{(2)}) \otimes i([r^{(1)}, r^{(2)}]) \\
- i(r^{(2)}) \otimes r^{(1)} \triangleright x^{(1)} \otimes x^{(2)} - i(r^{(2)}) \otimes x^{(1)} \otimes r^{(1)} \triangleright x^{(2)} \\
- i(r^{(2)}) \otimes r^{(1)} \triangleright x \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright x \otimes i([r^{(1)}, r^{(2)}]) \\
+ i(r^{(2)}) \otimes i([r^{(1)}, r^{(2)}]) \otimes r^{(1)} \triangleright x + i(r^{(2)}) \otimes i(r^{(2)}) \otimes r^{(1)} r^{(1)} \triangleright x + \text{cyclic} \\
\]

using the definition of \( \delta \) and the previous covariance result. Several of the resulting terms cancel immediately. Using the quasitriangular form of \( \delta \) on \( r^{(2)} \) and the further freedom to cyclically rotate all tensor products so that \( x \) appears in the first factor, our expression becomes

\[ = r^{(1)} \triangleright x \otimes i([r^{(2)}, r^{(1)}]) \otimes i(r^{(2)}) + r^{(1)} \triangleright x \otimes i(r^{(1)}) \otimes i([r^{(2)}, r^{(2)}]) \\
+ r^{(1)} \triangleright x \otimes i([r^{(1)}, r^{(2)}]) \otimes i(r^{(2)}) - r^{(1)} \triangleright x \otimes i(r^{(2)}) \otimes i([r^{(1)}, r^{(2)}]) \\
+ [r^{(1)}, r^{(1)}] \triangleright x \otimes i(r^{(2)}) \otimes i(r^{(2)}) \\
+ r^{(1)} \triangleright x \otimes i(r^{(2)}) \otimes i([r^{(1)}, r^{(2)}]) - r^{(1)} \triangleright x \otimes i([r^{(1)}, r^{(2)}]) \otimes i(r^{(2)}) + \text{cyclic} \\
= ([x \otimes i \otimes i] ([r^{(1)}, r^{(1)}] \otimes r^{(2)} \otimes r^{(2)} \\
+ r^{(1)} \otimes [r^{(2)}, r^{(1)}] \otimes r^{(2)} \otimes r^{(1)} \otimes [r^{(2)}, r^{(2)}]) + \text{cyclic} \\
= 0 \\
\]

by the CYBE (1).

Finally, we prove that \( d \delta = \psi \). Thus,

\[ \hat{\delta}(x, y) = \delta(x, y) + r^{(1)} \triangleright [x, y] \otimes i(r^{(2)}) - i(r^{(2)}) \otimes r^{(1)} \triangleright [x, y] \]
\[ \delta x = 2r_+^{(1)} \otimes [x, r_+^{(2)}]. \]

**Proof.** We take the identity map \( i = \text{id} : \mathfrak{g} \to \mathfrak{g} \) and \( \mathfrak{g} = \mathfrak{b}(\mathfrak{g}, \mathfrak{g}) \). Its braided-Lie cobracket from Theorem 3.1 is \( \delta x = [x, r_+^{(1)}] \otimes r_+^{(2)} + r_+^{(1)} \otimes [x, r_+^{(2)}] + r_+^{(1)} \otimes r_+^{(2)} - r_+^{(2)} \otimes r_+^{(1)} \otimes x \) using the quasitriangular form of \( \delta \).

The corollary ensures the existence of non-trivial braided-Lie bialgebras since nontrivial quasitriangular Lie bialgebras are certainly known.

**Example 3.3.** Let \( \mathfrak{g} \) be a finite-dimensional factorisable Lie bialgebra. Then \( \delta \) in Corollary 3.2 is equivalent under the isomorphism \( 2r_+ : \mathfrak{g}^* \cong \mathfrak{g} \) to the Kirillov-Kostant Lie cobracket on \( \mathfrak{g}^* \) (defined as the dualisation of the Lie bracket \( \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \)). The braided-Lie bialgebra \( \mathfrak{g} \) is self-dual.

**Proof.** It is well known that for any Lie algebra the vector space \( \mathfrak{g}^* \) acquires a natural Poisson bracket structure. Considering \( \mathfrak{g} \) as a subset of the functions on \( \mathfrak{g}^* \), this Kirillov-Kostant Poisson bracket is \( \{ \xi, \eta \}(\phi) = \langle \phi, [\xi, \eta] \rangle \) where \( \langle , \rangle \) denotes evaluation and \( \xi, \eta \in \mathfrak{g}, \phi \in \mathfrak{g}^* \). The associated Lie coalgebra structure \( \delta : \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^* \) is defined by \( \delta(\xi, \eta)\phi = \{ \xi, \eta \}/\phi = \{ \xi \otimes \eta, \delta\phi \} \) and is therefore the dualisation of the Lie bracket of \( \mathfrak{g} \). We call it the Kirillov-Kostant Lie coalgebra structure on \( \mathfrak{g}^* \).

Let \( K(\phi) = 2r_+^{(1)}\langle \phi, r_+^{(2)} \rangle \) denote the isomorphism \( K : \mathfrak{g}^* \cong \mathfrak{g} \) resulting from our factorisability assumption. Then

\[
\langle \xi \otimes \eta, (K^{-1} \otimes K^{-1})\delta K(\phi) \rangle = \langle \xi, K^{-1}(2r_+^{(1)})\rangle \langle \eta, K^{-1}([K(\phi), r_+^{(2)}]) \rangle
\]

\[
= \langle K^{-1}(\xi), 2r_+^{(1)} \rangle \langle \eta, [\phi, K^{-1}(r_+^{(2)})] \rangle
\]

\[
= \langle \eta, K(\phi) \rangle, K^{-1}(\xi) = \langle K(\phi), K^{-1}([\xi, \eta]) \rangle = \langle \phi, [\xi, \eta] \rangle.
\]

We used symmetry and ad-invariance of \( K \) as an element of \( \mathfrak{g} \), with its corresponding property \( \langle \eta, K^{-1}([\xi, \zeta]) \rangle = \langle [\eta, \xi], K^{-1}(\zeta) \rangle \forall \xi, \eta, \zeta \in \mathfrak{g} \), for the map \( K : \mathfrak{g}^* \to \mathfrak{g} \).
Next, we give $\mathfrak{g}^*$ with the above Kirillov-Kostant Lie cobracket $\delta \phi = \phi_{(1)} \otimes \phi_{(2)}$ (dual to the Lie algebra of $\mathfrak{g}$) a Lie bracket and $\mathfrak{g}$-module structure

$$\{\phi, \chi\} = \chi_{(1)} 2r_+ (\phi, \chi_{(2)})$$

for all $\xi \in \mathfrak{g}, \phi, \chi \in \mathfrak{g}^*$ and with $2r^+$ viewed as a map $\mathfrak{g}^* \otimes \mathfrak{g}^* \to k$. Then $\mathfrak{g}^*$ becomes a braided-Lie bialgebra in $\mathfrak{g}^* \mathcal{M}$, which we denote $\mathfrak{g}^*$. Its Lie cobracket is $\delta = \delta$ the dual of the Lie bracket of $\mathfrak{g}$ (since this is the same as that of $\mathfrak{g}$), and its Lie bracket is dual to the Lie cobracket of $\mathfrak{g}$ in Corollary 3.2 since

$$\langle \xi, [\phi, \chi]\rangle = \langle \xi, \chi_{(1)} \rangle \langle 2r_+ (\phi), \chi_{(2)} \rangle = \langle [\xi, 2r_+ (\phi)], \chi\rangle$$

for all $\xi \in \mathfrak{g}$ and $\phi, \chi \in \mathfrak{g}^*$. On the other hand,

$$\langle \xi, [\phi, \chi]\rangle = \langle \xi, \chi_{(1)} \rangle \langle \chi_{(2)}, K(\phi) \rangle = \langle [\xi, K(\phi)], \chi\rangle = \langle \xi, K^{-1} ([K(\phi), K(\chi)]) \rangle$$

hence $2r_+: \mathfrak{g}^* \to \mathfrak{g}$ is an isomorphism of braided-Lie bialgebras.

This is the Lie analogue of the theorem that braided groups obtained by full transmutation of factorisable quantum groups are self-dual via the quantum Killing form [14]. Also, the fact that the data corresponding to the original Lie cobracket on $\mathfrak{g}$ does not enter into $\mathfrak{g}$ corresponds in braided group theory to transmutation rendering a quasitriangular Hopf algebra braided-cocommutative. There is also a theory of quasitriangular braided-Lie bialgebras of which the more general $\mathfrak{b}(\mathfrak{g}, \mathfrak{f})$ are examples when $\mathfrak{f}$ is itself quasitriangular. The braided-quasitriangular structure is the difference of the quasitriangular structures on $\mathfrak{f}, \mathfrak{g}$ as the Lie version of results in [12].

For use later on, the general duality for braided-Lie bialgebras relevant to Example 3.3 is given by

**Lemma 3.4.** If $\mathfrak{b} \in g \mathcal{M}$ is a finite-dimensional braided-Lie bialgebra then there is a dual braided-Lie bialgebra $\mathfrak{b}^* \in g \mathcal{M}$. It is built on the vector space $\mathfrak{b}^*$ with action $(\xi \triangleright \phi)(x) = -\phi(\xi \triangleright x)$ and Lie (co)bracket structure maps given by dualisation.

**Proof.** The Jacobi and coJacobi (and antisymmetry) axioms are clear by dualisation, as is the specified left action on $\mathfrak{b}^*$. The induced infinitesimal braiding on the dual is the usual dual:

$$\langle x \otimes y, \psi \triangleright (\phi, \chi) \rangle = \langle x \otimes y, 2r_+ (\phi \otimes \chi - \chi \otimes \phi) \rangle = \langle \psi (x \otimes y), \phi \otimes \chi \rangle$$

for all $x, y \in \mathfrak{b}$ and $\phi, \chi \in \mathfrak{b}^*$. Moreover, the map $d\delta$ for $\mathfrak{b}$ dualises to $d\delta$ for $\mathfrak{b}^*$. The proof is identical to the proof that the dual of a usual Lie bialgebra is a Lie bialgebra (see [10] for details). Hence $d\delta = \psi$ for $\mathfrak{b}^*$ by dualisation of this relation for $\mathfrak{b}$. □
Note also that if $b \in \mathfrak{g}M$ is any braided-Lie bialgebra then so is $b^{\text{op/cop}}$ with opposite bracket and cobracket, in the same category. This is because the covariance conditions on the Lie bracket and cobracket are each linear in those structures and hence valid even with the additional minus signs in either case. Meanwhile, $d\delta$ is linear in $\delta$ and linear in the Lie bracket, hence invariant when both are changed by a minus sign. The infinitesimal braiding does not involve either the bracket or cobracket and is invariant. Applying this observation to $b^*$ gives us another dual, $b^\star$. This is the Lie analogue of the more braided-categorical dual which is more natural in the theory of braided groups. In the Lie setting, however, we have $b^* \cong b^\star$ by $x \mapsto -x$, so can work entirely with $b^\star$. We also conclude, in passing, that $b^{\text{op}}$ and $b^{\text{cop}}$ are braided-Lie bialgebras in the category of modules over the opposite quasitriangular Lie bialgebra (i.e., with quasitriangular structure $-r_{21}$ in place of $r$).

We consider now the adjoint direction to Theorem 3.1, to associate to a braided-Lie bialgebra an ordinary Lie bialgebra. The quantum group version [13] has been used to construct inhomogeneous quantum groups [7].

**Theorem 3.5.** Let $b \in \mathfrak{g}M$ be a braided-Lie bialgebra. Its bosonisation is the Lie bialgebra $b \triangleright \mathfrak{g}$ with $\mathfrak{g}$ as sub-Lie bialgebra, $b$ as sub-Lie algebra and

$$\{\xi, x\} = \xi \triangleright x, \quad \delta x = \delta x + r^{(2)} \otimes r^{(1)} \triangleright x - r^{(1)} \triangleright x \otimes r^{(2)}, \quad \forall \xi \in \mathfrak{g}, \ x \in b.$$  

**Proof.** The Lie algebra structure of $b \triangleright \mathfrak{g}$ is constructed as a semidirect sum by the given action of $\mathfrak{g}$ on $b$. The coassociativity of the Lie cobracket may be verified directly from the CYBE along the lines of the proof of coassociativity in Theorem 3.1. The line of deduction is reversed but the formulae are similar. That the result is a Lie bialgebra has three cases. For $\xi, \eta \in \mathfrak{g}$ we have $\delta(\{\xi, \eta\})$ as required since $\mathfrak{g}$ is a Lie bialgebra. For the mixed case we have

$$\delta(\{\xi, x\}) = \delta(\xi \triangleright x) = \delta(\xi \triangleright x) + r^{(2)} \otimes r^{(1)} \xi \triangleright x - r^{(1)} \xi \triangleright x \otimes r^{(2)}$$

$$-\text{ad}_\xi \delta x = -\xi \triangleright \delta x - [\xi, r^{(2)}] \otimes r^{(1)} \triangleright x - r^{(2)} \otimes \xi r^{(1)} \triangleright x + \xi r^{(1)} \triangleright x \otimes r^{(2)}$$

$$+ r^{(1)} \triangleright x \otimes [\xi, r^{(2)}]$$

$$\text{ad}_x \delta \xi = -\xi \triangleright \delta x \otimes \xi \triangleright (x) - \xi \triangleright (x) \otimes \xi \triangleright (x)$$

$$= [r^{(1)}, \xi] \triangleright x \otimes r^{(2)} + r^{(1)} \triangleright x \otimes [r^{(2)}, \xi] + [r^{(1)}, \xi] \otimes r^{(2)} \triangleright x$$

$$+ r^{(1)} \otimes [r^{(2)}, \xi] \triangleright x.$$ 

We used the definitions and, in the last line, the form of $\delta \xi$ as a quasitriangular Lie bialgebra. Adding these expressions, we obtain

$$\delta(\{\xi, x\}) - \text{ad}_\xi \delta x + \text{ad}_x \delta \xi$$

$$= \Delta \xi \triangleright x - \xi \triangleright \Delta x + [2r^{(2)}_+, \xi] \otimes r^{(1)}_+ \triangleright x + 2r^{(2)}_+ \otimes [r^{(2)}_+, \xi] \triangleright x = 0$$
by covariance of $\delta$ and ad-invariance of $2r_+$. The remaining case is

$$\delta([x, y]) = \delta([x, y]) + r^{(2)} \odot r^{(1)} \triangleright [x, y] - r^{(1)} \triangleright [x, y] \odot r^{(2)}$$

$$= (\text{ad}_x \delta y + r^{(2)} \odot \text{ad}_x (r^{(1)} \triangleright y) - \text{ad}_x (r^{(1)} \triangleright y) \odot r^{(2)} - (x \leftrightarrow y))$$

$$= \psi(x \otimes y)$$

$$= (\text{ad}_x \delta y - \text{ad}_x (r^{(2)} \odot r^{(1)} \triangleright y + r^{(1)} \triangleright y \odot \text{ad}_x (r^{(2)}))$$

$$= \psi(x \otimes y) - \psi(x \otimes y)$$

$$= \psi(x \otimes y)$$

on writing $\text{ad}_x (r^{(2)}) = -r^{(2)} \triangleright x$ and comparing with the definition of $\psi$. We used the braided-Lie bialgebra property of $\delta$. $\square$

The construction in the bosonisation theorem can also be viewed as a special case of a more general construction for Lie bialgebras which are semidirect sums as Lie algebras and Lie coalgebras by a simultaneous Lie action and Lie coaction. We call such Lie algebras *bisum* Lie algebras. They are the analogue of *biproduct* Hopf algebras in [20]. In the general case one only needs covariance under a Lie bialgebra, not necessarily quasitriangular. However, any Lie bialgebra has a Drinfeld double [2] which is quasitriangular. In order to explain these topics we need quite a bit more formalism. Firstly, if $f$ is any Lie coalgebra, we have a notion of left *Lie coaction* on a vector space $V$. This is a map $\beta : V \to f \otimes V$ such that

$$\delta \odot \text{id} \circ \beta = (\text{id} - \tau) \otimes \text{id} \circ (\text{id} \otimes \beta) \circ \beta.$$

The category of left Lie comodules is denoted $\mathcal{M}$ and is monoidal in the obvious derivation-like way. Morphisms are defined as linear maps intertwining the Lie coactions, again in the obvious way.

**Lemma 3.6.** Let $f$ be a Lie bialgebra. There is a monoidal category of Lie crossed modules $\mathcal{M}$ having as objects vector spaces $V$ which are simultaneously $f$-modules $\triangleright : f \otimes V \to \otimes V$ and $f$-comodules $\beta : V \to f \otimes V$ obeying $\forall \xi \in f, v \in V$,

$$\beta(\xi \triangleright v) = ([\xi, \cdot] \otimes \text{id} + \text{id} \otimes \xi \triangleright) \beta(v) + (\delta \xi) \triangleright v.$$

It can be identified when $f$ is finite-dimensional with the category $D(f, \mathcal{M})$ where $D(f)$ is the Drinfeld double [2]. Writing $\beta(v) = v^{(1)} \otimes v^{(2)}$, the corresponding infinitesimal braiding on any object $V \in \mathcal{M}$ is

$$\psi(v \otimes w) = w^{(1)} \triangleright v \otimes w^{(2)} - v^{(1)} \triangleright w \otimes v^{(2)} - w^{(2)} \otimes w^{(1)} \triangleright v + v^{(2)} \otimes v^{(1)} \triangleright w.$$

**Proof.** Morphisms in $\mathcal{M}$ are maps intertwining both the Lie action and the Lie coaction. We start with $f$ finite-dimensional and use Drinfeld’s formulae for $D(f)$ in the conventions in [10], where it contains the Lie algebras $f$ and $f^{\text{op}}$ with the cross relations $[\xi, \phi] = \phi_{(1)} \langle \phi_{(2)}, \xi \rangle + \xi_{(1)} \langle \phi, \xi_{(2)} \rangle$ for all $\xi \in f$ and
\( \phi \in \mathfrak{f}^* \). A left module of \( D(\mathfrak{f}) \) therefore means a vector space which is a left \( \mathfrak{f} \)-module and a right \( \mathfrak{f}^* \)-module, obeying \( \xi \triangleright (v \triangleleft \phi) = (\xi \triangleright v) \triangleleft \phi = v \triangleleft \phi(1) \langle \phi(2), \xi \rangle + \xi \triangleright v \langle \phi, \xi \rangle \rangle \). Next we view the right action of \( \mathfrak{f}^* \) as, equivalently, a left coaction of \( \mathfrak{f} \) by \( v \triangleleft \phi = \langle \phi, v^{(1)} \rangle v^{(2)} \). Inserting this, we have the condition
\[
\xi \triangleright v^{(2)} \langle \phi, v^{(1)} \rangle - \langle \phi, (\xi \triangleright v)^{(1)} \rangle (\xi \triangleright v)^{(2)} = \langle \phi, [v^{(1)}, \xi] \rangle v^{(2)} + \xi \triangleright v \langle \phi, \xi \rangle
\]
for all \( \phi \). We wrote the Lie cobracket of \( \mathfrak{f}^* \) in terms of the Lie algebra of \( \mathfrak{f} \) here. This is the condition stated for \( \beta \), which manifestly makes sense even for infinite-dimensional Lie algebras. It is easy to check that the category is well defined and monoidal even in this case. In the same spirit, \( \mathcal{D}(\mathfrak{f}) \) has a quasitriangular structure given by the canonical element for the duality pairing \( [\mathfrak{f}] \). Then \( 2r_+ = \sum_a f^a \otimes e_a + e_a \otimes f^a \) where \( \{e_a\} \) is a basis of \( \mathfrak{f} \) and \( \{f^a\} \) is a dual basis. Hence the infinitesimal braiding in \( D(\mathfrak{f})M \) is
\[
\psi(v \otimes w) = 2r_+ \triangleright (v \otimes w - w \otimes v) = \langle f^a, v^{(1)} \rangle v^{(2)} \otimes e_a \triangleright w + e_a \triangleright v \otimes (f^a, w^{(1)}) w^{(2)} - (v \leftrightarrow w)
\]
when the left action of \( \mathfrak{f}^\text{op} \) is of the form given by a left coaction of \( \mathfrak{f} \). This gives \( \psi \) as stated. Note that the use of a coaction to reformulate an action of the dual in the infinite dimensional case is a completely routine procedure in Hopf algebra theory; we have given the details here since the Lie version is less standard; the category of Lie crossed modules \( \mathcal{M} \) should not be viewed as anything other than a version of the ideas behind Drinfeld’s double construction.

The resulting map \( \psi \) is well-defined even in the infinite dimensional case; we call it the infinitesimal braiding of the category \( \mathcal{M} \) of crossed \( \mathfrak{f} \)-modules and define a braided-Lie bialgebra in \( \mathcal{M} \) with respect to this.

**Theorem 3.7.** Let \( \mathfrak{f} \) be a Lie bialgebra and let \( \mathfrak{b} \subset \mathcal{M} \) be a braided-Lie bialgebra. The bisum Lie bialgebra \( \mathfrak{b} \triangleright \mathfrak{f} \) has semidirect Lie bracket/cobracket and projects onto \( \mathfrak{f} \). Conversely, any Lie bialgebra \( \mathfrak{g} \) with a split Lie bialgebra projection \( \mathfrak{g} \rightarrow \mathfrak{f} \) is of this form, with \( \mathfrak{b} = \ker \pi \) and braided-Lie bialgebra structure given by \( \mathfrak{b} \subset \mathfrak{g} \) as a Lie algebra and
\[
\xi \triangleright = \text{ad}_{\xi(1)}, \quad \beta = (\pi \otimes \text{id}) \circ \delta, \quad \hat{\delta} = (\text{id} - i \circ \pi)^{\otimes 2} \circ \delta.
\]

**Proof.** In the forward direction, since \( \mathfrak{b} \) is covariant under an action of \( \mathfrak{f} \) we can make, as usual, a semidirect sum \( \mathfrak{b} \triangleright \mathfrak{f} \). The bracket on general elements of the direct sum vector space is \([x \oplus \xi, y \oplus \eta] = ([x, y] + \xi \triangleright y - \eta \triangleright x) \oplus [\xi, \eta]\) as usual. On the other hand, since the Lie coalgebra of \( \mathfrak{b} \) is covariant under a Lie coaction of \( \mathfrak{f} \), one may make a semidirect Lie coalgebra \( \mathfrak{b} \triangleright \mathfrak{f} \) with [10]
\[
\delta(x \oplus \xi) = \delta \xi + \hat{\delta}x + (\text{id} - \tau) \circ \beta(x)
\]
where $\delta$ is the Lie cobracket of $b$. The required covariance of the Lie coalgebra under the coaction here is

\begin{equation}
(\id \otimes \delta) \circ \beta = (\id \otimes (\id - \tau) \circ (\beta \otimes \id)) \circ \delta
\end{equation}

and ensures that $\delta$ on $b \bowtie f$ obeys the coJacobi identity.

The further covariance assumptions on $b$ are that its Lie bracket is covariant under the Lie coaction and its Lie cobracket is covariant under the Lie action. These assumptions are all needed to show that the semidirect Lie bialgebra in $\delta$ is a braided-Lie bialgebra in $\delta$. The case $\delta(\xi, \eta)$ is clear since $f$ is a Lie subalgebra and Lie subcoalgebra. The mixed case is

$$
\delta(\xi, x) = \delta(\xi \triangleright x) = \delta(\xi \triangleright x) + (\xi \triangleright x)(\delta(\xi, x))^{(1)} \otimes (\xi \triangleright x)^{(2)} - (\xi \triangleright x)^{(2)} \otimes (\xi \triangleright x)^{(1)}
$$

$$
= \xi \triangleright \delta x + \text{ad}_\xi((\id - \tau) \circ \beta(x)) + \xi^{(1)} \otimes \xi^{(2)} \triangleright x - \xi^{(2)} \triangleright x \otimes \xi^{(1)}
$$

$$
= \text{ad}_\xi \delta x - \text{ad}_x \delta \xi
$$

as required. We used the definition of $\delta$ for $b \bowtie f$, the covariance of $\delta$ under $\xi$, the crossed module condition, antisymmetry of $\delta \xi$ and $\text{ad}_\xi(x) = \xi \triangleright x = -\text{ad}_x(\xi)$ when viewed in the Lie algebra $b \bowtie f$. The remaining case is

$$
\delta([x, y])
$$

$$
= \delta([x, y]) + (\id - \tau) \beta([x, y])
$$

$$
= \text{ad}_x \delta y + y^{(2)} \otimes y^{(1)} \triangleright x - y^{(1)} \triangleright x \otimes y^{(2)} + x^{(1)} \otimes [x^{(2)}, y]
$$

$$
- [x^{(2)}, y] \otimes x^{(1)} - (x \triangleright y)
$$

$$
= \text{ad}_x \delta y - y^{(2)} \otimes y^{(1)} \triangleright x + y^{(1)} \otimes [x, y^{(2)}] - [x, y^{(2)}] \otimes y^{(1)}
$$

$$
+ y^{(2)} \otimes [x, y] - (x \triangleright y)
$$

$$
= \text{ad}_x \delta y + [x, y^{(1)}] \otimes y^{(2)} + y^{(1)} \otimes [x, y^{(2)}] - [x, y^{(2)}] \otimes y^{(1)}
$$

$$
- y^{(2)} \otimes [x, y^{(1)}] - (x \triangleright y)
$$

$$
= \text{ad}_x \delta y - \text{ad}_y \delta x
$$

where we used $\beta([x, y]) = y^{(2)} \otimes [x, y] - x^{(1)} \otimes [y, x]$ (covariance of the Lie bracket of $b$ under the Lie coaction) and the assumption that $\delta$ is a braided-Lie bialgebra in $M$ with infinitesimal braiding from Lemma 3.6. We then used the crossed module compatibility condition also from Lemma 3.6 and $\xi \triangleright x = -\text{ad}_x(\xi)$ to recognise the required result. It is easy to see that the projection $b \bowtie f \rightarrow f$ defined by setting elements of $b$ to zero is a Lie bialgebra map covering the inclusion $f \subset b \bowtie f$.

In the converse direction, we assume a split projection, i.e. a surjection $\pi : g \rightarrow f$ between Lie bialgebras covering an inclusion $i : f \rightarrow g$ of Lie bialgebras (so that $\pi \circ i = \id$). We define $b = \ker \pi$. Since this is a Lie ideal, it both forms a sub-Lie algebra of $g$ and is covariant under the action of $f$ given by pull-back along $i$ of $f$. Moreover, $g$ coacts on itself by its Lie cobracket $\delta$ (the adjoint coaction of any Lie bialgebra on itself) and hence...
push-out along $\pi$ is an $f$-coaction $\beta = (\pi \otimes \text{id}) \circ \delta$, which restricts to $\mathfrak{b}$ since $(\text{id} \otimes \pi)\beta(x) = (\pi \otimes \pi)\delta x = \delta \pi(x) = 0$ for $x \in \ker \pi$. This Lie action and Lie coaction fit together to form a Lie crossed module,

$$
\beta(\xi \triangleright x) = (\pi \otimes \text{id}) \circ \delta([i(\xi), x]) = (\pi \otimes \text{id})(\text{ad}(i(\xi))\delta x - \text{ad}_x \delta i(\xi))
$$

$$
= \pi([i(\xi), x_{(1)}]) \otimes x_{(2)} + \pi(x_{(1)}) \otimes [i(\xi), x_{(2)}] - \pi([x, i(\xi)_{(1)}]) \otimes i(\xi)_{(2)}
$$

$$
- \pi(i(\xi)_{(1)}) \otimes [x, i(\xi)_{(2)}]
$$

$$
= [x, \pi(x_{(1)})] \otimes x_{(2)} + \pi(x_{(1)}) \otimes \xi \triangleright x_{(2)} + \xi_{(1)} \otimes \xi_{(2)} \triangleright x
$$

$$
= ([\xi, ] \otimes \text{id} + \text{id} \otimes \xi \triangleright \beta(x) + (\delta \xi) \triangleright x
$$
as required. We used that $i$ is a Lie coalgebra map and $\pi$ a Lie algebra map, along with $x \in \ker \pi$ to kill the term with $\pi([x, i(\xi)_{(1)}])$.

Finally, we give $\mathfrak{b}$ a Lie cobracket $\delta$ as stated. Writing $p = \text{id} - i \circ \pi$, we have

$$
(\text{id} \otimes \delta) \circ \delta x
$$

$$
= p(x_{(1)}) \otimes p(p(x_{(2)}_{(1)}) \otimes p(x_{(2)}_{(2)}))
$$

$$
= (p \otimes p \otimes p)(\text{id} \otimes \delta) \delta x - p(x_{(1)}) \otimes p(i \circ \pi(x_{(2)}_{(1)}) \otimes p(i \circ \pi(x_{(2)}_{(2)}))
$$

$$
= (p \otimes p \otimes p)(\text{id} \otimes \delta) \delta x
$$
since $i \circ \pi$ is a Lie coalgebra map and $p \circ i \circ \pi = 0$. Hence $\delta$ obeys the coJacobi identity since $\delta$ does. Moreover, for all $x, y \in \ker \pi$,

$$
\delta([x, y]) = (\text{id} - i \circ \pi) \otimes (\text{id} - i \circ \pi) \delta x, y
$$

$$
= [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}] - [x, y_{(1)}] \otimes i \circ \pi(y_{(2)})
$$

$$
- i \circ \pi(y_{(1)}) \otimes [x, y_{(2)}] - (x \mapsto y)
$$
since $i \circ \pi([x, y_{(2)}]) = 0$ etc., as $i \circ \pi$ is a Lie algebra map. Also, from Lemma 3.6 and the form of $\beta$ and antisymmetry of $\delta$ we have

$$
\psi(x \otimes y) = [i \circ \pi(y_{(1)}), x] \otimes y_{(2)} + y_{(1)} \otimes [i \circ \pi(y_{(2)}), x] - (x \mapsto y).
$$

Then,

$$
\text{ad}_x \delta y - \text{ad}_y \delta x
$$

$$
= [x, (\text{id} - i \circ \pi)(y_{(1)})] \otimes (\text{id} - i \circ \pi)(y_{(2)})
$$

$$
+ (\text{id} - i \circ \pi)(y_{(1)}) \otimes [x, (\text{id} - i \circ \pi)(y_{(2)})] - (x \mapsto y)
$$

$$
= [x, y_{(1)}] \otimes y_{(2)} + y_{(1)} \otimes [x, y_{(2)}] - [x, i \circ \pi(y_{(1)})] \otimes y_{(2)} - [x, y_{(1)}] \otimes i \circ \pi(y_{(2)})
$$

$$
- i \circ \pi(y_{(1)}) \otimes [x, y_{(2)}] - y_{(1)} \otimes [x, i \circ \pi(y_{(2)})] - (x \mapsto y)
$$

$$
= \psi(x \otimes y) + \delta([x, y])
$$
as required. The additional terms $i \circ \pi(y_{(1)}) \otimes [x, i \circ \pi(y_{(2)})]$ etc. vanish as $i \circ \pi$ is a Lie coalgebra map and $x, y \in \ker \pi$. Hence $\mathfrak{b} = \ker \pi$ becomes a braided-Lie bialgebra in $\mathcal{M}$. One may then verify that the bisum Lie
bialgebra \( b \triangleright f \) coincides with \( g \) viewed as a direct sum \( b \oplus f \) of vector spaces according to the projection \( i \circ \pi \).

This is the Lie analogue of the braided groups interpretation \([14]\) of Radford’s theorem \([20]\). It tells us that braided-Lie bialgebras are rather common as they arise whenever we have a projection of ordinary Lie bialgebras. Finally, we provide the Lie analogue of the functor \([17]\) which connects biproducts and bosonisation.

**Lemma 3.8.** Let \( g \) be a quasitriangular Lie bialgebra. There is a monoidal functor \( gM \to g^0M \) respecting the infinitesimal braidings. It sends an action \( \triangleright \) to a pair \( (\triangleright, \beta) \) where \( \beta = r_{21} \triangleright \), the induced Lie coaction. The bosonisation of \( b \in gM \) in Theorem 3.5 can thereby be viewed as an example of a biproduct in Theorem 3.7.

**Proof.** We first verify that \( \beta(v) = r^{(2)} \otimes r^{(1)} \triangleright v \) defines a Lie coaction for any \( g \)-module \( V \ni v \). This follows immediately from the identity \((id \otimes \delta)r = [r^{(1)}, r^{(1)}] \otimes r^{(2)} \otimes r^{(2)}\) holding for any quasitriangular Lie bialgebra (following from the CYBE and \( \delta = dr \)). Thus,

\[
(id \otimes \delta)\beta(v) = r^{(2)} \otimes r^{(2)} \otimes r^{(1)}r^{(1)} \triangleright v - r^{(2)} \otimes r^{(2)} \otimes r^{(1)}r^{(1)} \triangleright v
\]

Thus,

\[
\beta(\xi \triangleright v) = r^{(2)} \otimes r^{(1)} \xi \triangleright v = r^{(2)} \otimes r^{(1)} \triangleright v + r^{(2)} \otimes \xi r^{(1)} \triangleright v
\]

as required. This fits together with the given action to form a Lie crossed module as

\[
\beta(\xi \triangleright v) = r^{(2)} \otimes r^{(1)} \xi \triangleright v = r^{(2)} \otimes r^{(1)} \triangleright v + r^{(2)} \otimes \xi r^{(1)} \triangleright v
\]

as required, using the quasitriangular form of \( \delta \xi \). More trivially, a morphism \( \phi : V \to W \) in \( gM \) is automatically an intertwiner of the induced coactions (since \( r^{(2)} \otimes r^{(1)} \triangleright \phi(v) = r^{(2)} \otimes \phi(r^{(1)} \triangleright v) \)) and hence a morphism in \( g^0M \). It is also clear that the functor respects tensor products. In this way, \( gM \) is a full monoidal subcategory.

Finally, we check that the infinitesimal braidings coincide. Computing \( \psi \) from Lemma 3.6 in the image of the functor, we have

\[
\psi(v \otimes w) = r^{(2)} \triangleright v \otimes r^{(1)} \triangleright w + r^{(1)} \triangleright v \otimes r^{(2)} \triangleright w - (v \leftrightarrow w)
\]

as required. From the form of the Lie cobracket in the bosonisation construction, it is clear that it can be viewed as a semidirect Lie coalgebra by the induced action, i.e., it can be viewed as a nontrivial construction for examples of bisum Lie algebras.

There is a dual theory of dual quasitriangular (or coquasitriangular) Lie bialgebras \([10]\) where the Lie bracket has a special form

\[
[\xi, \eta] = \xi^{(1)}r(\xi^{(2)}, \eta) + \eta^{(1)}r(\xi, \eta^{(2)}), \quad \forall \xi, \eta \in g,
\]
defined by a dual quasitriangular structure \( r : g \otimes g \to k \). This is required to obey the CYBE in a dual form

\[
(9) \quad r(\xi,\eta_{(1)})r(\eta_{(2)},\zeta) + r(\xi_{(1)},\eta)r(\xi_{(2)},\zeta) + r(\xi,\xi_{(1)})r(\eta,\zeta_{(2)}) = 0, \quad \forall \xi, \eta, \zeta \in g
\]

and \( 2r_+ \) is required to be invariant under the adjoint Lie coaction (= \( \delta \), the Lie cobracket) according to

\[
(10) \quad \psi(v \otimes w) = r(v^{(1)},w^{(1)})(v^{(2)} \otimes w^{(2)} - u^{(2)} \otimes v^{(2)})
\]

with respect to which we define a braided-Lie bialgebra in \( \mathcal{M} \). The Lie comodule transmutation theory associates to a map \( f \to g \) of Lie bialgebras with \( g \) dual quasitriangular, a braided-Lie bialgebra \( b(f,g) \in \mathcal{M} \).

For example, the Lie comodule version of Corollary 3.2 is \( g \in \mathcal{M} \) with the same Lie cobracket as \( g \), the adjoint coaction \( \delta \) and

\[
(11) \quad [\xi, \eta] = \eta_{(1)}2r_+(\xi,\eta_{(2)}) \quad \forall \xi, \eta \in g.
\]

A concrete example is provided by \( g^* \) when \( g \) is finite-dimensional quasitriangular. Then \( g^* \) is dual quasitriangular and its transmutation \( g^* \) coincides with \( (g^*)^* \) in (3) in Example 3.3.

Similarly, there is a functor \( \mathcal{M} \to \mathcal{M} \) sending a Lie coaction by \( g \) to a crossed module with an induced action \( \xi \triangleright v = r(v^{(1)},\xi)v^{(2)} \) and respecting the infinitesimal braiding. A braided-Lie bialgebra in \( b \in \mathcal{M} \) has a Lie bosonisation \( b \triangleright g \) given by a semidirect Lie cobracket by the given Lie coaction and semidirect Lie bracket given by the induced action. All of this dual theory follows rigorously and automatically by writing all constructions in terms of equalities of linear maps and then reversing all arrows. Such dualisation of theorems is completely routine in the theory of Hopf algebras, and similarly here. Hence we do not need to provide a separate proof of these assertions. Note that dualisation of theorems should not be confused with the dualisation of given algebras and coalgebras, which can be far from routine.

**Example 3.9.** Let \( g \) be a finite-dimensional quasitriangular Lie bialgebra and \( g^* \) the dual of its transmutation. Its bosonisation \( g^* \triangleright g \) is isomorphic as a Lie bialgebra to the Drinfeld double \( D(g) \).

**Proof.** The required isomorphism \( \theta : D(g) \to g^* \triangleright g \) is \( \theta(\phi) = \phi - r^{(2)}(\phi, r^{(1)}) \) and \( \theta(\xi) = \xi \) for \( \xi \in g \) and \( \phi \in g^* \). We check first that it is a Lie algebra map. The \([\xi, \eta]\) case is automatic as \( g \) is a sub-Lie algebra on both sides. The mixed case is

\[
[\theta(\xi), \theta(\phi)]_{\text{bos}} = [\xi, \phi - r^{(2)}(\phi, r^{(1)})]_{\text{bos}} = \xi \triangleright \phi - [\xi, r^{(2)}](\phi, r^{(1)}) = \phi^{(1)}(\phi^{(2)}), \xi + \xi^{(1)}(\phi, \xi^{(2)}) + r^{(2)}(\phi, [\xi, \phi^{(1)}])
\]
\[ \theta(\phi_{(1)}\langle \phi_{(2)}, \xi \rangle + \xi_{(1)}\langle \phi, \xi_{(2)} \rangle) = \theta([\xi, \phi]) \]

where \([ ]\)_{bos} is the Lie bracket of \(g^* \otimes g\). We use the definition of \(\theta\), the quasitriangular form of \(\delta\xi\), the action \(\xi \triangleright \phi = \phi_{(1)}\langle \phi_{(2)}, \xi \rangle\) for \(g^*\) and the cross relations in \(D(g)\) (as recalled in Lemma 3.6) to recognise the result. The remaining case is

\[ [\theta(\phi), \theta(\chi)]_{bos} \]
\[ = [(\phi - r^{(2)}(\phi, r^{(1)}), \chi - r^{(1)} \triangleright \chi)] \]
\[ = [r^{(2)}(\phi, r^{(1)}), \chi, r^{(1)}] - r^{(2)} \triangleright \chi \phi, r^{(1)} + r^{(2)} \triangleright \phi, r^{(1)} \]
\[ + \chi_{(1)}(2r_+, \phi \otimes \chi_{(2)}) \]
\[ = [r^{(2)}(\phi, r^{(1)}), \chi, r^{(1)}] + \chi_{(1)}(r, \chi_{(2)} \triangleright \phi) + \phi_{(1)}(r, \chi \otimes \phi_{(2)}) \]
\[ = [\chi, \phi] - r^{(2)}(\delta r^{(1)}, \chi \otimes \phi) = [\chi, \phi] - r^{(2)}([\chi, \phi], r^{(1)}) = \theta([\chi, \phi]) \]

as required since \(D(g)\) contains \(g^{*\text{op}}\) as a sub-Lie algebra. We used the definition of \(\theta\) and the Lie bracket (3) of \(g^*\) as a sub-Lie algebra of the bosonisation. We then used form of the action \(r^{(1)} \triangleright \chi\) etc. and combined the result with the \(2r_+\) term to recognise the Lie bracket \([\chi, \phi]\) (as in (8)) of the dual quasitriangular Lie bialgebra \(g^*\). We also use the quasitriangular form of \(g\) to recognise \(\delta r^{(1)}\).

Next, we verify that \(\theta\) is a Lie coalgebra map. This is automatic on \(\xi \in g\) as a sub-Lie bialgebra on both sides. The remaining case is

\[ \delta_{bos}(\delta(\phi)) \]
\[ = \delta_{bos}(\delta_+ \phi - \delta r^{(2)}(\phi, r^{(1)})) \]
\[ = \delta_+ \phi + r^{(2)} \otimes r^{(1)} \triangleright \phi - r^{(1)} \triangleright \phi \otimes r^{(2)} - r^{(2)} \otimes r^{(1)} \phi, \chi, r^{(1)} \]
\[ = \delta_+ \phi + r^{(2)} \otimes (\phi_{(1)}(\phi_{(2)}, r^{(1)})) - \phi_{(1)}(\phi_{(2)}, r^{(1)}) \otimes r^{(2)} - r^{(2)} \otimes r^{(1)} \phi, \chi, r^{(1)} \]
\[ = \phi_{(1)}(\phi_{(2)}, r^{(1)}) \otimes (\phi_{(2)} - r^{(2)}(\phi_{(2)}, r^{(1)})) = (\theta \otimes \theta) \delta \phi \]

using the Lie cobracket \(\delta_{bos}\) on \(g^* \otimes g\) from Theorem 3.5. The braided-Lie cobracket of \(g^*\) coincides with that of \(g^*\), i.e., \(\delta\phi = \delta \phi\). We also use the quasitriangular form of \(g\) to compute its Lie cobracket on \(r^{(2)}\).

Note that another way to present the result is that \(\pi(\xi) = \xi\) and \(\pi(\phi) = -r^{(2)}(\phi, r^{(1)})\) is a Lie bialgebra projection \(D(g) \rightarrow g\) split by the inclusion of \(g\), and recognise \(g^*\) as the image under \(\theta\) of the braided-Lie bialgebra kernel of this according to Theorem 3.7. The computations involved are similar to the above proofs for \(\theta\). Similar formulae are obtained if one takes \(\pi(\phi) = r^{(1)}(\phi, r^{(2)})\), corresponding to transmutation with respect to the conjugate quasitriangular structure. \(\square\)

This is the Lie version of the result for the quantum double of a quasitriangular Hopf algebra in [17]. It completes the partial result in [14] where, in the absence of a theory of braided-Lie bialgebras we could only give the
result $D(\mathfrak{g}) \cong \mathfrak{g} > \mathfrak{g}$ in the factorisable case (where $\mathfrak{g}^\ast \cong \mathfrak{g}$) and only as a Lie algebra isomorphism. Since $\mathfrak{g} > \mathfrak{g}$ by ad is easily seen to be isomorphic to a direct sum Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$, one recovers the result that $D(\mathfrak{g})$ in the factorisable case is a Lie algebra direct sum, but now with a certain Lie bialgebra structure (namely the double cross sum of $\mathfrak{g} \bowtie \mathfrak{g}$ in [10]).

More recently, we have obtained a more general ‘double bosonisation’ theorem [15] which yields as output quasitriangular Hopf algebras. It provides an inductive construction for factorisable quasitriangular Hopf algebras such as $U_q(\mathfrak{g})$. The Lie version of this is as follows. We suppose $\mathfrak{c}, \mathfrak{b}$ are dually paired in the sense of a morphism $(\cdot, \cdot) : \mathfrak{c} \otimes \mathfrak{b} \to k$ such that the Lie bracket of one is adjoint to the Lie cobracket of the other, and vice versa. The nicest case is where $\mathfrak{b}$ is finite-dimensional and $\mathfrak{c} = \mathfrak{b}^\ast$ as in Lemma 3.4, but we do not need to assume this for the main construction.

**Theorem 3.10.** For dually paired braided Lie bialgebras $\mathfrak{b}, \mathfrak{c} \in \mathfrak{g} \mathcal{M}$ the vector space $\mathfrak{b} \oplus \mathfrak{g} \oplus \mathfrak{c}$ has a unique Lie bialgebra structure $\mathfrak{b} \bowtie \mathfrak{g} \bowtie \mathfrak{c}^{op}$, the double-bosonisation, such that $\mathfrak{g}$ is a sub-Lie bialgebra, $\mathfrak{b}, \mathfrak{c}^{op}$ are sub-Lie algebras, and

\[
[x, \phi] = x \triangleright \phi, \quad [\xi, \phi] = \xi \triangleright \phi,
\]

\[
\delta x = \mathbf{d} x + r^{(2)} \otimes r^{(1)} \triangleright x - r^{(1)} \triangleright x \otimes r^{(2)},
\]

\[
\delta \phi = \mathbf{d} \phi + r^{(2)} \triangleright \phi \otimes r^{(1)} - r^{(1)} \otimes r^{(2)} \triangleright \phi
\]

$\forall x \in \mathfrak{b}, \xi \in \mathfrak{g}$ and $\phi \in \mathfrak{c}$. Here $\mathbf{d} x = x^{(1)} \otimes x^{(2)}$.

**Proof.** Here $\mathfrak{b}, \mathfrak{g}$ clearly form the bosonisation Lie bialgebra $\mathfrak{b} \bowtie \mathfrak{g}$ from Theorem 3.5. In the same way, we recognise $\mathfrak{c}^{op} \bowtie \mathfrak{g}$ as the bosonisation of $\mathfrak{c}^{op}$ as a braided-Lie bialgebra in the category of $\mathfrak{g}$-modules with opposite infinitesimal braiding (see the remark below Lemma 3.4). Since these are already known to form Lie bialgebras, the coJacobi identity for the double-bosonisation holds, as well as the 1-cocycle axiom for all cases except $\delta([x, \phi])$ mixing $\mathfrak{b}, \mathfrak{c}$. We outline the proof of this remaining case. From the definition of $\mathfrak{b} \bowtie \mathfrak{g} \bowtie \mathfrak{c}^{op}$, we have

\[
\delta([x, \phi]) = \delta(x^{(1)} \otimes \phi - \phi^{(1)} \otimes x) + 2r^{(1)} \langle \phi, r^{(2)} \triangleright x \rangle
\]

\[
= x^{(1)} \otimes x^{(2)} \langle \phi, r^{(2)} \triangleright x \rangle + r^{(2)} \otimes r^{(1)} \triangleright x^{(1)} \otimes r^{(2)} \langle \phi, r^{(2)} \triangleright x \rangle
\]

\[
+ 2\mathbf{d} r^{(1)} \langle \phi, r^{(2)} \triangleright x \rangle
\]

\[
\mathbf{d} \phi = \mathbf{d} x \otimes x \otimes [x, \phi] + [x, r^{(2)} \triangleright \phi] \otimes r^{(1)}
\]
+ \Delta r^2 \phi \otimes [x, \Delta r^1] - [x, \Delta r^1] \otimes \Delta r^2 \phi - r^2(1) \otimes [x, r^2 \phi] \\
= x_{(1)}(\phi_{(1)}, x_{(2)}) \otimes x_{(2)} + x_{(1)}(\phi_{(2)}, x_{(1)}) \otimes x_{(2)} + 2r^2_{+} \langle \phi_{(1)}, r^2_{+} \phi \otimes x_{(1)} \rangle \otimes \phi_{(2)} \\
= x_{(1)}(\phi_{(1)}, x_{(2)}) + x_{(1)}(\phi_{(2)}, x_{(1)}) + x_{(1)}(\phi_{(2)}, x_{(2)}) \\
= x_{(1)} \otimes 2r^2_{+} \langle \phi_{(1)}, r^2_{+} \phi \otimes x_{(1)} \rangle \\
= \Delta r^2 \phi \otimes r^2 \phi + r^2 \phi \otimes r^1 \phi x + r^1 \phi x \otimes r^2 \phi \\
= [x, r^2 \phi] \otimes r^1 - r^2 \otimes [x, r^2 \phi].

In a similar way, one has

\text{ad} \phi \delta x

\text{Ad} \phi \delta x

Adding the latter two expressions and comparing with $\delta([x, \phi])$ we see that the terms of the form $r^2 \phi \otimes r^1 \phi x$ etc. immediately cancel, the terms of the form $\phi_{(1)}(\phi_{(2)}, x_{(1)} \otimes x_{(2)}$ etc. (involving Lie cobrackets of both $x$ and $\phi$) cancel by antisymmetry of the Lie cobrackets, and the terms of the form $x_{(1)}(\phi, x_{(2)}) \otimes x_{(2)}$ etc. (involving iterated Lie cobrackets of either $x$ or $\phi$) cancel using antisymmetry of the Lie cobrackets and the cocompatibility of $\delta$.

Hence the 1-cocycle identity for this case reduces to the more manageable

\[ r^2 \otimes r^1 \phi x_{(1)}(\phi, x_{(2)}) + [2r^2_{+}, r^1] \otimes r^2 \phi_{(1)} \phi_{(2)}, x \]

where ‘-flip’ means to subtract all the same expressions with the opposite tensor product. We used antisymmetry of the Lie cobrackets and the quasitriangularity of $\mathfrak{g}$ for $\delta r^2_{+}$. One then has to put in the stated definitions of the Lie brackets $[x, r^2 \phi]$ and $r^1 \phi x$ and use $\mathfrak{g}$-cocompatibility of the pairing, and of the braided-Lie brackets and cobrackets to obtain equality.

Note that by comparing the Lie bosonisation formulæ with the braided group case, we can read off the Lie double-bosonisation formulæ from the braided group case given in the required left-module form in the appendix of [18].

The only subtlety is that in the Lie case we can eliminate the categorical pairing $\text{ev}$ (corresponding to the categorical dual $\text{b}^*$ in the finite-dimensional case): $\mathfrak{c}, \mathfrak{b}$ are categorically paired by $\text{ev} : \mathfrak{c} \otimes \mathfrak{b} \rightarrow k \iff \langle , \rangle = -\text{ev}$ is a $(\mathfrak{g}$-equivariant) ordinary duality pairing.
relations as stated. Finally, in [15] it is explicitly shown that the double-bosonisation is built on the tensor product vector space. The analogous arguments now prove that the Lie double bosonisation is built on the direct sum vector space.

Proposition 3.11. Let \( b \in \mathfrak{g} \mathcal{M} \) be a finite-dimensional braided-Lie bialgebra with dual \( b^\ast \). Then the double-bosonisation \( b \triangleright \mathfrak{g} \triangleleft b^{\ast \mathrm{op}} \) is quasitriangular, with

\[
r_{\mathrm{new}} = r + \sum_a f^a \otimes e_a,
\]

where \( \{e_a\} \) is a basis of \( b \) and \( \{f^a\} \) is a dual basis, and \( r \) is the quasitriangular structure of \( \mathfrak{g} \). If \( \mathfrak{g} \) is factorisable then so is the double-bosonisation.

**Proof.** We show first that the Lie cobracket of the double-bosonisation has the form \( \delta = dr_{\mathrm{new}} \). With summation over \( a \) understood, we have

\[
[\phi, r_{\mathrm{new}}^{(1)}] \otimes r_{\mathrm{new}}^{(2)} + r_{\mathrm{new}}^{(1)} \otimes [\phi, r_{\mathrm{new}}^{(2)}] = [\phi, r^{(1)}] \otimes r^{(2)} + r^{(1)} \otimes [\phi, r^{(2)}] - [\phi, f^a]^{b^\ast} \otimes e_a + f^a \otimes [\phi, e_a]
\]

\[
= -[\phi, f^a]^{b^\ast} \otimes e_a - r^{(1)} \triangleright [\phi \otimes r^{(2)}] - r^{(1)} \triangleright r^{(2)} \triangleright \phi
\]

\[
- f^a \otimes e_a(1) \langle \phi, e_a(2) \rangle - f^a \otimes \phi(1) \langle \phi(2), e_a \rangle - f^a \otimes 2r_+^{(1)} \langle \phi, r_+^{(2)} \triangleright e_a \rangle
\]

\[
= \delta \phi - r^{(1)} \triangleright \phi \otimes r^{(2)} - r^{(1)} \triangleright r^{(2)} \triangleright \phi + 2r_+^{(2)} \triangleright \phi \otimes r_+^{(1)} = \delta \phi
\]

as required. Here \( f^a, \phi \) is defined as \( f^a \otimes e_a(1) \langle \phi, e_a(2) \rangle \) since both evaluate against \( x \in b \) to \( x(1) \langle \phi, x(2) \rangle \). The suffix \( b^\ast \) is to avoid confusion with the Lie bracket inside the double-bosonisation, which is that of \( b^{\ast \mathrm{op}} \) on these elements. Similarly,

\[
[x, r_{\mathrm{new}}^{(1)}] \otimes r_{\mathrm{new}}^{(2)} + r_{\mathrm{new}}^{(1)} \otimes [x, r_{\mathrm{new}}^{(2)}] = -r^{(1)} \triangleright x \otimes r^{(2)} - r^{(1)} \otimes r^{(2)} \triangleright x + [x, f^a] \otimes e_a + f^a \otimes [x, e_a]
\]

\[
= -r^{(1)} \triangleright x \otimes r^{(2)} - r^{(1)} \otimes r^{(2)} \triangleright x + f^a \otimes [x, e_a] + x(1) \langle f^a, x(2) \rangle \otimes e_a
\]

\[
+ f^a(1) \langle f^a(2), x \rangle \otimes e_a + 2r_+^{(1)} \otimes r_+^{(2)} \triangleright x
\]

\[
= \delta x - r^{(1)} \triangleright x \otimes r^{(2)} + r^{(2)} \otimes r^{(1)} \triangleright x = \delta x.
\]

Here \( f^a(1) \langle f^a(2), x \rangle \otimes e_a = -f^a \otimes [x, e_a] \) as both evaluate against \( \phi \in b^\ast \) to \( \phi(1) \langle \phi(2), x \rangle \). Since the Lie cobracket of the double-bosonisation is antisymmetric, we conclude also that \( 2r_{\mathrm{new}} \) is ad-invariant.

Finally, we verify the CYBE for \( r_{\mathrm{new}} \). Actually, once \( \delta = dr_{\mathrm{new}} \) has been established, the CYBE is equivalent to

\[
(\delta \otimes \text{id}) r_{\mathrm{new}} = r^{(1)}_{\mathrm{new}} \otimes r^{(1)}_{\mathrm{new}} \otimes [r^{(2)}_{\mathrm{new}}, r^{(2)}_{\mathrm{new}}]
\]

(see [10]). Note that

\[
\delta f^a \otimes e_a = f^a \otimes f^b \otimes [e_a, e_b]
\]
(sum over $a,b$) since evaluation against $x,y \in b$ gives $[x,y]$ in both cases. Then

\[(\delta \otimes \text{id})r_{\text{new}} = (\delta \otimes \text{id})r + \delta f^a \otimes e_a\]

\[= r^{(1)} \otimes r^{(1)} \otimes [r^{(2)}, r^{(2)}] + f^a \otimes f^b \otimes [e_a, e_b] \]

\[+ r^{(2)} \triangleright f^a \otimes r^{(1)} \otimes e_a - r^{(1)} \otimes r^{(2)} \triangleright f^a \otimes e_a \]

\[+ r^{(1)} \otimes r^{(1)} \otimes [r^{(2)}, r^{(2)}] + f^a \otimes f^b \otimes [e_a, e_b] \]

\[+ f^a \otimes r^{(1)} \otimes [r^{(2)}, e_a] + r^{(1)} \otimes f^a \otimes [r^{(2)}, e_a] \]

as required. We used $g$-covariance of the pairing, so that $\xi \triangleright f^a \otimes e_a = -f^a \otimes \xi \triangleright e_a = -f^a \otimes [\xi, e_a]$ for all $\xi \in g$.

If $g$ is factorisable then $2r_{+\text{new}}$ as a map $(b \triangleleft \triangleright g \triangleright b^{*\text{op}})^* \rightarrow b \triangleleft \triangleright g \triangleright b^{*\text{op}}$ has $g$ in its image, by restricting to $g$. It has $b$ in its image by restricting to $b$, and $b^*$ in its image by restricting to $b^*$. So the double-bosonisation is again factorisable. Explicitly, if we denote by $K$ the bilinear form on $g$ corresponding to the inverse of $2r_+$ as a map, we have

\[K_{\text{new}}(x \oplus \xi \oplus \phi, y \oplus \eta \oplus \psi) = \langle \psi, x \rangle + K(\xi, \eta) + \langle \phi, y \rangle.\]

\[\square\]

There is also a more general double-bisum construction $b \triangleleft \triangleright f \triangleright c^{\text{op}}$ containing biproducts $b \triangleleft \triangleright f$ and $f \triangleright c^{\text{op}}$ (with $c, b \in \mathfrak{f} \mathcal{M}$ suitably paired braided-Lie bialgebras) and reducing to the double-bosonisation in the case when $c, b$ are in the image of the functor in Lemma 3.8.

Double bosonisation reduces to Drinfeld’s double $D(b)$ when $g = 0$ (then a braided-Lie bialgebra reduces to an ordinary Lie bialgebra). And because it preserves factorisability, it provides an inductive construction for new factorisable quasitriangular Lie bialgebras from old ones. We will see in the next section that it can be used as a coordinate free version of the idea of adjoining a node to a Dynkin diagram (adjoining a simple root vector in the Cartan-Weyl basis). Moreover, building up $g$ iteratively like this also builds up the quasitriangular structure $r$. Finally, the triangular decomposition implies, in particular, examples of Lie algebra splittings and hence of matched pairs of Lie algebras as in [9]. Thus, $b \triangleright \triangleleft g \triangleright b^{*\text{op}} = (b \triangleright \triangleleft g) \triangleright b^{*\text{op}}$ as Lie algebras, where $b \triangleright \triangleright g$ (the semidirect sum by the given action of $g$ on $b$) and $b^{*\text{op}}$ act on each other by

\[\phi \triangleright x = \langle \phi, x^{(1)} \rangle x^{(2)} - 2r_{+}^{(1)} \langle \phi, r_{+}^{(2)} \triangleright x \rangle,\]

\[\phi \triangleright \xi = 0, \quad \phi \triangleright x = \langle \phi^{(1)}, x \rangle \phi^{(2)}, \quad \phi \triangleright \xi = -\xi \triangleright \phi\]

for $x \in b$, $\phi \in b^*$, $\xi \in g$. This is immediate from the Lie bracket stated in Theorem 3.10.
4. Parabolic Lie bialgebras and Lie induction.

In this section we give some concrete examples and applications of the above theory. We work over \( \mathbb{C} \). We begin with the simplest example of a braided-Lie bialgebra, with zero Lie bracket and zero Lie cobracket. According to Definition 2.2 this means precisely modules of our background quasitriangular Lie bialgebras for which the infinitesimal braiding cocycle \( \psi \) vanishes.

Proposition 4.1. Let \( \mathfrak{g} \) be a semisimple factorisable (s.s.f) Lie bialgebra and \( \mathfrak{b} \) an isotypical representation such that \( \Lambda^2 \mathfrak{b} \) is isotypical. Then \( \mathfrak{b} \) with zero bracket and zero cobracket is a braided-Lie bialgebra in \( \tilde{\mathfrak{g}} \mathcal{M} \), where \( \tilde{\mathfrak{g}} \) is a central extension.

Proof. Let \( c = r_+^{(1)} r_+^{(2)} \) in \( U(\mathfrak{g}) \). Since \( r_+ \) is ad-invariant, \( c \) is central. Moreover, \( 2r_+ = \Delta c - (c \otimes 1 + 1 \otimes c) \) where \( \Delta \) is the coproduct of \( U(\mathfrak{g}) \) as a Hopf algebra. Since \( \mathfrak{b} \) is assumed isotypical, the action of \( c \) on it is by multiplication by a scalar, say \( \lambda_1 \). Since \( \Lambda^2 \mathfrak{b} \) is assumed isotypical, the action of \( c \) on it, which is the action of \( \Delta c \) in each factor, is also multiplication by a scalar, say \( \lambda_2 \). Then \( \psi(x \otimes y) = (\Delta c - (c \otimes 1 + 1 \otimes c)) \triangleright (x \otimes y - y \otimes x) = (\lambda_2 - 2\lambda_1)(x \otimes y - y \otimes x) = \lambda(x \otimes y - y \otimes x) \) say, where \( \lambda \) is a constant.

Now, \( \mathfrak{b} \) with the zero bracket and cobracket is not a braided group in \( \mathfrak{g} \mathcal{M} \) unless our cocycle \( \psi \) vanishes. However, in the present case we can neutralise the cocycle with a central extension. Thus, let \( \tilde{\mathfrak{g}} = \mathbb{C} \oplus \mathfrak{g} \) with \( \mathbb{C} \) spanned by \( \varsigma \), say. We take the Lie bracket, quasitriangular structure and Lie cobracket

\[
[\xi, \varsigma] = 0, \quad \tilde{r} = r - \frac{\lambda}{2} \varsigma \otimes \varsigma, \quad \delta \varsigma = 0
\]

for all \( \xi \in \mathfrak{g} \). In this way, \( \tilde{\mathfrak{g}} \) becomes a quasitriangular Lie bialgebra. We consider \( \mathfrak{b} \in \tilde{\mathfrak{g}} \mathcal{M} \) by \( c \triangleright x = x \) for all \( x \in \mathfrak{b} \). The infinitesimal braiding on \( \mathfrak{b} \) in this category is \( \tilde{\psi}(x \otimes y) = 2\tilde{r} \triangleright (x \otimes y - y \otimes x) = \psi(x \otimes y) - \lambda(x \otimes y - y \otimes x) = 0 \). So \( \mathfrak{b} \) is a braided-Lie bialgebra in this category.

The constant \( \lambda \) is the infinitesimal analogue of the so-called quantum group normalisation constant. The central extension is the analogue of the central extension by a ‘dilaton’ needed for the quantum planes to be viewed as braided groups [7]. We see now the infinitesimal analogue of this phenomenon.

Next, we can apply Theorem 3.5 and obtain a Lie bialgebra \( \mathfrak{b} \bowtie \tilde{\mathfrak{g}} \) as the bosonisation of \( \mathfrak{b} \). Moreover, double-bosonisation provides a still bigger and factorisable Lie algebra containing \( \mathfrak{b} \bowtie \tilde{\mathfrak{g}} \).

Corollary 4.2. Let \( \mathfrak{g} \) be simple and strictly quasitriangular, and \( \mathfrak{b} \) a finite-dimensional irreducible representation with \( \Lambda^2 \mathfrak{b} \) isotypical. Then the double bosonisation \( \mathfrak{b} \bowtie \tilde{\mathfrak{g}} \bowtie \mathfrak{b}^* \) from Theorem 3.10 is again simple, strictly quasitriangular and of strictly greater rank.
Proof. The Lie bracket in the double-bosonisation in Theorem 3.10, and the form of \( \tilde{r} \) are
\[
[\xi, x] = \xi \triangleright x, \quad [\varsigma, x] = x, \quad [\xi, \phi] = \xi \triangleright \phi, \quad [\varsigma, \phi] = -\phi, \quad [\xi, \varsigma] = 0
\]
for all \( \xi \in \mathfrak{g} \), \( x \in \mathfrak{b} \) and \( \phi \in \mathfrak{g}^* \). Consider \( I \subseteq \mathfrak{b} \oplus \mathfrak{g} \oplus \mathbb{C} \oplus \mathfrak{b}^* \) an ideal of the double-bosonisation. Let \( I_{\mathfrak{b}}, I_{\mathfrak{b}}^*, I_{\mathfrak{g}}, I_{\mathbb{C}} \) be the components of \( I \) in the direct sum. By the relation \([\xi, x] = \xi \triangleright x\), \( I_{\mathfrak{b}} \) is a subrepresentation under \( \mathfrak{g} \). Since \( \mathfrak{b} \) is irreducible, \( I_{\mathfrak{b}} \) is either zero or \( \mathfrak{b} \). Similarly for \( I_{\mathfrak{b}}^* \). Likewise \( I_{\mathfrak{g}} \) is zero or \( \mathfrak{g} \) as \( \mathfrak{g} \) is simple. Finally, \( I_{\mathbb{C}} \) is zero or \( \mathbb{C} \) by linearity. We therefore have 16 possibilities to consider for whether \( \mathbb{C}, \mathfrak{g}, \mathfrak{b}, \mathfrak{b}^* \) are contained or not in \( I \). (i) If \( \mathfrak{g} \) is contained, then since \( \mathfrak{b} \) is irreducible, the relation \([\xi, x] = \xi \triangleright x\) spans \( \mathfrak{b} \) for any fixed \( x \), and hence is certainly not always zero. So \( \mathfrak{b} \) is contained, and likewise \( \mathfrak{b}^* \) is contained if \( \mathfrak{g} \) is. In this case, the \([x, \phi]\) relation means that \( \mathbb{C} \) is contained and \( I \) is the whole space. (ii) If \( \mathfrak{b} \) is contained then the \([x, \phi]\) relation and \( 2r_+ \) nondegenerate means that \( \mathfrak{g} \) and \( \mathbb{C} \) are contained and hence \( I \) is the whole space. (iii) Similarly if \( \mathfrak{b}^* \) is contained. (iv) Finally, if \( \mathbb{C} \) is contained then the relation \([\varsigma, x] = x\) implies that \( \mathfrak{b} \) is contained and hence \( I \) is the whole space. Hence \( I \) is zero or the whole space, as required.

The new quasitriangular structure is non-zero since its component in \( \mathfrak{g} \otimes \mathfrak{g} \) is non-zero. The rank is clearly increased by at least 1 due to the addition of \( \varsigma \).

Thus the double-bosonisation in Theorem 3.10 provides an inductive construction for simple strictly quasitriangular Lie bialgebras. It is possible to see that the fundamental representations of \( su_n \) or \( so_n \) take us up to \( su_{n+1} \) and \( so_{n+1} \), i.e., precisely take us up the ABD series in the usual classification of Lie algebras. Moreover, we see the role of the single bosonisation in Theorem 3.5:

Example 4.3. Consider \( \mathfrak{g} = su_2 \) with the Drinfeld-Sklyanin quasitriangular structure. The 2-dimensional irreducible representation \( \mathfrak{b} \) is a braided-Lie bialgebra via Proposition 4.1. Its bosonisation \( \mathbb{C}^2 \bowtie \tilde{su}_2 \) is the maximal parabolic of the double bosonisation \( \mathbb{C}^2 \bowtie \tilde{\mathfrak{g}} \bowtie \mathbb{C}^2 = su_3 \). Explicitly, it is the Lie algebra of \( su_2 \) and
\[
[x, y] = 0, \quad [X_+, x] = 0, \quad [X_+, y] = x, \quad [X_-, x] = y, \quad [X_-, y] = 0
\]
\[
[H, x] = x, \quad [H, y] = -y, \quad [\varsigma, H] = 0, \quad [\varsigma, X_\pm] = 0,
\]
where \( \{x, y\} \) are a basis of \( \mathbb{C}^2 \) and \( H, X_\pm \) are the standard \( su_2 \) Chevalley generators. The Lie cobracket on the generators is
\[
\delta \varsigma = 0, \quad \delta X_\pm = \frac{1}{2} X_\pm \wedge H, \quad \delta x = \frac{1}{2} x \wedge h
\]
where $h = -\frac{1}{2}H - \frac{3}{2}\varsigma$ and $\wedge = (\text{id} - \tau) \circ \otimes$.

**Proof.** Note that we work over $\mathbb{C}$, but there is are natural real forms justifying the notation. Here $\Lambda^2 b$ is the 1-dimensional (i.e., spin 0) representation of $su_2$. The standard quasitriangular structure of $su_2$ is

$$r = \frac{1}{4}H \otimes H + X_+ \otimes X_-.$$

Then $c = r_+(1) r_+(2)$ is twice the quadratic Casimir in its usual normalisation. Hence its value in the $(2j + 1)$ dimensional (i.e., spin $j$) irreducible representation is $j(j + 1)$. In the present case, we have $\lambda = 0, (0 + 1) - 2 \frac{1}{2}(1 + 1) = -\frac{3}{2}$ in Proposition 4.1. We therefore make the central extension to $\tilde{\mathfrak{g}}$ and apply Theorem 3.5. The Lie algebra of the bosonisation is given by the action of $\tilde{\mathfrak{g}}$.

Its explicit form in the representation $\rho(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\rho(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is $X_+ \triangleright x = 0$, $X_+ \triangleright y = x$, $X_- \triangleright x = y$, $X_- \triangleright y = 0$, $H \triangleright x = x$ and $H \triangleright y = -y$, giving the Lie bracket stated. The Lie cobracket is $\delta x = 0 + \tilde{r}^{(2)} \wedge \tilde{r}^{(1)} \triangleright x = \frac{1}{4}H \wedge H \triangleright x + \frac{3}{2}x \wedge \triangleright x = \frac{1}{2}x \wedge h$ as stated. We identify $X_\pm = X_{\pm 1}$, $H = H_1$ as a sub-Lie algebra of $su_3$ and $x = X_{-2}, h = H_2$ as the remaining Chevalley generators of its standard maximal parabolic. Finally, let $\mathfrak{b}^*$ have dual basis $\{\phi, \psi\}$. By a similar computation to the above, we obtain $\tilde{su}_2 \otimes \mathbb{C}^2$ with Lie bracket

$$[\phi, \psi] = 0, \quad [X_+, \phi] = -\psi, \quad [X_+, \psi] = 0, \quad [X_-, \phi] = 0, \quad [X_-, \psi] = -\phi, \quad [H, \phi] = -\phi, \quad [H, \psi] = \psi, \quad [\varsigma, \phi] = -\phi, \quad [\varsigma, \psi] = -\psi.$$

Among the further $\mathfrak{b}, \mathfrak{b}^*$ brackets in the double bosonisation in Theorem 3.10, we have $[x, \phi] = 2r_+(1) \langle \phi, r_+(2) \triangleright x \rangle + \frac{3}{2} \varsigma \langle \phi, x \rangle = \frac{1}{2}H \langle \phi, H \triangleright x \rangle + 0 + \frac{3}{2} \varsigma \langle \phi, x \rangle = -h$. From these relations we find that $\phi = X_{+2}$ and $\psi = X_{+12}$ explicitly identifies the double bosonisation as $su_3$. The Lie cobracket on $\phi$ is $\delta \phi = \tilde{r}^{(2)} \triangleright \phi \wedge \tilde{r}^{(1)} = \frac{1}{4}H \triangleright \phi \wedge H + \frac{3}{2} \varsigma \triangleright \phi \wedge \varsigma = \frac{1}{2} \phi \wedge h$. This conforms with the standard Lie cobracket for $su_3$. Indeed, the quasitriangular structure of the double bosonisation in Theorem 3.10 reproduces the Drinfeld-Sklyanin quasitriangular structure of $su_3$. \hfill $\Box$

This is far from the only braided-Lie bialgebra in the category of $\tilde{su}_2$-modules, however.

**Example 4.4.** Consider $\mathfrak{g} = su_2$ with the Drinfeld-Sklyanin quasitriangular structure. The 3-dimensional irreducible representation $\mathfrak{b}$ is a braided-Lie bialgebra via Proposition 4.1. Its bosonisation $\mathbb{R}^3 \otimes \tilde{so}_3$ is the maximal parabolic of the double bosonisation $so_5$. Explicitly, it is the Lie algebra of $so_3$ and

$$[x_i, x_j] = 0, \quad [e_i, x_j] = \sum_k e_{ijk} x_k, \quad [\varsigma, x_j] = x_j$$
where \(i, j, k = 1, 2, 3\) and \(\epsilon\) is the totally antisymmetric tensor with \(\epsilon_{123} = 1\). Here \(\{e_i\}\) are the vector basis of \(so_3\). The Lie cobracket is

\[
\delta e_1 = ie_1 \wedge e_3, \quad \delta e_2 = ie_2 \wedge e_3, \quad \delta e_3 = 0, \quad \delta \varsigma = 0,
\]

\[
\delta x_1 = (ie_1 + e_2) \wedge x_3 + x_2 \wedge e_3 + \varsigma \wedge x_1,
\]

\[
\delta x_2 = x_3 \wedge (e_1 - ie_2) + e_3 \wedge x_1 + \varsigma \wedge x_2,
\]

\[
\delta x_3 = (e_1 - ie_2) \wedge x_2 + x_1 \wedge (ie_1 + e_2) + \varsigma \wedge x_3.
\]

**Proof.** Here \(\Lambda^2 b\) is also the 3-dimensional (i.e., spin 1) representation. Hence, from the first part of the proof of Proposition 4.1, we have \(\lambda = 1, (1 + 1) - 2.1. (1 + 1) = -2\). The Lie algebra \(so_3\) in the vector basis is \([e_1, e_2] = e_3\) and cyclic rotations of this, and the Drinfeld-Sklyanin quasitriangular structure in this basis is [10, Ex. 8.1.13]

\[
r = -\sum_i e_i \otimes e_i + i(e_1 \otimes e_2 - e_2 \otimes e_1).
\]

We add \(\varsigma \otimes \varsigma\) to give the quasitriangular structure \(\tilde{r}\). The action on \(\mathbb{C}^3\) with basis \(x_i\) is \([e_1, x_2] = x_3\) and cyclic rotations of this. This immediately provides the Lie algebra of the bosonisation. The Lie cobracket from Theorem 3.5 is

\[
\delta x_i = ie_2 \wedge [e_1, x_i] - ie_1 \wedge [e_2, x_i] + \sum_{j,k} e_j \wedge \epsilon_{ijk} x_k + \varsigma \wedge x_i
\]

with computes as stated. \(\square\)

This example is manifestly the Lie algebra of motions plus dilation of \(\mathbb{R}^3\), as a sub-Lie algebra of the conformal Lie algebra \(so(1, 4)\), equipped now with a Lie bialgebra structure. At the level of complex Lie algebra, it is the maximal parabolic of \(so_5\). The generator \(\varsigma\) is called the ‘dilaton’ in the corresponding quantum groups literature. We likewise obtain natural maximal parabolics for the whole ABD series by bosonisation of the fundamental representation \(b\).

On the other hand, these steps for other Lie algebras can involve less trivial braided-Lie bialgebras \(b\) (with non-zero bracket and cobracket). The general case is as follows. We consider simple Lie algebras \(g\) associated to root systems in the usual conventions. Positive roots are denoted \(\alpha\), with length \(d_{\alpha}\). The Cartan-Weyl basis has root vectors \(X_\pm \alpha\) and Cartan generators \(H_i\) corresponding to the simple roots \(\alpha_i\). We define \(d_{\alpha} H_\alpha = \sum_i n_i d_i H_i\) if \(\alpha = \sum_i n_i \alpha_i\). We take the Drinfeld-Sklyanin quasitriangular structure in its general form

\[
r = \sum_{\alpha} d_{\alpha} X_\alpha \otimes X_{-\alpha} + \frac{1}{2} \sum_{ij} A_{ij} H_i \otimes H_j,
\]
where $A_{ij} = d_i(a^{-1})_{ij}$. Here $a$ is the Cartan matrix. The corresponding Lie cobracket is $\delta X_{\pm i} = \frac{d_i}{2} X_{\pm i} \wedge H_i$ and $\delta H_i = 0$ on the generators.

**Proposition 4.5.** Let $i_0$ be a choice of simple root such that its deletion again generates the root system of a simple Lie algebra, $\mathfrak{g}_0$. Let $\mathfrak{b} \subset \mathfrak{g}$ be the standard (negative) Borel and let $\mathfrak{f} \subset \mathfrak{b}$ denote the sub-Lie algebra excluding all vectors generated by $X_{-i_0}$. Both $\mathfrak{b}$ and $\mathfrak{f}$ are sub-Lie bialgebras of $\mathfrak{g}$ and

$$\mathfrak{b} \xrightarrow{\pi} \mathfrak{f}, \quad \pi(H_i) = H_i, \quad \pi(X_{-\alpha}) = \begin{cases} 0 & \text{if } \alpha \text{ contains } \alpha_{i_0} \\ X_{-\alpha} & \text{else} \end{cases}$$

is a split Lie bialgebra projection. Then $\mathfrak{b} = \ker \pi$ is the Lie ideal generated by $X_{-i_0}$ in $\mathfrak{b}$ and is a braided-Lie bialgebra in $\mathfrak{f}\mathcal{M}$ by Theorem 3.7.

**Proof.** Here $\mathfrak{f}$ is generated by all the $H_i$ and only those $X_{-j}$ where $j \neq i_0$, i.e. spanned by the $H_i$ and $\{X_{-\alpha}\}$ such that $\alpha$ does not contain $\alpha_{i_0}$. It is clearly a sub-Lie algebra of $\mathfrak{b}$. We show first that it is a sub-Lie bialgebra. First of all, note that the Lie coproduct in $\mathfrak{g}$ has the general form

$$\delta X_{\pm \alpha} = \frac{d_\alpha}{2} X_{\pm \alpha} \wedge H_\alpha + \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} X_{\pm \beta} \wedge X_{\pm \gamma}$$

where the sum is over positive root $\beta, \gamma$ adding up to $\alpha$ and the $c$ are constants. The proof is by induction (being careful about signs). From the Lie bialgebra cocycle axiom and the induction hypothesis,

$$\delta([X_i, X_\alpha])$$

$$= \frac{d_\alpha}{2} [X_i, X_\alpha] \wedge H_\alpha + \frac{d_\alpha}{2} X_\alpha \wedge [X_i, H_\alpha] + \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} [X_i, X_\beta] \wedge X_\gamma$$

$$+ \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} X_\beta \wedge [X_i, X_\gamma] - \frac{d_i}{2} [X_\alpha, X_i] \wedge H_i - \frac{d_i}{2} X_i \wedge [X_\alpha, H_i]$$

$$= \frac{d_{\alpha + i}}{2} [X_i, X_\alpha] \wedge H_{\alpha + i} + \alpha(d_i H_i) X_i \wedge X_\alpha$$

$$+ \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} [X_i, X_\beta] \wedge X_\gamma + \sum_{\beta + \gamma = \alpha} c_{\beta, \gamma} X_\beta \wedge [X_i, X_\gamma]$$

if $\alpha + \alpha_i$ is a positive root. We used the identities $[d_\alpha H_\alpha, X_i] = \alpha(d_i H_i) X_i$ and $[d_i H_i, X_\alpha] = \alpha(d_\alpha H_\alpha) X_i$. Since all positive root vectors are obtained by iterated Lie brackets of the $X_i$, we conclude the result (the argument for negative roots is similar).

From this form, it is clear first of all that $\delta$ restricts to $\mathfrak{b} \rightarrow \mathfrak{b} \otimes \mathfrak{b}$, so this becomes a sub-Lie bialgebra of $\mathfrak{g}$ (this is well-known). Moreover, if $\alpha$ does not involve $\alpha_{i_0}$ then neither can positive $\beta, \gamma$ such that $\beta + \gamma = \alpha$. Hence $\mathfrak{f}$ is a Lie sub-bialgebra of $\mathfrak{b}$. 

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Finally, $\pi$ is clearly a Lie algebra map by considering the cases separately. For elements of $\mathfrak{f} \otimes \mathfrak{f}$ we know that $\pi \circ [\cdot, \cdot] = [\pi(\cdot), \pi(\cdot)]$ since $\mathfrak{f}$ is closed, while if $\alpha$ involves $\alpha_{i_0}$ then so does $\alpha + \beta$ and $\pi([X_{-\alpha}, X_{-\beta}]) = 0 = [\pi(X_{-\alpha}), \pi(X_{-\beta})]$. Moreover, $\pi$ is a Lie coalgebra map on $\mathfrak{f}$ since $\delta \mathfrak{f} \subset \mathfrak{f} \otimes \mathfrak{f}$ as shown above. Finally, $(\pi \otimes \pi)\delta X_{-i_0} = 0 = \delta \pi(\pi(X_{-i_0}))$ from the simple form of $\delta$ on the generators.

Therefore we may apply Theorem 3.7 and obtain a braided-Lie bialgebra $\mathfrak{b} = \ker \pi$. Here $\mathfrak{b} \subset \mathfrak{b}_-$ is the Lie ideal generated by $X_{-i_0}$, i.e., spanned by $\{X_{-\alpha}\}$ where $\alpha$ contains $\alpha_{i_0}$.

The braided-Lie cobracket of $\mathfrak{b}$ from Theorem 3.7 is

$$\delta X_{-\alpha} = \sum c_{-\beta, -\gamma} X_{-\beta} \wedge X_{-\gamma},$$

the part of the Lie cobracket $\delta X_{-\alpha}$ in which both $\beta, \gamma$ contain $\alpha_{i_0}$. The action of $\mathfrak{f}$ is by Lie bracket in $\mathfrak{b}_-$ and the Lie coaction of $\mathfrak{f}$ is $\beta(X_{-\alpha}) = -\frac{d_\beta}{2}H_{\alpha} \otimes X_{-\alpha} + \sum c_{-\beta, -\gamma} X_{-\beta} \otimes X_{-\gamma}$ where the sum is the part of $\delta X_{-\alpha}$ where $\beta$ does not contain $\alpha_{i_0}$. \hfill $\Box$

This constructs the required braided-Lie bialgebra for the general case. Although obtained here in the category of $D(\mathfrak{f})$-modules, this action is compatible with an action of the central extension $\tilde{\mathfrak{g}}_0 \subset \mathfrak{g}$. It is easy to see that there is a unique element $\zeta \in \mathfrak{g}$, $\zeta \notin \mathfrak{g}_0$ which commutes with the image of $\mathfrak{g}_0$. It is determined by the Cartan matrices of $\mathfrak{g}, \mathfrak{g}_0$. Viewed in $\mathfrak{g}$, this $\tilde{\mathfrak{g}}_0$ acts on $\mathfrak{g}$ by the adjoint action and this action restricts to $\mathfrak{b}$. In this way, $\mathfrak{b}$ becomes a braided-Lie bialgebra in $\tilde{\mathfrak{g}}_0 \mathcal{M}$. One may then recover $\mathfrak{g} = \mathfrak{b} \op \tilde{\mathfrak{g}}_0 \op \mathfrak{b}^{\op}$ from Theorem 3.10.

**Example 4.6.** When $\mathfrak{g} = \mathfrak{g}_2$ and $\mathfrak{g}_0 = \mathfrak{su}_2$, we obtain the 5-dimensional braided-Lie bialgebra where $\mathfrak{su}_2$ acts as the $4 \oplus 1$ dimensional (i.e., the spin $\frac{3}{2}$ and spin 0 representations). Both the Lie bracket and the Lie cobracket are not identically zero.

**Proof.** We take the Cartan matrix for $\mathfrak{g}$ as \(\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\). We take $i_0 = 1$ so that the required $\mathfrak{su}_2$ is spanned by $H_2, X_{\pm 2}$. The negative roots vectors $X_{-1}, X_{-21}, X_{-221}, X_{-2221}$ span the 4-dimensional representation of $\mathfrak{su}_2$, the eigenvalues of the adjoint action of $-\frac{1}{2}H_2$ being $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ respectively. These and the remaining negative root vector $X_{-12221}$ (which forms a 1-dimensional trivial representation of $\mathfrak{su}_2$) are a basis of $\mathfrak{b}$. We then restrict the Lie bracket to $\mathfrak{b}$, the only non-zero entries being

\[[X_{-1}, X_{-2221}] = X_{-12221} = [X_{-221}, X_{-21}]\]

This is a central extension (by a cocycle) of the zero bracket on the 4-dimensional representation. The Lie cobracket can then be computed by
projection of the Lie cobracket in $g_2$. Since (as one may easily verify) the infinitesimal braiding is nontrivial, both the braided-Lie bracket and braided-Lie cobracket on $\mathfrak{b}$ are not identically zero. The element $\zeta = -2H_1 - H_2$ commutes with $su_2$ and acts as the identity in the 4-dimensional part of $\mathfrak{b}$. □

**Example 4.7.** When $\mathfrak{g} = \mathfrak{sp}_6$ and $\mathfrak{g}_0 = \mathfrak{sp}_4$, we obtain the 5-dimensional braided-Lie bialgebra where $\mathfrak{sp}_4$ acts in the $4 \oplus 1$ dimensional representation. Here the 4-dimensional representation is the fundamental one of $\mathfrak{sp}_4$. Both the Lie bracket and the Lie cobracket are not identically zero.

**Proof.** We take the Cartan matrices for $\mathfrak{g}$ and $\mathfrak{g}_0$ as

$$a = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}, \quad a_0 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$

where $i_0 = 1$. We identify $\mathfrak{sp}_4 = C_2$ inside $\mathfrak{sp}_6$ as the root vectors $X_{\pm 2}, X_{\pm 3}, X_{\pm 23}$ and Cartan vectors $H_2, H_3$. The negative root vectors $X_{-1}, X_{-12}, X_{-123}, X_{-1223}$ form the 4-dimensional representation of $\mathfrak{sp}_4$. These and the remaining negative root vector $X_{-11223}$ (which forms a 1-dimensional trivial representation of $\mathfrak{sp}_4$) are a basis of $\mathfrak{b}$. We then restrict the Lie bracket to $\mathfrak{b}$ and find that this is again a cocycle central extension of the zero Lie bracket on the 4-dimensional representation, the only non-zero entries being

$$[X_{-12}, X_{-123}] = \frac{1}{2}X_{-11223}, \quad [X_{-1}, X_{-1223}] = X_{-11223}.$$ 

The infinitesimal braiding and the Lie cobracket are also nontrivial, as one may verify by further computation. The element $\zeta = -(H_1 + H_2 + H_3)$ commutes with $\mathfrak{sp}_4$ and acts as the identity in the 4-dimensional part of $\mathfrak{b}$. □

These examples show that the general case need not depart too far from the setting of Proposition 4.1 and Corollary 4.2; we need to make a central extension of the underlying irreducible representation to define $\mathfrak{b}$. By construction, $\mathfrak{b} \bowtie \mathfrak{g}_0$ is once again the maximal parabolic of $\mathfrak{g}$ associated to $\alpha_{i_0}$. A similar construction works for more roots missing, giving non-maximal parabolics of the double-bosonisation. We simply define $\pi$ setting to zero all the root vectors containing the roots to be deleted in defining $\mathfrak{g}_0$. Clearly, the extreme example of this is $\mathfrak{f} = \mathfrak{t}$ (the Cartan subalgebra) so that $\pi(H_i) = H_i$ and $\pi(X_{-\alpha}) = 0$. Then $\mathfrak{b} = \mathfrak{n}_-$ (the Lie algebra generated by the $X_{-1}$) is a braided-Lie bialgebra in $\mathfrak{t}\mathcal{M}$ with

$$\delta X_{-i} = 0, \quad \psi(X_{-\alpha} \otimes X_{-\beta}) = (\alpha, \beta)X_{-\alpha} \wedge X_{-\beta},$$

$$h \triangleright X_{-\alpha} = -\alpha(h), \quad \beta(X_{-\alpha}) = -\frac{d\alpha}{2}H_\alpha \otimes X_{-\alpha},$$

where $i_0 = 1$.
for all \( h \in \mathfrak{t} \). The coaction here is induced from the action as in Lemma 3.8, where \( \mathfrak{t} \) is a quasitriangular Lie algebra with zero bracket, zero cobracket and \( r = \frac{1}{2} \sum_{ij} A_{ij} H_i \otimes H_j \). In this way we can also view \( \mathfrak{n}_- \in \mathfrak{t} \mathcal{M} \) and \( \mathfrak{g} = \mathfrak{n}_- \bowtie \mathfrak{b} \bowtie \mathfrak{n}_+ \) via Theorem 3.10, where we identify \( (\mathfrak{n}_-)^{\text{op}} = \mathfrak{n}_+ \) via the Killing form.

5. Concluding remarks.

We have given here the basic theory of braided-Lie algebras, obtained by infinitesimalising the existing theory of braided groups. We also outlined in Section 4 its application to the inductive construction of simple Lie algebras with their standard quasitriangular structures. Further variations of these constructions are certainly possible, and by making them one should be able to also obtain the other strictly quasitriangular Lie bialgebras structures in the Belavin-Drinfeld classification [1]. For example, there is a twisting theory of quantum groups [4] and braided groups [19]. An infinitesimal version of the latter would allow one to introduce additional twists at each stage of the inductive construction of the simple Lie algebra.

Also, although we have (following common practice) named our Lie algebras by their natural real forms, our Lie algebras in Section 4 were complex ones. There is a theory of \(*\)-braided groups (real forms of braided groups) as well as their corresponding bosonisations and double-bosonisations [19], [18]. The infinitesimal version of these should yield, for example, \( \mathfrak{so}(1,4) \) as a real form arising from the double-bosonisation of the 3-dimensional braided-Lie bialgebra in Example 4.4. The construction of natural compact real forms and the classification of real forms would be a further goal. These are some directions for further work.

Finally, just as Lie bialgebras extend to Poisson-Lie groups, so braided-Lie bialgebra structures typically extend to the associated Lie group \( B \) of \( \mathfrak{b} \), at least locally. First, one needs to exponentiate \( \psi \in Z^2_{\text{ad}}(\mathfrak{b}, \mathfrak{b} \otimes \mathfrak{b}) \) to a group cocycle \( \Psi \in Z^2_{\text{Ad}}(B, \mathfrak{b} \otimes \mathfrak{b}) \). Since \( d\delta = \psi \), we should likewise exponentiate \( \delta \) to the group as a map \( D : B \to \mathfrak{b} \otimes \mathfrak{b} \) with coboundary \( \Psi \), and define from this a ‘braided-Poisson bracket’. The latter will not, however, respect the group product in the usual way but rather up to a ‘braiding’ obtained from \( \psi \). Details of these braided-Poisson-Lie groups and the example of the Kirillov-Kostant braided-Poisson bracket from Example 3.3 extended to the group manifold (e.g., to \( SU_2 \)) will be developed elsewhere.

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REDUCIBLE DEHN SURGERY AND ANNULAR DEHN SURGERY

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Let $M$ be a compact, orientable, irreducible, $\partial$-irreducible, anannular 3-manifold with one component $T$ of $\partial M$ a torus. Suppose that $r_1$ and $r_2$ are two slopes on $T$. In this paper, we shall show that if $M(r_1)$ is reducible while $M(r_2)$ contains an essential annulus, then $\Delta(r_1, r_2) \leq 2$.

1. Introduction.

Let $M$ be a compact, orientable, irreducible, $\partial$-irreducible, anannular 3-manifold with one component $T$ of $\partial M$ a torus. A slope $r$ on $T$ is a $T$-isotopy class of essential, unoriented, simple closed curves on $T$, and the distance between two slopes $r_1$ and $r_2$, denoted by $\Delta(r_1, r_2)$, is the minimal geometric intersection number among all the curves representing the slopes. For a slope $r$ on $T$, we denote by $M(r)$ the surgered manifold obtained by attaching a solid torus $J$ to $M$ along $T$ so that $r$ bounds a disk in $J$. Now consider two distinct slopes $r_1$, $r_2$ on $T$. There are many results showing how constraints on the topology of $M(r_1)$ and $M(r_2)$ put constraints on $\Delta(r_1, r_2)$. For example, C. Gordon and J. Luecke [5] have shown that if $M(r_1)$ and $M(r_2)$ are reducible, then $\Delta(r_1, r_2) \leq 1$. C. Gordon [3] has shown that if $M$ contains no essential torus, and $M(r_i)$ contains an essential torus, $i = 1, 2$, then $\Delta(r_1, r_2) \leq 8$. Y-Q Wu [8] has shown that if $M(r_1)$ and $M(r_2)$ are $\partial$-reducible, then $\Delta(r_1, r_2) \leq 1$. In this paper, we shall estimate $\Delta(r_1, r_2)$ when $M(r_1)$ is reducible, and $M(r_2)$ contains an essential annulus. The main result is the following theorem:

Theorem 1. Let $M$ be a compact, orientable, irreducible, $\partial$-irreducible, anannular 3-manifold with one component $T$ of $\partial M$ a torus. If $r_1$ and $r_2$ are two slopes on $T$, such that $M(r_1)$ is reducible while $M(r_2)$ contains an essential annulus, then $\Delta(r_1, r_2) \leq 2$.

An example has been given by Hayashi and Motegi [6] showing that the bound 2 in Theorem 1 is the best possible in general.

Another proof of Theorem 1 has been obtained independently by Y-Q Wu.

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2. Scharlemann cycle and parallel edges.

In what follows, we shall assume that $M(r_1)$ is reducible, and $M(r_2)$ contains an essential annulus. We may assume further that $M(r_2)$ is irreducible, $\partial$-irreducible (see [5], [7]).

Let $V_i$ be the solid torus attached to $M$ in forming $M(r_1)$, $i = 1, 2$. Consider the family of essential 2-spheres in $M(r_1)$ which intersect $V_1$ in a family of meridional discs, and let $S \subset M(r_1)$ be such a 2-sphere chosen so that $S \cap V_1$ has the minimal number, say $n_1$, of components. Similarly, let $A \subset M(r_2)$ be an essential annulus which intersects $V_2$ in a collection of meridian discs, the number of which, say $n_2$, is minimal among all such annuli. By assumptions, $n_1 > 2$ and $n_2 > 0$.

Now suppose that $F_1 = M \cap S$ and $F_2 = M \cap A$. Then $F_1$ is an essential planar surface in $M$ with boundary slope $r_1$ while $F_2$ is an essential punctured annulus in $M$ with boundary slope $r_2$. We may assume that the number of components of $F_1 \cap F_2$ is minimal subject to these conditions. Then no circle component of $F_1 \cap F_2$ bounds a disk in $F_1$ or $F_2$, and no arc component of $F_1 \cap F_2$ is boundary parallel in $F_1$ or $F_2$. Each component of $\partial F_i$ lying in $T$ is called an inner component of $\partial F_i$, $i = 1, 2$.

Let $\Gamma(1, 1)$ be the graph in $S(A)$ obtained by taking the arc components of $F_1 \cap F_2$ as edges and taking the inner components of $\partial F_1 (\partial F_2)$ as fat vertices.

We shall use the indices $\alpha$ and $\beta$ to denote 1 or 2, with the convention that, when they are used together, $\{\alpha, \beta\} = \{1, 2\}$.

Number the inner components of $\partial F_\alpha$, $\partial_1 F_\alpha$, $\ldots$, $\partial_{n_\alpha} F_\alpha$, so that they appear consecutively on $T$. By construction, each inner component $\partial_i F_\alpha$ of $\partial F_\alpha$ intersects each inner component $\partial_j F_\beta$ of $\partial F_\beta$ in exactly $\Delta(r_1, r_2)$ points. The ends of the edges in $\Gamma_\alpha$ may be labeled by an integer $k \in \{1, 2, \ldots, n_\beta\}$ as follows. Let $x$ be the intersection of an edge $e$ of $\Gamma_\alpha$ with one of its vertices $\partial_i F_\alpha$, then $x$ is labeled $k$, where $\partial_k F_\beta$ is the unique vertex of $\Gamma_\beta$, such that $x \in e \cap \partial_i F_\alpha \cap \partial_k F_\beta$. Thus when we travel around $\partial_i F_\alpha$, the labels appear in the order $1, \ldots, n_\beta, \ldots, 1, \ldots, n_\beta$ (repeated $\Delta(r_1, r_2)$ times).

Now fix an orientation on $F_\alpha$, and let each inner component $\partial_i F_\alpha$ of $\partial F_\alpha$ have the induced orientation. Two inner components of $\partial F_\alpha$ are said to be parallel if they, when given the induced orientation by $F_\alpha$, are homologous on $T$; otherwise they are antiparallel. Two vertices of $\Gamma_\alpha$ are said to be parallel if the corresponding inner components of $\partial F_\alpha$ are parallel; otherwise they are antiparallel.

Parity rule [2]. An edge connects parallel vertices of $\Gamma_\alpha$ if and only if it connects antiparallel vertices of $\Gamma_\beta$.

Two edges of $\Gamma_\alpha$ are said to be parallel if they, together with some arcs in $\partial F_\alpha$, bound a disk in $F_\alpha$. A cycle $\sigma$ in $\Gamma_\alpha$ is a subgraph of $\Gamma_\alpha$ homeomorphic to a circle. The length of a cycle is the number of edges contained in it. A
cycle $\sigma$ in $\Gamma_\alpha$ is called a Scharlemann cycle if it bounds a disk face of $\Gamma_\alpha$ and the edges of $\sigma$ connect parallel vertices of $\Gamma_\alpha$, and have the same two labels at their ends. A length two Scharlemann cycle is called an $S$-cycle. A length two cycle $\sigma' = \{e'_1, e'_2\}$ in $\Gamma_\alpha$ is called an extended $S$-cycle if there is an $S$-cycle $\sigma = \{e_1, e_2\}$ in $\Gamma_\alpha$ such that $e'_i$ and $e_i$ are adjacent parallel edges in $\Gamma_\alpha$, $i = 1, 2$.

Let $x$ be a vertex of $\Gamma_\alpha$. An edge of $\Gamma_\beta$ is called an $x$-edge if it has label $x$ at one of its two ends. We denote by $\Gamma_\beta$ the subgraph of $\Gamma_\beta$ consisting of all the vertices of $\Gamma_\beta$ and the $x$-edges connecting parallel vertices of $\Gamma_\beta$. A disk face $D$ of $\Gamma_\beta^0$ is called an $x$-face.

**Lemma 2.1.** Let $W$ be a compact, irreducible, $\partial$-irreducible $3$-manifold, and let $A_0$ be a non-separating annulus properly embedded in $W$. Then $A_0$ is essential in $W$.

**Proof.** Let $D$ be a compressing disk of $A_0$. If one component of $\partial A_0$ is essential on $\partial W$, then $W$ is $\partial$-reducible, a contradiction. If the two components of $\partial A_0$ are trivial on $\partial W$, then $W$ contains a non-separating $2$-sphere. Thus $W$ is reducible, a contradiction.

Now let $D$ be a $\partial$-compressing disk of $A_0$, such that $\partial D = a \cup b$, where $a$ is an arc on $A_0$, and $b$ is an arc on $\partial W$. If the two components of $\partial A_0$ bounds an annulus $A_1$ on $\partial W$, and $b \subset A_1$, then $A_0 \cup A_1$ is non-separating in $W$, and the surface obtained by doing a $2$-surgery on $A_0 \cup A_1$ along $D$ is a non-separating $2$-sphere in $W$. Thus $W$ is reducible, a contradiction. If not, then the band connected sum of the two components of $\partial A_0$ along $b$ on $\partial W$, say $C$, bounds a disk in $W$, and $C$ is essential on $\partial W$. Thus $W$ is $\partial$-reducible, a contradiction. $\square$

**Lemma 2.2.** If $\Gamma_\alpha$ contains a Scharlemann cycle, then $F_\beta$ is separating.

**Proof.** By Lemma 2.1 of [5], $F_1$ is separating when $\Gamma_2$ contains a Scharlemann cycle.

Now let $\sigma$ be a Scharlemann cycle of $\Gamma_1$ with label pair $\{1, 2\}$, $D$ be the disk face bounded by $\sigma$ in $\Gamma_1$, and let $A_1$ be the annulus bounded by $\partial_1 F_2$ and $\partial_2 F_2$ on $T$, such that the interior of $A_1$ is disjoint from $A$. Let $D_1$ be the disks in $A$ bounded by $\partial_2 F_2$, and let $T' = (A - D_1 \cup D_2) \cup A_1$. Then $T'$ is a punctured torus. Let $A'$ be the surface obtained by doing a $2$-surgery on $T'$ along $D$, then $A'$ is an annulus in $M(r_2)$, such that $|A' \cap V_2| < n_2$. If $F_2$ is non-separating, then $A'$ is also non-separating. By Lemma 2.1, $A'$ is essential, contradicting the minimality of $n_2$. $\square$

**Lemma 2.3.** Let $\sigma$ be a Scharlemann cycle of $\Gamma_1$, then the edges in $\sigma$ can not lie in a disk of $A$.

**Proof.** Suppose, otherwise, that the edges in $\sigma$ lie in a disk of $A$. Then $M(r_2)$ contains a lens space as a factor (by the proof of Lemma 2.8 of [1]).
Since ∂M(r_2) ≠ φ, M(r_2) is reducible, contradicting our assumptions on M(r_2).

\[ \Box \]

**Proposition 2.4.** \( \Gamma \) cannot contain two Scharlemann cycles with distinct label pairs.

**Proof.** By Theorem 2.4 of [5], \( \Gamma_2 \) can not contain two Scharlemann cycles with distinct label pairs.

Now suppose, otherwise, that \( \Gamma_1 \) contains two Scharlemann cycles \( \sigma_1 \) and \( \sigma_2 \), with label pairs \{x, y\} and \{x', y'\} respectively, such that \{x, y\} ≠ \{x', y'\}. By Lemma 2.2, \( n_2 \) is even.

Now consider the edges of \( \sigma_1 \) and \( \sigma_2 \) as they lie in \( \Gamma_2 \), joining the vertices \( x, y \) and \( x', y' \). By Lemma 2.3, there exists an annulus \( E \subset A \), such that

1) one component of \( \partial E \) is one component of \( \partial A \), say \( \partial_1 A \), and another component of \( \partial E \) is contained in \( \text{int}A \);

2) the number of vertices of \( \Gamma_2 \) in \( E \) is at most \( n_2/2 \);

3) \( \text{int}E \) contains the edges of one of the two Scharlemann cycles, say \( \sigma_1 \), and the corresponding vertices \( x, y \).

Let \( E' \) be an annulus containing \( x', y' \) and the edges of \( \sigma_2 \), such that one component of \( \partial E' \) is the remaining component of \( \partial A \), say \( \partial_2 A \), and another component of \( \partial E' \) is contained in \( \text{int}A \). If \( \{x, y\} \cap \{x', y'\} = \phi \), then we may assume \( \text{int}E \cap \text{int}E' = \phi \).

Now let \( D \) be the face of \( \Gamma_1 \) bounded by \( \sigma_1 \), let \( H \) be the 3-cell in \( V_2 \) between the consecutive meridional disks of \( V_2 \) corresponding to \( x \) and \( y \), and let \( N \) be a regular neighborhood of \( E \cup H \cup D \) in \( M(r_2) \). Then the frontier of \( N \) is an annulus \( A' \), properly embedded in \( M(r_2) \), whose two boundary components are \( \partial_1 A \times \{1\} \) and \( \partial_2 A \times \{1\} \), and the union of \( N \) and \( D_0 \times [-1,1] \) along \( \partial_1 A \times [-1,1] \) is a punctured lens space whose fundamental group has order the length of \( \sigma_1 \), where \( D_0 \) is a disk. Similarly, let \( D' \) be the face of \( \Gamma_1 \) bounded by \( \sigma_2 \), let \( H' \) be the 3-cell in \( V_2 \) between the consecutive meridional disks of \( V_2 \) corresponding to \( x', y' \), and let \( N' \) be a regular neighborhood of \( E' \cup H' \cup D' \). Then the frontier of \( N' \) is an annulus \( A'' \) properly embedded in \( M(r_2) \), whose two boundary components are \( \partial_2 A \times \{1\} \) and \( \partial_2 A \times \{1\} \), and the union of \( N' \) and \( D_0 \times [-1,1] \) along \( \partial_2 A \times [-1,1] \), say \( M_1 \), is a punctured lens space whose fundamental group has order the length of \( \sigma_2 \), where \( D_0 \) is a disk. We may assume that \( N \cap N' = \phi \) (moving \( \partial N \) slightly off \( A \) if \( \{x, y\} \cap \{x', y'\} = \phi \)). We claim that \( A' \) is essential in \( M(r_2) \).

Suppose, otherwise, that \( A' \) is not essential in \( M(r_2) \). Since \( M(r_2) \) is \( \partial \)-irreducible, \( A' \) is incompressible in \( M(r_2) \). Now let \( D_1 \) be a \( \partial \)-compressing disk of \( A' \), such that \( \partial D_1 = a \cup b, a \subset \partial M(r_2) \), and \( b \subset A' \).

Case 1. \( a \subset \partial_1 A \times [-1,1] \).
Now either $M(r_2)$ is reducible, or the union of $N$ and $D_0 \times [-1,1]$ along $\partial_1 A \times [-1,1]$ is a 3-cell, a contradiction.

Case 2. $a \subset \partial M(r_2) - \partial_1 A \times (-1,1)$.

If $\partial M - \partial_1 A \times (-1,1)$ is not an annulus, then either $M(r_2)$ is reducible, or $M(r_2)$ is $\partial$-reducible, a contradiction. If $\partial M - \partial_1 A \times (-1,1)$ is an annulus, then $\partial M(r_2)$ is a torus, and the union of $M(r_2) - \text{int} N$ and $D_0 \times [-1,1]$ along $\partial M - \partial_1 A \times (-1,1)$ is a 3-cell, but it contains $M_1$ as a factor, a contradiction.

By construction, $|A' \cap V_2| = |A \cap V_2| - 2$, contradicting the minimality of $n_2$. \hfill $\square$

**Lemma 2.5.**

(1) $\Gamma_2$ contains no extended $S$-cycle.

(2) $\Gamma_2$ contains at most $n_1/2 + 1$ mutually parallel edges connecting parallel vertices.

(3) $\Gamma_2$ contains at most $n_1 - 1$ mutually parallel edges.

(4) If $\Gamma_1$ contains a great cycle, then $\Gamma_1$ contains a Scharlemann cycle.

(5) If $n_\alpha \geq 3$, and $\Gamma_\beta$ contains two distinct Scharlemann cycles $\sigma_1$ and $\sigma_2$, then the edges of $\sigma_1$ are disjoint from the edges of $\sigma_2$.

**Proof.** (1) is Lemma 2.3 of [9]. (2) is Lemma 2.4 of [9]. (3) is Lemma 2.6 of [1]. See also [4, Proposition 1.3]. (4) is Lemma 2.6.2 of [2]. (5) Suppose, otherwise, that one edge of $\sigma_1$ is contained in $\sigma_2$. Then $n_\alpha = 2$, a contradiction. \hfill $\square$

**Lemma 2.6.** Let $y$ be a vertex of $\Gamma_\beta$.

(1) If $\Gamma_\alpha$ contains a $n$-sided $y$-face, such that $2 \leq n \leq 3$, then $\Gamma_\alpha$ contains a Scharlemann cycle.

(2) If $F_\beta$ is separating, and $\Gamma_\alpha$ contains a $y$-face $f$, then $\Gamma_\alpha$ contains a Scharlemann cycle in $f$.

**Proof.** (1) Suppose that $\Gamma_\alpha$ contains a $n$-sided $y$-face, such that $2 \leq n \leq 3$. Then $\Gamma_\alpha$ contains a great cycle. By Lemma 2.5(4), $\Gamma_\alpha$ contains a Scharlemann cycle. (2) is Lemma 2.2 of [5]. \hfill $\square$

### 3. Reduced graph.

Let $G$ be a graph in a surface $S$. The reduced graph of $G$ is the graph obtained from $G$ by amalgamating each complete set of mutually parallel edges of $G$ to a single edge.

**Lemma 3.1.** One of $\Gamma_1$ and $\Gamma_2$ satisfies

(*) Each vertex is incident to an edge connecting it to an antiparallel vertex.

This follows immediately from the proof of [9, Lemma 2.6].
Let $G_\alpha$ be the subgraph of $\Gamma_\alpha$ consisting of all the vertices of $\Gamma_\alpha$ and the edges connecting parallel vertices. We first suppose that $\Gamma_2$ has property $(\ast)$. A component $F'$ of $G_2$ is called an extremal component if there is a disc $D$ in $\hat{A}$ such that $D \cap G_2 = F'$, where $\hat{A}$ is the 2-sphere obtained by capping off the two components of $\partial A$ with disks. In this case $F = D \cap \Gamma_2$ is a graph in $D$. If $e$ is an edge in $\Gamma_2$ connecting a vertex of $F'$ to an antiparallel vertex, then $e \cap D$ is an edge of $F$ connecting that vertex to $\partial D$. Such an edge is called a boundary edge of $F$. Property $(\ast)$ means that each vertex of $F$ belongs to a boundary edge.

**Lemma 3.2.** Let $\Gamma$ be a graph in a disk with no 1-sided disk face or two sided disk face, such that every vertex of $\Gamma$ belongs to a boundary edge, then either $\Gamma$ contains only one vertex, or there are at least two vertices of valency at most 3, each of which belongs to a single boundary edge.

This follows immediately from the proof of [2, Lemma 2.6.5].

**Lemma 3.3.** If $\Gamma_2$ has property $(\ast)$, then there exists at least one vertex of $\Gamma_2$, such that among the families of ends around it, there are at most two families which are ends of edges connecting it to parallel vertices. Furthermore, if there are two such families, they are successive.

**Proof.** Since $G_2$ contains at least two extremal components, there is an extremal component $\tilde{F}'$ of $G_2$, such that the correspond disc $D$ of $\tilde{F}'$ in $\hat{A}$ contains at most one component of $\partial A$, say $\partial_1 A$, and the remaining component of $\partial A$ is disjoint from $D$. Thus $F$ contains at most one 1-sided disk face. Furthermore, if $F$ contains a 1-sided disk face $f_1$, then $\partial_1 A \subset \text{int} f_1$. Let $\tilde{F}$ be the reduced graph of $F$ in $D$. Let $S$ be the graph obtained from $\tilde{F}$ by removing the edge bounding $f_1$. (Let $S = F$ when $F$ contains no 1-sided disk face.) Then $S$ contains no 1-sided disk face, and it contains at most one 2-sided disk face. Furthermore, if $S$ contains a 2-sided disk face $f_2$, then $\partial_1 A \subset \text{int} f_1 \subset \text{int} f_2$. Let $\tilde{S}$ be the reduced graph of $S$ in $D$. Let $D_0$ be the disk bounded by $\partial_1 A$ in $D$, and let $\tilde{F}_0$ be the reduced graph of $F$ in $D - \text{int} D_0$.

To prove the lemma, we need only to prove that $\tilde{F}_0$ contains a vertex of valency at most 3, which belongs to a boundary edge. If $\tilde{F}'$ contains only one vertex $v$, then $v$ has valency 3 in $\tilde{F}_0$, and it belongs to a boundary edge. If $\tilde{F}'$ contains two vertices, then the one which does not belong to the edge bounding $f_1$, has valency at most 3 in $\tilde{F}_0$. Assume now that $\tilde{F}'$ contains at least three vertices.
Case 1. \( S \) contains no 2-sided disk face.

By Lemma 3.2, \( S \) contains at least two vertices of valency at most 3, say \( v_1 \) and \( v_2 \), each of which belongs to a single boundary edge.

Suppose that \( \partial_1 A \) is contained in a fat edge \( l \) of \( S \) (as in Fig. 1). If \( v_i \notin \{v', v''\}, i = 1, \) or 2, then \( v_i \) has valency at most 3 in \( \bar{F}_0 \). If \( \{v_1, v_2\} = \{v', v''\} \), then the edges incident to one of \( v_1 \) and \( v_2 \) are as in one of Figures 2-4.

(1) the edges incident to one of \( v_1 \) and \( v_2 \) are as in Fig. 2.

Let \( S' \) be the graph obtained from \( S \) by taking the union of \( v_1, l \) and \( v_2 \) as a fat vertex, say \( v \). Then \( S' \) contains no 1-sided disk face or 2-sided disk face. By Lemma 3.2, \( S' \) contains at least two vertices of valency at most 3, each of which belongs to a single boundary edge, and the one which is not equal to \( v \), has valency at most 3 in \( \bar{F}_0 \).
(2) the edges incident to one of $v_1$ and $v_2$ are as in one of Figures 3-4.

Now $l_1 \cup v_1 \cup l \cup v_2 \cup l_2$ separates $D$ into two discs $D'$ and $D''$. Since $\partial_1 A$ is contained in $l$, the graph $S \cap D'$ contains no 1-sided disk face or 2-sided disk face. If $S \cap D'$ contains only one vertex $v$, then $v$ has valency at most 3 in $\tilde{F}_0$. If $S \cap D'$ contains at least two vertices, then $S \cap D'$ contains at least two vertices of valency at most 3, each of which belongs to a single boundary edge, and the one which is not equal to $v$, has valency at most 3 in $\tilde{F}_0$.

Now we suppose that $\partial_1 A$ is not contained in a fat edge. Since $S$ contains no 2-sided disk face, the one of $v_1$ and $v_2$ which does not belong to the edge bounding $f_1$, has valency at most 3 in $\tilde{F}_0$.

Case 2. $S$ contains a 2-sided disk face $f_2$.

Now $\partial_1 A \subset \text{int} f_2$. That means that $\partial_1 A$ is contained in a fat edge of $\tilde{S}$. By Lemma 3.2, $\tilde{S}$ contains at least two vertices of valency at most 3, each
of which belongs to a single edge. Using $\bar{S}$ to take place of $S$ in Case 1, we can proof that $\bar{F}_0$ contains a vertex of valency at most 3, which belongs to a boundary edge.

**Proposition 3.4.** If $\Gamma_2$ has property $(\ast)$, then $\Gamma_2$ contains at least one vertex, such that around it, all the endpoints of edges connecting it to parallel vertices are successive, and there are at most $n_1 + 2$ of them.

**Proof.** This follows immediately from Lemma 2.5(2) and Lemma 3.3. □

**Proposition 3.5.** If $\Gamma_2$ does not have property $(\ast)$, then $\Gamma_1$ contains at least one vertex, such that around it, all the endpoints of edges connecting to parallel vertices are successive, and there are at most $n_2 - 1$ of them.

**Proof.** By Lemma 3.1, $\Gamma_1$ has property $(\ast)$. Now let $F'$ be an extremal component of $G_1$, and let $D$ be the corresponding disc. In this case $F = D \cap \Gamma_1$ is a graph in $D$. By Lemma 3.2, $\bar{F}$ contains at least one vertex, say $x$, of valency at most 3, which belongs to a single boundary edge. That implies that in $\Gamma_1$, there are at most two families of parallel edges connecting $x$ to parallel vertices, and if there are two such families, then they are successive. Since $\Gamma_2$ does not have property $(\ast)$, $\Gamma_2$ contains one vertex, such that each of the edges incident to it connects it to a parallel vertex. That implies that all edges in $\Gamma_1$ with this vertex as a label connect nonparallel vertices, hence the above two families of parallel edges contains at most $n_2 - 1$ edges. □

4. The proof of Theorem 1.

Let $G$ be a graph on a surface $S$. In this section, we shall denote by $V$ the number of vertices of $G$, $E$ the number of edges of $G$ and $F$ the number of disk faces of $G$. By the Euler characteristic formula, $V - E + F \geq \chi(S)$.

**Proposition 4.1.** If $n_2 = 1$, then $\Delta(r_1, r_2) \leq 1$.

**Proof.** Suppose, otherwise, that $\Delta(r_1, r_2) \geq 2$. Since $n_2 = 1$, $\Gamma_2$ contains only one vertex, say $x$. It is easy to see that $x$ has valency 2 in $\bar{\Gamma}_2$. Hence $\Gamma_2$ contains $n_1$ mutually parallel edges, contradicting Lemma 2.5(3). □

**Proposition 4.2.** If $n_2 = 2$, then $\Delta(r_1, r_2) \leq 2$.

**Proof.** Suppose, otherwise, that $\Delta(r_1, r_2) \geq 3$. Since $n_2 = 2$, each vertex of $\bar{\Gamma}_2$ has valency 4 as in Fig. 5 (otherwise $\bar{\Gamma}_2$ contains $n_1$ mutually parallel edges). By Lemma 2.5(2), $l_1$ and $l_2$ contains at most $n_1 + 2$ edges. If $l_1$ and $l_2$ occupy at most $n_1 + 1$ edges, then one of $l_3$ and $l_4$ occupies at least $n_1$ mutually parallel edges, contradicting Lemma 2.5(3).
Figure 5.

Now we suppose that that $l_1$ and $l_2$ occupy $n_1 + 2$ edges. Then $\Gamma_2$ contains an S-cycle. By Lemma 2.2, $F_1$ is separating, and $n_1 \geq 4$. By Lemma 2.5(3), each of $l_3$ and $l_4$ occupies $n_1 - 1$ edges.

**Case 1.** $\partial_1 F_2$ and $\partial_2 F_2$ are parallel.

Let $x$ be a vertex of $\Gamma_1$. By the parity rule, $\Gamma_2^x$ contains at least six $(3n_2)$ edges. Since $\Gamma_2$ has four edges, $\Gamma_2^x$ contains a 2-sided disk face. By Lemma 2.5(1) and Lemma 2.5(3), $\Gamma_2$ contains an S-cycle, one label of which is $x$, for any given vertex $x$ in $\Gamma_1$. Since $n_1 \geq 3$, $\Gamma_2$ contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4.

**Case 2.** $\partial_1 F_2$ and $\partial_2 F_2$ are antiparallel.

Let $\Gamma$ be the subgraph of $\Gamma_1$ consisting of all the vertices of $\Gamma_1$ and the edges in $l_3$. Since $F_1$ is separating, $\Gamma$ is not connected. By the Euler characteristic formula, $\Gamma$ contains a disk face. By the proof of Proposition 1.3 of [4], $M$ contains an essential annulus, a contradiction.

In the following arguments, we shall assume that $n_\alpha \geq 3$.

**Proposition 4.3.** If $\Gamma_2$ has property $(\ast)$, then $\triangle(r_1, r_2) \leq 2$.

*Proof.* Suppose, otherwise, that $\triangle(r_1, r_2) \geq 3$. By Proposition 3.4, there exists a vertex of $\Gamma_2$, say $y$, such that $\Gamma_1^y$ contains at least $2n_1 + l$ edges, where $l \geq -2$. By the Euler characteristic formula, $\Gamma_1^y$ contains at least $n_1 + l + 2 \geq n_1$ disk faces. Since there are $n_1$ adjacent edges at $y$ connecting it to antiparallel vertices, there is a great $y$-cycle in $\Gamma_1$. By Lemma 2.5(4), $\Gamma_1$ contains a Scharlemann cycle. By Lemma 2.2, $F_2$ is separating. By Lemma 2.6(2) and Proposition 2.4, $\Gamma_1$ contains at least $n_1$ Scharlemann cycles with the same label pair, say $\{1, 2\}$. Now suppose that $\Gamma_1$ contains $m$ Scharlemann cycles with label pair $\{1, 2\}$. Then $m \geq n_1$. By Lemma 2.5(5), $\Gamma_1^1$ contains at least $2m$ edges. By the Euler characteristic formula, $\Gamma_1^1$ contains
at least \(2m - n_1 + 2 \geq m + 2\) disk faces. By Lemma 2.6(2), \(\Gamma_1\) contains at least \(m + 2\) Scharlemann cycles. Thus \(\Gamma_1\) contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4.

**Proposition 4.4.** If \(\Gamma_2\) does not have property (*), then \(\Delta(r_1, r_2) \leq 2\).

**Proof.** Suppose, otherwise, that \(\Delta(r_1, r_2) \geq 3\). By Proposition 3.5, there exists a vertex of \(\Gamma_1\), say \(x\), such that \(\Gamma_2^x\) contains \(2n_2 + l\) edges, where \(l \geq 1\). By the Euler characteristic formula, \(\Gamma_2^x\) contains at least \(n_2 + l\) disk faces. By the parity rule and Proposition 3.5, there are \(n_2 - l\) vertices of \(\Gamma_2\), each of which is incident to an edge connecting it to an antiparallel vertex. That means that there are at least \(n_2 - l\) edges of \(\Gamma_2^x\), each of which is on the boundary of only one disk face of \(\Gamma_2^x\). We claim that \(\Gamma_2^x\) contains either a 2-sided disk face, or a 3-sided disk face.

If \(\Gamma_2^x\) contains no 2-sided disk face or 3-sided disk face. Then \(4F \leq 2(2n_2 + l) - (n_2 - l)\). Thus \(F \leq 3/4(n_2 + l) < n_2 + l\), a contradiction.

Now by Lemma 2.6(1), \(\Gamma_2\) contains a Scharlemann cycle. By Lemma 2.2, \(F_1\) is separating. By Lemma 2.6(2) and Proposition 2.4, \(\Gamma_2\) contains at least \(n_2 + l\) Scharlemann cycles with the same label pair, say \(\{1, 2\}\). Now suppose that \(\Gamma_2\) contains \(m\) Scharlemann cycles with label pair \(\{1, 2\}\). Then \(m \geq n_2 + l\), where \(l \geq 1\). By Lemma 2.5(5), \(\Gamma_2^1\) contains at least \(2m\) edges. By the Euler characteristic formula, \(\Gamma_2^1\) contains at least \(m + l\) disk faces. By Lemma 2.6(2), \(\Gamma_2\) contains at least \(m + l\) Scharlemann cycles. Thus \(\Gamma_2\) contains two Scharlemann cycles with distinct label pairs, contradicting Proposition 2.4.

Theorem 1 follows immediately from Propositions 4.1-4.4.

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**References**


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EXTENDING MAPS OF A CANTOR SET PRODUCT WITH AN ARC TO NEAR HOMEOMORPHISMS OF THE 2-DISK

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We prove that a positive entropy map of the product of a Cantor Set and an arc (which covers a homeomorphism) cannot be “embedded” into a near homeomorphism of the 2-disk. Thus a theorem of M. Brown cannot be used to embed the induced shift map on the corresponding inverse limit space into a 2-disk homeomorphism.

1. Introduction.

In 1990, M. Barge and J. Martin [BM90] proved that the shift map on the inverse limit space \(([0, 1], f)\), for any map \(f : [0, 1] \to [0, 1]\), can be realized as a global attractor in the plane. In 1960, M. Brown [Bro60] proved that the inverse limit space of any near homeomorphism (Definition 1.2) of a compact metric space is homeomorphic to the original space. M. Barge and J. Martin prove that, for all such \(f\), there exists an embedding \(h : [0, 1] \to D^2\) such that \(h \circ f \circ h^{-1}\) can be extended to a near homeomorphism of the 2-disk, \(D^2\). They then use M. Brown’s theorem to extend the induced shift homeomorphism on \(h([0, 1])\) to a homeomorphism of \(D^2\). With care in the construction of the near homeomorphism of \(D^2\), the inverse limit space \((h([0, 1]), h \circ f \circ h^{-1})\) becomes a global attractor.

The main goal of this paper is to show that analogous techniques for maps, \(F : C \times [0, 1] \to C \times [0, 1]\), where \(C\) is a Cantor set, \(F(x, y) = (F_1(x), F_2(x, y))\) is a surjective map with positive topological entropy (Definition 1.4), and \(F_1\) is a homeomorphism, do not work; no near homeomorphic extension of \(h \circ F \circ h^{-1}\) to \(D^2\) exists for any embedding \(h : C \times [0, 1] \to D^2\) (Theorem 3.1). In our terminology, such \(F\) cannot be “embedded” into any 2-disk homeomorphism (Definition 1.1). In the proof of Theorem 3.1 one first assumes that \(h\) is a “tame” embedding (Definition 3.1). But in recent work, R. Walker proves that all embeddings of \(C \times [0, 1]\) into \(D^2\) are tame [Wal].

Our study of maps of \(C \times [0, 1]\) and their embeddings has links to a central problem in the dynamical systems of positive entropy homeomorphisms of compact surfaces.

*Does there exist a \(C^1\) positive entropy 2-disk diffeomorphism without shifts?*
In 1980, A. Katok [Kat80] proved that all $C^{1+\alpha}$, $\alpha > 0$, positive entropy diffeomorphisms of a compact surface, have transverse homoclinic points. So some power of such a diffeomorphism restricts to a shift map of finite type. The next year M. Rees announced a minimal positive entropy homeomorphism of the 2-torus [Ree81]. So her homeomorphism has no periodic orbits thus no shifts. Though not in print, it appears that techniques M. Rees used can be adapted to build a positive entropy 2-disk homeomorphism which has a fixed point and no other periodic orbits. The $C^1$ case remains open. In 1993, M. Barge and R. Walker built a chainable continuum which is the inverse limit space of a map of a Cantor comb [BW93]. The map restricted to each “tooth” was a tent map over an adding machine base map. The induced shift homeomorphism has positive entropy but all periodic orbits are period a power of 2. Thus no shifts are present. All chainable continua can be embedded into the 2-disk [Bin62]. Although their Cantor comb map can be embedded into a near homeomorphism of the 3-ball, it cannot be embedded into a near homeomorphism of the 2-disk. (To prove this M. Barge and R. Walker rely on properties of the adding machine base map.) So their induced shift homeomorphism cannot be used to build a new Rees-type 2-disk homeomorphism. By our Theorem 3.1, a much larger class of maps (all positive entropy maps of $C \times [0,1]$ which cover any homeomorphism) has the same drawback.

In Section 2 we show that if $F : C \times [0,1] \to C \times [0,1]$ is a surjective map such that $F(x,y) = (F_1(x), F_2(x,y))$, $F_1$ is a homeomorphism and $F_2(x_0, \cdot) : [0,1] \to [0,1]$ is nonmonotone (Definition 1.3) for some $x_0$, then there exists no embedding of $F$ into a near homeomorphism (Definition of the 2-disk). We will show this by assuming such a near homeomorphism does exist and then obtaining a contradiction using a result of S. Schwartz [Sch92] (Theorem 1.1) concerning nonmonotone maps.

Unless otherwise specified $X$, and $Y$ are compact metric spaces. And $\pi_1$ and $\pi_2$ on $X \times Y$ are the first and second coordinate projection maps.

**Definition 1.1.** A map $f : X \to X$ can be embedded into the map $F : Y \to Y$ if there exists a topological embedding $h : X \to Y$ such that $F|_{h(X)} = h \circ f \circ h^{-1}$.

**Definition 1.2.** A map $f : X \to Y$ is called a near homeomorphism provided there exists a sequence $\{f_k : X \to Y\}_{k=1}^\infty$ of homeomorphisms which uniformly converge to $f$.

**Definition 1.3.** A map $f : X \to Y$ is monotone provided $f^{-1}(V)$ is connected, whenever $V \subset Y$ is connected.

**Theorem 1.1** (S. Schwartz [Sch92]). Suppose that $X$ is a locally connected compact metric space. If $f : X \to X$ is a near homeomorphism then $f$ is monotone.
As mentioned, in Section 3 we show that if $F : C \times [0, 1] \to C \times [0, 1]$ is a surjective map with positive topological entropy (Definition 1.4), which is embedded in the 2-disk, then $F$ cannot be extended to a near homeomorphism of the disk. The proof uses theorems of R. Bowen (Theorem 1.2) [Bow71] and M. Barge (Theorem 1.3) [Bar87].

**Definition 1.4** (Topological Entropy). Suppose that $F : X \times Y \to X \times Y$ is a surjective map and has the form $F(x, y) = (F_1(x), F_2(x, y))$. Fix $x_0$ and let $\epsilon > 0$. A set $E \subset Y$ is $(n, \epsilon)$-separated by $F|_{\pi_1^{-1}(x_0)}$ if for all $y_0, y_1 \in E$, $y_0 \neq y_1$, $d(\pi_2 F^k(x_0, y_0), \pi_2 F^k(x_0, y_1)) > \epsilon$ for some $k \in [0, n)$, where $d$ is the $Y$-metric. Since $Y$ is compact and $n < \infty$, card $E < \infty$. Let the maximum number of $(n, \epsilon)$-separated orbits for each $\epsilon$ be

$$s(n, \epsilon) = \max \left\{ \text{card } E \mid E \subset Y \text{ such that } E \text{ is } (n, \epsilon) \text{-separated by } F|_{\pi_1^{-1}(x_0)} \right\}.$$

Now, let the growth rate of $s(n, \epsilon)$ (or $\epsilon$-topological entropy) be

$$h_{\text{top}} \left( F|_{\pi_1^{-1}(x_0)}, \epsilon \right) = \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n}.$$

Lastly we let $\epsilon \to 0$ and define topological entropy for $F|_{\pi_1^{-1}(x_0)}$.

$$h_{\text{top}} \left( F|_{\pi_1^{-1}(x_0)} \right) = \lim_{\epsilon \to 0} h_{\text{top}} \left( F|_{\pi_1^{-1}(x_0)}, \epsilon \right).$$

The topological entropy $h_{\text{top}}(F_1)$ of the homeomorphism $F_1$ is defined similarly (see [Bow71]).

**Theorem 1.2** (R. Bowen [Bow71]). If $F : X \times Y \to X \times Y$ has the form $F(x, y) = (F_1(x), F_2(x, y))$ then

$$h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{x \in X} \left\{ h_{\text{top}} \left( F|_{\pi_1^{-1}(x)} \right) \right\}.$$

If $h_{\text{top}}(F_1) = 0$ then $h_{\text{top}}(F) = \sup_{x \in X} \left\{ h_{\text{top}} \left( F|_{\pi_1^{-1}(x)} \right) \right\}$.

**Theorem 1.3** (M. Barge [Bar87]). If $F : X \times [0, 1] \to X \times [0, 1]$ has the form $F(x, y) = (F_1(x), F_2(x, y))$, $F_2(x, \cdot) : [0, 1] \to [0, 1]$ is monotone for each $x$ and $h_{\text{top}}(F_1) = 0$, then $h_{\text{top}}(F) = 0$.

### 2. Nonmonotone Maps of the Cantor Set Cross the Interval

#### 2.1. Preliminaries

Let $C \subset [0, 1]$ be a Cantor set. Let $C \times [0, 1]$ and $\{\alpha\} \times [0, 1] \subset \mathbb{R}^2$ for $\alpha \in C$. The goal of this section is to prove Theorem 2.1 to follow. But first some preliminaries.

2.1.0.1. Assume $F : C \times [0, 1] \to C \times [0, 1]$ is a surjective map that has the form

$$F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y))$$
where $F_1 : C \to C$ is a homeomorphism. Furthermore, for a given $\alpha_0 \in C$, $F_2(\alpha_0, y) = t(y)$ where $t : [0, 1] \to [0, 1]$ is a continuous nonmonotone map (see Figure 1 for an example). Let $\lambda_0 = F_1(\alpha_0)$.

![Figure 1. Example of a nonmonotone map.](image)

It will be needed later, that because $t$ is nonmonotone we can find a point that has at least two points in the the pre-image that can be separated by disjoint epsilon balls. We introduce this idea at this point so that we can use the notation developed here throughout.

2.1.0.2. Since $t$ is nonmonotone and continuous, there is an $a \in (0, 1)$ such that $t^{-1}(a)$ is closed and not connected. Thus, there is an interval $(m, M) \subset [0, 1] \setminus t^{-1}(a)$ such that $a = t(m) = t(M)$, and $t([m, M]) = [a, b]$ (or $[b, a]$) for some $b \neq a$. Without loss of generality we will assume that $a < b$. Let $\tau \in t^{-1}(b)$. By the intermediate value theorem, $t([m, M]) = [t(m), t(\tau)]$. Now choose $c = \frac{1}{2}(a + b)$. Since $t$ is continuous there are $s_1 \in (m, \tau)$ and $s_2 \in (\tau, M)$ such that $c = t(s_1) = t(s_2)$ (see Figure 1).

2.1.0.3. By the continuity of $F$, for any $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that $F(x, y) \in B_c(\lambda_0, \lambda(0))$ when $d(\alpha_0, x) < \delta_1$ and $y \in [0, 1]$. Suppose $K_1 = K_1(\epsilon) \in \mathbb{N}$ is such that $\frac{1}{K_1} < \delta_1$.

Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\}$. Now let $h_0 : C \times [0, 1] \to D$ be an arbitrary topological embedding. Then there is a homeomorphism $h_1 : D \to D$ such that $(h_1 \circ h_0)(\alpha_0, y) = (\alpha_0, y)$ and $(h_1 \circ h_0)(\lambda_0, y) = (\lambda_0, y)$ for all $y \in [0, 1]$. So $h_1$ “straightens out” $h_0(\alpha_0 \times [0, 1])$ and $h_0(\lambda_0 \times [0, 1])$ in a strong sense. Notice that $C \times [0, 1] \to D$.

2.1.0.4. By the uniform continuity of $h_1 \circ h_0$, for all $\epsilon > 0$ there is a $\delta_2 = \delta_2(\epsilon) > 0$ such that for all $y \in [0, 1]$, $h_1 \circ h_0(x, y) \in B_c(\alpha_0, y)$ and $h_1 \circ h_0(x', y) \in B_c(\lambda_0, y)$, for all $(x, y) \in B_{\delta_2}^c(\alpha_0, y)$ and $(x', y) \in B_{\delta_2}^c(\lambda_0, y)$. Let $K_2 = K_2(\epsilon) \in \mathbb{N}$ be such that $\frac{1}{K_2} < \delta_2$. 
2.1.0.5. With \( a, b \) defined as in [2.1.0.2], let \( \hat{d} = \min\{a, 1-b, |a-b|\} \). For
\( 0 < \epsilon_0 < \frac{\hat{d}}{100} \) choose 0 < \( \delta_0 \leq \min \{ \delta_1(\epsilon_0), \delta_2(\epsilon_0), \frac{M-m}{100} \} \). So in particular
[2.1.0.3] and [2.1.0.4] are satisfied. Note that
\( \epsilon < \epsilon_0 \) is a sequence \( \{K_n\} \). Let \( \Lambda(\alpha) \) is defined as in [2.1.0.2], let \( \hat{\alpha} \) be the horizontal line
\( \{\alpha\} \times [0, 1] \), for all \( k > K_0 \). Let \( \lambda_k = F_1(\alpha_k) \). (Note that \( \mathcal{N}_\delta(S) \) is a \( \delta \)-neighborhood of \( S \).) It follows that \( \lambda_k \rightarrow \lambda_0 \) as \( k \rightarrow \infty \) and \( \lambda_k \times [0, 1] \subset \mathcal{N}_\delta(\lambda_0 \times [0, 1]) \) for all \( k > K_0 \). For a possibly larger \( K_0 \), also called
\( K_0 \), and \( o_k \in \mathcal{B}_\delta(\lambda_0, c) \), \( k > K_0 \), there exist \( q_1(k) \) and \( q_2(k) \) such that
\( \{q_1(k), q_2(k)\} \subset \mathcal{F}^{-1}(o_k) \), \( q_1(k) \in \mathcal{B}_\delta(\alpha_0, s_1) \) and \( q_2(k) \in \mathcal{B}_\delta(\alpha_0, s_2) \).

We now state our first theorem.

2.1.1. Nonmonotone Nonextension Theorem.

**Theorem 2.1.** Let \( F : C \times [0, 1] \rightarrow C \times [0, 1] \) be a map of the form \( F(\alpha, y) = (F_1(\alpha), F_2(\alpha, y)) \) where \( F_1 : C \rightarrow C \) is a homeomorphism. Furthermore,
assume \( F_2(\alpha_0, \cdot) : [0, 1] \rightarrow [0, 1] \) is surjective but not monotone for some \( \alpha_0 \).
Then there exists no extension of \( h_0 \circ F \circ h_0^{-1} \) to a near homeomorphism of the disk \( D \), for any topological embedding \( h_0 : C \times [0, 1] \rightarrow D \).

**Proof.** Assume \( h, K_0, \epsilon_0, \delta_0 \), \( \{\alpha_k\} \), \( \{\lambda_k\} \), \( q_1(k) \), and \( q_2(k) \) are defined as in
[2.1.0.1-5]. Suppose that \( H_0 : D \rightarrow D \) is a near homeomorphism such
that \( H_0 |_{h_0(C \times [0, 1])} = h_0 \circ F \circ h_0^{-1} \). And let \( H_1 : D \rightarrow D \) be given by
\( H_1 = h_1 \circ H_0 \circ h_1^{-1} \). Thus \( H_1 \) is also a near homeomorphism. So the
diagram in Figure 2 commutes.

\[
\begin{array}{ccc}
C \times [0, 1] & \xrightarrow{F} & C \times [0, 1] \\
h_0 \downarrow & & h_0 \downarrow \\
D & \xrightarrow{H_0} & D \\
h_1 \downarrow & & h_1 \downarrow \\
D & \xrightarrow{H_1} & D \\
\end{array}
\]

**Figure 2.** Commuting Diagram.

2.1.1.1. Let \( \Lambda(\alpha) = h_1 \circ h_0 (\alpha \times [0, 1]) \) for all \( \alpha \in C \). By [2.1.0.3] \( h_1 \circ h_0 \) is a homeomorphism and if \( \{\alpha\} \times [0, 1] \cap \{\lambda\} \times [0, 1] \) = \( \emptyset \) (when \( \alpha \neq \lambda \)), then
\( \Lambda(\alpha) \cap \Lambda(\lambda) = \emptyset \). Let \( \ell_\beta \) be the horizontal line \( \{y = \beta\} \). And let \( \ell_\beta(k) = \)
\( \Lambda(\alpha_k) \cap \ell_\beta = \Lambda(\lambda_k) \cap \ell_\beta \). Because \( h_1 \circ h_0 (\alpha_k, 0) \in B_{\delta_0}(\alpha_0, 0) \), \( h_1 \circ h_0 (\alpha_k, 1) \in B_{\delta_0}(\alpha_0, 1) \), and \( \Lambda(\alpha_k) \) is connected, then \( \ell_\beta^\alpha(k) \neq \emptyset \) and all \( k \geq K_0 \) (see Figure 3 and [2.1.0.2]). Similarly \( \ell_\beta^\lambda(k) \neq \emptyset \), for all \( \beta \in [\epsilon_0, 1-\epsilon_0] \) and \( k \geq K_0 \). Note that if \( p \in \ell_\beta^\lambda(k) \) for given \( k \geq K_0 \) then \( p \in B_\epsilon(\lambda_0, \beta) \).

**Figure 3.** Intersection of \( \Lambda(\alpha_k) \) with \( \ell_\beta \).

Lemma 2.1 follows from the continuity of \( h_1, h_0 \), and \( \pi_1 \).

**Lemma 2.1.** Choose \( p_k \in \ell_\beta^\alpha(k) \) for each \( k \). Then \( \pi_1 p_k \to \alpha_0 \) as \( k \to \infty \).

Notice that \( \pi_1 (h_1 \circ h_0 (\alpha_k, \frac{1}{2})) \neq \alpha_0 \) for sufficiently large \( k \). So either

\[
\text{card} \left\{ k \mid \pi_1 \left( h_1 \circ h_0 \left( \alpha_k, \frac{1}{2} \right) \right) > \alpha_0 \right\} = \infty
\]

or

\[
\text{card} \left\{ k \mid \pi_1 \left( h_1 \circ h_0 \left( \alpha_k, \frac{1}{2} \right) \right) < \alpha_0 \right\} = \infty.
\]

2.1.1.2. So without loss of generality we may assume there exist distinct \( \{k_n\}_{n=1}^\infty \) such that \( k_n \to \infty \) as \( n \to \infty \), and

\[
\pi_1 \left( h_1 \circ h_0 \left( \alpha_{k_n}, \frac{1}{2} \right) \right) > \alpha_0.
\]

2.1.1.3. For the sake of simplicity we denote \( \alpha_{k_n} \) by \( \alpha_n \), \( \Lambda(\alpha_{k_n}) \) by \( \Lambda(\alpha_n) \), \( \Lambda(\lambda_{k_n}) \) by \( \Lambda(\lambda_n) \) and \( \ell_\beta^\lambda(k) \) by \( \ell_\beta^\alpha \).
Lemma 2.2. Let \( N_0 \) be such that \( k_n \geq K_0 \) for all \( n \geq N_0 \). Then
\[
\Lambda(\alpha_n) \cap \left\{ \left( x, \frac{1}{2} \right) \mid x < \alpha_0 \right\} = \emptyset.
\]

Proof. Fix \( n \geq N_0 \) and assume there exists
\[
p_1 \in \Lambda(\alpha_n) \cap \left\{ \left( x, \frac{1}{2} \right) \mid x < \alpha_0 \right\},
\]
and let \( p_2 = (h_1 \circ h_0(\alpha_0, \frac{1}{2})) \). By [2.1.1.2] \( \pi_1(p_2) > 0 \). Let \( A \) be the arc in \( \Lambda(\alpha_n) \) with end points \( p_1 \) and \( p_2 \). By [2.1.1.1], \( p_1, p_2 \in \mathcal{B}_{\varepsilon_0}(\alpha_0, \frac{1}{2}) \). So by [2.1.0.5],
\[
d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_2)) < \delta_0.
\]
Since \( \Lambda(\alpha_n) \cap \Lambda(\alpha_0) = \emptyset \), then using a Jordan Curve argument, it follows
\[
A \cap \{(0, y) \mid y > 1 \text{ or } y < 0\} \neq \emptyset.
\]

Let \( p_3 \in A \cap \{(0, y) \mid y > 1 \text{ or } y < 0\} \). So \( d(p_1, p_3) > \frac{1}{2} \). But because \( p_3 \in A \), either
\[
\pi_2 \circ (h_1 \circ h_0)^{-1}(p_1) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_2)
\]
or
\[
\pi_2 \circ (h_1 \circ h_0)^{-1}(p_2) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_3) < \pi_2 \circ (h_1 \circ h_0)^{-1}(p_1).
\]
In either case \( d((h_1 \circ h_0)^{-1}(p_1), (h_1 \circ h_0)^{-1}(p_3)) < \delta_0 \). And so \( d(p_1, p_3) < \epsilon_0 \) which is a contradiction. \( \square \)

2.1.1.4. Assume \( n \geq N_0 \). Let \( g_n : [0, 1] \to \Lambda(\alpha_n) \) be the parameterization of \( \Lambda(\alpha_n) \) defined by \( g_n(\beta) = h_1 \circ h_0(\alpha_n, \beta) \). Using \( \tau, m \) from [2.1.0.2] \( \Lambda(\alpha_n) \cap \ell_{\tau} \neq \emptyset \) and \( \Lambda(\alpha_n) \cap \ell_m \neq \emptyset \) (see Figure 4) so by Lemma [2.2] and the connectivity of \( \Lambda(\alpha_n) \) there is a largest \( \beta \), call it \( \beta_n^- \), such that \( g_n(\beta_n^-) \in \ell_m \). Let \( \alpha_n = g_n(\beta_n^-) \) (see Figure 4). Similarly there is a smallest \( \beta \), call it \( \beta_n^+ \), such that \( g_n(\beta_n^+) \in \ell_m \). Let \( b_n = g_n(\beta_n^+) \).

2.1.1.5. If necessary, renumber the \( k_n \)'s so that if \( k_n < k_{n+1} \) then \( \pi_1(a_n) > \pi_1(a_{n+1}) \). It follows by an argument similar to that of Lemma [2.2] that \( \pi_1(b_n) > \pi_1(b_{n+1}) \). (Because \( h_1 \circ h_0 \) may have scrambled the \( C \times [0, 1] \) order in the first coordinate, it may be necessary to relabel the \( k_n \)'s so that \( \Lambda(\alpha_{k_n}) \) to be “between” \( \Lambda(\alpha_{k_{n-1}}) \) and \( \Lambda(\alpha_{k_{n+1}}) \).)

Considering [2.1.1.5] and [2.1.1.2] and to simplify the notation assume
\[
\text{card} \left\{ k \mid \pi_1(h_1 \circ h_0(\alpha_k, \frac{1}{2})) > \alpha_0 \right\} = \infty
\]
and \( \pi_1(a_k) > \pi_1(a_{k+1}) \) for all \( k \).
Figure 4. First and Last Intersections.

Using [2.1.1.4], for \( k \geq N_0 \) define the four curves \( I(k,m) \), \( I(k,\tau) \), \( J_{k-1} \), and \( J_{k+1} \) in the following manner (see Figure 5). Let \( I(k,m) \) be the line segment in \( \ell_m \) between \( a_{k+1} \) and \( a_{k-1} \) and \( I(k,\tau) \) be the line segment in \( \ell_\tau \) between \( b_{k+1} \) and \( b_{k-1} \). Let

\[
J_{k-1} = \{ g_{k-1}(\beta) \mid \beta_{k-1}^- \leq \beta \leq \beta_{k-1}^+ \}, \quad \text{and} \\
J_{k+1} = \{ g_{k+1}(\beta) \mid \beta_{k+1}^- \leq \beta \leq \beta_{k+1}^+ \}.
\]

Lemma 2.3. \( I(k,\tau) \cup J_{k-1} \cup I(k,m) \cup J_{k+1} \) is a simple closed curve.

Proof. Since \( \Lambda(\alpha_{k-1}) \cap \Lambda(\alpha_{k+1}) = \emptyset \) we have \( J_{k-1} \cap J_{k+1} = \emptyset \). By [2.1.1.1] \( I(k,m) \cap I(k,\tau) = \emptyset \). And by [2.1.1.4] we have that

\[
a_{k-1} = J_{k-1} \cap I(k,m) \quad \text{and} \quad a_{k+1} = J_{k+1} \cap I(k,m)
\]
and 
\[ b_{k-1} = J_{k-1} \cap I(k, \tau) \text{ and } b_{k+1} = J_{k+1} \cap I(k, \tau). \]
And so the lemma follows. \( \square \)

Let \( R_k \) be the closed and bounded set with boundary 
\[ I(k, m) \cup J_{k-1} \cup I(k, \tau) \cup J_{k+1} \]
(see Figure 5). Recall from [2.1.0.2] that \( s_1 \in [m, \tau] \) and from [2.1.1.1] that \( \ell_{s_1}^\alpha(k) = \Lambda(\alpha_k) \cap \ell_{s_1} \) (see Figure 3).

**Lemma 2.4.** \( R_k \cap \ell_{s_1}^\alpha(k) \neq \emptyset. \)

**Proof.** Let \( \gamma_k \) be the arc \( \{ g_k(\beta) | 0 \leq \beta \leq \beta_k^+ \} \). Let \( S_k = R_k \cap \pi_2^{-1}[s_1, \tau] \) (see Figure 6). So \( \partial S_k \supset I(k, \tau) \) and by [2.1.1.5] \( b_k \in I(k, \tau) \). But \( b_k \) is not an endpoint of \( I(k, \tau) \) because the endpoints of \( I(k, \tau) \) are \( b_{k-1} \) and \( b_{k+1} \). And so there is an \( \eta > 0 \) such that if \( p \in B_\eta(b_k) \) and \( \pi_2(p) < \tau \) then \( p \in \operatorname{int} S_k \). Now, if \( q \in \gamma_k \setminus \{ b_k \} \) then \( \pi_2(q) < \frac{1}{2} \). And since \( \gamma_k \) connects \( h_1 \circ h_0(\alpha_k, 0) \) to \( b_k \), we have that \( (\gamma_k \cap B_\eta(b_k)) \setminus \{ b_k \} \neq \emptyset \). Thus there exists \( p_0 \in \gamma_k \cap B_\eta(b_k) \cap \operatorname{int} S_k \). Let \( A_k \subset \gamma_k \) be the arc with endpoints \( p_0 \) and \( h_1 \circ h_0(\alpha_k, 0) \) (see Figure 6).

![Figure 6. The Arc \( A_k \).](image)

Because \( p_0 \in \operatorname{int} S_k \) and \( h_1 \circ h_0(\alpha_k, 0) \not\subset S_k \) then \( A_k \cap \partial S_k \neq \emptyset \). Since \( A_k \cap \Lambda(\alpha_{k-1}) = \emptyset, A_k \cap \Lambda(\alpha_{k+1}) = \emptyset, A_k \cap I(n, \tau) = \emptyset \) and \( \ell_{s_1} \cap R_k \subset \partial S_k \), we have that \( A_k \cap \ell_{s_1} \cap R_k \neq \emptyset, \) or \( R_k \cap \ell_{s_1}^\alpha(k) \neq \emptyset. \) \( \square \)

2.1.1.6. Note that since \( \ell_{s_1}^\alpha(k) \cap \partial R_k = \emptyset \) then \( \ell_{s_1}^\alpha(k) \subset \operatorname{int} R_k. \)

**Lemma 2.5.** \( [\Lambda(\alpha_l) \cap H^{-1}_1(h_1 \circ h_0(\lambda_k, y))] = \emptyset \) for \( k \neq l. \)
Proof. Suppose that \( \rho \in \Lambda(\alpha_l) \cap H_{1}^{-1}(h_{1} \circ h_{0} (\lambda_{k}, y)) \) for \( k \neq l \). Then \( H_{1}(\rho) = h_{1} \circ h_{0} (\lambda_{k}, y) \) but \( H_{1} \Lambda(\alpha_l) = \Lambda(\lambda_l) \). Thus \( H_{1}(\rho) \in \Lambda(\lambda_l) \) so \( h_{1} \circ h_{0} (\lambda_{k}, y) \in \Lambda(\alpha_l) \) or \( (\lambda_{k}, y) \in C_{\lambda_l} \). Which contradicts, \([2.1.1.1]\) since \( k \neq l \). \( \square \)

Proof of Theorem 2.1 continued. By Lemma \([2.4]\) there exists \( p_{1}(k) \in R_{k} \cap \ell_{s_{i}}(k) \) for all \( k \geq N_0 \). By \([2.1.0.5]\) \( (h_{1} \circ h_{0})^{-1}(p_{1}(k)) \in \mathcal{B}_{\delta_{0}}(\alpha_{0}, s_{1}) \). Using \([2.1.0.5]\), let \( \alpha_{k} = F \circ (h_{1} \circ h_{0})^{-1}(p_{1}(k)) \). So there exists \( \{q_{1}(k), q_{2}(k)\} \subset F^{-1}(\alpha_{k}) \) such that \( q_{1}(k) \in \mathcal{B}_{\delta_{0}}(\alpha_{0}, s_{1}) \) and \( q_{2}(k) \in \mathcal{B}_{\delta_{0}}(\alpha_{0}, s_{2}) \). Choose \( q_{1}(k) \) so that \( p_{1}(k) = h_{1} \circ h_{0}(q_{1}(k)) \). And let \( p_{2}(k) = h_{1} \circ h_{0}(q_{2}(k)) \) and \( r_{k} = h_{1} \circ h_{0}(\alpha_{k}) \) (see Figure 7). Because \( H_{1} \circ h_{1} \circ h_{0} = h_{1} \circ h_{0} \circ F \) then \( \{p_{1}(k), p_{2}(k)\} \in H_{1}^{-1}(r_{k}) \). By the size of \( \delta_{0} \) chosen in \([2.1.0.5]\), \( p_{2}(k) \in \mathcal{B}_{\delta_{0}}(\alpha_{0}, s_{2}) \not\subseteq R_{k} \).

Recall that \( H_{0} \) and \( H_{1} \) are near homeomorphisms. Near homeomorphisms are monotone on locally connected compact metric spaces (\([\text{Sch92}]\)). Thus pre-images of connected sets under \( H_{1} \) are connected. So \( H_{1}^{-1}(r_{k}) \) is a connected set which contains \( p_{2}(k) \) \( \not\subseteq R_{k} \) and by \([2.1.1.6]\) \( p_{1}(k) \in \text{int}R_{k} \). Then \( H_{1}^{-1}(r_{k}) \cap \partial R_{k} \neq \emptyset \). By Lemma \([2.5]\) either \( H_{1}^{-1}(r_{k}) \cap I(k, \tau) \neq \emptyset \) or \( H_{1}^{-1}(r_{k}) \cap I(k, m) \neq \emptyset \). So there is an infinite sequence \( \{\rho_{k_{j}}\} \) such that either \( \rho_{k_{j}} \in I(k, \tau) \cap H_{1}^{-1}(r_{k_{j}}) \) or \( \rho_{k_{j}} \in I(k, m) \cap H_{1}^{-1}(r_{k_{j}}) \) for all \( j \) (see Figure 7).

![Figure 7. Subsequence and Pre-image.](image)

Now by Lemma \([2.1]\) either \( \rho_{k_{j}} \rightarrow h_{1} \circ h_{0} (\alpha_{0}, \tau) \) or \( \rho_{k_{j}} \rightarrow h_{1} \circ h_{0} (\alpha_{0}, m) \) as \( j \rightarrow \infty \). Since \( H_{1} \) is continuous for all \( j \), either

\[
H_{1}\rho_{k_{j}} \rightarrow H_{1} \circ h_{1} \circ h_{0} (\alpha_{0}, \tau) \quad \text{or} \quad H_{1}\rho_{k_{j}} \rightarrow H_{1} \circ h_{1} \circ h_{0} (\alpha_{0}, m).
\]

Because \( H_{1} \circ h_{1} \circ h_{0} = h_{1} \circ h_{0} \circ F \), then either

\[
r_{k_{j}} \rightarrow h_{1} \circ h_{0} (\lambda_{0}, t(\tau)) \quad \text{or} \quad r_{k_{j}} \rightarrow h_{1} \circ h_{0} (\lambda_{0}, t(m)).
\]
Since \( h_1 \circ h_0 \) is a homeomorphism either
\[
\alpha_{k_j} \rightarrow (\lambda_0, b) \quad \text{or} \quad \alpha_{k_j} \rightarrow (\lambda_0, a)
\]
which is a contradiction since \( \{\alpha_{k_j}\} \subset B_{r_0}(\lambda_0, c) \).

\[\square\]

3. Positive Entropy Maps of \( C \times [0,1] \).

3.1. Introduction. Let \( C \subset \mathbb{R} \) be a Cantor set. In this chapter we use the results of Chapter 2 to prove the following:

**Theorem 3.1.** Let \( F : C \times [0,1] \rightarrow C \times [0,1] \) be a surjective map such that \( F(a,y) = (F_1(a), F_2(a,y)) \), where \( F_1 : C \rightarrow C \) is a homeomorphism. If \( h_{\text{top}}(F) > 0 \) then there exists no topological embedding \( h_0 : C \times [0,1] \rightarrow D \subset \mathbb{R}^2 \) such that \( h_0 \circ F \circ h_0^{-1} \) extends to a near homeomorphism of the disk \( D \).

Recall that \( \pi_1 : C \times [0,1] \rightarrow C \) is the projection map onto the first coordinate. By work of R. Bowen [Bow71] we know that \( h_{\text{top}}(F) \leq h_{\text{top}}(F_1) + \sup_{a \in C} \{ h_{\text{top}}(F|_{\pi_1^{-1}(a)}) \} \). It has been shown by M. Barge and R. Walker [BW93] that any near homeomorphism that extends \( h_0 \circ F \circ h_0^{-1} \) to the disk must preserve a certain local order on the set of fibers \( \{h_0(a \times [0,1]) | a \in C \} \). But we will show that if \( h_{\text{top}}(F_1) > 0 \) no such local order is preserved. So in fact \( h_{\text{top}}(F_1) = 0 \). Using [Bow71] and a result of M. Barge [Bar87], if \( h_{\text{top}}(F) > 0 \) then for some \( a_0 \in C, F_2(a_0, \cdot) \) is a nonmonotone map. Thus by Theorem 2.1, \( h_0 \circ F \circ h_0^{-1} \) cannot be extended to a near homeomorphism of the disk.

3.2. Proof of Theorem 3.1.

**Definition 3.1** (Tame Embedding). \( h_0 : C \times [0,1] \rightarrow D \subset \mathbb{R} \) is a tame embedding provided there is a homeomorphism \( h_1 : D \rightarrow D \) such that for all \( a \in C, h_1 \circ h_0 (\{a\} \times [0,1]) = (\{a'\} \times [0,1]) \) for some \( \{a'\} \). If \( h_0 \) is a tame embedding, using a theorem of E. Moise [Moi77], we may further require that \( h_1 \) has the property: \( h_1 \circ h_0 (\{a\}, i) = (\{a'\}, i) \) for all \( a \) and \( i = 0, 1 \).

For more information concerning tame embeddings see [Rus73] or [Bin54].

3.2.1. **Proof of Theorem 3.1.** All topological embeddings of \( C \times [0,1] \) into \( D^2 \) are tame [Wal]. So it is enough to prove the theorem for all tame embeddings, \( h_0 \).

Let \( h_1 \) be as in Definition 3.1. Denote by \( \Lambda \) the set \( h_1 \circ h_0 (C \times [0,1]) \) and by \( \Lambda(a) \) the set \( h_1 \circ h_0 (a \times [0,1]) \). Note that \( \pi_1(\Lambda(a)) = a' \) for some \( a' \in \mathbb{R} \). Assume there is a near homeomorphism \( H : D \rightarrow D \) such that on \( C \times I, h_1 \circ h_0 \circ F = H \circ h_1 \circ h_0 \). Before continuing with the proof, we stop to define a local ordering on \( \{\Lambda(a) | a \in C \} \) and prove a lemma.
3.2.2. Order Definitions and Lemmas. Here we show that $H$ preserves the local order of fibers as defined by M. Barge and R. Walker [BW93], which we will write as $<_{bw}$. And it will follow that $F_1 : C \to C$ is a “local order preserving homeomorphism.”

Note: Since $h_0$ is tame one could use the order on $\{\Lambda(a) | a \in C\}$ induced by $\pi_1$ in place of $<_{bw}$. That is, $\Lambda(a) < \Lambda(b)$ if $\pi_1 \Lambda(a) < \pi_1 \Lambda(b)$. Although $h_1$-dependent, this order may be more natural than the $<_{bw}$ order, and is locally equivalent to it. But in order to show that $H$ preserves such a local order on fibers, one must cycle through the definition of $<_{bw}$ in any case.

Barge-Walker order:

Definition 3.2. For $a, b \in C$ suppose that $\gamma_-$ and $\gamma_+$ are arcs in the plane with the properties:

$\gamma_-$ has endpoints $h_1 \circ h_0 (a, 0)$ and $h_1 \circ h_0 (b, 0)$, and $\gamma_-$ is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; $\gamma_+$ has endpoints $h_1 \circ h_0 (a, 1)$ and $h_1 \circ h_0 (b, 1)$ and $\gamma_+$ is otherwise disjoint from $\Lambda(a) \cup \Lambda(b)$; and

$$\left(\gamma_- \cup \gamma_+\right) \cap \left( [0, 2] \times \left\{ \frac{1}{2} \right\} \right) = \emptyset.$$ 

Such arcs $\gamma_-$ and $\gamma_+$ will be called admissible arcs joining $\Lambda(a)$ and $\Lambda(b)$.

Definition 3.3. Given $a, b \in C$, $a \neq b$, then $\Lambda(a) <_{bw} \Lambda(b)$ if there are admissible arcs joining $\Lambda(a)$ and $\Lambda(b)$, as above, and the orientation $\gamma_- \to \Lambda(b) \to \gamma_+ \to \Lambda(a)$ is positive (counterclockwise) on the simple closed curve $\gamma_- \cup \Lambda(b) \cup \gamma_+ \cup \Lambda(a)$. (See Figure 8.)

Figure 8. Barge-Walker Ordering on Cantor Fibers.

Definition 3.4. $<_X$ is a local ordering on $X$ if for all $x \in X$ there is a $\delta > 0$ such that $<_X$ is an order relation on $B_{\delta}(x)$. $(X, <_X)$ is a locally ordered metric space.
Lemma 3.1. Given $R$ which contradicts $x < \pi z$ negative orientation or $\Lambda(\pi z)$.

Proof. Choose $\delta > 0$ such that if $x_0, x_1 \in X, |x_0 - x_1| < \delta$, and $x_0 < x_1$, then $G(x_0) <_Y G(x_1)$.

Denote by $[x, y] = \{z \in C | x \leq_C z \leq_C y\}$. We next show $<_C$ on $C$ is $\mathbb{R}$-like in the following sense.

**Lemma 3.1.** Given $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in C$ and $|x - y| < \delta$, then for all $z \in [x, y], |x - z| < \epsilon$ and $|y - z| < \epsilon$.

**Proof.** Suppose that $x, y, z \in C$ and $x <_C z <_C y$. By Definition 3.5 there are admissible arcs $\gamma^+_1, \gamma^-_1, \gamma^+_2$, and $\gamma^-_2$ such that $\Lambda(z) \rightarrow \gamma^+_1 \rightarrow \Lambda(x) \rightarrow \gamma^-_1$ and $\Lambda(y) \rightarrow \gamma^+_2 \rightarrow \Lambda(z) \rightarrow \gamma^-_2$ have positive orientation.

**Sublemma 3.1.** For $\epsilon > 0$ there is a $\delta_1 > 0$ such that if

$$\Lambda(z) \bigcap N_{\delta_1}(\Lambda(x)) \neq \emptyset$$

then $|x - z| < \epsilon$.

**Proof.** By the continuity of $(h_1 \circ h_0)^{-1}$, if $\epsilon > 0$ there is a $\delta_1 > 0$ such that if $d(p, q) < \delta_1$ where $p, q \in \Lambda$ then $d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$. So if $\Lambda(z) \bigcap N_{\delta_1}(\Lambda(x)) \neq \emptyset$ there is $p \in \Lambda(x), q \in \Lambda(z)$ such that $d(p, q) < \delta_1$. Thus $|x - z| = |\pi_1((h_1 \circ h_0)^{-1}(p)) - \pi_1((h_1 \circ h_0)^{-1}(q))| \leq d((h_1 \circ h_0)^{-1}(p), (h_1 \circ h_0)^{-1}(q)) < \epsilon$.

Choose $\delta_1 > 0$ smaller so that if

$$\Lambda(z) \bigcap N_{\delta_1}(\Lambda(x)) \neq \emptyset \text{ and } \Lambda(z) \bigcap N_{\delta_1}(\Lambda(y)) \neq \emptyset$$

then $|x - z| < \epsilon$ and $|y - z| < \epsilon$.

By the continuity of $(h_1 \circ h_0)$ there is $\delta > 0$ such that if $|x - y| < \delta$ then $\Lambda(x) \subset N_{\delta_1}(\Lambda(y))$ and $\Lambda(y) \subset N_{\delta_1}(\Lambda(x))$.

Suppose that $\Lambda(z) \bigcap N_{\delta_1}(\Lambda(x)) \bigcap N_{\delta_1}(\Lambda(y)) = \emptyset$. So either $\pi_1 \Lambda(z) < \pi_1 \Lambda(x)$ or $\pi_1 \Lambda(y) < \pi_1 \Lambda(z)$. Thus either $\Lambda(z) \rightarrow \gamma^+_1 \rightarrow \Lambda(x) \rightarrow \gamma^-_1$ has negative orientation or $\Lambda(z) \rightarrow \gamma^+_2 \rightarrow \Lambda(y) \rightarrow \gamma^-_2$ has positive orientation which contradicts $x <_C z <_C y$.
Thus \( \Lambda(z) \cap \mathcal{N}_{\delta_1}(\Lambda(x)) \neq \emptyset \) and \( \Lambda(z) \cap \mathcal{N}_{\delta_1}(\Lambda(y)) \neq \emptyset \). So by the choice of \( \delta \) then \( |x - z| < \epsilon \) and \( |y - z| < \epsilon \) as desired. \( \square \)

**Lemma 3.2.** Let \( f : (C, <_C) \rightarrow (C, <_C) \) be a local order preserving homeomorphism. Then there is a \( \delta > 0 \) such that if \( |x - y| < \delta \) then \( f([x, y]) = [f(x), f(y)] \).

**Proof.** By Definition 3.6 there is an \( \epsilon > 0 \) such that for any \( x, y \in C \) if \( |x - y| < \epsilon \), and \( x <_C y \) then \( f(x) <_C f(y) \). By Lemma 3.1 there is a \( \delta > 0 \) such that if \( x <_C z <_C y \) and \( |x - y| < \delta \) then \( |x - z| < \epsilon \) and \( |y - z| < \epsilon \). Thus \( f(x) <_C f(z) \) and \( f(z) <_C f(y) \). \( \square \)

The proof of the following lemma was suggested by M. Barge.

**Lemma 3.3.** Let \( f : (C, <) \rightarrow (C, <) \) be a local order preserving homeomorphism. Then \( h_{top} (f) = 0 \).

**Proof.** Recall that \( S \subset C \) is an \((n, \epsilon)\)-spanning set, for \( f \) if for all \( x \in C \) there is a \( y \in S \) such that \( |f^k(x) - f^k(y)| < \epsilon \) for all \( k = 0, 1, 2, \ldots n - 1 \). Then \( (h_{top})_\epsilon (f) = \lim_{n \to \infty} \frac{\log \text{card } S(n, \epsilon)}{n} \), and \( h_{top} (f) = \lim_{\epsilon \to 0} (h_{top})_\epsilon (f) \).

Choose \( \delta \) as in Lemma 3.2 and suppose that \( S \subset C \) is an \((n, \epsilon)\)-spanning set where \( 0 < \epsilon \leq \delta \) (\( \delta \) from the lemma). Let \( X \) be a finite set of \( C \) that is \( \epsilon \)-dense, let \( N = \text{card } X \). Before proceeding with the proof of Lemma 3.3 we prove the following sublemma.

**Sublemma 3.2.** \( S \bigcup f^{-n}(X) \) is an \((n + 1, \epsilon)\)-spanning set.

**Proof.** Let \( x \in C \). Suppose that \( y \in S \) is such that \( |f^k(x) - f^k(y)| < \epsilon \) for \( k = 0, 1, 2, \ldots n - 1 \). There is a \( z \in X \) such that either \( z \in [f^n(x), f^n(y)] \) or \( z \in [f^n(y), f^n(x)] \), and such that \( |f^n(x) - z| < \epsilon \). Then we have that \( f^{-n}(z) \in S \bigcup f^{-n}(X) \) and \( z \) satisfies \( |f^k(x) - f^k(z)| < \epsilon \) for \( k = 0, 1, 2, \ldots n \) as desired. \( \square \)

Continuing with proof of Lemma 3.3, it follows from Sublemma 3.2 that there exists a constant \( K > 0 \) such that for all \( n \), \( \text{card } S(n, \epsilon) \leq K + nN \). Thus,

\[
\begin{align*}
h_{top} (f) &= \lim_{\epsilon \to 0} (h_{top})_\epsilon (f) \\
&= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log \text{card } S(n, \epsilon)}{n} \\
&= \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{\log(K + nN)}{n} = 0.
\end{align*}
\]

\( \square \)

**Lemma 3.4.** Either \( H \) or \( H^2 \) locally preserves \( <_{bw} \) on \( \{\Lambda(a) | a \in C\} \).
**Proof.** By Theorem 2.1, $H|_{\Lambda(c)}$ is monotone for all $c \in C$. Fix $a_0 \in C$ and assume that $h_1 \circ h_0(a) \times \{i\} \subset \ell_i$ and $H \circ h_1 \circ h_0(a) \times \{i\} \subset \ell_i$ for $i = 0$ or 1. (The other cases are similar.) For all $a \neq a_0$ there exists an admissible arc, $\gamma_a^-$ linking $h_1 \circ h_0(a) \times \{0\}$ to $h_1 \circ h_0(a) \times \{0\}$ and an admissible arc, $\gamma_a^+$ linking $h_1 \circ h_0(a) \times \{1\}$ to $h_1 \circ h_0(a) \times \{1\}$. Now $H$ is monotone on the simple closed curve $\Gamma = \Lambda(a_0) \cup \gamma_a^- \cup \Lambda(a) \cup \gamma_a^+$. Thus $H$ can be approximated by a homeomorphism $H': D \to D$ such that $H'(\Lambda(a)) = H(\Lambda(a_0))$, $H'(\Lambda(a)) = H(\Lambda(a))$, $H\gamma_a^- = H(\gamma_a^-)$, and $H\gamma_a^+ = H(\gamma_a^+)$. So the orientation of $H(\Gamma)$ is identical to the orientation of $H'(\Gamma)$. For a sufficiently close to $a_0$ $H'$ (or $(H')^2$) preserves $<_{bw}$ between $\Lambda(a_0)$ and $\Lambda(a)$ [BW93]. Thus $H$ (or $(H)^2$) does so as well. \hfill $\Box$

**Proof of Theorem 3.1 continued.** We now complete the proof of Theorem 3.1. First suppose that $F_1$ and $F_2^2$ do not locally preserve $<_{C}$. Then by Definition 3.5 $H$ and $H^2$ cannot locally preserve $<_{bw}$ on the fibers $\{\Lambda(a)\} a \in C$, contradicting Lemma 3.4.

Next suppose $F_1$ locally order preserves $<_{C}$. Then by Lemma 3.3 we have that $h_{top}(F_1) = 0$. And if $F_1^2$ locally preserves $<_{C}$, then $h_{top}(F_1^2) = 0$, thus $h_{top}(F_1) = 0$. So by [Bow71] $h_{top}(F) = h_{top}(F_1) + \sup_{a \in C} \left\{h_{top}(\left|F\right|_{\pi^{-1}_1(a)})\right\} = \sup_{a \in C} \left\{h_{top}(\left|F\right|_{\pi^{-1}_1(a)})\right\}$. But if $h_{top}(F) > 0$ there is an $a_0 \in C$ such that $h_{top}(\left|F\right|_{\pi^{-1}_1(a_0)}) > 0$. Thus by Theorem 1.3 ([Bar87]) $F_2|_{a_0 \times [0,1]}$ is not monotone. So by Theorem 2.1 no such near homeomorphism extension $H$ of $h_1 \circ h_0 \circ F \circ (h_1 \circ h_0)^{-1}$ exists. \hfill $\Box$

**References**


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**K-TYPES OF SU(1, n) REPRESENTATIONS AND RESTRICTION OF COHOMOLOGY**

**Mark R. Sepanski**

This paper shows that the highest weights of the $K$-types of any irreducible admissible representation of $SU(1, n)$ are determined by certain restriction maps from $u$ to $u \cap t$ cohomology. In particular, the image of these maps determines a set of points in a Cartan subalgebra. It is proved that the highest weights of the $K$-types are given by intersecting a translate of the root lattice with the closed convex hull of the points determined by the restriction maps.

1. Introduction.

A basic idea often employed in the study of representations of real reductive Lie groups is the notion of a $K$-type. In particular, if $G$ is a real reductive Lie group, $K$ its maximally compact subgroup, and $X$ an admissible representation of $G$, then the representations of $K$ appearing in $X$ are called the $K$-types of $X$. The point is that compact groups are well understood and provide a powerful tool in the analysis of noncompact groups. The classical application of these ideas is Bargmann’s description in [1] of the representations of $SL(2, \mathbb{R})$ based on the study on its $SO(2)$-types (see also [4], [2], [3], and especially [6] for an extensive list).

In many cases formulas for the $K$-types are known. For instance Blattner’s formula ([5]) provides a wonderfully explicit description of the $K$-types of the discrete series (see [8] for a generalization). Unfortunately, even in these cases the formulas are combinatorially complex and it is often hard to determine whether a particular representation is a $K$-type.

A different approach, suggested by D. Vogan, is followed in [13]. There the object of study is the closed convex hull of the set of highest weights of the $K$-types. In the case of finite dimensional representations when $G$ is $SU(1, n)$ or $SO(1, n)$, an algebraic method is developed for finding the “edges” of this closed convex hull. The point is that knowledge of just the edges is enough to reconstruct the whole closed convex hull. This already provides fairly sharp control of which representations can be $K$-types. Moreover, it can be seen in [13] that intersecting a translate of the root lattice with the closed convex hull recovers all the $K$-types.
Let $X$ be an irreducible admissible representation of $G$, write $\mathfrak{g}$ for its complexified Lie algebra, and choose a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. The main algebraic tool used to construct the edges in [13] is a restriction map on cohomology

$$\tau : H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u} \cap \mathfrak{k}, X).$$

In a sense that is made precise, it is shown there that the image of $\tau$ (as $\mathfrak{q}$ varies) determines all the edges (not lying in a Weyl chamber wall of $K$) and therefore determines all the $K$-types.

This paper generalizes [13]. The main result is Theorem 9. It says that the edges of the closed convex hull of the set of $K$-types of any irreducible admissible representation of $G = SU(1, n)$ are completely determined by the image of $\tau$ (in fact, only two parabolics are needed). Corollary 3 completes the circle by showing that all $K$-types may be recovered from this closed convex hull by intersecting it with a translate of the root lattice of $K$. All notation necessary to understand the precise result is contained in Section 6.

The layout of the paper is as follows: Section 2 sets up the notation, Section 3 lists the $K$-types of the induced representations of $G$, Section 4 lists the infinitesimal characters and a reducibility criterion, Section 5 gives the $K$-types of all irreducible representations of $G$, and Section 6 constructs the $K$-types in terms of the image of $\tau$.

### 2. Notation.

Let $G = SU(1, n), n > 1$, and write $K \cong U(n)$ for its maximally compact subgroup embedded into $G$ as

$$K = \left\{ \begin{pmatrix} x & \  \\ X & \end{pmatrix} \mid x \in U(1), X \in U(n), x \det(X) = 1 \right\}.$$

Let $\mathfrak{g}_0 = \mathfrak{su}(1, n)$ be the Lie algebra of $G$ and write $\mathfrak{g}$ for its complexification. This convention will be followed throughout the paper. For example, $\mathfrak{k}_0$ is the Lie algebra of $K$, isomorphic to $\mathfrak{u}(n)$, and $\mathfrak{k}$ is its complexification. Also write $\theta$ for the standard Cartan involution and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the corresponding Cartan decomposition.

Let $T$ be the Cartan subgroup of $K$ (and $G$) consisting of all diagonal matrices in $G$. If $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, define its trace by $\text{tr}(x) = x_1 + \ldots + x_m$. With this notation and the identification of $i\mathfrak{t}_0^*$ with $i\mathfrak{t}_0$ via the standard dot product, the set of analytically integral weights on $\mathfrak{t}$ is

$$\hat{T} = \left\{ (\mu_0, \mu_1, \ldots, \mu_n) \in \left( \frac{1}{n+1} \mathbb{Z} \right)^{n+1} \mid \text{tr}(\mu) = 0, \mu_i - \mu_j \in \mathbb{Z} \text{ for } 0 \leq i < j \leq n \right\}.$$
Viewing $T$ as a Cartan subgroup of $K$, we say $\mu \in \hat{T}$ is positive and write $\mu \in \hat{T}^+$ if $\mu_1 \geq \mu_2 \geq \ldots \mu_n$. By taking highest weights, $T^+$ parameterizes the irreducible representations of $K$. We also write $W_K$ for the Weyl group of $K$ and $W_G$ for the Weyl group of $G$ with respect to $i\ell_0$. $W_K$ acts on $i\ell_0$ as the set of all permutations of the last $n$ coordinates and $W_G$ acts on $i\ell_0$ as the set of all permutations.

Let $A$ be the subalgebra of $G$ defined by exp($a_0$) where $a_0 \subseteq p$ is the subalgebra given by

$$a_0 = \left\{ a_\nu \equiv \begin{pmatrix} 0 & \nu \\ \vdots & \ddots \\ \nu & 0 \end{pmatrix} \mid \nu \in \mathbb{R} \right\}.$$ 

By conjugation, we may pull back the standard trace form on the diagonal matrices to $a$ so that $a_{\nu_1} \cdot a_{\nu_2} = 2\nu_1\nu_2$. We use this to identify $a$ and $a^*$. We further identify $\mathbb{C}$ with $a$ by mapping $z \in \mathbb{C}$ to $a_z \in a$. By the identification of $a$ and $a^*$, $z$ acts on $a$ by $z \cdot a_\nu = 2z\nu$.

Let $\Sigma = \Sigma(g, a)$ be the restricted root system so $\Sigma = \{\pm \frac{1}{2}, \pm 1\}$ with multiplicities $2(n-1)$ and $1$, respectively. Set $\Sigma^+ = \{\frac{1}{2}, 1\}$ and let $P$ be the corresponding parabolic subgroup with $P = MAN$ its Langlands decomposition. In particular,

$$M = \left\{ \begin{pmatrix} x & X \\ X & x \end{pmatrix} \mid x \in U(1), X \in U(n-1), x^2 \det(X) = 1 \right\}$$

and is a double cover of $U(n-1)$.

Let $S$ be the Cartan subgroup of $M$ consisting of all diagonal matrices in $M$ and write $H = SA$ as a Cartan subgroup of $G$. The set of analytically integral weights on $S$ is

$$\hat{S} = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_0 = x_n, \text{tr}(x) = 0, x_0 - x_1 \in \frac{1}{2}\mathbb{Z}, \right.$$  

$$x_i - x_j \in \mathbb{Z} \text{ for } 1 \leq i < j \leq n-1 \}$$

$$= \left\{ (x_0, x_1, \ldots, x_n) \mid x_1 \in \frac{1}{n+1}\mathbb{Z}, x_i - x_j \in \mathbb{Z} \text{ for } 1 \leq i < j \leq n-1, \right.$$  

$$x_0 = x_n = -\frac{1}{2} \sum_{j=1}^{n-1} x_j \}.$$ 

We say $x \in \hat{S}$ is positive and write $x \in \hat{S}^+$ if $x_1 \geq x_2 \geq \ldots x_{n-1}$. By taking highest weights, $\hat{S}^+$ parameterizes the irreducible representations of $M$. 

For $\mu \in \hat{T}$, let $V^K_\mu$ be the irreducible representation of $K$ with extremal weight $\mu$. Use similar notation for irreducible representations of $M$. We also write $V^K_\mu |_M$ to signify restriction to $M$. Since the branching law for restriction from $U(n)$ to $U(n-1)$ is well known ([14]), it is easy to determine how restriction works from $K$ to $M$:

**Theorem 1.** Let $\mu = (\mu_0, \mu_1, \ldots, \mu_n) \in \hat{T}^+$. Then

$$V^K_\mu |_M = \bigoplus_{x \in \Phi_\mu} V^M_x,$$

where

$$\Phi_\mu = \{(x_0, x_1, \ldots, x_n) \mid x_0 = x_n, \tr(x) = 0, x_i - \mu_0 \in \mathbb{Z} \text{ for } 1 \leq i \leq n-1, \mu_1 \geq x_1 \geq \mu_2 \geq x_2 \geq \ldots \geq \mu_{n-1} \geq x_{n-1} \geq \mu_n\}.$$

By Frobenius reciprocity, we then have

**Corollary 1.** Let $x = (x_0, x_1, \ldots, x_n) \in \hat{S}^+$. Then

$$\text{Ind}^K_M(V^M_x) = \bigoplus_{\mu \in \Phi_x} V^K_\mu,$$

where

$$\Phi_x = \{(\mu_0, \mu_1, \ldots, \mu_n) \mid \tr(\mu) = 0, \mu_i - x_1 \in \mathbb{Z} \text{ for } 0 \leq i \leq n, \mu_1 \geq x_1 \geq \mu_2 \geq x_2 \geq \ldots \geq \mu_{n-1} \geq x_{n-1} \geq \mu_n\}.$$

For Langlands parameters $x \in \hat{S}$ and $\nu \in \mathfrak{a}^*$, write $I(x, \nu)$ for the normalized induced module $\text{Ind}^G_P(V^M_x \otimes e^\nu)$. Since $I(x, \nu)|_K \cong \text{Ind}^K_M(V^M_x)$, Corollary 1 describes the $K$-types of $I(x, \nu)$. If $\text{Re}(\nu) > 0$, write $J(x, \nu)$ for the unique irreducible Langlands quotient of $I(x, \nu)$.

4. Character Equalities.

For $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, write $\lambda$ for the infinitesimal character $x + \rho_M + \nu$ of $I(x, \nu)$. After conjugating $\mathfrak{h}$ to $\mathfrak{t}$, we may take $\lambda \in \mathfrak{t}$ as

$$\lambda = (x_0 + \nu, x_1 + \frac{n}{2} - 1, x_2 + \frac{n}{2} - 2, \ldots, x_{n-1} - \frac{n}{2} + 1, x_n - \nu). \quad (4.1)$$

We say $\lambda$ is *nonsingular* if no two coordinates are the same.

Using the action of the Weyl group, it is straightforward to write down all induced modules with the same infinitesimal character. The following notation simplifies the results.
**Definition 1.** Let \( s_1 : \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{R}^{n+1} \) by \( s_1(x, \nu) = \hat{x} \) where

\[
\hat{x}_i = \begin{cases} 
  x_0 + \nu - \frac{n}{2} & \text{if } i = 0 \\
  x_i & \text{if } 1 \leq i \leq n - 1 \\\n  x_n - \nu + \frac{n}{2} & \text{if } i = n.
\end{cases}
\]

For each \( a, b \in \mathbb{Z} \) with \( 0 \leq a < b \leq n \), define \( s_{2,a,b} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1} \times \mathbb{C} \) by \( s_{2,a,b}(\hat{x}) = (x', \nu') \) where

\[
x'_i = \begin{cases} 
  \frac{1}{2}(\hat{x}_a + \hat{x}_b - a + n - b) & \text{if } i = 0, n \\
  \hat{x}_{i-1} + 1 & \text{if } 1 \leq i \leq a \\
  \hat{x}_i & \text{if } a + 1 \leq i \leq b - 1 \\
  \hat{x}_{i+1} - 1 & \text{if } b \leq i \leq n - 1,
\end{cases}
\]

\[\nu' = \frac{1}{2}(\hat{x}_a - \hat{x}_b - a + b)\]

and define \( s_{a,b} : \mathbb{R}^{n+1} \times \mathbb{C} \to \mathbb{R}^{n+1} \times \mathbb{C} \) by the composition \( s_{2,a,b} \circ s_1 \). It is easy to verify that the inverse, \( s_{2,a,b}^{-1} \), is determined by defining \( s_{2,a,b}^{-1}(y', w') = \hat{y} \) where

\[
\hat{y}_i = \begin{cases} 
  y'_{i+1} - 1 & \text{if } 0 \leq i \leq a - 1 \\
  y'_0 + w' - \frac{n}{2} + a & \text{if } i = a \\
  y'_i & \text{if } a + 1 \leq i \leq b - 1 \\
  y'_n - w' + \frac{n}{2} + b - n & \text{if } i = b \\
  y'_{i-1} + 1 & \text{if } b + 1 \leq i \leq n
\end{cases}
\]

and defining \( s_1^{-1}(\hat{y}) = (y, w) \) where

\[
w = \frac{1}{2}(\hat{y}_0 - \hat{y}_n + n),
\]

\[
y_i = \begin{cases} 
  \hat{y}_0 - w + \frac{n}{2} & \text{if } i = 0 \\
  \hat{y}_i & \text{if } 1 \leq i \leq n - 1 \\
  \hat{y}_n + w - \frac{n}{2} & \text{if } i = n.
\end{cases}
\]

**Definition 2.** For \( x \in \hat{\mathbb{H}}^+ \) and \( \nu \in \mathbb{C} \), define \( \lambda = \lambda(x, \nu) \) to be

\[
\lambda = \left( x_0 + \nu, x_1 + \frac{n}{2} - 1, x_2 + \frac{n}{2} - 2, \ldots, x_{n-1} - \frac{n}{2} + 1, x_n - \nu \right).
\]

Observe that

\[
\lambda_1 > \lambda_2 > \ldots > \lambda_{n-1}.
\]

If \( \nu \in \mathbb{R}_{\geq 0} \), define \( c, d \in \mathbb{Z}, 0 \leq c < d \leq n \), so that \( \lambda_c > \lambda_0 \geq \lambda_{c+1} \) and \( \lambda_{d-1} \geq \lambda_n > \lambda_d \). Define

\[
(\bar{x}, \bar{\nu}) = s_{c,d}^{-1}(x, \nu).
\]

Observe that if \( \bar{\lambda} = \lambda(\bar{x}, \bar{\nu}) \), then

\[
\bar{\lambda}_0 \geq \bar{\lambda}_1 > \bar{\lambda}_2 > \ldots > \bar{\lambda}_{n-1} \geq \bar{\lambda}_n.
\]
We say that $\nu \in \mathbb{C}$ is non-negative if either $\text{Re}(\nu) > 0$ or both $\text{Re}(\nu) = 0$ and $\text{Im}(\nu) \geq 0$. Since it is easy to see that $I(x, \nu)$ and $I(x, -\nu)$ have the same infinitesimal character for $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, we may assume in the theorem below that $\nu, \nu'$ are non-negative without loss of generality. We will use the notation from Definitions 1 and 2 without further comment. It is then easy to check that:

**Theorem 2.** Fix $x, x' \in \hat{S}^+$ and $\nu, \nu' \in \mathbb{C}$ both non-negative. If $\lambda_0 - \lambda_1 \in \mathbb{Z}$, then $I(x, \nu)$ and $I(x', \nu')$ have the same infinitesimal character if and only if $x = x'$ and $\nu = \nu'$. If $\lambda_0 - \lambda_1 \in \mathbb{Z}$, then $I(x, \nu)$ and $I(x', \nu')$ have the same infinitesimal character if and only if $(x', \nu') = s_{a,b}(x, \nu)$ for some $a, b$ with $0 \leq a < b \leq n$.

Using Theorem 2, the Subrepresentation theorem, and the Langland’s classification, it is easy to identify almost all the induced modules $I(x, \nu)$ that are irreducible. The calculations later in this paper or a few $R$-group calculations suffice to clear up the remaining ambiguities. But since this is known (Kraljevic, [11], Proposition 1, §3 from [12], Theorems 7.5 and 8.7) and not so important for the purpose of this paper, we simply state the result.

**Theorem 3.** For $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$, let $\lambda$ be the character $\lambda(x, \nu)$. Then $I(x, \nu)$ is reducible if and only if $\lambda_0 - \lambda_1 \in \mathbb{Z}$ and either $\lambda_0 - \lambda_{c+1} \neq 0$ or $\lambda_0 - \lambda_{d-1} \neq 0$.

Note that reducibility always implies $\nu \in \frac{1}{2}\mathbb{Z}$.

5. **$K$-types of Langlands quotients.**

In this section we record the $K$-types of each irreducible representation of $G$. The Langland’s classification says that every irreducible representation is a discrete series representation, limit of discrete series representation, an irreducible tempered representation of the form $I(x, i\nu)$ with $x \in \hat{S}^+$ and $\nu \in \mathbb{R}^{\geq 0}$, or one of the $J(x, \nu)$ with $x \in \hat{S}^+$ and $\nu \in \mathbb{C}$ with $\text{Re}(\nu) > 0$. Hence Corollary 1 and Theorem 3 yield the $K$-types of most irreducible representations. The only ones yet to be determined are the discrete series, limit of discrete series, and the irreducible quotients $J(x, \nu)$ in the cases where $I(x, \nu)$ is reducible.

We begin by studying the reducible $I(x, \nu)$ with nonsingular character. They may be parameterized as follows. Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots > \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Choose the unique $x \in S^+$ and $\nu \in \frac{1}{2}\mathbb{Z}^{>0}$ so that $\lambda = \lambda(x, \nu)$. Write

$$I_{a,b}(\lambda) = I(s_{a,b}(x, \nu))$$

and

$$J_{a,b}(\lambda) = J(s_{a,b}(x, \nu))$$
for $a, b \in \mathbb{Z}$ with $0 \leq a < b \leq n$. In particular, $I_{0,n}(\lambda) = I(x, \nu)$. By Section 4, the set of all $I_{a,b}(\lambda)$ encompass the set of all reducible principal series representations ($\text{Re}(\nu) > 0$) with nonsingular character.

The set of discrete series representations of $G$ may be parameterized as follows. Let $\omega_a \in W_G$, $0 \leq a \leq n$, be defined by

$$\omega_a(t_0, t_1, \ldots, t_n) = (t_0, t_0, t_1 \ldots t_{a-1}, t_{a+1}, \ldots t_n).$$

For each $\lambda$ from the previous paragraph, write

$$J_{a,a}(\lambda)$$

for the discrete series representation with infinitesimal character $\omega_a \lambda$ associated to the $G$ chamber determined by $\omega_a \lambda$. The set of all $J_{a,a}(\lambda)$ encompass all discrete series representations. Define $\Lambda_a$ to be the highest weight of its lowest $K$-type. It is easy to verify that

$$(\Lambda_i) = \begin{cases} \hat{x}_a + n - 2a & \text{if } i = 0 \\ \hat{x}_{i-1} + 1 & \text{if } 1 \leq i < a \\ \hat{x}_i - 1 & \text{if } a + 1 \leq i \leq n \end{cases}$$

and that the $K$-types of $J_{a,a}(\lambda)$ are contained in the cone $\Lambda_a + C_a$ where

$$(5.2) \quad C_a = \{ (t_0, t_1, \ldots, t_n) | \text{tr}(t) = 0, t_1, \ldots, t_a \in \mathbb{R}_{\geq 0}, t_{a+1}, \ldots, t_n \in \mathbb{R}_{\leq 0} \}.$$

Using Corollary 1, it easy to explicitly write the $K$-types for each $I_{a,b}(\lambda)$ and to check the following.

**Corollary 2.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots > \lambda_n$ and $\lambda_i = \lambda_j \in \mathbb{Z}, 0 \leq i < j \leq n$. Let $a, b \in \mathbb{Z}$ with $0 \leq a < b \leq n$. The $K$-types of $I_{a,b}(\lambda)$ occur with multiplicity one. $I_{a,b}(\lambda)$ and $I_{a',b'}(\lambda)$ have $K$-types in common if and only if $(a', b') \in \{ (a + \varepsilon_1, b + \varepsilon_2) | \varepsilon_1, \varepsilon_2 \in \{0, \pm 1\}, 0 \leq a' < b' \leq n \}.$

In the case $n = 2$, using only the Langlands classification, the information about $K$-types already determined, and a basic embedding result on discrete series ([9]), it is possible to give the semisimplification of each $I_{a,b}(\lambda)$ and to deduce the $K$-types of each $J_{a,b}(\lambda)$. However, things become too complicated for this line of reasoning to be sufficient for larger $n$. Thus we use the following well known description of the composition series of $I_{a,b}(\lambda)$ (see [11] Proposition 3, §7, [12] Theorem 7.5, [15]).

**Theorem 4.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots > \lambda_n$ and $\lambda_i = \lambda_j \in \mathbb{Z}, 0 \leq i < j \leq n$. Let $a, b \in \mathbb{Z}$ with $0 \leq a < b \leq n$. The socle filtration of $I_{a,b}(\lambda)$ is

$$\begin{array}{c}
J_{a,b}(\lambda) \\
\downarrow \\
J_{a,b-1}(\lambda) \\
\downarrow \\
J_{a+1,b-1}(\lambda)
\end{array}$$

where the bottom row does not occur if $a + 1 = b$. 

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Combining this theorem with our knowledge of the $K$-types allows us to prove:

**Lemma 1.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Let $0 \leq a \leq b \leq n$. Then $J_{a,b}(\lambda)$ and $J_{a',b'}(\lambda)$ have a $K$-type in common if and only if $a = a'$ and $b = b'$.

**Proof.** Corollary 1 and Equations 5.1 to 5.2 imply that if either $a$ and $a'$ or $b$ and $b'$ differ by more than one, then they have no $K$-types in common. On the other hand, if either differs by one, then Theorem 4 allows us to embed $J_{a,b}$ and $J_{a',b'}$ into some $I_{a'',b''}$ where either $a''$ and $a'''$ or $b''$ and $b'''$ differ by more than one so that Corollary 1 again says that they have no $K$-types in common. \(\square\)

This allows us to determine the $K$-types for each $J_{a,b}(\lambda)$ (which includes the discrete series).

**Theorem 5.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots \lambda_n$ and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Let $a, b \in \mathbb{Z}$ with $0 \leq a \leq b \leq n$. The $K$-types of $J_{a,b}(\lambda)$ appear with multiplicity one. Choose the unique $x \in \hat{S}^+$ and $\nu \in \frac{1}{2}\mathbb{Z}_{>0}$ such that $\lambda = \lambda(x, \nu)$. The highest weights of the $K$-types of $J_{a,b}(\lambda)$ are

\[
\{(\mu_0, \mu_1, \ldots, \mu_n) \mid \text{tr}(\mu) = 0, \mu_i - x_1 \in \mathbb{Z} \text{ for } 0 \leq i \leq n, \\
\mu_1 \geq \hat{x}_0 + 1 \geq \ldots \mu_a \geq \hat{x}_{a-1} + 1 > \hat{x}_a \geq \mu_{a+1} \geq \hat{x}_{a+1} \geq \ldots \\
\geq \mu_{b} \geq \hat{x}_b > \hat{x}_{b+1} - 1 \geq \mu_{b+1} \geq \hat{x}_{b+2} - 1 \geq \ldots \mu_{n-1} \geq \hat{x}_n - 1 \geq \mu_n\}.
\]

This notation includes the natural collapsing of certain $\mu$. For instance, if $a = b$ the above inequalities reduce to

\[
\mu_1 \geq \hat{x}_0 + 1 \geq \ldots \mu_a \geq \hat{x}_{a-1} + 1 > \hat{x}_a
\]

\[
\hat{x}_a > \hat{x}_{a+1} - 1 \geq \mu_{a+1} \geq \hat{x}_{a+2} - 1 \geq \ldots \mu_{n-1} \geq \hat{x}_n - 1 \geq \mu_n.
\]

**Proof.** This follows using Lemma 1, Theorem 4, and Corollary 1. For instance, the $K$-types of $J_{a,b}$ for $0 < a \leq b < n$ are the $K$-types that occur in both $I_{a-1,b}$ and $I_{a,b+1}$. The other cases are handled similarly. \(\square\)

We turn our attention to the reducible $I(x, \nu)$ with singular character. They may be parameterized as follows. Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots \lambda_c = \lambda_{c+1} > \ldots \lambda_n$, $0 \leq c \leq n - 1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Choose the unique $x \in \mathbb{R}^{n+1}$ and $\nu \in \frac{1}{2}\mathbb{Z}_{>0}$ so that $\lambda = \lambda(x, \nu)$. Write

\[
I_{a,c+1}(\lambda) = I(s_{a,c+1}(x, \nu)) \\
J_{a,c+1}(\lambda) = J(s_{a,c+1}(x, \nu))
\]

for each $0 \leq a < c$ and

\[
I_{c,b}(\lambda) = I(s_{c,b}(x, \nu)) \\
J_{c,b}(\lambda) = J(s_{c,b}(x, \nu))
\]
for each $c + 1 < b < n$.

The set of discrete series representations of $G$ may be parameterized as follows. Continue with the same $\lambda$ from the previous paragraph and recall the elements $\omega_a \in W_G$ from the discussion of the discrete series. Write

$$J^-_{c,c+1}(\lambda)$$

for the limit of discrete series representation with infinitesimal character $\lambda$ corresponding to the chamber determined by $\omega_c$. It is immediate that Equation 5.1 (with $a = c$) gives the lowest $K$-type and that Equation 5.2 describes a cone containing all of its $K$-types. Similarly write

$$J^+_{c,c+1}(\lambda)$$

for the limit of discrete series representation with infinitesimal character $\lambda$ corresponding to the chamber determined by $\omega_{c+1}$. It is immediate that Equations 5.1 and 5.2 (with $a = c + 1$) describe its $K$-types. The set of all $J^-_{c,c+1}(\lambda)$ encompass all the limits of discrete series. Moreover, it is possible to check that $I(s_{c,c+1}(x,\nu))$ (notice $s_{c,c+1}(\nu) = 0$) splits as a direct sum of $J^-_{c,c+1}(\lambda) \bigoplus J^+_{c,c+1}(\lambda)$.

The composition series of $I(x,\nu)$ is also well known in the singular setting (see [11] Proposition 3, §7, [12] Theorem 7.5) and closely parallels Theorem 4.

**Theorem 6.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots \lambda_c = \lambda_{c+1} > \ldots \lambda_n$, $0 \leq c \leq n - 1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. For $0 \leq a < c$, the socle filtration of $I_{a,c+1}(\lambda)$ is

$$J^-_{a,c+1}(\lambda) \ldots J^-_{a+1,c+1}(\lambda).$$

For $c + 1 < b \leq n$, the socle filtration of $I_{c,b}^+(\lambda)$ is

$$J^+_{c,b}(\lambda) \ldots J^+_{c,b-1}(\lambda).$$

As in Theorem 5, Theorem 6 allows us to immediately determine the $K$-types for each $J^-_{a,c+1}(\lambda)$ and $J^+_{c,b}(\lambda)$ (which includes the limits of discrete series).

**Theorem 7.** Fix $\lambda$ with $\lambda_0 > \lambda_1 > \ldots \lambda_c = \lambda_{c+1} > \ldots \lambda_n$, $0 \leq c \leq n - 1$, and $\lambda_i - \lambda_j \in \mathbb{Z}$, $0 \leq i < j \leq n$. Choose the unique $x \in \mathbb{R}^{n+1}$ and $\nu \in \frac{1}{2}\mathbb{Z}^+$.
so that $\lambda = \lambda(x, \nu)$. For $0 \leq a < c$, the $K$-types of $J_{a,c+1}^-(\lambda)$ are

$$\{(\mu_0, \mu_1, \ldots, \mu_n) \mid \text{tr}(\mu) = 0, \mu_i - x_1 \in \mathbb{Z} \text{ for } 0 \leq i \leq n, \mu_1 \geq \hat{x}_0 + 1 \geq \cdots \mu_a \geq \hat{x}_{a-1} + 1 > \hat{x}_a \geq \mu_{a+1} \geq \hat{x}_{a+1} \geq \cdots \mu_{c} \geq \hat{x}_{c} \geq \mu_{c+1} \geq \hat{x}_{c+1} - 1 \geq \cdots \mu_{n-1} \geq \hat{x}_{n-1} - 1 \geq \mu_n\}.$$  

For $c + 1 \leq b \leq n$, the $K$-types of $J_{c,b}^+(\lambda)$ are

$$\{(\mu_0, \mu_1, \ldots, \mu_n) \mid \text{tr}(\mu) = 0, \mu_i - x_1 \in \mathbb{Z} \text{ for } 0 \leq i \leq n, \mu_1 \geq \hat{x}_0 + 1 \geq \cdots \mu_c \geq \hat{x}_{c-1} + 1 \geq \mu_{c+1} \geq \hat{x}_{c+1} \geq \cdots \mu_b \geq \hat{x}_b > \hat{x}_{b+1} - 1 \geq \mu_{b+1} \geq \hat{x}_{b+2} - 1 \geq \cdots \mu_{n-1} \geq \hat{x}_{n-1} - 1 \geq \mu_n\}.$$  

Corollary 1 and Theorems 5 and 7 give the $K$-types for all irreducible representations of $G$.

6. Restriction of Cohomology.

We begin this section by recalling some notation from [13].

**Definition 3.** Fix a $(\mathfrak{g}, K)$ module $X$ and a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$ where $\mathfrak{l}$ is the Levi component and $\mathfrak{u}$ is the nilradical of $\mathfrak{q}$. Let $\tau$ be the map on cohomology

$$\tau : H^b(\mathfrak{u}, X) \to H^b(\mathfrak{u} \cap \mathfrak{k}, X)$$

induced by restricting $\text{Hom}(\bigwedge^b \mathfrak{u}, X) \to \text{Hom}(\bigwedge^b (\mathfrak{u} \cap \mathfrak{k}), X)$.

Write $\Delta^+(\mathfrak{k}, \mathfrak{t})$ for the positive roots of $\mathfrak{t}$ corresponding to the choice of $\hat{T}^+$ and write $\rho_K$ for the half sum these roots.

**Definition 4.** Fix a $(\mathfrak{g}, K)$ module $X$ and a $\theta$-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$. Let $\nu$ be the highest weight of an $L \cap K$ representation appearing in $H^b(\mathfrak{u} \cap \mathfrak{k}, X)$ and choose $w \in W_K$ so that $w(\nu + \rho_K)$ is positive. Define

$$\nu_K = w(\nu + \rho_K) - \rho_K.$$  

By Kostant’s Borel-Weil theorem ([10]), $V_{\nu_K}^K$ appears in $X|_K$. We say that $\nu_K$ is the associated $K$-type to the $L \cap K$-type $\nu$.

**Definition 5.** Fix a $(\mathfrak{g}, K)$ module $X$. Let $\mathcal{C}$ be the closed convex hull in $i\mathfrak{t}_0^*$ of the set of highest weights of the $K$-types appearing in $X$. Given $\mu \in \hat{T}^+$ a $K$-type of $X$, we say that $\mu$ lies on the geometric edge of the set of $K$-types of $X$ if it lies on the boundary of $\mathcal{C}$ as a subset of $i\mathfrak{t}_0^*$.

It is hoped that the associated $K$-types to the $L \cap K$-types appearing in the image of $\tau$ describe the $K$-types lying on geometrical edges (as long the edge is not completely contained in a Weyl chamber wall of $K$). Thus it is hoped that knowledge of the image of $\tau$ completely determines $\mathcal{C}$ and therefore goes a long way towards describing all the $K$-types.
Definition 6. Fix a \((g, K)\) module, \(X\). If \(\mu \in \hat{T}^+\), the multiplicity of \(\mu\) in \(X\), \(m(\mu)\), is the multiplicity of \(V^K_\mu\) in \(X|_K\). Extend this definition as follows. For \(\mu \in \hat{T}\) with \(\mu + \rho_K\) singular, define \(m_e(\mu) = 0\). For \(\mu \in \hat{T}\) with \(\mu + \rho_K\) nonsingular, there exists a unique \(w \in W_K\) so that \(w(\mu + \rho_K) - \rho_K \in \hat{T}^+\). Define
\[
m_e(\mu) = (-1)^{l(w)}m(w(\mu + \rho_K) - \rho_K)
\]
where \(l(w)\) is the length of \(w\) in \(W_K\).

If \(b\) is any Lie algebra and \(b'\) is a toral subalgebra, write \(\Delta(b, b')\) for the set of roots of \(b\) with respect to \(b'\).

Definition 7. Fix a \((g, K)\) module \(X\) and a \(\theta\)-stable parabolic subalgebra \(q = l + u\) of \(g\). Choose any \(w \in W_K\) so that \(w\Delta(u \cap \mathfrak{t}, \mathfrak{t}) \subseteq \Delta^+(\mathfrak{t}, \mathfrak{t})\) and write
\[
m = \min\{\dim(u \cap \mathfrak{p}), l(w) + 1\}.
\]
Given a \(K\)-type \(\mu \in \hat{T}\) of \(X\), we say that \(\mu\) lies on an algebraic \(q\)-edge of the set of \(K\)-types if
\[
m_e(\mu + 2\rho_A) = 0
\]
for every nonempty collection \(A\) consisting of elements in \(\Delta(u \cap \mathfrak{p}, \mathfrak{t})\) of order at most \(m\) (here \(2\rho_A\) is the sum of the roots in \(A\)).

Theorem 8. Fix a \((g, K)\) module \(X\) and a \(\theta\)-stable parabolic subalgebra \(q = l + u\) of \(g\). If \(\mu \in \hat{T}\) is a \(K\)-type of \(X\) lying on an algebraic \(q\)-edge, then there is another \(\theta\)-stable parabolic and an \(L \cap K\)-type \(\nu\) in the image of \(\tau\) whose associated \(K\)-type is \(\mu\).

Proof. By Theorem 3.4 in \([13]\) (using the notation in Definition 7 above),
\[
\tau : H^l(w)(wu, X) \to H^l(w)(wu \cap \mathfrak{t}, X)
\]
is surjective on the \(L \cap K\)-types \(\nu = w(\mu + \rho_K) - \rho_K\). Choosing \(w\) of minimal length, we may assume that \(\nu\) appears in \(H^l(w)(wu \cap \mathfrak{t}, X)\) (by Lemma 2.2 in \([13]\)) and therefore that \(\nu\) appears in the image of \(\tau\). But since \(\nu_K = \mu\), we are done. \(\square\)

We now apply Theorem 8 to the irreducible representations of \(G\) to show that the geometric edge can be constructed from the image of \(\tau\).

Theorem 9. Let \(X\) be an irreducible representation of \(G = SU(1, n)\). Then any \(K\)-type of \(X\) lying on a geometric edge is the associated \(K\)-type to an \(L \cap K\)-type lying in the image of \(\tau : H^*(u, X) \to H^*(u \cap \mathfrak{t}, X)\) for some \(\theta\)-stable parabolic subalgebra \(q = l + u\) of \(g\).
Proof. The idea of the proof is to use the explicit description of $K$-types (Corollary 1 and Theorems 5, 7, and 3) and to show that every $K$-type of $X$ lying on a geometric edge lies on an algebraic $q$-edge for some $\theta$-stable parabolic subalgebra $q = 1 + u$ of $g$. Theorem 8 then finishes the proof. Since this is merely a matter of coming up with $q$ and checking the appropriate definitions, we only give the details in the case of $X = J_{a,b}(\lambda)$ with $\lambda_0 > \lambda_1 > \ldots > \lambda_n$, $\lambda_i - \lambda_j \in \mathbb{Z}$ for $0 \leq i < j \leq n$, and $a, b \in \mathbb{Z}$ with $0 \leq a \leq b \leq n$. The argument in the other cases is identical. Let $\mu \in \tilde{T}^+$. Then $\mu$ is a $K$-type of $J_{a,b}(\lambda)$ if and only if it satisfies the integrality condition in Theorem 5 and $\xi_i^- \geq \mu_i \geq \xi_i^+$, $1 \leq i \leq n$, where $\xi_i^\pm$ (possibly equal to $\pm \infty$) are identified explicitly in Theorem 5. It is easy to verify that $\xi_{i-1}^+ \geq \xi_i^-$. Thus $\mu$ lies on a geometric edge if and only if either $\xi_i^- = \mu_i$ or $\mu_i = \xi_i^+$, for some $i$, $1 \leq i \leq n$. Write $E_i^\pm$, respectively, for the set of $K$-types lying on a geometric edge satisfying either $\xi_i^\pm = \mu_i$, respectively. Let $\varepsilon_i = (0, \ldots, 0, 1, 0, \ldots 0)$ with $i$ zeros before the one and let $\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j$ be the usual root vector in $\Delta(g, t)$. For each edge define $q_i^\pm = t_i^\pm + u_i^\pm$ to be the $\theta$-stable parabolic subalgebra of $g$ generated by $\pm \varepsilon_i$. In particular,

$$\Delta(q_i^\pm, t) = \{\pm \varepsilon_i \cdot \varepsilon_{r,s} \geq 0 \text{ where } 0 \leq r, s \leq n, r \neq s\}$$

and $\Delta(u_i^\pm \cap p, t) = \{\pm \varepsilon_{0,i}\}$ so that $\dim(u_i^\pm \cap p) = 1$. There are two motivating factors behind this choice of $q_i^\pm$. The first condition is that $t_i^\pm$ is supposed to describe the direction of $E_i^\pm$; i.e., if $\mu, \mu' \in E_i^\pm$ then their difference should be in the span of the roots of $t_i^\pm$. The second condition is that $u_i^\pm$ should point towards the outside of the $K$-types; i.e., if $\mu \in E_i^\pm$ then the sum of $\mu$ and any non-zero root in $u_i^\pm$ should not be a $K$-type. It is easy to check that $q_i^\pm$ is the largest parabolic satisfying these both conditions.

Let $w_i^\pm \in W_K$ be defined by the cyclic permutations $w^+ = (1, 2, \ldots, i)$ and $w^- = (n, n - 1, \ldots, i)$. Then $l(w^+) = i - 1$ and $l(w^-) = n - i$. These elements are chosen so that $w_i^\pm \Delta(u_i^\pm \cap t, t) \subseteq \Delta^+(t, t)$. In particular, $\omega^+ q^\pm$ is generated by $e_1$ and $\omega^- q^-$ is generated by $e_n$. Let $\mu$ be a $K$-type in $E_i^\pm$. Then $\mu$ lies on the algebraic $q_i^\pm$-edge if and only if $m_c(\mu \mp \varepsilon_{0,i}) = 0$. In fact, we prove a much stronger statement that is special to $SU(1, n)$: that $m_c(\mu \mp r \varepsilon_{0,i}) = 0$ for any $r \in \mathbb{R}^>0$. For this it suffices to set $y = \omega(\mu \mp r \varepsilon_{0,i} + \rho_K) - \rho_K$ for any $\omega \in W_G$ and show that $y$ cannot be a $K$-type of $J_{a,b}$.

It is convenient to shift everything by $\rho_K$. Therefore write $\tilde{\mu}$ for $\mu + \rho_K$ and employ similar notation for $\xi$ and $y$. Thus assume we have $\tilde{\xi}^\pm$ satisfying $\tilde{\xi}_{k-1}^+ > \xi_k^+$, $1 \leq k \leq n$, $\tilde{\mu}$ satisfying $\xi_k^- \geq \tilde{\mu}_k \geq \xi_k^+$ and either $\xi_k^\pm = \tilde{\mu}_k$, and $\tilde{y} = \omega(\tilde{\mu} + r \varepsilon_{0,i})$ with $r > 0$. We show there is always some $k$, $1 \leq k \leq n$, so that $\tilde{y}_k$ fails to lie between $\tilde{\xi}_k^\pm$. View $\omega$ as a permutation and consider the case where $\omega(i) = i$ first. In this case $\tilde{y}_i = \tilde{\mu}_i \mp r$ and therefore fails to lie between $\tilde{\xi}_i^-$ and $\tilde{\xi}_i^+$. On the other hand, say $\omega(i) = j \neq i$. Then
\( \tilde{y}_j = \tilde{\mu}_i \) and therefore lies between \( \tilde{\xi}_i^\pm \). However, this makes it impossible for \( \tilde{y}_j \) to lie between \( \tilde{\xi}_j^\pm \) since \( \tilde{\xi}_i^\pm \) and \( \tilde{\xi}_j^\pm \) are disjoint intervals. This finishes the proof.

\[ \square \]

**Corollary 3.** Let \( X \) be any irreducible representation of \( SU(1, n) \). Let \( q_i = l_i + u_i, \ i = 1, n, \) be the two maximal proper \( \theta \)-stable parabolic subalgebras of \( g \) generated by \( \varepsilon_1 \) and \( -\varepsilon_n \), respectively. Let \( E \) be the set of associated \( K \)-types to the \( L \cap K \)-types in the images of \( \tau : H^*(u_i, X) \to H^*(u_i \cap k, X) \).

Then the closed convex hull of the set of highest weights of the \( K \)-types of \( X \) is equal to the closed convex hull of \( E \). Moreover, \( \mu \in \mathfrak{k}^* \) is a \( K \)-type of \( X \) if and only if \( \mu \) lies in the closed convex hull of \( E \) and differs from some element of \( E \) by an element of the root lattice of \( g \).

**Proof.** This follows immediately from the Theorem 9 (noting that all the parabolics used in the proof were \( K \) conjugate to either \( q_1 \) or \( q_n \)) and the explicit description of \( K \)-types.

\[ \square \]

While this is a strong statement, the generalization to other groups cannot always be as nice. In particular, “gaps” may appear in the set of \( K \)-types. But in any case, it is still conjectured that the image of \( \tau \) is enough to describe the closed convex hull of the set of \( K \)-types.

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VERONESE SUBRINGS AND TIGHT CLOSURE

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We determine when an \( \mathbb{N} \)-graded ring has Veronese subrings which are F-rational or F-regular. The results obtained here give a better understanding of these properties, and include various techniques of constructing F-rational rings which are not F-regular.

1. Introduction.

Throughout this paper, all rings are commutative, Noetherian, and have an identity element. By a graded ring, we mean a ring \( R = \oplus_{n \geq 0} R_n \), which is finitely generated over a field \( R_0 = K \).

The theory of tight closure was developed by Melvin Hochster and Craig Huneke in [HH1], and has yielded many elegant and powerful results in commutative algebra and related fields. The theory draws attention to rings which have the property that all their ideals are tightly closed, called weakly F-regular rings, and rings with the weaker property that parameter ideals are tightly closed, called F-rational rings. The term F-regular is reserved for rings with the property that all their localizations are weakly F-regular. (The recent work of Lyubeznik and Smith shows that for graded rings the properties of weak F-regularity and F-regularity are equivalent, see [LS].) These properties turn out to be of significant importance, for instance the Hochster-Roberts theorem of invariant theory that direct summands of polynomial rings are Cohen-Macaulay ([HR1]), can actually be proved for the much larger class of F-regular rings.

While the property of F-rationality provides an algebraic analogue of the notion of rational singularities, F-regularity, in general, is not so well understood geometrically. One approach is to study the variety \( X = \text{Proj} \ R \) for a graded F-regular ring \( R \). The Veronese subrings of \( R \) are also homogeneous coordinate rings for \( X \), and so it is interesting to determine when graded rings have F-rational or F-regular Veronese subrings. The question regarding F-rational Veronese subrings is easily answered: Let \( (R, m, K) \) be a Cohen-Macaulay graded domain of dimension \( d \), with an isolated singularity at \( m \). We show that there exists a positive integer \( n \) such that the Veronese subring \( R^{(n)} \) is F-rational if and only if \( [H^d_m(R)]_0 = 0 \). With regard to F-regular Veronese subrings, we show that if \( R \) is a normal ring generated by degree one elements over a field, then either \( R \) is F-regular, or else no Veronese
subring of $R$ is $F$-regular. This leads us to the question: If $(R, m, K)$ is a normal graded domain, generated by degree one elements, with an isolated singularity at $m$, then under what conditions is $R$ $F$-regular? It is easily seen that $F$-regularity forces the $a$-invariant, $a(R)$, to be negative. For rings of dimension two (although not in higher dimensions) this is also a sufficient condition for $F$-regularity. We construct rings $R$ of dimension $d \geq 3$ with $a(R) = 2 - d$ which are not $F$-regular, while if $a(R) < 2 - d$, Smith has pointed out that $\text{Proj } R$ is a variety of minimal degree, and $R$ is indeed $F$-regular. We also construct a rich family of $F$-rational rings of characteristic zero, with isolated singularities, which have no $F$-regular Veronese subrings.

We would like to point out that although tight closure is primarily a characteristic $p$ notion, it has strong connections with the study of singularities of algebraic varieties over fields of characteristic zero. Specifically, let $R$ be a ring essentially of finite type over a field of characteristic zero. Then $R$ is $F$-regular if and only if it is of $F$-rational type, see [Ha, Sm3]. In the $\mathbb{Q}$-Gorenstein case, we have some even more remarkable connections: $F$-regular type is equivalent to having log-terminal singularities and $F$-pure type implies (and is conjectured to be equivalent to) log-canonical singularities, see [Sm5, Wa4].

2. Preliminaries.

Let $R$ be a Noetherian ring of characteristic $p > 0$. We shall always use the letter $e$ to denote a variable nonnegative integer, and $q$ to denote the $e$th power of $p$, i.e., $q = p^e$. We shall denote by $F^e$, the Frobenius endomorphism of $R$, and by $F^{e}$, its $e$th iteration, i.e., $F^{e}(r) = r^q$. For an ideal $I = (x_1, \ldots, x_n) \subseteq R$, we let $I^{[q]} = (x_1^q, \ldots, x_n^q)$. Note that $F^{e}(I)R = I^{[q]}$, where $q = p^e$, as always. Let $S$ denote the ring $R$ viewed as an $R$-algebra via $F^{e}$. Then $S \otimes_R -$ is a covariant functor from $R$-modules to $S$-modules, and so is a covariant functor from $R$-modules to $R$-modules! If we consider a map of free modules $R^n \to R^m$ given by the matrix $(r_{ij})$, applying $F^e$ we get a map $R^n \to R^m$ given by the matrix $(r^q_{ij})$. For an $R$-module $M$, note that the $R$-module structure on $F^e(M)$ is $r'(r \otimes m) = r' \otimes m$, and $r' \otimes rm = r' r^q \otimes m$. For $R$-modules $N \subseteq M$, we use $N_M^{[q]}$ to denote $\text{Im}(F^e(N) \to F^e(M))$.

For a reduced ring $R$ of characteristic $p > 0$, $R^{1/p}$ shall denote the ring obtained by adjoining all $q$th roots of elements of $R$. The ring $R$ is said to be $F$-finite if $R^{1/p}$ is module-finite over $R$. Note that a finitely generated algebra $R$ over a field $K$ is $F$-finite if and only if $K^{1/p}$ is a finite field extension of $K$.

We shall denote by $R^e$ the complement of the union of the minimal primes of $R$. We say $I = (x_1, \ldots, x_n) \subseteq R$ is a parameter ideal if the images of $x_1, \ldots, x_n$ form part of a system of parameters in the local ring $R_P$, for every prime ideal $P$ containing $I$. 
Definition 2.1. Let $R$ be a ring of characteristic $p$, and $I$ an ideal of $R$. An element $x$ of $R$ is said to be in $I^F$, the Frobenius closure of $I$, if there exists some $q = p^e$ such that $x^q \in I[q]$.

For $R$-modules $N \subseteq M$ and $u \in M$, we say that $u \in N^*_M$, the tight closure of $N$ in $M$, if there exists $c \in R^*\cap M$ such that $cu^q \in N[q]_M$ for all $q = p^e \gg 0$. If $I^* = I$ we say that the ideal $I$ is tightly closed.

A ring $R$ is said to be $F$-pure if for all $R$-modules $M$, the Frobenius homomorphism $F: M \to F(M)$ is injective. A ring $R$ is weakly $F$-regular if every ideal of $R$ is tightly closed, and is $F$-regular if every localization is weakly $F$-regular. Lastly, $R$ is said to be $F$-rational if every parameter ideal of $R$ is tightly closed.

It is easily verified that $I \subseteq I^F \subseteq I^*$. Furthermore, $I^*$ is always contained in the integral closure of $I$, and is frequently much smaller. A weakly $F$-regular ring is $F$-rational as well as $F$-pure. We next record some useful results.

Theorem 2.2.

(1) Regular rings are $F$-regular. A ring which is a direct summand of an $F$-regular ring is itself $F$-regular.

(2) An $F$-rational ring $R$ is normal. If, in addition, $R$ is the homomorphic image of a Cohen-Macaulay ring, then it is Cohen-Macaulay.

(3) An $F$-rational Gorenstein ring is $F$-regular.

(4) Let $(R, m)$ be a reduced excellent local ring of dimension $d$ and characteristic $p > 0$. If $c \in R^*$ is an element such that $R_c$ is $F$-rational, then there exists a positive integer $N$ such that $c^N(0^*_{H^d_0(R)}) = 0$.

(5) Let $R$ be a graded ring. Then $R$ is weakly $F$-regular if and only if it is $F$-regular.

Proof. For assertions (1)-(3), see [HH2, Theorem 4.2]. Part (4) is a result of Velez, [Ve], and (5) is [LS, Corollary 4.4]. □

Remark 2.3. The equivalence of weak $F$-regularity and $F$-regularity, in general, is a formidable open question. However in the light of Theorem 2.2 (5) above, we frequently have no reason to distinguish between these notions.

By a graded ring $(R, m, K)$, we shall always mean a ring $R = \oplus_{n \geq 0}R_n$ finitely generated over a field $R_0 = K$. We shall denote by $m = R_+$, the homogeneous maximal ideal of $R$. The punctured spectrum of $R$ refers to the set $\text{Spec } R - \{m\}$. By a system of parameters for $R$, we shall mean a sequence of homogeneous elements of $R$ whose images form a system of parameters for $R_m$. In specific examples involving homomorphic images of polynomial rings,
lower case letters shall denote the images of the corresponding variables, the variables being denoted by upper case letters.

For conventions regarding graded modules and homomorphisms, we follow [GW]. For a graded $R$-module $M$, we shall denote by $[M]_i$, the $i$-th graded piece of $M$.

**Definition 2.4.** Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring, and $n$ be a positive integer. We shall denote by $R^{(n)}$, the Veronese subring of $R$ spanned by all elements of $R$ which have degree a multiple of $n$, i.e., $R^{(n)} = \bigoplus_{i \geq 0} R_{in}$.

Note that the ring $R^{(n)}$ is a direct summand of $R$ as an $R^{(n)}$-module and that $R$ is integral over $R^{(n)}$. Hence whenever $R$ is Cohen-Macaulay or normal, so is $R^{(n)}$. We record the following result, see [EGA, Lemme 2.1.6] or [Mu, page 282] for a proof.

**Lemma 2.5.** Let $R$ be a graded ring. Then there exists a positive integer $n$ such that the Veronese subring $R^{(n)}$ is generated over $K$ by forms of equal degree.

Recall that the highest local cohomology module $H^d_m(R)$ of $R$, where $\dim R = d$, may be identified with $\lim_{\to} R/(x_1^{t_1}, \ldots, x_d^{t_d})$ where $x_1, \ldots, x_d$ is a system of parameters for $R$ and the maps are induced by multiplication by $x_1 \cdots x_d$. If $R$ is Cohen-Macaulay, these maps are injective. The $R$-module $H^d_m(R)$ carries a natural graded structure, namely $\deg [r + (x_1^{t_1}, \ldots, x_d^{t_d})] = \deg r - t \sum_{i=1}^d x_i$, where $r$ and $x_i$ are homogeneous elements of $R$.

**Definition 2.6.** In the above setting, Goto and Watanabe define the $a$-invariant of $R$ as the highest integer $a(R) = a$ such that $[H^d_m(R)]_a$ is nonzero.

When $R$ is a ring of characteristic $p$, the Frobenius homomorphism of $R$ gives a natural Frobenius action on $H^d_m(R)$ where

$$F: [r + (x_1^{t_1}, \ldots, x_d^{t_d})] \mapsto [r^p + (x_1^{pt_1}, \ldots, x_d^{pt_d})],$$

see [FW] or [Sm2]. For a graded $R$-module $M$, define $M^{(n)} = \bigoplus_{i \in \mathbb{Z}} [M]_{in}$. With this notation, it follows from [GW, Theorem 3.1.1] that

$$H^d_m_{R^{(n)}}(R^{(n)}) \cong (H^d_m(R))^{(n)}.$$

The following theorem, [HH3, Theorem 7.12], indicates the importance of the $a$-invariant in the study of graded F-rational rings.

**Theorem 2.7.** A graded Cohen-Macaulay normal ring $R$ over a field of prime characteristic $p$ is F-rational if and only if $a(R) < 0$ and the ideal generated by some homogeneous system of parameters for $R$ is Frobenius closed.
3. F-rationality of Veronese subrings.

The following proposition, well-known to the experts, addresses the existence of F-rational Veronese subrings.

**Proposition 3.1.** Let $R$ be a graded Cohen-Macaulay domain of dimension $d$, which is locally F-rational on the punctured spectrum $\text{Spec } R - m$. (This is satisfied, in particular, if $R$ has an isolated singularity.) Then $[H^d_m(R)]_0 = 0$ if and only if the Veronese subring $R^{(n)}$ is F-rational for all integers $n \gg 0$. In particular if $a(R) < 0$, then $R^{(n)}$ is F-rational for all integers $n \gg 0$.

**Proof.** Note that we have $[H^d_m(R)]_0 \subseteq 0^{*}_{H^d_m(R)}$, since for $z \in [H^d_m(R)]_0$ we get $cz^q = 0$ for all $q = p^e$, when $c \in m$ is of a sufficiently large degree. Consequently if $R^{(n)}$ is F-rational for some $n$, we must have $a(R^{(n)}) < 0$, but then $[H^d_m(R)]_0 = 0$.

For the converse first note that since $R$ is F-rational on the punctured spectrum, Theorem 2.2 (4) says that $0^{*}_{H^d_m(R)}$ must be killed by a power of the maximal ideal $m$, and so is of finite length. As $[H^d_m(R)]_0 = 0$, for large positive integers $n$ we see that $H^d_{m'}(R^{(n)}) \cong (H^d_m(R))^{(n)}$ contains no nonzero element of $0^{*}_{H^d_m(R)}$ where $m'$ denotes the homogeneous maximal ideal of $R^{(n)}$. If $u \in 0^{*}_{H^d_m(R^{(n)})}$ then $u \in 0^{*}_{H^d_m(R)} \cap H^d_{m'}(R^{(n)})$ and so $u = 0$. Hence $R^{(n)}$ is F-rational for $n \gg 0$. \[\square\]

**Example 3.2.** Let $R = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$ where $K$ is a field of prime characteristic $p$. We make this a graded ring by setting the weights of $x$, $y$ and $z$ to be 15, 10 and 6 respectively. We determine the positive integers $n$ for which the Veronese subring $R^{(n)}$ is F-rational. This shall, of course, depend on the characteristic $p$ of $R$.

First note that $a(R) = -1$ with this grading. If $p \geq 7$, it is easy to verify that the ring $R$ is F-regular. Consequently every Veronese subring of $R$, being a direct summand of $R$, is also F-regular. For $p = 2$, 3 or 5, $x^p \in (y^p, z^p)$, and so $R$ is not F-rational. It is also easily checked that the action of the Frobenius on $H^2_{m}(R)$ is injective in degree $\leq -2$ with the one exception of $p = 2$ where elements in degree $-7$ are mapped to zero under the action of the Frobenius, specifically $F(xy^{-1}z^{-2}) = 0$ in $H^2_{m}(R)$. Recall that $H^2_{m_{R^{(n)}}}(R^{(n)})$ is generated by elements of $H^2_{m}(R)$ whose degree is a multiple of $n$. Consequently for $n \geq 2$ the action of the Frobenius on $H^2_{m_{R^{(n)}}}(R^{(n)})$ is injective, with the one exception. Using the arguments in the proof of the above proposition, we see that $R^{(n)}$ is F-rational for all $n \geq 2$, excluding the case when $p = 2$ and $n = 7$. 

4. Rational coefficient Weil divisors.

We review some notation and results from [De], [Wa1] and [Wa3], as well as make a few observations which we shall find useful later in our study.

Definition 4.1. By a rational coefficient Weil divisor (or a $\mathbb{Q}$-divisor) on a normal projective variety $X$, we mean a $\mathbb{Q}$-linear combination of codimension one irreducible subvarieties of $X$. For $D = \sum n_i V_i$ with $n_i \in \mathbb{Q}$, we set $[D] = \sum [n_i] V_i$, where $[n]$ denotes the greatest integer less than or equal to $n$, and define $\mathcal{O}_X(D) = \mathcal{O}_X([D])$.

Let $D = \sum (p_i/q_i) V_i$ where the integers $p_i$ and $q_i$ are relatively prime and $q_i > 0$. We define $D' = \sum ((q_i - 1)/q_i) V_i$ to be the fractional part of $D$. Note that with this definition of $D'$ we have $-\lfloor -nD \rfloor = \lfloor nD + D' \rfloor$ for any integer $n$.

Given an ample $\mathbb{Q}$-divisor $D$ (i.e., such that $ND$ is an ample Cartier divisor for some $N \in \mathbb{N}$), we construct the generalized section ring:

$$R = R(X, D) = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T].$$

With this notation, Demazure’s result ([De, 3.5]) is:

Theorem 4.2. Let $R = \oplus_{n \geq 0} R_n$ be a graded normal ring. Then there exists an ample $\mathbb{Q}$-divisor $D$ on $X = \text{Proj} R$ such that $R = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T]$, where $T$ is a homogeneous element of degree one in the quotient field of $R$.

Example 4.3. Take the $\mathbb{Q}$-divisor

$$D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S + T)$$

on $\mathbb{P}^1 = \text{Proj} K[S, T]$ where $V(S)$, e.g., denotes the irreducible subvariety defined by the vanishing of $S$. Fix $T$ as the degree one element. Then

$$R = \oplus_{n \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(nD))T^n = K[X, Y, Z]/(X^2 + Y^3 + Z^5), \quad \text{where} \quad X = (S^8T^{10})/(S + T)^3, Y = (S^5T^7)/(S + T)^2, \quad \text{and} \quad Z = (-S^3T^4)/(S + T).$$

Remark 4.4. Let $R = R(X, D)$ be as above. Then the Veronese subring $R^{(n)}$ is given by $R^{(n)} \cong R(X, nD)$. For a rational function $f \in K(X)$ we have an isomorphism $R(X, D) \cong R(X, \text{div}(f) + D)$. If $R$ is generated over $K$ by its elements of degree one, we have $R = R(X, [D])$. Note that $[D]$ is a Weil divisor, i.e., has integer coefficients.

5. Results in dimension two.

In the following theorem, we summarize some familiar results about graded rings of dimension two.
**Theorem 5.1.** Let $R$ be a graded normal ring of dimension two, which is generated by degree one elements over an algebraically closed field. Then the following statements are equivalent:

1. $R$ is isomorphic to a Veronese subring of a polynomial ring in two variables.
2. $R$ is $F$-regular.
3. $R$ is $F$-rational.
4. $R$ has a negative $a$-invariant.

**Proof.** The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow easily. For (4) $\Rightarrow$ (1) note that $X = \text{Proj } R$ is a nonsingular projective curve. Since $[H^2_m(R)]_0 = 0$, we have $H^1(X, \mathcal{O}_X) = 0$ and so $X$ is of genus zero, i.e., $\mathbb{P}^1$. Consequently $R \cong R(\mathbb{P}^1, D)$ where $D$ is a Weil divisor on $\mathbb{P}^1$. Hence $D$ is linearly equivalent to $\mathcal{O}(m)$ for some $m \in \mathbb{N}$ and $R \cong R(\mathbb{P}^1, \mathcal{O}(m)) \cong (K[X_0, X_1])^{(m)}$. 

As an easy consequence of the above, we have:

**Theorem 5.2.** Let $R$ be a graded domain of dimension two, with an isolated singularity, which is finitely generated over an algebraically closed field. If $a(R) < 0$, there exists a positive integer $n$ such that $R^{(n)}$ is isomorphic to a Veronese subring of a polynomial ring in two variables over $K$. In particular, some Veronese subring of $R$ is $F$-regular.

**Proof.** Note that $R$ is excellent and so $R'$, the integral closure $R$ in its fraction field, is module-finite over $R$. Since $R$ has an isolated singularity, the conductor (i.e., the largest common ideal of $R$ and $R'$) is primary to the maximal ideal of $R'$, by which $R_i = R'_i$ for all $i \gg 0$. We may therefore choose a positive integer $k$ such that $R^{(k)}$ is normal, and then choose an appropriate multiple $n$ of $k$, by Lemma 2.5, such that $R^{(n)}$ is generated by elements of equal degree. We are now in a position to apply the above theorem to conclude that $R^{(n)}$ is isomorphic to a Veronese subring of a polynomial ring in two variables.

**Example 5.3.** Let $S = K[X, Y, Z]/(X^3 - YZ(Y + Z))$ where $K$ is a field of characteristic $p \equiv 1 \pmod{3}$ and consider the subring

$$R = K[X, Y^3, Y^2Z, YZ^2, Z^3]/(X^3 - YZ(Y + Z)).$$

It is proved in [HH3] that $R$ is $F$-rational but not $F$-regular, see also [Wa3]. Since $R^{(3)}$ is generated by elements of equal degree, it must be isomorphic to a Veronese subring of a polynomial ring by Theorem 5.1. Indeed,

$$R^{(3)} = K[Y^3, Y^2Z, YZ^2, Z^3].$$

**Example 5.4.** Let $R = K[t, t^4x, t^4x^{-1}, t^4(x + 1)^{-1}]$ where $K$ is a field of prime characteristic $p$. This is one of the examples in [Wa2] of rings which
are F-rational but not F-pure; for a different proof see [HH13]. By mapping a polynomial ring onto it, we may write $R$ as

$$R = K[T, U, V, W]/(T^8 - UV, T^4(V - W) - VW, U(V - W) - T^4W).$$

This is graded by setting the weights of $t, u, v$ and $w$ to be $1, 4, 4$ and $4$ respectively. Note that

$$R^{(4)} = K[S, U, V, W]/(S^2 - UV, S(V - W) - VW, U(V - W) - SW)$$

where we relabel $T^4$ as $S$. Then $R^{(4)}$ is generated by elements of equal degree, and is isomorphic to $K[X^3, X^2Y, XY^2, Y^3]$ by setting $S = XY(X - Y)$, $U = XY^2$, $V = X(X - Y)^2$, and $W = Y(X - Y)^2$.

By Theorem 5.2 we know that a graded normal ring $R$ of dimension two over an algebraically closed field has a Veronese subring $R^{(n)}$ which is F-regular. We next show that if $R$ is a hypersurface, there exists $n$ such that $R^{(n)}$ is actually an F-regular hypersurface.

**Theorem 5.5.** Let $R$ be a graded normal hypersurface of dimension two with $a(R) < 0$. Then there exists a positive integer $n$ such that the Veronese subring $R^{(n)}$ is an F-regular hypersurface.

**Proof.** Let $R = K[X, Y, Z]/(f)$ where $x, y$ and $z$ have weights $m, n$ and $r$ respectively. We may assume without any loss of generality that $m, n$ and $r$ have no common factor. If $d = \gcd(m, n)$, then by our assumption $d$ and $r$ are relatively prime. Therefore $f$ must be a polynomial in $x, y$ and $z^d$. Consequently $R^{(n)}$ is again a hypersurface, and satisfies all the initial hypotheses, and so we may assume that $R$ satisfies the extra hypothesis that $m, n$ and $r$ are pairwise relatively prime. Assume further that $m \geq n \geq r$.

We consider the two cases: a) $n = 1$ and $r = 1$, and b) $m > n > r$. Note that it suffices to show that $R$ is F-rational, since it is indeed a hypersurface.

We first eliminate the case (#) when $f$ is of the form $XH(Y, Z) + G(Y, Z)$. We may take a system of parameters of $R$ of the form $x, t$ where $t$ is the image in $R$ of a polynomial $T \in K[X, Y, Z]$ involving only $Y$ and $Z$. If $R$ is not F-rational, then since $a(R) < 0$, $(x, t)$ cannot be F-pure. Hence for some $q = p^k$, we have $s^q \in (x^q, t^q)$ while $s \notin (x, t)$. Again, we may assume that $s$ is the image in $R$ of a polynomial $S \in K[X, Y, Z]$ involving only $Y$ and $Z$. This means that in $K[X, Y, Z]$, we have $S^q \in (x^q, t^q, XH + G)$ but then $S^q \in (T^q, G^q)$ and so $S \in (T, G)$ in $K[X, Y, Z]$, giving us the contradiction $s \in (x, t)$.

a) We have $a(R) = \deg f - (m + n + r) < 0$, and so $\deg f < m + 2$ since $n = r = 1$. This forces $f$ to be of the form (#).

b) Since $a(R) = \deg f - (m + n + r) < 0$, we have $\deg f < m + n + r < 3m$. Hence up to a scalar multiple, $f$ is of the form $XH(Y, Z) + G(Y, Z)$ or $X^2 + G(Y, Z)$. Note that the first case has already been handled.
Now suppose \( f = X^2 + G(Y, Z) \). Then \( \deg f = 2m < m + n + r \) and so \( 3 < m < n + r \), consequently \( G \) cannot involve a term of the form \( Y^2Z^l \) where \( l \geq 2 \). If \( G \) has a term \( Y^k \), then \( 2m = kn \) and so \( n = 1 \) or \( 2 \). Since \( n > r \), we can only have \( n = 2 \) and \( r = 1 \), but this too is impossible. Hence \( f \) can only be of the form \( f = X^2 + aZ^k + bYZ^l + cY^2Z \) where \( a, b \) and \( c \) are scalars. \( R \) is normal, and so \( c \) must be non-zero since \( l \geq 2 \) and \( k \geq 2 \).

It follows that \( 2m = 2n + r \). If \( a \) is non-zero, \( 2m = rk \) and since \( r \) is even, we can only have \( r = 2 \). But then \( m = n + 1 \), and so \( r \) divides either \( m \) or \( n \), a contradiction. Hence \( a = 0 \), and so \( f = X^2 + bYZ^l + cY^2Z \). If \( b \) were non-zero, then we would have \( n + rl = 2n + r \), i.e., \( n = r(l - 1) \), which forces \( r = 1 \). However we know \( r \) to be even, and so \( b = 0 \). We are left with \( f = X^2 + cY^2Z \) but this is ruled out since \( R \) is normal.

□

6. F-regular Veronese subrings.

We begin by recalling a theorem of Watanabe, [Wa3, Theorem 3.4]:

**Theorem 6.1.** Let \( D_1 \) and \( D_2 \) be ample \( \mathbb{Q} \)-divisors on a normal projective variety \( X \). If the fractional parts \( D'_1 \) and \( D'_2 \) are equal, then the ring \( R(X, D_1) \) is F-regular (F-pure) if and only if the ring \( R(X, D_2) \) is F-regular (F-pure).

A complete proof of the theorem, as stated above, relies on the characterization of strong F-regularity in terms of the tight closure of the zero submodule of the injective hull of the residue field, [Sm1, Proposition 7.1.2], as well as the results of [LS].

**Corollary 6.2.** Let \( R \) be a graded normal ring which is generated by degree one elements over a field. Then either \( R \) is F-regular (F-pure), or else no Veronese subring of \( R \) is F-regular (F-pure).

**Proof.** Since \( R \) is generated by its elements of degree one, we have \( R = R(X, D) \), where \( D \) is a Weil divisor, i.e., has \( D' = 0 \). Also, \( (nD)' = 0 \) where \( n \) is any positive integer. By the above Theorem, \( R = R(X, D) \) is F-regular (F-pure) if and only if \( R^{(n)} \cong R(X, nD) \) is F-regular (F-pure). □

As an application of this result, we now construct a family of rings with negative \( a \)-invariants, which have no F-pure Veronese subrings. This shows that a result corresponding to Theorem 5.2 is no longer true in higher dimensions.

**Example 6.3.** Let \( R = K[X_0, \ldots, X_d]/(X_0^3 + \cdots + X_d^3) \) with \( d \geq 3 \), where \( K \) is a field of characteristic 2. It is readily seen that \( x_0^3 \in (x_1, \ldots, x_d)^* \), since \( x_i^3 \in (x_1, \ldots, x_d) \). Hence \( R \) is not F-pure, and since it is generated by elements of degree one, Corollary 6.2 shows that \( R \) has no F-regular or F-pure Veronese subrings. Note that \( a(R) = 2 - d < 0 \).
We can also see that $R^{(n)}$ is not F-pure (for any $n > 0$) by showing that the element $x_0^d(x_1 \cdots x_d)^{n-1}$ is in the Frobenius closure of the ideal
\[
(x_0^{d-2} x_1^{n-2} \cdots x_{d-1}^{n-1}, x_0^{d-2} x_2 x_3^{n-1} \cdots x_d^{n-1}, \ldots, x_0^{d-2} x_d x_1^{n-1} \cdots x_{d-2}^{n-1}),
\]
although not in the ideal itself.

For all $n \geq 2$, the ring $R^{(n)}$ is an example of a graded ring generated by degree one elements (with an isolated singularity and a negative $a$-invariant) which is F-rational but not F-pure.

**Remark 6.4.** The examples above are not completely satisfactory as they are not valid in the characteristic zero setting: in fact, for $d \geq 3$, the ring $R = \mathbb{Q}[x_0, \ldots, x_d]/(x_0^d + \cdots + x_d^5)$ is of F-regular type. Characteristic zero examples turn out to be much more subtle, and we construct these in the next section.

We again return to the ring $R = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$, and this time determine its F-regular and F-pure Veronese subrings.

**Example 6.5.** Let $R = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$ where $K$ is a field of prime characteristic $p$, and the grading is as before. For $p \geq 7$ the ring $R$ is F-regular, and therefore so is any Veronese subring $R^{(n)}$. We now determine when $R^{(n)}$ is F-regular assuming $p$ is either 2, 3 or 5.

Note that the Veronese subrings $R^{(2)}$, $R^{(3)}$ and $R^{(5)}$ are in fact polynomial rings. Therefore when $n$ is divisible by one of 2, 3 or 5, $R^{(n)}$ is a direct summand of a polynomial ring, and so is F-regular. We show that these are the only instances when $R^{(n)}$ is F-regular, or even F-pure.

Recall from Example 4.3 that $R = R(X,D)$ where $X = \text{Proj} K[S,T]$ and $D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S+T)$. If $n$ is relatively prime to 30, the $\mathbb{Q}$-divisor $nD$ has the same fractional part as $D$, and so $R^{(n)} \cong R(X,nD)$ is not F-pure or F-regular by Theorem 6.1.

We can also construct explicit instances of Frobenius closure to illustrate why $R^{(n)}$ is not F-pure when $n$ is relatively prime to 30. Since $n$ is relatively prime to the weight of $y$, the ring $R^{(n)}$ has a unique monomial of the form $x^l y^m$ with $0 < l < n$. Similarly there is a unique integer $m$ with $0 \leq m < n$ such that $y^{l+1} z^m \in R^{(n)}$, and a unique integer $r$ with $0 < r < n$ such that $x^r z \in R^{(n)}$. We claim that
\[
x^{r+1} y^{l+1} z^d \in (x^r y^{r+1+l+1} z^m, x^r z^{d+1})^F, \quad \text{and}
\]
\[
x^{r+1} y^{l+1} z^d \notin (x^r y^{r+1+l+1} z^m, x^r z^{d+1}).
\]
The second statement is true in $R$ and so also in $R^{(n)}$, while the first assertion follows from
\[
(x^{r+1} y^{r+1} z^d)^p \in ((x^r y^{r+1+l+1} z^m)^p, (x^r z^{d+1})^p) \quad \text{for } p = 2, 3 \text{ or } 5.
\]
Example 6.6. We saw that the F-purity and F-regularity of a ring \( R = R(X,D) \) depend only on the fractional part \( D' \) of the \( \mathbb{Q} \)-divisor \( D \). This is by no means true of F-rationality and F-injectivity (i.e., the injectivity of the Frobenius action on the highest local cohomology module). As an example of this, consider the \( \mathbb{Q} \)-divisors on \( \text{Proj} K[S,T] \)

\[
E = (1/2)V(S) + (1/3)V(T) + (1/5)V(S+T) \quad \text{and} \quad \quad D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S+T),
\]

which have the same fractional part. Then

\[
S = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nE)) T^n \cong K[A,B,C,T]/I
\]

where \( I = (AB-T^5, BC+CT^3-BT^5, AC+CT^2-ABT^2) \) and \( A = T^3/S, B = ST^2 \) and \( C = ST^5/(S+T) \). If the characteristic of \( K \) is 2, 3 or 5, the ring \( R = R(X,D) = K[X,Y,Z]/(X^2 + Y^3 + Z^5) \) is not F-rational (or F-injective) as we saw in Example 3.2. We claim that the ring \( S \) is however F-rational. To see this note that \( a(S) < 0 \), and so it suffices by Theorem 2.7 to verify that the ideal \( I \) generated by the homogeneous system of parameters \( t, a_{15}b^{10} + c^6 \) is Frobenius closed. However this is easily verified: The ring \( S/tS \cong K[A,B,C]/(AB,BC,CA) \) is F-pure since the ideal \((AB,BC,CA)\) is generated by square free monomials, see \([HR2, Proposition 5.38]\).

Remark 6.7. Let \( R \) be a Cohen-Macaulay ring with an isolated singularity, which is generated by degree one elements over an algebraically closed field. For a two dimensional ring \( R \), a negative \( a \)-invariant forces \( R \) to be F-regular, although for rings of higher dimensions this is no longer true: In Example 6.3 we constructed rings \( R \) of dimension \( d > 3 \), with \( a(R) = 2 - d \), which were not F-regular. Smith has pointed out that if \( R \) satisfies the stronger condition that \( a(R) \leq 1 - d \), then \( \text{Proj} R \) is a variety of minimal degree. These are completely classified (see, for example, \([EH]\)) and it is easily verified that in this case \( R \) is F-regular, see \([Sm4, Remark 4.3.1]\).

7. The case of characteristic zero.

Hochster and Huneke have defined analogous notions of tight closure for rings essentially of finite type over a field of characteristic zero, see \([HH1, HH4]\). However we can also define notions corresponding to F-regularity, F-purity, and F-rationality in characteristic zero, without using a closure operation.

Consider the ring \( R = K[X_1, \ldots, X_n]/I \) where \( K \) is a field of characteristic zero. Choose a finitely generated \( \mathbb{Z} \)-algebra \( A \) such that \( R_A = A[X_1, \ldots, X_n]/I_A \) is a free \( A \)-algebra, with \( R \cong R_A \otimes_A K \). Note that the fibers of the homomorphism \( A \to R_A \) over maximal ideals of \( A \) are finitely generated algebras over fields of prime characteristic.
Definition 7.1. Let $R$ be a ring finitely generated over a field of characteristic zero. Then $R$ is said to be of F-regular type if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq K$ and a finitely generated $A$-algebra $R_A$ such that $R \cong R_A \otimes_A K$, and for all maximal ideals $\mu$ in a Zariski dense subset of Spec $A$, the fiber rings $R_A \otimes_A A/\mu$ are F-regular.

Similarly, $R$ is said to be of F-pure type if for all maximal ideals $\mu$ in a Zariski dense subset of Spec $A$, the fiber rings $R_A \otimes_A A/\mu$ are F-pure.

Remark 7.2. Some authors use the term F-pure type (F-regular type) to mean that $R_A \otimes_A A/\mu$ is F-pure (F-regular) for all maximal ideals $\mu$ in a Zariski dense open subset of Spec $A$.

All our positive results towards the existence of F-rational and F-regular Veronese subrings in prime characteristic do have corresponding statements in the characteristic zero situation. However we have so far not exhibited a normal Cohen-Macaulay ring, generated by degree one elements over a field of characteristic zero, which has an isolated singularity and a negative $a$-invariant but is not of F-regular type. N. Hara has pointed out to us a geometric argument for the existence of such rings using a blow-up of $\mathbb{P}^2$ at nine points. In this section, we construct a large family of explicit examples of such rings of dimension $d \geq 3$.

Example 7.3. Take two relatively prime homogeneous polynomials $F$ and $G$ of degree $d$ in the ring $\mathbb{Z}[X_1, \ldots, X_k]$, where $k \geq 3$, such that $G$ is monic in $X_k$ and the monomial $X_k^d$ does not occur in $F$. Using $F$ and $G$, construct the hypersurface $S = \mathbb{Q}[S, T, X_1, \ldots, X_k]/(SF - TG)$ and let $R$ be the subring generated by the elements $sx_1, \ldots, sx_k, tx_1, \ldots, tx_k$.

For suitably general choices of the polynomials $F$ and $G$ of degree $d = k$ the ring $R$ has only isolated singularities, and we show that it is Cohen-Macaulay with $a(R) = -1$, and is not of F-regular type. For an explicit example, take $k = 3$, $F = X_1X_2X_3$ and $G = X_1^3 + X_2^3 + X_3^3$.

We shall prove that $R$ is Cohen-Macaulay whenever $d \leq k$. We first show that the Hilbert polynomial multiplicity of $R$ is $d(k-1)+1$, and then construct a system of parameters such that the ring obtained by killing this system of parameters has length $d(k-1)+1$.

We construct a basis for the vector space generated by the monomials of degree $n \gg 0$, $s^{kn-i}x_1^{j_1}x_2^{j_2} \cdots x_k^{j_k}$, where the $j_r$ are nonnegative integers which add up to $n$. The relations permit us to express $tx_k^d$ in terms of other monomials. Let $[u_1, \ldots, u_m]^i$ denote the set $S$ of monomials of degree $i$ in $u_1, \ldots, u_m$, and for two such sets, let $S \cdot T$ denote the product of all possible pairs from $S$ and $T$. In this notation, for $n \gg 0$, the following monomials...
constitute a basis for \( R_n \):
\[
[s, t]^n \cdot [x_1, \ldots, x_{k-1}]^n,
\]
\[
[s, t]^n \cdot [x_1, \ldots, x_{k-1}]^{n-1} \cdot [x_k],
\]
\[
\ldots
\]
\[
[s, t]^n \cdot [x_1, \ldots, x_{k-1}]^{n-d+1} \cdot [x_k]^{d-1},
\]
\[
[s]^n \cdot [x_1, \ldots, x_k]^{n-d} \cdot [x_k]^d.
\]
Consequently for large \( n \) the vector space dimension of \( R_n \) is
\[
(n + 1) \left\{ \binom{n + k - 2}{k - 2} + \cdots + \binom{n - d + 1 + k - 2}{k - 2} \right\} + \binom{n - d + k - 1}{k - 1}.
\]
As a polynomial in \( n \), the leading term of this expression is
\[
n \left\{ \frac{n^{k-2}}{(k-2)!} + \cdots + \frac{n^{k-2}}{(k-2)!} \right\} + \frac{n^{k-1}}{(k-1)!} = \frac{n^{k-1}(d(k-1)+1)}{(k-1)!},
\]
and so the Hilbert polynomial multiplicity of \( R \) is \( d(k-1)+1 \).

The sequence of elements \( sx_1, sx_2 - tx_1, sx_3 - tx_2, \ldots, sx_k - tx_{k-1} \) is a system of parameters for \( R \). Since we have already verified that the Hilbert polynomial multiplicity of \( R \) is \( d(k-1)+1 \), to prove that \( R \) is Cohen-Macaulay when \( d \leq k \), it suffices to show that the length of the ring \( T \) obtained by killing this system of parameters is at most \( d(k-1)+1 \).

Relabel the generators of \( T \) as \( a_2 = s x_2, a_3 = s x_3, \ldots, a_k = s x_k, a_{k+1} = t x_k \). Note that the relations amongst the \( a_i \) include the size two minors of the matrix
\[
\begin{pmatrix}
0 & a_2 & \cdots & a_{k-1} & a_k \\
a_2 & a_3 & \cdots & a_k & a_{k+1}
\end{pmatrix}.
\]
Consequently a generating set for \([ T ]_{<d}\) is given by
\[
\begin{align*}
deg 0 : & \ 1, \\
deg 1 : & \ a_2, a_3, \ldots, a_{k+1}, \\
deg 2 : & \ a_2 a_{k+1}, a_3 a_{k+1}, \ldots, a_{k+1}^2, \\
deg 3 : & \ a_2 a_{k+1}^2, a_3 a_{k+1}^2, \ldots, a_{k+1}^3, \\
\ldots
\end{align*}
\]
\[
\begin{align*}
deg d - 1 : & \ a_2 a_{k+1}^{d-2}, a_3 a_{k+1}^{d-2}, \ldots, a_{k+1}^{d-1}.
\end{align*}
\]
In degree \( d \) the ring \( T \) has \( d \) additional independent relations coming from the equations \( s^i t^{d-i} f - s^{i-1} t^{d-i+1} g \), for \( 1 \leq i \leq d \). Consequently we need \( k - d \) generators for the degree \( d \) piece of \( T \), and one can check that there are no nonzero elements in degree \( d + 1 \). Hence the length of \( T \) is bounded by \( d(k-1)+1 \), and this completes the proof that \( R \) is Cohen-Macaulay.

It only remains to show that \( R \) is not of \( F \)-regular type when \( k \leq d \). Consider the fiber \( A \) of the map \( Z \mapsto R_Z \) over an arbitrary closed point...
$p\mathbb{Z}$. Then $A$ is a finitely generated algebra over the finite field $\mathbb{Z}/p\mathbb{Z}$, and it suffices to show that $A$ is not $F$-regular. Take the ideal

$$I = (sx_1, sx_2, \ldots, sx_{k-1}, tx_1, tx_2, \ldots, tx_{k-1})A.$$ 

It is easily verified that $(tx_k)^{d-1} \notin I$, and we show that $(tx_k)^{d-1} \in I^*$. To see $(tx_k)^{d-1} \in I^*$ it suffices to check that $\alpha_q = (tx_k)^{(d-1)(q+1)} \in I[j]$. Using the relation $t^d q - t^{d-1} s f$ where $1 \leq i \leq d$, we may rewrite $\alpha_q$ with lower powers of $x_k$ occurring in the expressions involved. We can proceed in this manner till we are left with terms which involve powers of $x_k$ not greater then $d-1$. Hence $\alpha_q$ is a sum of terms which are multiples of

$$s^i t^{q(d-1)-i} x_1^{j_1} x_2^{j_2} \cdots x_{k-1}^{j_{k-1}}, \text{ where } i \leq q(d-1), \text{ and } \sum_{r=1}^{k-1} j_r = q(d-1).$$

If $\alpha_q \notin I[j]$, then $j_r < q$ for $1 \leq r \leq k-1$. However on summing these inequalities we get $q(d-1) < q(k-1)$, a contradiction.

**Remark 7.4.** Consider the polynomial ring $K[X_1, \ldots, X_k]$ where $k \geq 3$. It is worth noting that the ring $R$, as above, is isomorphic to a subring of $K[X_1, \ldots, X_k]$,

$$R = K[X_1 F, X_2 F, \ldots, X_k F, X_1 G, X_2 G, \ldots, X_k G].$$

We can show that $R$ is Cohen-Macaulay precisely when the degree $d$ of $F$ and $G$ is less than or equal to $k$. It would certainly be interesting to explore generalizations of this construction.

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RATIONAL CURVES ON A COMPLETE INTERSECTION
CALABI–YAU VARIETY IN $\mathbb{P}^3 \times \mathbb{P}^3$

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We study rational curves on the Tian-Yau complete inter-
section Calabi–Yau threefold (CICY) in $\mathbb{P}^3 \times \mathbb{P}^3$. Existence
of positive dimensional families of nonsingular rational curves
is proved for every degree $\geq 4$. The number of nonsingu-
lar rational curves of degree $1, 2, 3$ on a general Tian–Yau
CICY is finite and enumerated. The number of curves of
these degrees are also enumerated for the special Tian–Yau
CICY. There are two 1-dimensional families of singular ratio-
nal curves of degree 3 on a general Tian–Yau CICY, making
this degree a turning point between finite and infinite num-
ber of curves. We also introduce a notion of equivalence of a
family of rational curves, and determine the equivalences of
the two 1-dimensional families on the Tian–Yau CICY. The
equivalences equal the predicted numbers of curves obtained
by a power series expansion of the solution of a Picard-Fuchs
equation that arises in superconformal field theory.

1. Introduction and basic definitions.

In the 1980’s the physicists started considering supersymmetric theories for
a 10-dimensional universe. In these theories a Calabi–Yau threefold is at-
tached to every point of the Minkowski time-space. Moreover, certain invari-
ants of this Calabi–Yau threefold are linked to observables in our universe.
For example, the number of generations of elementary particles is 3 (elec-
tron, muon, tauon) in our universe. The superstring theory yields that the
absolute value of the Euler number of the manifold must be twice the number
of generations. So physicists were hoping to find relatively easy examples of
manifolds with Euler number $\pm 6$. The first example was found by G. Tian
and S.-T. Yau ([23], [24]). Their starting point was the following complete
intersection Calabi–Yau threefold in $\mathbb{P}^3 \times \mathbb{P}^3$:

$$X = Z \left( \sum x_i^3, \sum x_i y_i, \sum y_i^3 \right).$$

We shall call this the special Tian–Yau CICY. This variety has Euler number
$-18$. Furthermore, this variety allows a free action by a group $G$ of order 3.
Hence, the quotient variety $X/G$ is CY with Euler number $-6$. 

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More generally, a Tian–Yau CICY is defined by:

\[ X = Z(f_1, f_2, g) \subseteq \mathbb{P}^3_1 \times \mathbb{P}^3_2 \]

where \( f_1, f_2, g \) are polynomials of bidegrees \((3, 0), (0, 3)\) and \((1, 1)\) respectively, and such that \( X \) is nonsingular. By a general Tian–Yau variety we shall mean a generic choice of the polynomials \( f_1, f_2, g \). We introduce the following notation: \( F_1 \) (resp. \( F_2 \)) is the cubic surface in \( \mathbb{P}^3_1 \) (resp. \( \mathbb{P}^3_2 \)) defined by the polynomial of bidegree \((3, 0)\) (resp. \((0, 3)\)). The variety \( G \) is defined by the polynomial of bidegree \((1, 1)\). In other words:

\[ X = F_1 \times F_2 \cap G. \]

All deformations of a general Tian–Yau variety as an abstract variety, are realisable as polynomial deformations of the defining equations. The family of Tian–Yau varieties in \( \mathbb{P}^3_1 \times \mathbb{P}^3_2 \) is complete in this sense.

After the example of G. Tian and S.-T. Yau was found, an intensive search for more examples of complete intersections in multiprojective space was undertaken ([2], [3], [8], [9]). It is proven that no complete intersection Calabi–Yau threefold \( X \) in multiprojective space would have \(|\chi(X)| = 6\). As in the example given by G. Tian and S.-T. Yau one may look for some group acting freely on the variety. Such groups are of course hard to find and in many cases it is quite easy to prove that such groups cannot exist. In fact, starting with the list of all the CICY threefold types (approximately 10,000) in multiprojective space, the final result was that there were at most 3 types (including the Tian–Yau CICY type) ([3]) that possibly could have a free action by a group acting of the desired cardinality. Moreover, to the present day a such group has only been found in the Tian–Yau case.

The quintic in \( \mathbb{P}^4 \) is the most studied Calabi–Yau threefold. Clemens conjectured that there are only finitely many smooth rational curves on a general quintic threefold for every degree. This has been proven for degrees less than 10 ([13], [12]). The numbers have been computed for degrees less than or equal to 4 using algebro-geometric techniques ([13], [6], [15]). Using conformal field theory, one is able to predict the number of rational curves of every degree. More precisely, the predicted numbers appear in a power series expansion of the solution of a Picard-Fuchs equation. In the case of the quintic in \( \mathbb{P}^4 \), the power series looks like:

\[ F(q) = 5 + \sum_d n_d d^3 \frac{q^d}{1 - q^d}, \]

where \( n_d \) is the (conjectured) number of rational curves of degree \( d \). This expansion first appeared in ([4]), and started off a new branch of mathematics trying to understand the mathematical implications of mirror symmetry. Recently progress has been made in achieving this goal ([7], [16]). For a further discussion and a more complete reference list, see ([17]).
In general it is hard to determine when the number of curves of a given degree is finite. We address this question in case of a general Tian–Yau CICY. Every rational curve on a Tian–Yau variety has a bidegree, and a degree via the Segre embedding in $\mathbb{P}^{15}$. Furthermore the Hilbert scheme $\text{Hilb}^{m+1}_X$ has a natural partition in open-closed disjoint subschemes $\text{Hilb}^{(i,j)m+1}_X$ with $i + j = d$.

Our main result is the following:

**Theorem 1.1.** Let $X$ be a general Tian–Yau CICY. Let $m > 3$ be an integer, and $i \in \{0, 1, 2, 3\}$. Then there exist positive dimensional families of nonsingular rational curves of bidegree $(m, m-i)$.

There exist also positive dimensional families of nonsingular rational curves of bidegrees $(2, 2)$, $(3, 3)$, $(3, 2)$.

This has the following corollary:

**Corollary 1.2.** There exist positive dimensional families of nonsingular rational curves on a general Tian–Yau CICY for every degree $n$, $n \geq 4$.

However, this abundance of curves for infinitely many bidegrees, does not extend to all bidegrees, on the contrary, there are infinitely many bidegrees that do not allow any rational curves at all:

**Theorem 1.3.** There are no curves of bidegree $(m, 1)$ or $(m, 0)$ on a general Tian–Yau CICY for $m \geq 4$.

The number rational curves on a general Tian–Yau CICY is finite only for degrees 1 and 2. An explicit enumeration of rational curves of the various bidegrees shows that these numbers are in agreement with the numbers worked out by S. Hosono, A. Klemm, S. Theisen, S.-T. Yau ([11]) and by V.V. Batyrev and D. van Straten ([1]) using the Picard-Fuchs equation. The number of nonsingular rational curves of degree 3 is finite, but there are also two 1-dimensional families of singular rational curves on a general Tian–Yau CICY. We give an algebro-geometric definition of the equivalence of a 1-dimensional family of rational curves, and apply this definition to the two 1-dimensional families of degree 3 curves on a general Tian–Yau CICY.

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2. Preliminaries.

In this section we study the geometry and the rational curves of the variety $G = Z(\sum \alpha_{ij}x_iy_j)$. We start this section with two lemmas.
Lemma 2.1. Set $G = Z(\sum \alpha_{ij} x_iy_j) \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$ and let $L$ be a line in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ (resp. $\mathbb{P}_1^3$). Then there exists a unique maximal linear space $V(L)$ in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ (resp. $\mathbb{P}_1^3$) such that $V(L) \times L$ (resp. $L \times V(L)$) is contained in $G$.

Proof. We prove the assertion in the case where the line $L$ is in $\mathbb{P}_1^3$. Assume $L$ is defined by $Z(y_2, y_3) \subseteq \mathbb{P}_2^3$. Set $G = G|_{\mathbb{P}_1^3 \times L}$, then $G$ is defined in $\mathbb{P}_1^3 \times L$ by the following equation:

$$\sum \alpha_{i0}x_iy_0 + \sum \alpha_{i1}x_1y_1 = 0.$$ 

Obviously

$$V(L) = Z\left( \sum \alpha_{i0}x_i, \sum \alpha_{i1}x_1 \right)$$

is both maximal and unique.

Since we made no assumption on the $\alpha_{ij}$'s, we can always reduce to the case where $L$ is as above. \hfill \square

Remark 2.2. In fact we proved more: Every point $a \in \mathbb{P}_1^3$ with the property that $a \times L \subseteq G$, is contained in $V(L)$. Note also that all cases dim$V(L) = 1, 2, 3$ occur. The general case is clearly dim$V(L) = 1$. The definition of $V(L)$ depends on $L$ as well as on $G$. We are primarily interested in the case when dim$V(L) = 1$ for all $L \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$, $i = 1, 2$.

Lemma 2.3. Let $G = Z(\sum \alpha_{ij} x_iy_j)$. The matrix $[\alpha_{ij}]$ is invertible if and only if dim$V(L) = 1$ for all $L \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3$, $i = 1, 2$.

Proof. Assume that $[\alpha_{ij}]$ is invertible. Introduce the notation $xAy^t = \sum \alpha_{ij}x_iy_j$, where $x = (x_0, \ldots, x_3)$, $y = (y_0, \ldots, y_3)$, and

$$A = \begin{pmatrix} \alpha_{00} & \ldots & \alpha_{03} \\ \vdots & \ddots & \vdots \\ \alpha_{30} & \ldots & \alpha_{33} \end{pmatrix}.$$ 

We have to prove that for every line $L$ in $\mathbb{P}_1^3$, $V(L)$ is of minimal dimension.

Consider first the special case where $L = Z(y_2, y_3) \subseteq \mathbb{P}_2^3$. This gives

$$V(L) = Z\left( \sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i \right) \times L \subseteq \mathbb{P}_1^3 \times \mathbb{P}_2^3.$$ 

Assume that $V(L)$ is not of minimal dimension, i.e., dim$Z(\sum \alpha_{i1}x_i, \sum \alpha_{i0}x_i) \geq 2$. This implies that $\sum \alpha_{i1}x_i = \lambda \sum \alpha_{i0}x_i$, giving $\alpha_{i1} = \lambda \alpha_{i0}$. In other words, the first two columns are proportional, which contradicts that $A$ is of maximal rank.

The final step is reducing the general situation to the special case considered above. This is done in the following way: Choose any line $L$ in $\mathbb{P}_1^3$. It is possible to change the coordinates on the second factor, such that $L$ is equal to $Z(y'_2, y'_3)$. Call this coordinate change matrix $P$ (i.e.,
(y_1, \ldots, y_3)^t = P \cdot (y_1, \ldots, y_3)^t$. We make the following change of coordinates on the first factor:

\[ x'^t = (A^{-1})^t P^t A^t x^t. \]

This gives $G = Z(\sum \alpha_{ij} x_i' y_j')$ with respect to the new coordinates, since

\[ \sum \alpha_{ij} x_i y_j = x A y^t = (x' A P A^{-1}) A(P^{-1} y'^t) = x' A y'^t = \sum \alpha_{ij} x_i' y_j'. \]

The result now follows from the special case considered above.

We also observe that $\dim V(L) = 1$ for all $L$, implies that all columns must be linearly independent (by considering the lines $Z(y_2, y_3)$, $Z(y_0, y_3)$, and $Z(y_0, y_2)$). This gives the implication the other way. \[\square\]

When $G$ is as in the previous proposition, we have a map:

\[ l : \text{Grass}(1, \mathbb{P}^3) \rightarrow \text{Grass}(1, \mathbb{P}^3) \]

defined by sending $L$ to $V(L)$. This map is obviously bijective, since $l(l(L)) = L$ by definition. In this case, we shall write $l(L)$ for $V(L)$ to signify that it is a line. Moreover, note that then the defining equation can be brought to diagonal form $\sum_{i=0}^3 x_i y_i$ by a suitable change of coordinates on $\mathbb{P}_1^3$ and $\mathbb{P}_2^3$.

**Proposition 2.4.** Let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and set $\bar{G} = G|_{\mathbb{P}^3 \times L}$. Then $\bar{G}$ is isomorphic to the blow-up of $\mathbb{P}_1^3$ in $l(L)$.

**Proof.** We can, without loss of generality, assume that $\bar{G}$ is defined by

\[ Z(x_1 y_2 - x_2 y_1) \subseteq \mathbb{P}^3(x_0, \ldots, x_3) \times \mathbb{P}^1(y_1, y_2) \]

(by change of coordinates). Then $l(L)$ is defined by $x_1 = x_2 = 0$. It is enough to check the statement locally, take for instance $x_0 = 1$. Then we have

\[ Z(x_1 y_2 - x_2 y_1) \subseteq \mathbb{A}^3 \times \mathbb{P}^1. \]

This is in fact the blow-up of $\mathbb{A}^3$ with center $Z(x_1, x_2)$ ([10], II.7.12.1, p. 163). \[\square\]

**Corollary 2.5.** Let $G = Z(\sum \alpha_{ij} x_i y_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and let $\bar{G} = G|_{H \times L}$, where $H$ is a hyperplane and $L$ is a line. Let $\pi : G \rightarrow \mathbb{P}_1^3$ denote the blow-up of $\mathbb{P}_1^3$ in $l(L)$. Then $\bar{G}$ is isomorphic to $\pi^{-1}(H)$. If $l(L) \not\subseteq H$, then $\bar{G}$ is isomorphic to $H$ blown up in the point $H \cap l(L)$.

We end this section by a proposition that we will use extensively in the following sections. For its formulation we need a definition.

**Definition 2.6.** A rational curve in $\mathbb{P}_1^3 \times \mathbb{P}_2^3$ is of type $(\bar{m}, \bar{n})$ if the image of the first (resp. second) projection is of degree $m$ (resp. $n$).
Proposition 2.7. Let $L$ be a line in $\mathbb{P}^3$, and let $C_1$ be a rational curve of degree $m$ in $\mathbb{P}^3$. Furthermore, let $G = Z(\sum \alpha_{ij}x_ix_j)$, where the matrix $[\alpha_{ij}]$ is invertible, and denote $G|_{\mathbb{P}^3 \times L}$ by $\bar{G}$. Let $C$ be the unique irreducible component of $D = C_1 \times L \cap \bar{G}$ such that $\pi_1(C) = C_1$, where $\pi_1$ is the projection map on the first factor. Let $i = lg(C_1 \cap l(L))$. Then $C$ is a rational curve of bidegree $(m, m-i)$ and of type $(\bar{m}, 1)$. Furthermore, every rational curve of bidegree $(m, n)$ and of type $(\bar{m}, 1)$ must arise in this way.

Proof. The variety $\bar{G}$ is isomorphic to the blow-up of $\mathbb{P}^3$ with center $l(L)$, so $C$ is by definition the strict transform of $C_1$. Moreover, $D = C \cup E_1 \cup \cdots \cup E_i$, where $E_1, \cdots, E_i$ are the exceptional fibers corresponding to the intersection points $p_1, \ldots, p_i$ in $C_1 \cap l(L)$. The curve $C$ is rational ([10], V.3.7, p. 389). The degree on the second factor drops by one for each intersection point counted with multiplicity, giving the desired bidegree.

The converse follows by reversal of the above argument. □

We have the following important corollary:

Proposition 2.8. Let $G = Z(\sum \alpha_{ij}x_ix_j)$, where the matrix $[\alpha_{ij}]$ is invertible. Then there are no nonsingular rational curves of bidegree $(m, 0)$, $m \geq 3$, on $G$.

Proof. Assume for contradiction that $C$ is a nonsingular rational curve of bidegree $(m, 0)$, $m \geq 3$ on $G$, i.e., $C = C_1 \times \{p\}$, where $C_1$ is a nonsingular rational curve in $\mathbb{P}^3$ and $p$ is a point in $\mathbb{P}^2$. Fix a line $L$ in $\mathbb{P}^3$ passing through $p$. By the proof Proposition 2.7 $l(L)$ has to be an $m$-secant to the curve $C_1$. This is impossible since a nonsingular rational curve of degree $m$ has at most an $(m-1)$ secant for $m \geq 3$. □

3. The number of rational curves of degree less than 4 on a general Tian–Yau CICY.

In this section we compute the number of nonsingular rational curves on a general Tian–Yau CICY for degrees less than 4. In the end of this section we describe two 1-dimensional families of singular rational curves of degree 3.

Proposition 3.1. The numbers $N_{i,j}$ of nonsingular rational curves of bidegree $(i, j)$ on a general Tian–Yau variety are finite for $i + j \leq 3$ and are given by:

\[
N_{0,1} = N_{1,0} = 81 \\
N_{0,2} = N_{2,0} = 81 \\
N_{1,1} = 729 \\
N_{1,2} = N_{2,1} = 2187 \\
N_{3,0} = N_{0,3} = 0.
\]
Proof. All curves of bidegree \((1,0)\) have to be of the form \(L \times \{b\}\) where \(L\) is a line on \(F_1\) and \(b\) is a point on \(F_2\). The condition that this \((1,0)\) also should be contained in \(G\), gives three possible values for the point \(b\) given a line \(L\) on \(F_1\). Since \(F_1\) has 27 lines, the number of \((1,0)\) curves is \(3 \cdot 27 = 81\).

A \((2,0)\) curve is of the form \(C_1 \times \{b\} \subseteq H \times \{b\}\), where \(C_1\) is a conic in a hyperplane \(H\) in \(\mathbb{P}^3\). It follows that \(H \cap F_1 = C_1 \cup L\), since \(F_1\) is of degree 3. Hence, the number of \((2,0)\) curves has to be equal to the number of \((1,0)\) curves, which is 81. The intersection \(L' \times L \cap G\) is obviously an irreducible rational curve of bidegree \((1,1)\), and every rational curve of bidegree \((1,1)\) has to arise in this way. Hence, the number of bidegree \((1,1)\) curves is equal to the number of pairs \((L',L)\) where \(L' \subseteq F_1\) and \(L \subseteq F_2\). The number of such pairs is \(27 \cdot 27 = 729\).

We now want to find the number of rational curves of bidegree \((2,1)\). Denote the lines on \(F_i\) by \(L_k^i\) where \(k \in \{1, \ldots, 27\}\). In the generic situation \(l(L_k^i) \cap L_k^{i'} = \emptyset\) for all \(k,k' \in \{1, \ldots, 27\}\). Since there are only finitely many planes \(H\) in \(\mathbb{P}^3\) such that \(H \cap F_1\) is the union of three lines, the \(l(L_k^i)\) are in the general situation not contained in any of these planes. This gives that for each 1-dimensional family of conics and for each \(l(L_k^i)\) we get three planes such that \(l(L_k^i)\) intersects a conic contained in the intersection of \(F_1\) and the plane. By Proposition 2.7 each of these cases gives rise to one bidegree \((2,1)\) curve and every bidegree \((2,1)\) curve has to arise this way. This gives the total number of bidegree \((2,1)\) curves: \(3 \cdot 27 \cdot 27 = 2187\).

Finally, there are no nonsingular rational curves of bidegree \((3,0)\) or \((0,3)\) by Proposition 2.8.

\[\text{Corollary 3.2. Let } N_d \text{ denote the number of nonsingular rational curves of degree } d \text{ on a general Tian–Yau variety. Then}\]

\[N_1 = 162, \quad N_2 = 891, \quad N_3 = 4374.\]

\[\text{Remark 3.3. In Section 4 we will prove that there are positive dimensional families of nonsingular rational curves for every degree higher than 3. Hence the list in Corollary 3.2 is complete. The curves of degree 1 and 2 are necessarily nonsingular. However, there exist singular rational curves of degree 3. These have to be of bidegree \((3,0)\), since the \((2,1)\) curves necessarily had to be nonsingular by the proof of Proposition 3.1.}\]

We shall show that a general Tian–Yau variety contains two 1-dimensional families of singular rational curves of degree 3. Suppose \(C\) is a rational curve of bidegree \((3,0)\). By Proposition 2.8, the curve \(C\) has to be singular. Hence, \(C\) has to be a plane rational curve of degree 3, i.e., a nodal or cuspidal cubic curve. After a suitable change of coordinates we may assume that \(G = Z(\sum_i x_i y_i)\). The curve \(C\) is of the form \(C_1 \times \{p\}\), where \(C_1\) is on \(F_1\) and \(p = (p_0, p_1, p_2, p_3)\) is a point on \(F_2\). Furthermore, \(C_1 \subseteq H_p = Z(\sum_i p_i x_i)\). By Bezout’s theorem, \(F_1 \cap H_p\) is of degree 3, so \(C_1 = F_1 \cap H_p\). Finally, that
$F_1 \cap H_p$ is a singular cubic implies that $H_p$ is a tangent hyperplane, i.e., $H_p \in F'_1 \subseteq (\mathbb{P}^3)'$, the dual variety of $F_1$. We may identify the set of $(3,0)$ curves (not necessary irreducible) on $X$ with the set:

$$\{(H_p,p) \subseteq F_1' \times F_2 \} \subseteq \mathbb{P}^3' \times \mathbb{P}^3 \cong \mathbb{P}^3_1 \times \mathbb{P}^3_2$$

hence with $F_1' \times F_2 \cap \Delta$, where $\Delta$ denotes the diagonal. This set is isomorphic to $F'_1 \cap F_2 \in \mathbb{P}^3_2$.

Hence, we can represent the complete family of bidegree $(3,0)$ rational curves as a curve in $\mathbb{P}^3$. We denote this curve by $\Gamma$.

Since $\deg F'_1 = 12$, $\deg \Gamma = 3 \cdot 12 = 36$. Note that $\Gamma$ is a (local) complete intersection, so the dualising sheaf is given by:

$$\omega_{\Gamma} \cong \wedge^2 N_{\Gamma/\mathbb{P}^3} \otimes \omega_{\mathbb{P}^3}|_\Gamma.$$ 

Using $N_{\Gamma}|_{\mathbb{P}^3} \cong \mathcal{O}_\Gamma(d_1) \oplus \mathcal{O}_\Gamma(d_2)$, where $d_1 = 12$ and $d_2 = 3$ are the degrees of $F'_1$ and $F_2$ respectively, we get

$$\deg \omega_{\Gamma} = 2p_a - 2 = (d_1 + d_2 - 4)d_1d_2 = 396$$

so that the arithmetic genus, $p_a$, of $\Gamma$ is 199.

Finally, we want to determine the singularities of the curve $\Gamma$. The dual surface $F'_1$ has a double curve of degree $\frac{1}{2}d(d-1)(d-2)(d^3-d^2-d-12) = 27$ (see e.g. [18]), where $d = \deg F_1 = 3$. The nodes on $\Gamma$ are precisely the intersection points between $F_2$ and the double curve on $F'_1$. Let $\delta$ denote the number of nodes. Then

$$\delta = 3 \cdot 27 = 81.$$ 

The surface $F'_1$ also has a cuspidal edge of degree $4d(d-1)(d-2) = 24 ([18])$. The cusps on $\Gamma$ are the intersection points between $F_2$ and this curve. Let $\kappa$ denote the number of cusps, then

$$\kappa = 3 \cdot 24 = 72.$$ 

Hence the geometric genus, $p_g$, of $\Gamma$ is given by $p_g = p_a - \delta - \kappa = 46$.

In the end of this section we will give a definition of the equivalence, that apply to our two 1-dimensional families. We start out by reviewing the definition given by S. Katz of the equivalence of a family of rational curves ([14]). Let $X$ be a Calabi–Yau threefold and let $H$ be a $k$-dimensional nonsingular family of nonsingular rational curves on $X$. Consider

$$\mathbb{D} \longrightarrow H \times X$$

$$\pi \downarrow$$

$$H$$

where $\mathbb{D}$ is the total space of the family and $\pi$ is the projection on the first factor. Let $\mathcal{N}_{\mathbb{D}/H \times X}$ denote the normal bundle of $\mathbb{D}$ in $H \times X$. The
equivalence, $e(H)$, of the family $H$ is then defined to be:

$$e(H) = \deg c_k(R^1\pi_*\mathcal{N}_{\mathbb{D}/H \times X}).$$

We are interested in determining the equivalence of a family of mildly singular curves, say local complete intersection curves with at most node and cusp singularities. Furthermore, we shall allow the base space of the family to have the same kind of mild singularities. Because of the singularities, we can not apply Katz’ notion of equivalence directly. It is easy to see that $R^1\pi_*\mathcal{N}_{\mathbb{D}/H \times X}$ is not a vector bundle in general.

Let $C$ be a local complete intersection curve in $X$. Let $I$ be its defining ideal. The sheaf $I/I^2$ is locally free, hence so is the normal sheaf $\mathcal{N}_{C/X} = (I/I^2)'$. The adjunction formula says:

$$\wedge^2 \mathcal{N}_{C/X} \otimes \omega_X|_C \cong \omega_C$$

where $\omega_C$ is the dualising sheaf. Since $X$ is Calabi–Yau, this gives:

$$\wedge^2 \mathcal{N}_{C/X} \cong \omega_C.$$

Furthermore, since $\mathcal{N}_{C/X}$ is a rank 2 bundle, we have the following perfect pairing $\mathcal{N}_{C/X} \cong \mathcal{N}_{C/X}' \otimes \wedge^2 \mathcal{N}_{C/X}$. Hence

$$\mathcal{N}_{C/X} \cong \mathcal{N}_{C/X}' \otimes \omega_C.$$

By Serre Duality we get:

$$(5) \quad H^1(\mathcal{N}_{C/X}) \cong H^{1-1}(\mathcal{N}_{C/X}' \otimes \omega_C)' \cong H^0(\mathcal{N}_{C/X})'.$$

Consider a family $\pi : \mathbb{D} \to H$ of local complete intersection curves with at most cusps and nodes on $X$. The relative version of the isomorphism (5) is

$$(6) \quad R^1\pi_*\mathcal{N}_{\mathbb{D}/H \times X} \cong \text{Hom}(\pi_*\mathcal{N}_{\mathbb{D}/H \times X}, \mathcal{O}_H).$$

Assuming that (6) holds and that the Kodaira-Spencer map

$$\Omega^1_H' \longrightarrow \pi_*\mathcal{N}_{\mathbb{D}/H \times X}$$

is an isomorphism (e.g., if $H$ is a component of the Hilbert scheme), we obtain Kodaira-Spencer

$$R^1\pi_*\mathcal{N}_{\mathbb{D}/H \times X} \cong \text{Hom}(\pi_*\mathcal{N}_{\mathbb{D}/H \times X}, \mathcal{O}_H) \cong \text{Hom}(\Omega^1_H', \mathcal{O}_H) = \Omega^1_H''.$$

In the cases we are interested in, $R^1\pi_*\mathcal{N}_{\mathbb{D}/H \times X}$ is not necessarily isomorphic to $\Omega^1_H''$, and $\Omega^1_H''$ is not a vector bundle. When $H = \Gamma$ is a curve, we can modify $\Omega^1_H''$ so as to obtain a vector bundle on the normalisation of $\Gamma$, and it is this bundle we shall use to define the equivalence of $\Gamma$.

We want to associate a number to our family of $(3,0)$-curves. This family is not a component of the Hilbert scheme of curves. However, it parametrises all equivalence classes of maps from $\mathbb{P}^1 \longrightarrow X$ of degree 3, when we identify maps with the same image. We would like to define the equivalence using a vector bundle on $\Gamma$. 

We shall associate a vector bundle to $\Omega^1_{\Gamma''}$ in a natural way. The curve $\Gamma$ is singular and $\Omega^1_{\Gamma''}$ is not isomorphic to $\Omega^1_\Gamma$. However, the canonical map $\Omega^1_\Gamma \to \Omega^1_{\Gamma''}$ is surjective. (This is easily seen by local computations at cusps and nodes of $\Gamma$.) Let $\tilde{\Gamma}$ denote the normalisation of $\Gamma$ and let $\psi$ be the natural map:

$$\psi: \tilde{\Gamma} \to \Gamma.$$ 

We have the following exact sequence of sheaves on $\tilde{\Gamma}$:

$$\psi^*\Omega^1_\Gamma \to \Omega^1_{\tilde{\Gamma}} \to \Omega^1_{\Gamma/\Gamma} \to 0. \tag{7}$$

Let $\Omega$ denote the image of $\psi^*\Omega^1_\Gamma$ in $\Omega^1_{\tilde{\Gamma}}$, i.e.,

$$0 \to \Omega \to \Omega^1_{\tilde{\Gamma}} \to \Omega^1_{\Gamma/\Gamma} \to 0. \tag{8}$$

The sheaf $\Omega$ can be considered as a “modification” of $\Omega^1_{\Gamma''}$ in the following way. Consider the following commutative diagram:

$$\begin{align*}
\psi^*\Omega^1_\Gamma & \longrightarrow \psi^*\Omega^1_{\Gamma''} \\
\downarrow & \quad \downarrow \\
\Omega^1_{\tilde{\Gamma}} & \sim \quad \Omega^1_{\Gamma''}
\end{align*}$$

We define the equivalence $e(\Gamma)$ of $\Gamma$ as the (degree of the) first Chern class of the image of $\psi^*\Omega^1_{\Gamma''}$ in $\Omega^1_{\Gamma''}$. Note that $\psi^*\Omega^1_\Gamma \to \psi^*\Omega^1_{\Gamma''}$ is surjective and $\Omega^1_{\tilde{\Gamma}} \sim \Omega^1_{\Gamma''}$, hence the image is isomorphic to $\Omega$. We get

$$e(\Gamma) = c_1(\Omega) = c_1(\Omega^1_{\tilde{\Gamma}}) - \deg \Omega^1_{\Gamma/\Gamma}$$

$$= 2p_g(\tilde{\Gamma}) - 2 - \#\text{cusps}. \tag{9}$$

Since the geometric genus is 46 and the number of cusps is 72, we get,

$$e(\Gamma) = 2 \cdot 46 - 2 - 72 = 18.$$ 

**Remark 3.4.** This number is equal to the predicted number of curves calculated by S. Hosono, A. Klemm, S. Theisen and S.-T. Yau ([11], p. 521) and by D. van Straten and V. V. Batyrev ([1]). A general hypersurface of bidegree $(3,3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ also has a 1-dimensional family of singular rational curves of bidegree $(3,0)$. Applying the above definition of equivalence to this family gives the number 162. This agrees with the conjectured number in [11].
4. Rational Curves of higher degree on a general Tian–Yau CICY.

In this section we study rational curves of degree higher than 3 on a general Tian–Yau CICY. We consider certain linear systems on $\mathbb{P}^2$, and use them to prove the existence of positive dimensional families of curves of every degree greater than 3 on a general Tian–Yau CICY. In the first part of the section we refine this study, and give results concerning existence of rational curves of various bidegrees.

We want to give a constructive proof of Theorem 1.1, and we start out by two preliminary lemmas.

Lemma 4.1. Fix a point $p$ in $\mathbb{P}^2$ and let $d \geq 3$. The linear system of curves of degree $d$, with a point of order $(d-1)$ at $p$, is of dimension $2d$, and a generic member is an irreducible rational curve.

Proof. The dimension of the linear system of curves of degree $d$ is $(d+2)/2 - 1$. The condition that a curve has a given point $p$ as a multiple point of order $(d-1)$, is equivalent to the vanishing of the $(d-1)$ first partial derivatives at $p$. This gives $1 + \cdots + (d-1)$ conditions on the coefficients, and the first statement follows. To prove the second statement it suffices to show that there exists an irreducible rational curve in the linear system. One can construct one in the following way: Let

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^d,$$

$$f(u,v) = (u^d, u^{d-1}v, \ldots, v^d).$$

Let $C = f(\mathbb{P}^1) \subseteq \mathbb{P}^d$. Choose $d-1$ points on $C$. These points span a linear subspace $L$ of dimension $(d-2)$. Let $L' \subset L$ be a linear subspace with the following properties: $\dim L' = d-3$ and $L' \cap C = \emptyset$, and let $\pi : \mathbb{P}^d \longrightarrow \mathbb{P}^2$ be the projection from the linear subspace $L'$. Then $\hat{C} = \pi(C)$ is a curve with the desired properties. $\square$

Lemma 4.2. Let $F$ be a nonsingular cubic surface in $\mathbb{P}^3$. For every natural number $m \geq 3$, there exists a 2-dimensional family of nonsingular rational curves of degree $m$ on $F$.

Proof. The hypersurface $F$ is isomorphic to $\mathbb{P}^2$ blown up in six points $p_0, \ldots, p_5$. Consider the linear system $\sigma^0$ of curves of degree $d \geq 3$, and with a multiple point of order $(d-1)$ at $p_0$ in $\mathbb{P}^2$. We denote a generic curve of $\sigma^0$ by $C_0$. The strict transform of $C_0$ is a rational curve $C_1$ of degree $2d+1$. Since the dimensions of the linear systems considered down on $\mathbb{P}^2$ is at least 6 by the preceding lemma, the statement is proved for odd degrees $7, 9, 11, \ldots$. For even degrees we take a sublinear system $\sigma^1$ of $\sigma^0$, by demanding the curve to pass through $p_1$ once. The strict transform of a generic curve is a rational curve of degree $2d$. The dimensions of these
families of curves are at least 5. In the same manner we can take curves that in addition to the requirements above also pass through $p_2$ and so on. In each case the dimension drops by no more than one. Hence, we have inclusions $\sigma^0 \supset \sigma^1 \supset \ldots \supset \sigma^t \supset \ldots \supset \sigma^5$. (The linear system $\sigma^t$ consist of curves of degree $d$ passing through the points $p_1, \ldots, p_t$.) This gives the desired results for the remaining degrees 3, 4, 5. In the case $m = 3$ (corresponding to $d = 3$ and $t = 4$) the dimension is at least equal to 2. □

Now we give a constructive proof of Theorem 1.1.

Proof of the theorem. Let $X = F_1 \times F_2 \cap G$ be a general Tian–Yau CICY, and let $L$ be one of the 27 lines on $F_2$. Let $q_1, q_2, q_3$ be the points of intersection of $l(L)$ and $F_1$. Furthermore, fix a blowing down of the exceptional divisors $\pi : F_1 \to \mathbb{P}^2$, and let $\bar{q}_i = \pi(q_i)$ for $i = 1, 2, 3$. We shall use the linear systems of curves in $\mathbb{P}^2$ considered in Lemma 4.1 and in Lemma 4.2.

Consider first $m \geq 3$ and $i = 0$. By Lemma 4.2 we have linear systems $\sigma^t_i$ with $t$ base points $p_1, \ldots, p_t$. A general member of this linear system does not pass through any of the $\bar{q}_i$, i.e., it gives rise to a rational curve of bidegree $(m, m)$ on $X$ by Proposition 2.7. Since these linear systems are all positive dimensional, we get positive dimensional families of bidegree $(m, m)$, for $m > 2$, on $X$.

In order to prove the statement in the case $m \geq 3$ and $i = 1$, we take sublinear systems of $\sigma^t_i$ considered above, by assigning the basepoint $q_1$. The dimension of $\sigma^t_i$ is $\dim \sigma^t - 1$. Lemma 4.2 then gives $\dim \sigma^t_i \geq 1$, and the result follows.

For $i = 2$ we take sublinear systems of $\sigma^t_i$, by assigning $\bar{q}_2$ as an additional base point. By the same reasoning as above this gives positive dimensional families, using Prop. 2.7, Lemma 4.1 and Lemma 4.2 for $m > 3$.

Finally, the case $i = 3$ is treated analogously by considering sublinear systems of $\sigma^t$ by assigning $\bar{q}_1, \bar{q}_2, \bar{q}_3$ as base points. Using Prop. 2.7, Lemma 4.1 and Lemma 4.2 we obtain positive dimensional families of bidegree $(m, m-3)$ curves for $m > 3$.

In the proof of Proposition 3.1 we gave all rational curves of bidegree $(2, 1)$. They were realised as degenerations of a 1-dimensional family of bidegree $(2, 2)$ rational curves of type $(\bar{2}, \bar{1})$, constructed from a pencil of planes in $\mathbb{P}^3$ containing a line in $F_1$. (In fact, this shows that there are exactly 27 1-dimensional families of bidegree $(2, 2)$ and of type $(\bar{2}, \bar{1})$.) □

Theorem 4.3. A general Tian–Yau CICY contains no nonsingular rational curves of bidegree $(m, m - i)$ and of type $(\bar{m}, \bar{1})$, for $m \geq i \geq 4$.

Proof. An $i$-secant of a curve when $i \geq 4$, has to be contained in $F_1$, by Bezout’s theorem. In other words, it has to be one of the 27 lines, but for a general Tian–Yau CICY, none of the 27 $l(L)$’s are among the 27 lines on $F_1$. □
Remark 4.4. Note that Theorem 1.3 follows directly from the proof of Theorem 4.3.

The last theorem states the non-existence of rational curves of certain bidegrees on a general Tian–Yau CICY. However, there may exist nongeneral Tian–Yau CICYs with rational curves of these bidegrees.

Proposition 4.5. Let $d \geq 3$ and let $t \in \{0, 1, 2, 3, 4, 5\}$. There exist varieties of Tian–Yau CICY type with positive dimensional families of nonsingular rational curves of bidegree $(2d + 1 - t, d + 2 - t)$.

Proof. In Lemma 4.1 and Lemma 4.2 we constructed the linear systems of curves $\sigma^t$. The strict transforms of these curves have a $(d - 1)$-secant, $E_0$, the exceptional divisor corresponding to $p_0$. Furthermore, the degree of the strict transform of a general member of $\sigma^t$ is $2d + 1 - t$. If $F_i$ is a nonsingular cubic surface in $\mathbb{P}^3$, denote by $L^i_k$, $k \in \{1, \ldots, 27\}$, the 27 lines on $F_i$. Now, choose a pair of cubic surfaces $F_1, F_2$ and a $G = Z(\sum \alpha_{ij} x_i y_j)$, such that the matrix $[\alpha_{ij}]$ is invertible, and with the property that there exists a pair of lines $L^1_{j_1}$ and $L^2_{n_2}$ such that $l(L^1_{j_1}) = L^2_{n_2}$. Applying Proposition 2.7 gives the desired result. □

Remark 4.6. All of these curves are of type $(\bar{m}, \bar{1})$, except for the case $d = 3, t = 5$, which gives a bidegree $(2, 0)$ curve.

Clemens’ conjecture states that a general quintic threefold in $\mathbb{P}^4$ contains a finite number of smooth rational curves for every degree. The conjecture has been proven for degrees less than 10 ([13], [12]). Corollary 1.2 states the existence of positive dimensional families of nonsingular rational curves for every degree $\geq 4$ on a general Tian–Yau CICY.

5. Curves of degree 1, 2 and 3 on the special Tian–Yau CICY.

In this section we are going to study rational curves of degree less than 4 on the special Tian–Yau variety.

Proposition 5.1. The numbers $N_{i,j}$ of nonsingular rational curves of bidegree $(i, j)$ on a general Tian–Yau variety are finite for $i + j \leq 3$ and are given by:

$$
N_{0,1} = N_{1,0} = 81 \\
N_{0,2} = N_{2,0} = 81 \\
N_{1,1} = 567 \\
N_{1,2} = N_{2,1} = 972 \\
N_{3,0} = N_{0,3} = 0.
$$

Proof. The number of rational curves of bidegree $(1, 0)$ and $(2, 0)$ are computed in the same way as in Proposition 3.1. The number of $(1, 1)$ curves is
equal to the number of irreducible intersections \( L' \times L \cap G \), where \( L' \subseteq F_1 \) and \( L \subseteq F_2 \), by Proposition 2.7. We have 729 pairs of lines to consider. Computation gives 567 irreducible intersections, hence 567 \((1,1)\) curves. The number of rational curves of bidegree \((2,1)\) is a little more complicated to obtain. First, notice that there is no rational curves of bidegree \((2,1)\) and of type \((1,1)\). Assume for contradiction that there is one, and denote it by \( C \). Let \( L_i = \pi_i(C) \). Choose a plane \( H \) in \( \mathbb{P}^3 \) such that \( L_1 \subseteq H \). Since \( H \times L_2 \cap G \) is the blow-up of \( H \) in \( l(L_2) \) and \( L_1 \neq l(L_2) \), \( D = L_1 \times L_2 \cap G \) is of dimension one. Assume that \( L_1 \cap l(L_2) = \emptyset \), then \( L_1 \cong D \). This implies that \( C \)'s degree on the first factor is at most 1. Assume that \( L_1 \cap L_2 = \emptyset \), then \( D \) consists of a curve \( C \) the strict transform of \( L_1 \), and an exceptional divisor \( E \). The curve \( C \) has to be contained in \( C \), since it is dominant on the first factor. This implies that \( C = C \). This implies that \( C \)'s degree on the first factor is 1.

In view of Proposition 2.7, to give a rational curve of bidegree \((2,1)\) is equivalent to giving a conic \( C \) in \( F_1 \) and a line \( L \) in \( F_2 \) such that \( l(L) \) intersect \( C \) in one point. Any conic \( C \) in \( \mathbb{P}^3 \) is contained in a unique hyperplane \( H \), so \( F_1 \cap H = C \cup L' \), where \( L' \) is a line in \( F_1 \). Conversely, every line \( L' \) determines a pencil of planes. The lines on the Fermat cubic \( F_2 \) are:

\[
y_{n_0} + \alpha^i y_{n_1} = y_{n_2} + \alpha^j y_{n_3} = 0,
\]

where the \( n_i \in \{0,1,2,3\} \) and are all different. Let \( L \) be \( y_0 + \alpha^i y_1 = y_2 + \alpha^j y_3 = 0 \), then \( l(L) \cap F_1 = (1, \alpha^i, -\alpha^j, -\alpha^{i+j}) \) for \( l \in \{0,1,2\} \).

Consider one line in \( F_1 \), say: \( L' = Z(x_0 + x_3 = x_1 + ax_3) \). The pencil is then given by: \( H_a = Z(ax_0 + \alpha^2 x_1 + x_2 + aX_3) \) (where we allow \( a = \infty \)). Let \( C_a \) be defined by: \( H_a \cap F_1 = C_a \cup L \).

Demanding a point in \( l(L) \cap F_1 \) to be in \( H_a \), gives the following condition on \( a \): \( a(1 - \alpha^{i+j}) = \alpha^i - \alpha^{i+j} \). There are 27 cases to consider: \( i, j, l \) may take values in \( \{0,1,2\} \). A case by case study gives that only 12 of these give a rational curve of bidegree \((2,1)\), i.e., where \( l(L) \) intersects an irreducible \( C_a \) once. This accounts for 9 lines on \( F_1 \) (9 different pairs \((i,j)\)). A similar study of the remaining 18 lines, gives 24 rational curves of the desired bidegree (out of 54 candidates). Hence the number of rational curves of bidegree \((2,1)\) is 36. Note that all these curves are mapped to \( L \subseteq F_2 \) by the second projection. By symmetry the total number of rational curves of bidegree \((2,1)\) is \( 27 \cdot 36 = 972 \). \( \square \)

**Corollary 5.2.** Let \( N_d \) denote the number of nonsingular rational curves of degree \( d \) on a general Tian–Yau variety. Then

\[
N_1 = 162, \quad N_2 = 729, \quad N_3 = 1944.
\]

**Remark 5.3.** The numbers of curves of bidegree \((0,1),(1,0)\) and \((1,1)\) have been calculated previously, using similar techniques ([5], [19]).
Corollary 5.4. There exist positive dimensional families of rational curves on the special Tian–Yau CICY for every degree \( n \), \( n \geq 4 \).

Proof. This is a corollary of the proof of Corollary 1.2. The construction of curves relied on the use of Proposition 2.7, i.e. on the fact that \( G = Z(\sum_{ij} \alpha_{ij} x_i y_j) \), where the matrix \([\alpha_{ij}]\) is invertible, which is clearly satisfied in this case. The last necessary ingredient in the proof is that none of the \( l(L) \), where \( L \) is one of the 27 lines of \( F_2 \), are tangent to the surface \( F_2 \). This is easily checked for the special Tian–Yau CICY. The rest of the proof is identical to the proof of Theorem 1.1. \( \square \)

Remark 5.5. By comparing Corollary 5.2 with Corollary 3.2, we see that the number of curves of degree 2 and 3 on the special Tian–Yau variety are different from the corresponding numbers for the general Tian–Yau variety. For example for degree 2, the difference stems from the number of \((1,1)\) curves. The difference between the numbers of \((1,1)\) curves on the general and the special is \(729 - 567 = 162\). This difference is explained by the fact that we have 162 reducible intersections of type \( L_1 \times L_2 \cap G \) (where \( L_i \subseteq F_i \)) on the special Tian–Yau variety, and none on the general Tian–Yau variety. A reducible intersection \( L_1 \times L_2 \cap G \) gives one \((1,0)\) curve and one \((0,1)\) curve. It is easy to see that every rational curve of bidegree \((1,0)\) is contained in precisely two distinct intersections of the type \( L_1 \times L_2 \cap G \). Hence, we get 81 \((1,0)\) curves. By Proposition 5.1 we know that these are in fact the only ones.

References


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GENERALIZED CARTAN TYPE S LIE ALGEBRAS IN CHARACTERISTIC 0 (II)

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In this paper, we introduce a class of Lie algebras which are subalgebras of generalized Cartan type S Lie algebras of characteristic 0. We determine the necessary and sufficient conditions for such Lie algebras to be simple. And we give all derivations of such simple Lie algebras.

1. Introduction.

This paper is a sequel to the paper [7] in which generalized Cartan type S Lie algebras \( t^2S(A, T, \varphi) \) over a field \( F \) of characteristic 0 were studied. We have tried to make this paper independent of other papers. So in Section 2, we give a description of relevant Lie algebras and some basic facts which will be used in this paper. In Section 3 we introduce a class of Lie algebras which are subalgebras of generalized Cartan type S Lie algebras, and determine the necessary and sufficient conditions for such Lie algebras to be simple. We give all derivations of such simple Lie algebras in Section 4.

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In this section, for the convenience of the reader, we recall the relevant Lie algebra definitions and some basic facts which will be used later in this paper. Throughout this paper we assume that \( F \) is a field of characteristic 0, and that \( A \) is a nonzero abelian group written additively.

2.1 Generalized Witt algebras.

Let \( n \) be a positive integer, and \( t_1, \ldots, t_n \) independent and commuting indeterminates over \( F \). Denote by \( P_n \) and \( Q_n \) the polynomial algebra \( F[t_1, \ldots, t_n] \), and the Laurent polynomial algebra \( F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) respectively. By \( W_n = W_n(F) \) we denote the Witt algebra, i.e., the Lie algebra of
all formal vector fields

\[ \sum_{i=1}^{n} f_i \frac{\partial}{\partial t_i} \]

with coefficients \( f_i \in Q_n \). The bracket in \( W_n \) is

\[ \left[ f \frac{\partial}{\partial t_i}, g \frac{\partial}{\partial t_j} \right] = f \frac{\partial(g)}{\partial t_i} \frac{\partial}{\partial t_j} - g \frac{\partial(f)}{\partial t_j} \frac{\partial}{\partial t_i}, \]

where \( f, g \in Q_n \), and \( i, j \in \{1, 2, \ldots, n\} \). The subalgebra \( W_n^+ = W_n^+(F) \) of \( W_n \) consisting of all vector fields (2.1) with polynomial coefficients, i.e., \( f_i \in P_n \), is known as the \textit{general Lie algebra}, or the \textit{Lie algebra of Cartan type} \( W \). For more details, please refer to [13]. It is well known that \( W_n \) and \( W_n^+ \) are simple Lie algebras.

Let \( T \) be a vector space over \( F \). We denote by \( FA \) the group algebra of \( A \) over \( F \). The elements \( t^x, x \in A \), form a basis of this algebra, and the multiplication is defined by \( t^x \cdot t^y = t^{x+y} \). We shall write \( 1 \) instead of \( t^0 \). The tensor product \( W = FA \otimes_F T \) is a free \( FA \)-module in the natural way. We denote an arbitrary element of \( T \) by \( \partial \) (to remind us of differential operators). For the sake of simplicity, we shall write \( t^x \partial \) instead of \( t^x \otimes \partial \).

We now choose a pairing \( \varphi : T \times A \to F \) which is \( F \)-linear in the first variable and additive in the second one. For convenience we shall also use the following notations:

\[ \varphi(\partial, x) = \langle \partial, x \rangle = \partial(x) \]

for arbitrary \( \partial \in T \) and \( x \in A \). \( W \) becomes a Lie algebra under the following bracket:

\[ [t^x \partial_1, t^y \partial_2] := t^{x+y} (\partial_1(y)\partial_2 - \partial_2(x)\partial_1), \]

for arbitrary \( x, y \in A \) and \( \partial_1, \partial_2 \in T \). We refer to this algebra \( W = W(A, T, \varphi) \) as a \textit{generalized Witt algebra}.

The subspaces \( W_x = t^x T, \ x \in A \), define an \( A \)-gradation of \( W \), i.e., \( W \) is the direct sum of the \( W_x \)'s, and \( [W_x, W_y] \subset W_{x+y} \) for all \( x, y \in A \).

It follows from (2.2) that \( \text{ad}(\partial) \) acts on \( W_x \) as a scalar \( \partial(x) \). Hence each \( \partial \in T \) is ad-semisimple, and \( T \) is a torus (i.e., an abelian subalgebra consisting of ad-semisimple elements). In fact \( T \) is the only maximal torus of \( W \) (see [3, Lemma 4.1]). Kawamoto proved in [11] that the Lie algebra \( W = W(A, T, \varphi) \) is simple if and only if \( A \neq 0 \) and \( \varphi \) is nondegenerate in the sense that the conditions

\[ \langle \partial, x \rangle = 0, \forall \partial \in T \Rightarrow x = 0 \]

and

\[ \langle \partial, x \rangle = 0, \forall x \in A \Rightarrow \partial = 0 \]

hold.
Note that (2.3) implies that $A$ is torsion free. This implies that $FA$ is an integral domain and it implies that the invertible elements of $FA$ have the form $at^x$, where $a \in F^*, x \in A$.

There is a natural structure of a left $W$-module on $FA$, namely the structure is such that

$$(2.5) \quad t^x \partial \cdot t^y = \partial(y)t^{x+y}$$

for $x, y \in A$ and $\partial \in T$. Also we have the natural left $FA$-module structure on $W$. These two module structures are related by the identity

$$(2.6) \quad [fu, gv] = f(u \cdot g)v - g(v \cdot f)u + fg[u, v]$$

where $f, g \in FA$ and $u, v \in W$ are arbitrary. The $W$-module structure on $FA$ gives rise to a homomorphism

$$(2.7) \quad W \to \text{Der}(FA)$$

because each $w \in W$ acts on $FA$ as a derivation. Clearly (2.7) is also a homomorphism of $FA$-modules. For more details about $W(A, T, \varphi)$, please refer to [3].

2.2 Generalized Cartan type W Lie algebras.

Suppose that $W = W(A, T, \varphi)$ denotes a simple generalized Witt algebra. Let $I$ be an index set, $d : I \to T$ an injective map, and write $d_i = d(i)$ for $i \in I$. We say that $d$ is admissible if the following two conditions hold:

(Ind) $d_i, i \in I$, are linearly independent;

(Int) $d_i(A) = \mathbb{Z}$ for all $i \in I$.

We assume throughout that an admissible $d$ has been fixed. We set

$$A_d^+ = \{x \in A : d_i(x) \geq 0, \; \forall i \in I\},$$

$$A_d^0 = \{x \in A : d_i(x) = 0, \; \forall i \in I\},$$

$$A_{d,i} = \{x \in A : d_i(x) = -1; \; d_j(x) \geq 0, \; \forall j \in I \setminus \{i\}\},$$

$$A_{d,i}^\# = \{x \in A : d_i(x) = -1; \; d_j(x) = 0, \; \forall j \in I \setminus \{i\}\},$$

$$A_d = A_d^+ \cup (\bigcup_{i \in I} A_{d,i}).$$

We now introduce some subalgebras of $W$:

$$W_d^+ = \sum_{x \in A_d^+} W_x,$$

$$W_{d,i} = \left(\sum_{x \in A_{d,i}} Ft^x\right) \cdot d_i, \; i \in I;$$
and

\[ W_d = W_d(A, T, \varphi) = W_d^+ + \sum_{i \in I} W_{d,i}. \]

We also introduce the subalgebra \( FA_d^+ \) of \( FA \), which is the span of all elements \( t^x \) with \( x \in A_d^+ \). Since \( W \) is a left \( FA \)-module, we can view \( W \) also as a left \( FA_d^+ \)-module. Then it is easy to see that the subspaces \( W_d^+ \) and \( W_d \) are \( FA_d^+ \)-submodules of \( W_d \).

By restricting the action of \( W \) on \( FA \), we can view \( FA \) as a left \( W_d \)-module, and then \( FA_d^+ \) is a \( W_d \)-submodule of \( FA \). When \( d \) is fixed, and there is no danger of confusion, we shall write

\[ A^+, A_i, A_i^#, W^+, W_i, FA^+ \]

instead of

\[ A_d^+, A_{d,i}, A_{d,i}^#, W_d^+, W_{d,i}, FA_d^+, \]

respectively. The following Theorem is proved in [5].

**Theorem 2.1.** The Lie algebra \( W_d \) is simple if and only if the following conditions hold:

(i) if \( \partial \in T \) and \( \partial(x) = 0 \) for all \( x \in A_d \), then \( \partial = 0 \);
(ii) if \( x \in A_d \), then \( d_i(x) = 0 \) for almost all \( i \in I \);
(iii) \( A_i^# \neq \emptyset \) for all \( i \in I \).

The simple Lie algebra \( W_d \) is called an algebra of generalized Cartan type \( W \). For more details on the Lie algebra \( W_d \), please refer to the papers [5] and [12].

### 2.3 Generalized Cartan type \( S \) Lie algebras.

It is well known that the classical *divergence* \( \text{Div}: W_n \to F[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \) maps \( \sum_{i=1}^n f_i \frac{\partial}{\partial t_i} \) to \( \sum_{i=1}^n \frac{\partial f_i}{\partial t_i} \). The kernel \( \mathcal{S}_n \) of \( \text{Div} \) is a subalgebra of \( W_n \). The algebra \( S_n = (\mathcal{S}_n)'' \) and \( S_n^+ = S_n \cap W_n^+ \) are called *Lie algebras of Cartan type \( S \).*

Suppose that \( W = W(A, T, \varphi) \) is a simple generalized Witt algebra. The *divergence* \( \text{div}: W \to FA \) is the \( F \)-linear map such that

\[ \text{div} (t^x \partial) = \partial(x)t^x \]

holds for all \( x \in A \) and \( \partial \in T \). It has the following two properties:

\[ \text{div} (fw) = f \text{div} (w) + w \cdot f \]

and

\[ \text{div} [u, v] = u \cdot \text{div} (v) - v \cdot \text{div} (u) \]
where \( u, v, w \in W \) and \( f \in FA \) are arbitrary. The latter property shows that \( \text{div} \) is a derivation of \( W \) with values in the \( W \)-module \( FA \). Since \( \text{div} : W \to FA \) is a derivation of degree 0, its kernel \( \tilde{S} := \ker(\text{div}) \) is a homogeneous subalgebra of \( W \):

\[
\tilde{S} = \bigoplus_{x \in A} \tilde{S}_x, \quad \tilde{S}_x := \tilde{S} \cap W_x.
\]

For \( x \in A \) define the \( F \)-linear function \( \tilde{x} : T \to F \) by \( \tilde{x}(\partial) = \partial(x) \). The condition (2.3) shows that, if \( T^* \) is the dual space of \( T \), the \( \mathbb{Z} \)-linear map \( A \to T^* \) sending \( x \mapsto \tilde{x} \) is injective. If \( T_x := \ker(\tilde{x}) \), then we have \( \tilde{S}_x = t_x^x T_x \). Hence \( \tilde{S}_0 = W_0 = T \) and, for \( x \neq 0 \), \( \tilde{S}_x \) is a hyperplane of \( W_x \). In particular, if \( \dim T = 1 \), then \( \tilde{S} = T \). To avoid trivialities, we shall assume always that \( \dim T > 1 \).

Let \( \bar{S} := (\tilde{S})' \) be the derived algebra of \( \tilde{S} \). Note that the notation here is different from that in [7]. We know (see [7]) that \( \bar{S} = \bigoplus_{x \neq 0} \tilde{S}_x \).

More generally, the subspaces \( t^z \tilde{S}, \ z \in A \), are subalgebras of \( W \) and their derived algebras are given by

\[
(t^z \tilde{S})' = t^z \bar{S} = \sum_{x \neq z} t^x T_{x-z}.
\]

If \( \dim T \geq 3 \), all the subalgebras \( t^z \tilde{S} \) are simple. If \( \dim T = 2 \), then \( \tilde{S} \) itself is simple while the shifted algebras \( t^z \tilde{S}, z \neq 0 \), are not. Their derived algebras

\[
(t^z \tilde{S})' = \sum_{x \neq z, 2z} t^x T_{x-z}, \ z \neq 0,
\]

are simple.

We shall refer to the subalgebras \( S(A, T, \varphi, z) := t^z \tilde{S} \) if \( \dim T \geq 3 \), and \( S(A, T, \varphi, z) := (t^z \tilde{S})' \) if \( \dim T = 2 \), as Lie algebras of generalized Cartan type \( S \). The Lie algebras \( S(A, T, \varphi, z) \) have the \( A \)-gradation:

\[
S(A, T, \varphi, z) = \begin{cases} \bigoplus_{x \in A \setminus \{z\}} t^x T_{x-z}, & \text{if } \dim T > 2, \\ \bigoplus_{x \in A \setminus \{z, 2z\}} t^x T_{x-z}, & \text{if } \dim T = 2. \end{cases}
\]

These algebras were studied in papers: [6] when \( \dim T = 2 \) and \( z = 0 \), [4] when \( \dim T = 2 \) and \( z \neq 0 \), and [7] when \( \dim T \geq 3 \).

### 2.4 Generalized Block algebras.

We shall denote by \( \text{Hom} (A, F) \) the \( F \)-vector space of all additive (i.e., \( \mathbb{Z} \)-linear) maps \( A \to F \). We now fix an additive map \( \alpha : A \to F \) and a skew-symmetric bi-additive map \( \varphi : A \times A \to F \).

Let \( L = L(A, \alpha, \varphi) \) be the vector space over \( F \) having a basis consisting of all symbols \( e_x, x \in A \). We make \( L \) into a (non-associative) algebra over
by defining $F$-bilinear multiplication $L \times L \to L$ by
\begin{equation}
[e_x, e_y] = f(x, y)e_{x+y}, \quad x, y \in A,
\end{equation}
where
\begin{equation}
f(x, y) = \varphi(x, y) + \alpha(x - y).
\end{equation}
If $\alpha = 0$, then $L$ is a Lie algebra. These Lie algebras were studied in [8]. It was shown by Albert and Frank [1] that, under the assumption that $\alpha \neq 0$, $L$ is a Lie algebra if and only if there exists another additive map $\beta : A \to F$ such that $\varphi = \alpha \wedge \beta$, i.e.,
\begin{equation}
\varphi(x, y) = \alpha(x)\beta(y) - \beta(x)\alpha(y).
\end{equation}
We shall assume throughout that such a $\beta$ exists, i.e. that $L$ is a Lie algebra. We also assume that
\begin{equation}
K_\alpha \cap K_\beta = 0,
\end{equation}
where $K_\mu$ denotes the kernel of $\mu$ for any additive map $\mu : A \to F$. Let $L^2 = [L, L]$ be the derived subalgebra of $L$ and $Z$ the center of $L$. The Lie algebra $L = L^2/Z$ is simple [4, Theorem 2.5], and we shall also write $\mathcal{L}(A, \alpha, \varphi)$ for $L$.

We refer to the Lie algebras $L = \mathcal{L}(A, \alpha, \varphi)$ as generalized Block algebras.

Suppose that $\beta \in \text{Hom}(A, F)$ can be chosen so that (2.15) holds and
\begin{equation}
\beta(A) = \mathbb{Z}.
\end{equation}
We now define the subset $A_\beta \subset A$ by
\begin{equation}
A_\beta = \{ x \in A : \beta(x) \geq -1 \},
\end{equation}
and denote by $L_\beta$ the subspace of $L$ with a basis consisting of all $e_x$ with $x \in A_\beta$. It follows that $L_\beta$ is a subalgebra of $L$.

We shall denote by $\mathcal{L}_\beta$ or $\mathcal{L}_\beta(A, \alpha, \varphi)$ the quotient algebra $L_\beta/Z$. The Lie algebra $\mathcal{L}_\beta$ is simple. For more details about $\mathcal{L}(A, \alpha, \varphi)$ and $\mathcal{L}_\beta(A, \alpha, \varphi)$, please refer to [4] and [8].

3. Lie algebra $S_d(A, T, \varphi, z)$.

Now we are ready to introduce our main object, the Lie algebra $S_d(A, T, \varphi, z)$. We assume that a generalized Witt algebra $W = W(A, T, \varphi)$, a generalized Cartan type W Lie algebra $W_d = W_d(A, T, \varphi) \subset W$, a generalized Cartan type S Lie algebra $S(A, T, \varphi, z) \subset W$ are given, and we also assume throughout the paper that all of these algebras are simple. Our hypotheses here imply that $\dim T \geq 2$. We define
\[ S_d(A, T, \varphi, z) := W_d \cap S(A, T, \varphi, z). \]
Then \( S_d = S_d(A, T, \varphi, z) \) is a subalgebra of \( W \). We shall also call the algebra \( S_d \) a generalized Cartan type \( S \) Lie algebra. It is clear that \( S_d \) has an \( A \)-graduation with the following components of degree \( x \in A \):

\[
(S_d)_x = \begin{cases} 
0, & \text{for } x = z \\
(t^x T_{x-z}) \cap W_d, & \text{for } x \in A \setminus \{z\}
\end{cases}
\]

if \( \dim T > 2 \),

\[
(S_d)_x = \begin{cases} 
0, & \text{for } x = z \text{ or } x = 2z \\
(t^x T_{x-z}) \cap W_d, & \text{for } x \in A \setminus \{z, 2z\}
\end{cases}
\]

if \( \dim T = 2 \). It follows that \( (S_d)_x = 0 \) for all \( x \notin A_d \).

For convenience, by \( \langle U_1 \rangle \) we denote the subspace of the vector space \( V \) generated by \( U_1 \subset V \).

**Theorem 3.1.** Suppose that the Lie algebras \( W = W(A, T, \varphi, z) \), \( W_d = W_d(A, T, \varphi) \subset W \), and \( S = S(A, T, \varphi, z) \subset W \) are given, and that all of them are simple. Then \( S_d(A, T, \varphi, z) \) is simple if and only if the following conditions hold:

(a) \( d_i(z) = -1 \) for all \( i \in I \);

(b) \( I \) is finite.

**Proof.** For simplicity we write \( L = S_d(A, T, \varphi, z) \), so \( L_x = (S_d)_x \) for \( x \in A \).

(\( \Rightarrow \)) Suppose that \( L \) is simple. We shall first show that (a) is true. For contradiction we assume that there exists an element in \( I \), say 1, such that \( d_1(z) \neq -1 \). If \( t^x d_1 \in W_1 \cap L \) (for definition of \( W_1 \) see Section 2.2), we know that \( d_1(x) = -1 \) and \( d_1(x - z) = 0 \), so \( -1 = d_1(x) = d_1(z) \neq -1 \). It is a contradiction. Then \( W_1 \cap L = 0 \), i.e., \( L_x = 0 \) for all \( x \in A_1 \). Theorem 2.1(iii) assures that we can choose \( y \in A \) with \( -y \in A^+_1 \). So \( y \in A^+_d \). If \( z \neq y \), then \( L_y = t^y T_{y-z} \neq 0 \). Let \( J = \oplus_{x: d_1(x) > 1} L_x \). If \( z = y \), we see that \( L_{3y} \neq 0 \), let \( J = \oplus_{x: d_1(x) > 3} L_x \). It is easy to verify that \( J \) is a nonzero proper ideal of \( L \) in both cases, which contradicts the simplicity of \( L \). Consequently (a) holds.

On the other side we have \(-z \in A^+_d \). From Theorem 2.1(ii) it follows that \( I \) is finite. So (b) is true.

(\( \Leftarrow \)) Suppose (a) and (b) hold. We write \( I = \{1, 2, \ldots, n\} \). We know that \( L_x \neq 0 \) if and only if \( x \in A_d \setminus \{z\} \). Fix \( \{u_1, u_2, \ldots, u_n\} \subset A \) such that \( d_i(u_j) = \delta_{i,j} \). Let \( A(d) = Zu_1 \oplus Zu_2 \oplus \cdots \oplus Zu_n \), and \( A'(d) = \{x \in A | d_i(x) = 0 \forall i \in I \} \). Then it follows from (b) that \( A = A(d) \oplus A'(d) \).

Case 1. Suppose that \( \dim T = 2 \) and \( |I| = 1 \), say \( I = \{1\} \). Choose \( d_2 \in T \) such that \( T = Fd_1 \oplus Fd_2 \). Denote \( e_x = t^x (d_1(x-z) d_2 - d_2(x-z) d_1) \) for
\(x \in A_d\). It follows that \(\{e_x| x \in A_d \setminus \{z\}\}\) is a basis of \(L = S_d(A, T, \varphi, z)\).
For any \(x, y \in A_d\) we have
\[
[e_x, e_y] = [t^x(d_1(x-z)d_2-d_2(x-z)d_1), t^y(d_1(y-z)d_2-d_2(y-z)d_1)]
\]
\[
= t^{x+y}((d_1(x-z)d_2(y)-d_2(x-z)d_1(y))
\cdot t^x(d_1(x-z)d_2 - d_2(x-z)d_1)
- (d_1(y-z)d_2(x) - d_2(y-z)d_1(x))
\cdot t^y(d_1(y-z)d_2 - d_2(y-z)d_1))
= t^{x+y}(d_1(x-z)d_2(y-z) - d_2(x-z)d_1(y-z))
\cdot (d_1(x+y-z)d_2 - d_2(x+y-z)d_1)
= \begin{vmatrix}
  d_1(x-z) & d_1(y-z) \\
  d_2(x-z) & d_2(y-z)
\end{vmatrix} e_{x+y}.
\]

Denote \(\alpha(x) = -d_2(x) - d_2(z)d_1(x), \beta(x) = d_1(x)\) for \(x, y \in A\). It is clear that \(\beta(A) = \mathbb{Z}\). By Section 2.4 we know that \(L \simeq \mathcal{L}_\beta(A, \alpha, \beta)\). Consequently \(L\) is simple.

**Case 2.** Suppose that \(\dim T = 2\) and \(|I| = 2\), say \(I = \{1, 2\}\). Since \(\varphi\) is nondegenerate, we get that \(A = A(d) = d_1(A) \otimes d_2(A) \simeq \mathbb{Z} \otimes \mathbb{Z}\).
We may assume that \(d_i(x) = x_i\) for \(x = (x_1, x_2) \in \mathbb{Z} \otimes \mathbb{Z}\). Thus \(A_d = \left((\mathbb{Z}+1) \otimes (\mathbb{Z}+1)\right) \setminus \{(-1, -1)\}\). Same as Case 1, we define \(e_x, x \in A_d\). Then we also have
\[
[e_x, e_y] = \begin{vmatrix}
  d_1(x-z) & d_1(y-z) \\
  d_2(x-z) & d_2(y-z)
\end{vmatrix} e_{x+y},
\]
i.e.,
\[
[e_x, e_y] = \begin{vmatrix}
  x_1 + 1 & y_1 + 1 \\
  x_2 + 1 & y_2 + 1
\end{vmatrix} e_{x+y}, \forall x, y \in A_d.
\]

It is well known that the Lie algebra \(S_2^+ \subset W_2^+\) has basis \(\{t_1^{x_1}t_2^{x_2}(t_2 \frac{\partial}{\partial t_1} - t_1 \frac{\partial}{\partial t_2})| (x_1, x_2) \in A_d\}\). It is easy to verify that the following linear map is an isomorphism of Lie algebras:
\[
S_d \rightarrow S_2^+, \ e_x \mapsto t_1^{x_1}t_2^{x_2} \left(t_2 \frac{\partial}{\partial t_1} - t_1 \frac{\partial}{\partial t_2}\right).
\]
Thus \(S_d\) is also simple in this case.

**Case 3.** Suppose that \(\dim T > 2\).

If \(|I| = \dim T = n\), as we did in Case 2 we can deduce that \(L \simeq S_n^+\), the special algebra of rank \(n\). Thus in this case \(S_d\) is simple. Next we assume that \(\dim T > |I| = n > 0\).
Let $J$ be a nonzero ideal of $L$. It is suffices to show that $J = L$. Choose a nonzero element $u \in J$, say

\begin{equation}
(3.1) \quad u = \sum_{i=1}^{m} t^{x_i} \partial_i, \ x_i \in A_d \setminus \{z\}, \ \partial_i \in T_{x_i - z}
\end{equation}

with $m$ is minimal. Then $x_1, \ldots, x_m$ are distinct and $\partial_i \neq 0$.

**Claim 1.** We have that $m = 1$.

Otherwise we suppose $m > 1$. Since $T_z \subset L$, from the minimality of $m$ it follows that

\[ (\hat{x}_1)|_{T_z} = (\hat{x}_2)|_{T_z} = \cdots = (\hat{x}_m)|_{T_z}. \]

Thus

\begin{equation}
(3.2) \quad \hat{x}_i - \hat{x}_j \in F\hat{\mathbb{Z}}, \ \forall \ i, j \in \{1, 2, \ldots, m\}.
\end{equation}

**Subclaim.** We can choose such an element $u$ in (3.1) such that $d_1(x_1) = -1$.

Suppose $d_1(x_1) \geq 0$. If $\partial_1 \in Fd_1$, let $y_1 \in A^\#_1$, then

\[ u' = [t^{y_1} d_1, u] = \sum_{i=1}^{m} t^{y_1 + x_i} (d_1(x_i) \partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\} \]

and $d_1(y_1 + x_1) < d_1(x_1)$. If $\partial_1 \notin Fd_1$, then $A_{d_1} \not\subset \ker(\partial_1)$. Otherwise we can show that $\partial_1(A_{d_1}) = 0$, it is impossible. We choose $y_1 \in A_{d_1} \setminus \ker(\partial_1)$. We deduce that

\[ u' = [t^{y_1} d_1, u] = \sum_{i=1}^{m} t^{y_1 + x_i} (d_1(x_i) \partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\}, \]

and also $d_1(y_1 + x_1) < d_1(x_1)$. After finitely many steps of this kind process, we get a nonzero element $u$ in (3.1) such that $d_1(x_1) = -1$. Our subclaim follows.

Without loss of generality we may assume that $\partial_1 = d_1$. From (3.2) it follows that there exists $\lambda_i \in F^*$ such that

\[ \hat{x}_i = \lambda_i \hat{z} + \hat{x}_i, \ \forall \ i \in \{2, \ldots, m\}. \]

Since $x_i \in A_d$, we have $d_1(x_i) \geq -1$, then $-\lambda_i \in \mathbb{N}$, the set of natural numbers. Thus $d_1(x_2) \geq 0$. If $\partial_2 \in Fd_1$, then

\[ [t^{x_1} d_1, t^{x_2} \partial_2] = t^{x_1 + x_2} (d_1(x_i) \partial_i - \partial_i(y_1)d_1) \neq 0. \]

It follows that

\[ u' = [t^{x_1} d_1, u] = \sum_{i=2}^{m} t^{x_1 + x_i} (d_1(x_i) \partial_i - \partial_i(y_1)d_1) \in J \setminus \{0\}. \]
This contradicts the minimality of $m$. Consequently $\partial_2 \not\in Fd_1$. Choose $z_1 \in A_{d,1} \setminus \ker(\partial_2)$, then $[t^{x_1}d_1, t^{z_1}d_1] = 0$ and $[t^{z_1}d_1, t^{x_2}\partial_2] = t^{z_1+x_2}(d_1(x_2)\partial_2 - \partial_2(z_1)d_1) \neq 0$. Thus

$$u' = [t^{z_1}d_1, u] = \sum_{i=2}^{m} t^{z_1+x_i}(d_1(x_i)\partial_i - \partial_i(z_1)d_1) \in J \setminus \{0\}.$$ 

It again contradicts the minimality of $m$. Therefore $m = 1$. Claim 1 follows.

**Claim 2.** We have $T_z \subset J$.

From Claim 1 we know that there exists a nonzero element $t^x \partial_0 \in J$, where $x \in A_d$, $\partial_0 \in T_{x-z}$.

**Subcase 1.** Suppose $x \in A_{d,1}^\#$, for some $i \in I$, say $x \in A_{d,1}^\#$. Then $\partial_0 \in Fd_1$. Thus $t^x d_1 \in J$. For any $\partial \in T_{x-z}$, from

$$[t^x d_1, t^{-x}\partial] = \partial - \partial(x)d_1 \in J,$$

and $d_1 \notin T_{x-z}$, we know that $\langle d_1, \partial - \partial(x)d_1 | \partial \in T_{x-z} \rangle = T$. Hence $\langle \partial - \partial(x)d_1 | \partial \in T_{x-z} \rangle = T_z = T$. Therefore $T_z \subset J$.

**Subcase 2.** Suppose $x \in A_d^\#$. Define $d(x) := \sum_{i=1}^{m} d_i(x)$. If $d_1(x) > 0$, choose $y_1 \in A_{d,1}^\#$. Since $d_1(x-z) \neq 0$ and $\partial_0(x-z) = 0$, we know that $d_1$ and $\partial_0$ are linearly independent. From the computation

$$[t^{y_1}d_1, t^x \partial_0] = t^{x+y_1}(d_1(x)\partial_0 - \partial_0(y_1)d_1) \neq 0,$$

we get a nonzero element $t^{x+y_1}(d_1(x)\partial_0 - \partial_0(y_1)d_1) \in J$ with $d(x+y_1) = d(x) - 1$. By repeatedly using this method, after finitely many steps we deduce that there exists a nonzero element $t^y \partial \in J$ with $y \in A_d^0$. If $A_{d,i}^\# \subset \ker(\partial)$ for all $i \in I$, we infer that $\partial(A_d) = 0$. It contradicts Theorem 2.1(i). Thus there exists an $i \in I$ such that $A_{d,i}^\# \not\subset \ker(\partial)$, say $A_{d,1}^\# \not\subset \ker(\partial)$. Choose $z_1 \in A_{d,1}^\# \setminus \ker(\partial)$, then

$$[t^{z_1}d_1, t^y \partial] = -t^{z_1+y}(\partial(z_1))d_1 \in J \setminus \{0\}.$$

By Subcase 1 we obtain again that $T_z \subset J$. Similarly $T_z \subset J$ for $x \in A_{d,i}$. Thus Claim 2 is proved.

If $x \in A_d$ and $\hat{x} \notin F \hat{z}$, then $T_z \not\subset \ker \hat{x}$. Choose $\partial \in T_z \setminus \ker \hat{x}$. From $[\partial, t^x \partial'] = \partial(x)t_x \partial' \in J$ we know that $t^x T_{x-z} \subset J$.

If $y \in A_d \setminus \{0\}$ and $\hat{y} \in F \hat{z}$, it follows that $y = -kz$ for some positive integer $k$. Since $\{\hat{x} | x \in A_{d,1}^\# \} \not\subset F \hat{z}$, we choose $x_1 \in A_{d,1}^\#$ with $\hat{x}_1 \notin F \hat{z}$. Since $y \neq z$, then $d_i(y) > 0$ for all $i \in I$. Note that $t^{y-x_1}T_{y-x_1-z} \subset J$. Then, for $t^{x_1}d_1, t^{y-x_1} \partial_1 \in L$, we have

$$[t^{y-x_1} \partial_1, t^{x_1}d_1] = -t^y(d_1(y-x_1)\partial_1 - \partial_1(x)d_1) \in J.$$
Because \(\langle d_1, d_1(y-x_1)\partial_1 - \partial_1 d_1|\partial_1 \in T_{y-x_1}z \rangle = T\). Then \(\langle d_1(y-x_1)\partial_1 - \partial_1 d_1|\partial_1 \in T_{y-x_1}z \rangle = T_{y-z} = T_{z}\), hence \(L_y \in J\). Therefore \(J = L\). This completes the proof of this theorem. \(\square\)

Note that \(z \notin A_d\) if \(|I| > 1\), and \(z \in A_d\) if \(|I| = 1\). The following corollary follows directly from the above theorem.

**Corollary 3.2.** Suppose that \(S_d = S_d(A,T,\varphi, z)\) is simple. Then \((S_d)_x \neq 0\) if and only if \(x \in A_d \setminus \{z\}\).

### 4. Derivations of \(S_d(A,T,\varphi, z)\).

In this section we assume that the Lie algebra \(S_d(A,T,\varphi, z)\) is simple, and we shall determine all the derivations of \(S_d(A,T,\varphi, z)\). From the proof of Theorem 3.1 we know that:

(a) If \(I = \emptyset\), \(S_d(A,T,\varphi, z) = S(A,T,\varphi, z)\), which was thoroughly studied in [7];

(b) If \(|I| = \dim T = n\), \(S_d(A,T,\varphi, z) \simeq S_n^+\), which was studied in many references, for example [12];

(c) If \(\dim T = 2\) and \(|I| = 1\), \(S_d(A,T,\varphi, z) \simeq L_d(A,\alpha, \beta)\) for some suitable \(\alpha, \beta \in \text{hom}(A,F)\), which was thoroughly studied in [8].

So from now on we **always assume** that \(0 < |I| < \dim T\) and \(\dim T \geq 3\).

Write \(L = S_d(A,T,\varphi, z)\), \(I = \{1, 2, \cdots, n\}\) and \(S_d^+ = \sum_{x \in A_d^+} (S_d)_x\). Recall that \(L\) has an \(A\)-gradation with the following components of degree \(x \in A\):

\[
L_x = \begin{cases} 
0, & \text{for } x = z \\
(t^xT_x) \cap W_d, & \text{for } x \in A \setminus \{z\}.
\end{cases}
\]

A derivation \(D\) of \(L\) is called **homogeneous of degree** \(x \in A\) if \(D(L_y) \subset L_{x+y}\) for all \(y \in A\).

From direct computation we can easily obtain the following lemma.

**Lemma 4.1.** Every \(D \in \text{Der}(L)\) has the form

\[
D = \sum_{y \in A} D_y,
\]

where \(D_y\) is a derivation of \(L\) of degree \(y\), such that for each \(u \in L\) there are only finitely many \(y \in A\) with \(D_y(u) \neq 0\).

First we construct some derivations of \(S_d\). For any additive function \(\mu : A \rightarrow F\) the linear map

\[
D_{\mu}(X) = \mu(x)X, \quad \forall X \in (S_d)_x
\]

is a derivation of \(S_d\) of degree 0.

The following Lemmas 4.2-4.5 will be useful in the sequel.
Lemma 4.2. Let \( x, y \in A \setminus \{0\} \). Then \( T_x = T_y \) if and only if \( \hat{x} \in F\hat{y} \).

This lemma is obvious.

Lemma 4.3. Let \( x_1, x_2 \in A_d \setminus \{z\} \). If one of the following conditions holds:

(a) \( x_2 \in A_d^+ \) and \( x_1 \in A_{d,i} \) for certain \( i \in I \),

(b) \( x_1, x_2 \in A_d^+ \) and \( T_{x_1-z} \neq T_{x_2-z} \),

then

\[
[\mathcal{L}_{x_1}, \mathcal{L}_{x_2}] = \mathcal{L}_{x_1+x_2}.
\]

Proof. Suppose (a) is satisfied. If \( d_i(x_2) > 0 \), then \( d_i \in T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z}) \), hence \( Fd_i + \langle d_i(x_2) \partial - \partial(x_1) d_i | \partial \rangle \in T_{x_2} \). We deduce that \( (d_i(x_2) \partial - \partial(x_1) d_i | \partial) \in T_{x_2-z} \). From

\[
[t^{x_1} d_i, t^{x_2} \partial] = t^{x_1+x_2}(d_i(x_2) \partial - \partial(x_1) d_i), \quad \forall \partial \in T_{x_2-z},
\]

we see that (4.2) follows.

If \( d_i(x_2) = 0 \), then \( x_1 + x_2 \in A_{d,i} \). If \( x_1 + x_2 = z \) further, it follows from \( (S_d)_x = 0 \) that (4.2) follows. Suppose \( x_1 + x_2 \neq z \), by Lemma 4.2 then \( T_{x_2-z} \neq T_{x_1} \). Choose \( \partial \in T_{x_2-z} \setminus T_{x_1} \), so \( [t^{x_1} d_i, t^{x_2} \partial] = -\partial(x_1) t^{x_1+x_2} d_i \). Since \( (S_d)_{x_1+x_2} = Ft^{x_1+x_2} d_i \), hence (4.2) follows again.

Suppose (b) is satisfied. We have

\[
(t^{x_1} \partial_1, t^{x_2} \partial_2) = t^{x_1+x_2}(\partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1)
\]

for all \( \partial_1 \in T_{x_1-z}, \partial_2 \in T_{x_2-z} \).

If \( x_1 = 0 \), it follows from \( T_{x_1-z} \neq T_{x_2-z} \) that \( \hat{x}_2 \notin F\hat{z} \). Then \( T_{x_2} \neq T_z \). Choose \( \partial_1 \in T_{x_2} \setminus T_z \). Then (4.2) follows from (4.3). Now we assume that \( x_1, x_2 \in A_d^+ \setminus \{0\} \).

We claim that \( T_{x_1-z} \neq T_{x_2} \) or \( T_{x_1} \neq T_{x_2-z} \). Otherwise from \( T_{x_1-z} = T_{x_2} \) and \( T_{x_1} = T_{x_2-z} \), by Lemma 4.2 we obtain

\[
d_1(x_2)(x_1-z) = (d_1(x_1) + 1)x_2, \quad d_1(x_1)(x_2-z) = (d_1(x_2) + 1)x_1.
\]

Then \( x_1 = -d_1(x_1)z \) and \( x_2 = -d_1(x_2)z \), it contradicts \( T_{x_1-z} \neq T_{x_2-z} \). Hence our claim holds.

We assume that \( T_{x_1-z} \neq T_{x_2} \). If \( T_{x_1-z} \setminus T_{x_2} \subset T_{x_2-z} \), then we deduce that \( T_{x_1-z} = T_{x_2} \), it is impossible. So \( T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z}) \neq \emptyset \). Choose \( \partial_1 \in T_{x_1-z} \setminus (T_{x_2} \cup T_{x_2-z}) \), then \( F\partial_1 + \langle \partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1 | \partial_2 \in T_{x_2-z} \rangle = T \).

Hence \( \langle \partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1 | \partial_2 \in T_{x_2-z} \rangle = T_{x_1+x_2-z} \). From (4.3) it follows that (4.2) holds.

Lemma 4.4. For a fixed \( i \in I \) and a fixed \( x_0 \in A_d \setminus \{z\} \) with \( d_i(x_0) = 1 \), the subspace

\[
(S_d)_{0} + \sum_{x \in A_d, d_i(x) \leq 0} (S_d)_x
\]

generates \( S_d \) as a Lie algebra.
Proof. Denote by $M$ the subalgebra generated by the above subspace. We shall show that $(S_d)_x \subset M$ for $x \in A_d$ by induction on $k = d_i(x)$. By definition of $M$ this is true for $x \in A_d$ with $d_i(x) < 1$.

Claim 1. We can assume that $x_0 \in A_d^0$.

If $d_j(x_0) = -1$ for one $j \in I \setminus \{i\}$, choose $u_j \in A_d^{i_j}$, by Lemma 4.3 we deduce that $[(S_d)_x, (S_d)_{-u_j}] = (S_d)_{x_0 - u_j} \subset M$. Note that $d_j(x_0 - u_j) = 0$. Then we may assume that $x_0 \in A_d^+$.

If $d_j(x_0) > 0$ for some $j \in I \setminus \{i\}$, also using $u_j \in A_d^{i_j}$, by Lemma 4.3 we obtain that $[(S_d)_x, (S_d)_{u_j}] = (S_d)_{x_0 + u_j} \subset M$. Note that $d_j(x_0 + u_j) = d_j(x_0) - 1$. Thus after finitely many such steps we may assume that $x_0 \in A_d^0$. This is our Claim 1.

Claim 2. There exists $y_0 \in A_d^0 \setminus \{0\}$ such that $(S_d)_x, (S_d)_{x + y_0} \subset M$.

Since $1 < |I| < \dim T$ we know that $A_d^0 \neq 0$. Choose $y' \in A_d^0 \setminus \{0\}$. If $T_{x_0 - z} = T_{y'} - z$, then $x_0 - z = 2y' - 2z$, i.e., $x_0 - 2y' + z = 0$, we set $y_0 = 2y'$.

If $T_{x_0 - z} \neq T_{y'} - z$ we set $y_0 = y'$. Thus $y_0 \in A_d^0 \setminus \{0\}$ and $T_{x_0 - z} \neq T_{y'} - z$.

Since $(S_d)_x, (S_d)_{y_0} \subset M$ and $[(S_d)_x, (S_d)_{y_0}] = (S_d)_{x_0 + y_0}$ (Lemma 4.3), then $(S_d)_x, (S_d)_{x + y_0} \subset M$. Claim 2 follows.

Suppose that $x \in A_d$ with $d_i(x) = 1$. Note that $T_{x_0 - z} = T_{x - x_0 - z}$ implies $x - x_0 - z = 2(x_0 - z)$, i.e., $x - 3x_0 - z = 0$, and that $T_{x_0 + y_0 - z} = T_{x - x_0 - y_0 - z}$ implies $x - 3x_0 - 3y_0 + z = 0$. Then $T_{x_0 - z} \neq T_{x - x_0 - z}$ or $T_{x_0 + y_0 - z} \neq T_{x - x_0 - y_0 - z}$. Without loss of generality we assume that $T_{x_0 - z} \neq T_{x - x_0 - z}$.

By Lemma 4.3 we know that $[(S_d)_{x - x_0}, (S_d)_x] = (S_d)_x$. By noting that $(S_d)_{x - x_0}, (S_d)_x \subset M$, we get that $(S_d)_x \subset M$. So $(S_d)_x \subset M$ for $x \in A_d$ with $d_i(x) \leq 1$.

Suppose that $(S_d)_x \subset M$ for all $x \in A_d$ with $d_i(x) = k$ for a fixed $k \geq 1$. Consider $x \in A_d$ with $d_i(x) = k + 1$. Similar to the above argument we know that $T_{x_0 - z} \neq T_{x - x_0 - z}$ or $T_{x_0 + y_0 - z} \neq T_{x - x_0 - y_0 - z}$, say $T_{x_0 - z} \neq T_{x - x_0 - z}$.

By Lemma 4.3 we obtain that $[(S_d)_{x - x_0}, (S_d)_x] = (S_d)_x$. By noting that $(S_d)_{x - x_0}, (S_d)_x \subset M$, we get that $(S_d)_x \subset M$.

By induction we obtain that $(S_d)_x \subset M$ for all $x \in A_d$. This completes the proof of Lemma 4.4.

Lemma 4.5. (a) Suppose that $D_1, D_2$ are derivations of a Lie algebra $g$, and that $M \subset g$ generates $g$. If $D_1|_M = D_2|_M$, then $D_1 = D_2$.
(b) Suppose that $D_1, D_2 \in \text{Der}(S_d)$ are homogeneous of degree $y \in A_d$. If $D_1(u) = D_2(u)$ for all $u \in (S_d)_x$ with $x \in A_d^+$, i.e., $D_1|_{S_d^+} = D_2|_{S_d^+}$, then $D_1 = D_2$.

Proof. (a) is obvious.
(b) It suffices to show that $D_1(t^x d_1) = D_2(t^y d_1)$ for $x \in A_d \setminus \{z\}$. If $x + y \notin A_d \setminus \{z\}$, we see that $D_1(t^x d_1) = D_2(t^y d_1) = 0$. Suppose that $x + y \in A_d \setminus \{z\}$, and that $D_1(t^x d_1) = t^x + y \partial \neq 0$ and $D_2(t^y d_1) = t^x + y \partial'$. If $\partial \neq \partial'$ we choose $x' \in A_d^+ \setminus x$ such that $\partial(x') - \partial'(x') \neq 0$. For any $\partial_1 \in T_{x' - z}$, we have

$$[t^{x'} \partial_1, t^x d_1] = t^{x'} + x (\partial_1(x) d_1 - d_1(x') \partial_1).$$

Then $[t^{x'} \partial_1, t^x + y \partial] = [t^{x'} \partial_1, t^x + y \partial']$, i.e.,

$$\partial_1(x + y) \partial - \partial(x') \partial_1 = \partial_1(x + y) \partial' - \partial'(x') \partial_1,$$

so, we deduce that

$$\partial_1(x + y) (\partial - \partial') = (\partial(x') - \partial'(x')) \partial_1, \ \forall \ \partial_1 \in T_{x' - z}.$$

This is impossible since $\dim T_{x' - z} > 1$. Thus we get a contradiction. Consequently $\partial = \partial'$, i.e., $D_1(t^x d_1) = D_2(t^y d_1)$.

Now we are ready to describe the homogeneous derivations $D_y$ in (4.1).

**Proposition 4.6.** If $y \notin A_d$, then every homogeneous derivation $D$ of $S_d$ of degree $y$ is $0$.

**Proof.** We shall divide the proof into three cases.

**Case 1.** Suppose $d_i(y) \leq -3$ for some $i \in I$.

From Corollary 3.2 we see that $D((S_d)_x) = 0$ for all $x \in A_d$ with $d_i(x) \leq 1$. By Lemmas 4.4 and 4.5 it follow that $D = 0$.

**Case 2.** Suppose $d_1(y) = -2$ and $d_i(y) \geq -2$ for all $i \in I$.

Then $D((S_d)_x) = 0$ for $x \in A_d$ with $d_1(x) \leq 0$. If $|I| = 1$, since $(S_d)_z = 0$ then $D((S_d)_{z - y}) \subset (S_d)_z = 0$ and $d_1(z - y) = 1$. By Lemma 4.4 it follows that $D = 0$.

Suppose $|I| > 1$. Assume that $\{u_1, u_2, \ldots, u_n\} \subset A$ such that $d_i(u_j) = -\delta_{i,j}$. If $d_i(y) \leq 0$ for some $i \in I \setminus \{1\}$, it follows from $d_1(u_i - u_1 + y) = -1, d_i(u_i - u_1 + y) < 0$ that $D((S_d)_{u_i - u_1}) \subset (S_d)_{u_i - u_1 + y} = 0$. By Lemmas 4.4 and 4.5, we also have $D = 0$.

Suppose $|I| > 1$ and $d_i(y) > 0$ for all $i \in I \setminus \{1\}$. Choose $v_1 \in A_1^\circ, v_2 \in A_2^\circ$. Let $D(t^{v_2 - v_1} d_2) = \lambda t^{v_2 - v_1 + y} d_1$ for some $\lambda \in F$. From $[t^{v_2 - v_1} d_2, d_1 - d_2] = -2t^{v_2 - v_1} d_2$ and $D(d_1 - d_2) = 0$ we get that $\lambda t^{v_2 - v_1 + y} d_1, d_1 - d_2] = -2\lambda t^{v_2 - v_1 + y} d_1$. Then $\lambda(d_1(y) - d_2(y)) = 0$. Since $d_1(y) - d_2(y) < -3$ we obtain that $\lambda = 0$, i.e., $D((S_d)_{v_1 - v_1}) = 0$. By Lemmas 4.4 and 4.5, it follows that $D = 0$.

**Case 3.** Suppose that $|I| \geq 2$, that $d_i(y) \geq -1$ for all $i \in I$ and that $d_1(y) = d_2(y) = -1$. We first show that $D((S_d)_x) = 0$ for all $x \in A_d$ with $d_1(x) \leq 0$. If $x + y \notin A_d$ then $D((S_d)_x) \subset (S_d)_{x + y} = 0$. In particular, $D(T_z) = 0$. Suppose $x \in A_d$ with $d_1(x) = 0$ and $x + y \in A_d$. Then $x + y \in A_1$ and $d_2(x) \geq 1, d_i(x) \geq 0, \forall i > 1$. If $y \neq z$ then $\hat{y} \notin F \hat{z}$. Choose $\partial_1 \in T_\hat{z} \setminus T_y$. 
Let $D(t^x \partial) = \lambda t^{x+y} d_1$ where $\lambda \in F$. Applying $D$ to $[\partial_1, t^x \partial] = \partial_1(x) t^x \partial$, we obtain that $\lambda \partial_1(y) = 0$. Thus $\lambda = 0$. Consequently $D((S_d) x) = 0$ in this case.

Suppose that $y = z$. Recall that $x, x + y (= x + z) \in A_d$, $d_1(x) = 0$ and $d_i(x) > 0$ for all $i \in I \setminus \{1\}$. Choose $u_j \in A^\#_j$, $\forall j \in I$. For any $x_0 \in A^0_d \setminus \{0\}$ we have

\begin{equation}
[t^{x_0} \partial, t^{-u_1} \partial_1] = -t^{x_0-u_1} (\partial(u_1) \partial_1 + \partial_1(x_0) \partial),
\end{equation}

where $\partial \in T_{x_0-z}, \partial_1 \in T_{u_1+z}$. Let $D(t^{-u_1} \partial_1) = \lambda_1 t^{x_1} d_2$, where $\lambda_1 \in F$. Since $t_0, t, u_1$ are linearly independent, we can choose $\partial \in T_{x_0-z} \cap T_{z-u_1} \setminus T_z$.

Then $\langle \partial(u_1) \partial_1 + \partial_1(x_0) \partial_1 \in T_{u_1+z} \rangle = T_{x_0-u_1-z}$ since $\partial \notin T_{u_1+z}$ and $\langle \partial, \partial(u_1) \partial_1 + \partial_1(x_0) \partial_1 \in T_{u_1+z} \rangle = T$. Applying $D$ to (4.4) we obtain that

\begin{align*}
-D(t^{x_0-u_1} (\partial(u_1) \partial_1 + \partial_1(x_0) \partial)) &= [t^{x_0} \partial, \lambda_1 t^{x_1} d_2] \\
&= \lambda_1 \partial(z - u_1) t^{x_1} d_2 = 0,
\end{align*}

for all $\partial_1 \in T_{u_1+z}$. Then $D((S_d) x_{0-u_2}) = 0$. Similarly we can get $D((S_d) x_{0-u_2}) = 0$. By $[(S_d) x_{0-u_2} : (S_d) x_{0-u_2}] = (S_d)_{u_2}$ we have $D((S_d)_{u_2}) = 0$. By induction on $k = d_2(x) \geq 2$, and using $[(S_d)_{u_2} : (S_d) x_{u_2}] = (S_d)_x$, we can obtain that $D((S_d) x) = 0$ for $x \in A_d$ with $d_1(x) = 0$.

Next we claim that $D((S_d)_{u_1}) = 0$ for $u_1 \in A$. If $y - u_1 \notin A_d$ we have $D((S_d)_{u_1}) \subset (S_d)_{y-u_1} = 0$.

Suppose that $y - u_1 \in A_d$. Then $d_i(y) \geq 0$ for all $i \geq 3$. If $y = z$, from the above argument we know that $D((S_d)_y) = 0$. Suppose also that $y \neq z$. Choose $\partial_1 \in T_z \setminus T_y$. Let $D(t^{y-u_1} \partial) = \lambda t^{y-u_1} d_2$ for $\partial \in T_{z+u_1}$, where $\lambda_1 \in F$. Then by $[\partial_1, t^{y-u_1} \partial] = -\partial(u_1) t^{y-u_1} \partial$ we obtain that $[\partial_1, \lambda t^{y-u_1} d_2] = -\lambda \partial(t^{y-u_1} d_2) = \partial_1(x_0) \partial$. Thus $\lambda \partial_1(y) = 0$. Since $\partial_1(y) \neq 0$ we infer that $\lambda = 0$. Consequently $D((S_d)_{u_1}) = 0$ also. Therefore our claim is true. By Lemmas 4.4 and 4.5 we conclude that $D = 0$ in this case. Hence we have proved that $D = 0$ when $y \notin A_d$. \hfill \square

**Proposition 4.7.** Suppose that $y \in A_d \setminus \{0\}$, and that $D \in \text{Der}(S_d)$ is homogeneous of degree $y$.

(a) If $y \neq z$, there exists $t^y \partial_0 \in (S_d)_y$ such that $D = \text{ad} (t^y \partial_0)$.

(b) If $y = z$, we have $|I| = 1$ and $D \in F \cdot \text{ad} (t^y d_1)$.

**Proof.** For any $x \in A_d \setminus \{z\}$, we define the linear map $D_x : T_{x-z} \rightarrow T_{x+y-z}$ (or $D_x : F d_i \rightarrow T_{x+y-z}$ if $x \in A_i$) by $D(t^x \partial) = t^{x+y}(D_x \partial)$. By applying $D$ to

\[ [t^{x_1} \partial_1, t^{x_2} \partial_2] = t^{x_1+x_2} (\partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1) \]

where $x_1, x_2 \in A_d \setminus \{z\}$ and $\partial_1 \in T_{x_1-z}, \partial_2 \in T_{x_2-z}$, we obtain that

\begin{equation}
D_x (\partial_1 \partial_2) \partial - D_x (\partial_2 \partial_1) \partial + \partial_2 \partial_1 \partial + \partial_1 \partial_2 \partial = D_{x_1+x_2} (\partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1)
\end{equation}
holds for \( \partial_1 \in T_{x_1-z} \), \( \partial_2 \in T_{x_2-z} \), and \( x_1, x_2 \in A_d \setminus \{ z \} \).

Case 1. Suppose that \( \hat{y} \) and \( \hat{z} \) are linearly independent. By setting \( x_2 = 0 \) in (4.5) we obtain
\[
(4.6) \quad \partial_1(y)D_0(\partial_2) = \partial_2(y)D_{x_1}(\partial_1) + \langle D_0(\partial_2), x_1 \rangle \partial_1.
\]
By setting here \( x_2 = 0 \) in (4.6) we obtain that
\[
\partial_1(y)D_0(\partial_2) = \partial_2(y)D_0(\partial_1).
\]
Choose \( \partial_2 \in T_z \setminus T_y \), and denote \( \partial_0 = \partial_2(y)^{-1}D_0(\partial_2) \). Then we have \( \partial_0 \in T_{y-z} \) and
\[
(4.7) \quad D_0(\partial_1) = -\partial_1(y)\partial_0, \ \forall \partial_1 \in T_z.
\]
Hence we can rewrite (4.6) as
\[
\partial_2(y)(D_{x_1}(\partial_1) - \partial_1(y)\partial_0 + \partial_0(x_1)\partial_1) = 0.
\]
Thus we deduce that
\[
D_{x_1}(\partial_1) = \partial_1(y)\partial_0 - \partial_0(x_1)\partial_1.
\]
It follows that \( D = -\text{ad}(t^y\partial_0) \). Note that by now we have not known that \( t^y\partial_0 \in S_d \) yet. If \( \partial_0 = 0 \) or \( y \in A^+_d \), from \( \partial_0 \in T_{y-z} \) then \( t^y\partial_0 \in S_d \). If \( \partial_0 \neq 0 \) and \( y \in A_i \) for some \( i \in I \), since \( A_i \nsubseteq \ker(\partial_0) \), choose \( x_0 \in A_i \setminus \ker(\partial_0) \). Since \( D((S_d)_{x_0}) = 0 \) and \( D = \text{ad}(t^y\partial_0) \), we deduce that
\[
[t^y\partial_0, t^{x_0}d_i] = t^{x_0+y}(\partial_0(x_0)d_i + \partial_0) = 0.
\]
Thus \( \partial_0 = -\partial_0(x_0)d_i \). Hence \( t^y\partial_0 \in S_d \).

Case 2. Suppose that \( y = z \). Then \( |I| = 1 \) and \( z \in A^+_d \). Since \( D_0((S_d)_{x_0}) \subset (S_d)_z = 0 \), we know that \( D_0 = 0 \).

Claim 1. For \( x_1 \in A_d \setminus \{ z \} \) with \( \hat{x}_1 \notin F\hat{z} \), there exists a constant \( a_{x_1} \in F \) such that
\[
(4.8) \quad D_{x_1}\partial = a_{x_1}\partial, \ \forall \partial \in T_{x_1} \cap T_z.
\]
If \( x_1 \in A_d \setminus A^+_d \), clearly (4.8) is true. Next we suppose that \( x_1 \in A^+_d \). By setting \( \partial_1 = \partial_2 = \partial \in T_{x_1} \cap T_z \) and \( x_2 = -z \) in (4.5), we obtain that
\[
-\langle D_{x_1}\partial, z \rangle \partial - \langle D_{-z}\partial, x_1 \rangle \partial = 0,
\]
and so
\[
(4.9) \quad \langle D_{-z}\partial, x_1 \rangle = -\langle D_{x_1}\partial, z \rangle
\]
holds. On the other hand, for \( x_2 = -z \), \( \partial_2 = \partial \in T_{x_1} \cap T_z \), and arbitrary \( \partial_1 \in T_{x_1-z} \), (note that we allow \( x_1 \in A_1 \) here), (4.5) gives that
\[
(4.10) \quad \langle D_{x_1}\partial_1, z \rangle \partial + \langle D_{-z}\partial, x_1 \rangle \partial_1 = \partial_1(z)D_{x_1-z}\partial.
\]
By evaluating both sides at $x_1$ and using $\partial_1(x_1) = \partial_1(z)$, we obtain that
\[
\partial_1(z)[\langle D_{x_1-z}\partial, x_1 \rangle - \langle D_{-z}\partial, x_1 \rangle] = 0.
\]
As $\hat{x}_1 \notin F \hat{z}$, we can choose $\partial_1 \in T_{x_1-z} \setminus T_{\hat{z}}$, and so
\begin{equation}
\langle D_{x_1-z}\partial, x_1 \rangle = \langle D_{-z}\partial, x_1 \rangle, \quad \forall \partial \in T_{x_1} \cap T_{\hat{z}}.
\end{equation}
By substituting $x_1+z$ for $x_1$ in (4.9), and using $\langle D_{-z}\partial, z \rangle = 0$, we infer that
\[-\langle D_{x_1-z}\partial, x_1 \rangle = \langle D_{-z}\partial, x_1 \rangle
\]
holds for $\partial \in T_{x_1} \cap T_{\hat{z}}$. By comparing this equation with (4.11), we conclude that
\begin{equation}
\langle D_{-z}\partial, x_1 \rangle = 0, \quad \forall x_1 \in A_d, \ \partial \in T_{x_1} \cap T_{\hat{z}}.
\end{equation}
Now (4.10) gives that
\[
\partial_1(z)D_{x_1-z}\partial = \langle D_{x_1}\partial_1, z \rangle \partial
\]
for $\partial_1 \in T_{x_1-z}$, $\partial \in T_{x_1} \cap T_{\hat{z}}$, $x_1 \in A_d$, with $\hat{x}_1 \notin F \hat{z}$. By choosing $\partial_1 \in T_{x_1-z} \setminus T_{\hat{z}}$ and setting $a_{x_1-z} = \frac{\langle D_{x_1}\partial_1, z \rangle}{\partial_1(z)}$, then we have $D_{x_1-z}\partial = a_{x_1-z}\partial$, thus
\[
D_{x_1}\partial = a_{x_1}\partial, \quad \forall x_1 \in A_d, \ \partial \in T_{x_1} \cap T_{\hat{z}}.
\]
Hence our first claim is proved.

**Claim 2.** If $x_1, x_2 \in A_d^+$ with $\hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin F \hat{z}$, then
\begin{equation}
\hat{a}_{x_1+x_2} = a_{x_1} + a_{x_2}.
\end{equation}

In order to prove this claim we shall consider first the case where $\hat{x}_1, \hat{x}_2$, and $\hat{z}$ are linearly independent. Then we can choose $\partial_1 \in (T_{x_1} \cap T_{\hat{z}}) \setminus T_{x_2}$ and $\partial_2 \in (T_{x_2} \cap T_{\hat{z}}) \setminus T_{x_1}$. It follows that $\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1$ is a nonzero vector in $T_{x_1+x_2} \cap T_{\hat{z}}$. By using the first claim, (4.5) gives that
\begin{equation}
(a_{x_1+x_2} - a_{x_1} - a_{x_2})[\partial_1(x_2)\partial_2 - \partial_2(x_1)\partial_1] = 0,
\end{equation}
and so (4.13) holds in this case.

Now assume that $\hat{x}_1, \hat{x}_2$ and $\hat{z}$ are linearly dependent. Since $\dim T \geq 3$ we can choose $w \in A$ such that $\hat{x}_1, \hat{w}$ and $\hat{z}$ are linearly independent. By using the case already established, we have
\[
a_{x_1} = a_{x_1+w} + a_{-w} = a_{x_1} + a_{w} + a_{-w},
\]
and
\[
a_{x_1+x_2} = a_{x_1+w} + a_{x_2-w} = a_{x_1} + a_{w} + a_{x_2} + a_{-w},
\]
and conclude again that (4.13) holds. Hence our second claim is proved.

Now let $x \in A_d^+$ with $\hat{x} \in F \hat{z}$ and set $a_x = a_{x+v} - a_v$ where $v \in A_d^+$ with $\hat{v} \notin F \hat{z}$. By our second claim, $a_{x+v} - a_v$ is independent of the choice of $v$. 
With $a_x$ now defined for all $x \in A_d^+$, it is easy to see that (4.13) is valid for all $x_1, x_2 \in A_d^+$.

We shall now remove the restriction $\hat{x}_1 \notin Fz$ in (4.8). Thus assume that $\hat{x}_1 \in Fz$ and $x_1 \in A_d^+$. We choose $x_2 \in A_d^+$ so that $\hat{x}_2 \notin Fz$, and let $\partial_1 \in T_x$ and $\partial_2 \in T_x \cap T_z$. By using the first claim and $\partial_2(x_1) = \partial_1(y) = 0$, the equation (4.5) gives $\langle D_{x_1} \partial_1, x_2 \rangle \partial_2 = a_{x_1} \partial_1(x_2) \partial_2$. Hence $\langle (D_{x_1} - a_{x_1}) \partial_1, x_2 \rangle = 0$ for all $x_2 \in A_d^+$ with $\hat{x}_2 \notin Fz$, and so (4.8) holds also for $\hat{x}_1 \in Fz$.

For $x \in A_d^+$, we define linear maps $D'_x : T_{x-z} \to T$ by $D'_x \partial = D_x \partial - a_x \partial$.

By Claim 1, $T_z \cap T_z$ is contained in the kernel of $D'_x$. In particular, if $\hat{x} \in Fz$, then $\partial_x = 0$. If $\hat{x} \notin Fz$, then $T_x \cap T_z$ is a hyperplane in $T_{x-z}$, and so the vector

$$
(4.14) \quad \partial_x := \frac{D'_x \partial}{\partial(y)}
$$

is independent of the choice of $\partial \in T_{x-z} \setminus T_z$ (note that $y = z$). Thus we have

$$
(4.15) \quad D'_x \partial = \partial(y) \partial_x, \quad \forall \ x \in A_d^+, \partial \in T_{x-z}.
$$

If $\hat{x} \in Fz$, then $\partial_x$ is not defined but (4.15) is also valid because $D'_x = 0$ and $\partial(y) = 0$ for $\partial \in T_{x-z} = T_z$.

By substituting $D'_{x_1} + a_{x_1}$ for $D_{x_1}$ and making similar substitutions for $D_{x_2}$ and $D_{x_1+x_2}$ in (4.5), we obtain that

$$
\langle D'_{x_1} \partial_1, x_2 \rangle \partial_2 - \langle D'_{x_2} \partial_2, x_1 \rangle \partial_1 + \partial_1(x_2 + y) D'_{x_2} \partial_2 \\
- \partial_2(x_1 + y) D'_{x_1} \partial_1 + a_{x_1} \partial_1(x_2) \partial_2 - a_{x_1} \partial_2(x_1) \partial_1 \\
= D'_{x_1+x_2} (\partial_1(x_2) \partial_2 - \partial_2(x_1) \partial_1)
$$

holds for $x_1, x_2 \in A_d^+$, $\partial_1 \in T_{x_1-z}$ and $\partial_2 \in T_{x_2-z}$.

By using (4.15) and similar expressions for $D'_{x_2}$ and $D'_{x_1+x_2}$, the last equation can be rewritten as follows

$$
(4.16) \quad \partial_1(y)[a_{x_2} + \partial_1(x_2)] \partial_2 - \partial_2(y)[a_{x_1} + \partial_2(x_1)] \partial_1 \\
= [\partial_1(x_2) \partial_2(y) - \partial_2(x_1) \partial_1(y)] \partial_{x_1+x_2} \\
+ \partial_1(y) \partial_2(x_1 + y) \partial_{x_1} - \partial_2(y) \partial_1(x_2 + y) \partial_{x_2}.
$$

Claim 3. The vector $\partial_x$ for $x \in A_d^+$ with $\hat{x} \notin Fz$ are independent of $x$.

Suppose $x_1, x_2 \in A_d^+$ with $\hat{x}_1, \hat{x}_2, \hat{x}_1 + \hat{x}_2 \notin Fz$. For $\partial_1 \in T_{x_1} \cap T_z \setminus \{0\}$ and $\partial_2 \in T_{x_2-z} \setminus T_z$, (4.16) gives

$$
(4.17) \quad \partial_1(x_2)(\partial_{x_2} - \partial_{x_1+x_2}) = [a_{x_2} + \partial_{x_2}(x_1)] \partial_1.
$$
If \( \hat{x}_1, \hat{x}_2, \hat{z} \) are linearly dependent, we see that \( \partial_1(x_2) = 0 \). Thus we obtain from (4.17) that \( a_{x_1} = -\partial_{x_2}(x_1) \).

Assume that \( \hat{x}_1, \hat{x}_2, \hat{z} \) are linearly independent. Then \( \partial_1 \in T_{x_1} \cap T_z \) can be chosen so that \( \partial_1(x_2) \neq 0 \). By evaluating both sides of (4.17) at \( x_1 \), we obtain that

\[
\partial_1(x_2)(\partial_{x_4}(x_1) - \partial_{x_1+x_2}(x_1)) = 0.
\]

Since \( \partial_1(x_2) \neq 0 \) we deduce that

\[
\partial_{x_1+x_2}(x_1) = \partial_{x_1}(x_1).
\]

By evaluating (4.16) at \( x \), we find

\[
\partial_1(z)\partial_2(z)[\partial_{x_1}(x_2) - \partial_{x_2}(x_1) + a_{x_2} - a_{x_1}] = 0.
\]

Since we can choose \( \partial_1 \in T_{x_1-z} \) and \( \partial_2 \in T_{x_2-z} \) such that \( \partial_1(z)\partial_2(z) \neq 0 \), we infer that

\[
a_{x_1} - a_{x_2} = \partial_{x_1}(x_2) - \partial_{x_2}(x_1).
\]

By replacing \( x_1 \) with \( x_1 + x_2 \) and using Claim 2, we obtain the equation

\[
a_{x_1} = \partial_{x_1+x_2}(x_2) - \partial_{x_2}(x_1 + x_2) = \partial_{x_1}(x_2) - \partial_{x_2}(x_1 + x_2).
\]

Combining this with (4.19), we deduce that \( \partial_{x_1}(x_2) = \partial_{x_2}(x_2) \). Considering (4.19) also we have \( a_{x_1} = -\partial_{x_1}(x_1) \). So \( \partial_{x_1}(x_2) \) is independent of \( x_1 \), consequently \( \partial_{x_1} \) is independent of \( x_1 \). Then Claim 3 is proved.

Denote \(-\partial_x \ (x \in A_+^d \text{ with } \hat{x} \notin F\hat{z}) \) by \( \partial_0 \). So we have

\[
a_x = \partial_0(x), \ \forall \ x \in A_+^d, \text{ with } \hat{x} \notin F\hat{z}.
\]

By the definition of \( a_{x_1} \) for \( x \in A_+^d \) with \( \hat{x} \notin F\hat{z} \), we also have

\[
a_x = \partial_0(x), \ \forall \ x \in A_+^d.
\]

Combining this with (4.15), we deduce that

\[
D_x\partial = -\partial(y)\partial_0 + \partial_0(x)\partial, \ \forall \ x \in A_+^d, \partial \in T_{x-z}.
\]

So \( D|_{S^+_d} = \text{ad} (t^y\partial_0)|_{S^+_d} \). By Lemma 4.5 we see that \( D = \text{ad} (t^y\partial_0) \).
Choose $x \in A^0_d \setminus \{0\}$, and $\partial \in T_{x-z} \setminus T_z$. We see that $D((S_d)_{x}) \subset (S_d)_{x+z} = F^{x+z}d_1$, i.e., $D_x \partial = \lambda_\partial d_1$ for some $\lambda_\partial \in F$. Combining this with (4.21), we infer that

$$\lambda_\partial d_1 = -\partial(y)\partial_0 + \partial_0(x)\partial.$$ 

Since $d_1 \notin T_{x-z}$ and $\dim T_{x-z} \geq 2$ we conclude from the above equation that $\partial_0(x) = 0$. Furthermore we deduce that $\partial_0 = \lambda_\partial \partial(y)^{-1}d_1$, i.e., $\partial_0 \in Fd_1$.

**Case 3.** Suppose that $y = -\lambda z$, where $\lambda \in \mathbb{N} = \{1, 2, 3, \ldots \}$.

By setting $x_2 = 0$ in (4.5) we obtain

$$\partial_1(y)D_0(\partial_2) = \langle D_0(\partial_2), x_1 \rangle \partial_1,$$

for $\partial_1 \in T_{x_1-z}, \partial_2 \in T_z$. Since $\dim T_{x_1-z} \geq 2$ we deduce that $\langle D_0(\partial_2), x_1 \rangle = 0$ for all $x \in A_d$. It follows that $D_0(\partial_2) = 0$ for $\partial_2 \in T_z$, i.e., $D_0 = 0$.

Now we show that Claim 1 is also true in this case.

First suppose that $x_1 \in A^+_d$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$. Since $\hat{x} \notin F\hat{z}$ and $-x_1 - z \in A^+_d$, we can choose $\partial_2 \in T_{-x_1-z} \setminus T_z$. By setting $x_2 = -x_1$ and $\partial_1 = \partial \in T_{x_1} \cap T_z$ in (4.5) and using $D_0 = 0$ we obtain that

$$\partial_2(x_1 + y)D_{x_1}(\partial) + \langle D_{x_1}(\partial), x_1 \rangle \partial_2 + \langle D_{-x_1}(\partial_2), x_1 \rangle \partial = 0.$$ 

By evaluating the above equation at $x_1 + y - z$ and by using $\langle D_{x_1}(\partial), x_1 + y - z \rangle = 0$, $\partial(x_1) = \partial(y) = \partial(z) = 0$, we conclude that $\langle D_{x_1}(\partial), x_1 \rangle = 0$, and consequently

$$\partial_2(x_1 + y)D_{x_1}(\partial) = -\langle D_{-x_1}(\partial_2), x_1 \rangle \partial,$$

holds for all $\partial \in T_{x_1} \cap T_z$ and $\partial_2 \in T_{-x_1-z}$. Since $\partial_2(x_1+y) = -(\lambda+1)\partial_2(z) \neq 0$, then (4.8) holds for $x_1 \in A^+_d$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$.

We shall show (4.8) for $x_1 \in A^+_d$ by induction on $d(x_1) := \sum_{i \in I} d_i(x_1)$.

This has been proved for all $x_1$ with $d(x_1) \leq 1$. Now suppose (4.8) holds for all $x_1$ with $d(x_1) \leq k$ ($\geq 1$). Consider $x_2 \in A^+_d$ with $d(x_2) = k + 1$. Choose $x_0 \in A^+_d$ with $d(x_0) = 1$ and with $\hat{x}_0, \hat{x}_2, \hat{z}$ being linearly independent. By inductive hypothesis we have

$$D_{x_0} \partial' = a_{x_0} \partial', \ \forall \partial' \in T_{x_0} \cap T_z,$$

$$D_{x_2-x_0} \partial = a_{x_2-x_0} \partial, \ \forall \partial \in T_{x_2-x_0} \cap T_z.$$

By replacing $x_1$ with $x_0$, $x_2$ with $x_2 - x_0$, $\partial_1$ with $\partial'_0$, $\partial_2$ with $\partial$ respectively in (4.5), we obtain that

$$D_{x_2}(\partial'_0(x_2) \partial - \partial(x_0) \partial'_0) = (a_{x_0} + a_{x_2-x_0})(\partial'_0(x_2) \partial - \partial(x_0) \partial'_0),$$

for all $\partial'_0 \in T_{x_0} \cap T_z$, $\partial \in T_{x_2-x_0} \cap T_z$. It suffices to show that

$$\langle \partial'_0(x_2) \partial - \partial(x_0) \partial'_0 \partial \rangle_{x_0} \partial'_0 \in T_{x_0} \cap T_z, \partial \in T_{x_2-x_0} \cap T_z \cap T_{x_2} \cap T_z.$$ 

Choose $\partial'_0 \in (T_{x_0} \cap T_z) \setminus T_{x_2}$, then

$$F \partial'_0 + \langle \partial'_0(x_2) \partial - \partial(x_0) \partial'_0 \partial \rangle_{x_0} \partial \in T_{x_2-x_0} \cap T_z = F \partial'_0 + T_{x_2-x_0} \cap T_z.$$
is codimension 1 in $T$. Hence the subspace
\[
\langle \partial_x(x_2) \partial - \partial(x_0) \partial_0 \partial \in T_{x_2-x_0} \cap T_z \rangle \subset T_{x_2} \cap T_z
\]
is codimension 2 in $T$. Therefore (4.22) holds. Thus there exists $a_{x_2} \in F$ such that $D_{x_2} \partial = a_{x_2} \partial$ for all $\partial \in T_{x_2} \cap T_z$. Consequently Claim 1 is true.

Exactly the same as that in Case 2, Claim 2 is true in this case also.

Now same as in Case 2, we define $a_x$ for $x \in A_2^d$ with $\hat{x} \notin F \hat{z}$ and we can remove the restriction $\hat{x} \notin F \hat{z}$ in (4.8), then we can also define the linear map $D_x'$, the vector $\partial_x$ in (4.14) for $x \in A_2^d$ with $\hat{x} \notin F \hat{z}$, and same as before we also get equations (4.15), (4.16).

Now we claim that Claim 3 is true in this case also. The proof is exactly the same as it was in Case 2.

We also denote $-\partial_x (x \in A_2^d)$ with $\hat{x} \notin F \hat{z})$ by $\partial_0$. So we have
\[
a_x = \partial_0(x), \quad \forall x \in A_2^d, \text{ with } \hat{x} \notin F \hat{z}.
\]
The same as in Case 2, Equations (4.20) and (4.21) are true. So $D|_{S_2^+} = \text{ad}(t^y \partial_0)|_{S_2^+}$. By Lemma 4.5 we see that $D = \text{ad}(t^y \partial_0)$. By definition of $\partial_0$ we know that $t^y \partial_0 \in (S_d)_y$.

By now we have completed the proof of Proposition 4.7. $\square$

**Proposition 4.8.** Suppose that $D \in \text{Der}(A_d)$ is homogeneous of degree 0. Then there exists a $\nu \in \text{hom}(A,F)$ such that $D = D_\nu$.

**Proof.** For any $x \in A_2^d$, we define the linear map $D_x : T_{x-z} \rightarrow T_{x-z}$ (or $D_x : Fd_i \rightarrow Fd_i$ if $x \in A_i$) by $D(t^x \partial) = t^{x+y}(D_x \partial)$. If $\partial \in T_{x_1-x_2}$, then $\partial(x_1) = \partial(z)$ and $\langle D_{x_1} \partial, x_1 \rangle = \langle D_{x_1} \partial, z \rangle$. As $x = 0$, the equation (4.5) takes the form
\[
\langle D_{x_1} \partial_1, x_2 \rangle \partial_2 - \langle D_{x_2} \partial_2, x_1 \rangle \partial_1 + \partial_1(x_2)D_{x_2} \partial_2 - \partial_2(x_1)D_{x_1} \partial_1 \quad \text{where } \partial_1 \in T_{x_1-x_2}, \partial_2 \in T_{x_2-x_2}, \text{ and } x_1, x_2 \in A_2^d.
\]
By setting $v = 0$ in (4.23), we obtain that $\langle D_0 \partial_2, x_1 \rangle \partial_1 = 0$. Hence $D_0 = 0$.

We claim that (4.8) holds for $x_1 \in A_d \setminus \{z\}$ and $\partial \in T_{x_1} \cap T_z$.

If $x_1 \in A_d \setminus A_2^d$, (4.8) holds clearly. Next suppose that $x \in A_2^d$.

First suppose that $x_1 \in A_2^d$ with $-x_1 \in A_d$ and $\hat{x} \notin F \hat{z}$ and $-x_1 - z \in A_2^d$, we can choose $\partial \in T_{-x_1-z} \setminus T_z$. By setting $x_2 = -x_1$ and $\partial_1 = \partial \in T_{x_1} \cap T_z$ in (4.23) and using $D_0 = 0$, we obtain that
\[
\partial_2(x_1)D_{x_1} \partial_1 + \langle D_{x_1} \partial, x_1 \rangle \partial_2 + \langle D_{-x_1} \partial_2, x_1 \rangle \partial = 0.
\]
By evaluating the above equation at $x_1 - z$ and by using $\langle D_{x_1}(\partial), x_1 - z \rangle = 0$, $
abla(x_1) = \nabla(z) = 0$, we conclude that $\langle D_{x_1}(\partial), x_1 \rangle = 0$, and consequently

$$\partial_2(x_1)D_{x_1}(\partial) = -\langle D_{x_1}(\partial_2), x_1 \rangle \partial,$$

holds for all $\partial \in T_{x_1} \cap T_z$ and $\partial_2 \in T_{x_1 - z}$. Since $\partial_2(x_1) = -\partial_2(z) \neq 0$, then (4.8) holds for $x_1 \in A_d^+$ with $-x_1 \in A_d$ and $\hat{x} \notin F\hat{z}$.

We shall show (4.8) for $x_1 \in A_d^+$ by induction on $d(x_1) := \sum_{i \in I} d_i(x_1)$.

This has been proved for all $x_1$ with $d(x_1) \leq 1$. Now suppose (4.8) holds for all $x_1$ with $d(x_1) \leq k \geq 1$. Consider $x_2 \in A_d^+$ with $d(x_2) = k + 1$. Choose $x_0 \in A_d^+$ with $d(x_0) = 1$ and with $\hat{x}_0, \hat{x}_2, \hat{z}$ being linearly independent. By inductive hypothesis we have

$$D_{x_0} \partial' = a_{x_0} \partial', \quad \forall \partial' \in T_{x_0} \cap T_z,$$

$$D_{x_2 - x_0} \partial = a_{x_2 - x_0} \partial, \quad \forall \partial \in T_{x_2 - x_0} \cap T_z.$$

By replacing $x_1$ with $x_0$, $x_2$ with $x_2 - x_0$, $\partial_1$ with $\partial'_0$, $\partial_2$ with $\partial$ respectively in (4.23), we obtain that

$$D_{x_2}(\partial'_0(x_2) \partial - \partial(x_0) \partial'_0) = (a_{x_0} + a_{x_2 - x_0})(\partial'_0(x_2) \partial - \partial(x_0) \partial'_0)$$

for all $\partial'_0 \in T_{x_0} \cap T_z$, $\partial \in T_{x_2 - x_0} \cap T_z$. It suffices to show that

$$\langle \partial'_0(x_2) \partial - \partial(x_0) \partial'_0 \partial, \partial \rangle \subset T_{x_0} \cap T_z, \partial \in T_{x_2 - x_0} \cap T_z = T_{x_2} \cap T_z.$$

Choose $\partial'_0 \in (T_{x_0} \cap T_z) \cap T_{x_2} \cap T_z$, then

$$F \partial'_0 + \langle \partial'_0(x_2) \partial - \partial(x_0) \partial'_0 \partial \rangle \subset T_{x_2 - x_0} \cap T_z = T_{x_2} \cap T_z$$

is codimension 1 in $T$. Hence the subspace

$$\langle \partial'_0(x_2) \partial - \partial(x_0) \partial'_0 \partial \rangle \subset T_{x_2 - x_0} \cap T_z \subset T_{x_2} \cap T_z$$

is codimension 2 in $T$. Therefore (4.23) holds. Thus there exists $a_{x_2} \in F$ such that $D_{x_2} \partial = a_{x_2} \partial$ for all $\partial \in T_{x_2} \cap T_z$. Consequently our claim about (4.8) is true.

Exactly the same as that in Case 2 in the proof of Proposition 4.7, Claim 2 in the proof of Proposition 4.7 is true in this case also.

Now as in Case 2 of the proof of Proposition 4.7, we define $a_x$ for $x \in A_d \setminus \{z\}$ with $\hat{x} \in F\hat{z}$ and we can remove the restriction $\hat{x} \notin F\hat{z}$ in (4.8). Then we have obtained that:

(a) For any $x \in A_d \setminus \{z\}$, there exists a constant $a_x \in F$ such that

(4.8')

$$D_x \partial = a_x \partial, \quad \forall \partial \in T_x \cap T_z.$$

(b) For all $x_1, x_2 \in A_d^+$,

$$a_{x_1 + x_2} = a_{x_1} + a_{x_2}.$$

Now we claim that, for any $x \in A_d \setminus \{z\}$,

(4.24)  

$$D_x \partial = a_x \partial, \quad \forall \partial \in T_{x - z}.$$
If \( \hat{x} \in F \hat{z} \), this follows from (4.8') since \( T_x \cap T_z = T_x \cap T_z = T_z \). If \( x \in A_d \setminus (A_d^+ \cup \{ z \}) \), (4.24) is clear. Next we suppose \( x \in A_d^+ \) with \( \hat{x} \notin F \hat{z} \). We shall show this by induction on \( d(x) := \sum_{i \in I} d_i(x) \).

If \( d(x) = 0 \), by replacing \( D \) with \( D + D_\mu \) for a suitable \( \mu \in \text{hom}(A, F) \), we may assume that \( x \neq 0 \). For \( \partial \in T_{x-z} \setminus T_z \) let \( D(t^x \partial) = ax t^x \partial' \). For any \( i \in I \), \( u_i \in A_i \), applying \( D \) to \( [t^x \partial, t^{u_i} \partial_i] = t^{x+u_i} \partial(u_i) d_i \), we obtain that
\[
[a_x t^x \partial', t^{u_i} \partial_i] + [t^x \partial, a_{u_i} t^{u_i} \partial_i] = (a_x + a_{u_i}) t^{x+u_i} \partial(u_i) d_i.
\]
We deduce that \( a_x (\partial - \partial')(u_i) d_i = 0 \). Thus \( \partial(u_i) = \partial'(u_i) \) for all \( u_i \in A_i \) and any \( i \in I \). So we obtain that \( \partial = \partial' \).

Suppose (4.24) holds for any \( x \in A_d \setminus \{ z \} \) with \( d(x) \leq k \) where \( k \geq 0 \). Consider \( x_1 \in A_d^+ \setminus (\mathbb{Z} z) \) with \( d(x_1) = k + 1 \). Suppose \( t^{x_1} \partial \in (S_d)_{x_1} \) with \( \partial \in T_{x_1-z} \setminus T_z \). For any \( i \in I \) and any \( u_i \in A_i \), we know that \( D|_{(S_d)_{x_1+u_i}} = a_{x_1+u_i} (S_d)_{x_1+u_i} \). By replacing \( D \) with \( D + D_\mu \) for a suitable \( \mu \in \text{hom}(A, F) \), we may assume that \( a_{x_1+u_i} \neq 0 \). Let \( D(t^{x_1} \partial) = a_{x_1} t^{x_1} \partial' \). Applying \( D \) to
\[
[t^{x_1} \partial, t^{u_i} \partial_i] = t^{x_1+u_i} (\partial(u_i) d_i - d_i(x_1) d_i),
\]
we obtain that
\[
a_{x_1}[t^{x_1} \partial', t^{u_i} \partial_i] + [t^{x_1} \partial, a_{u_i} t^{u_i} \partial_i] = (a_{x_1} + a_{u_i}) t^{x_1+u_i} (\partial(u_i) d_i - d_i(x_1) d_i).
\]
Then we infer that \( d_i(x_1)(\partial - \partial') = (\partial - \partial')(u_i) d_i \). If \( (\partial - \partial')(u_i) \neq 0 \), we deduce that \( d_i \in T_{x_1-z} \). This contradicts the fact that \( x_1 \in A_d^+ \). So we deduce that \( \partial(u_i) = \partial'(u_i) \) for all \( u_i \in A_i \), any \( i \in I \). Thus \( \partial = \partial' \).

By induction we see that (4.24) is true. Define \( \nu \in \text{hom}(A, F) \) so that \( \nu(x) = a_x \) for all \( x \in A_d^+ \). Then we see that \( D|_{S_d^+} = \text{ad} \left( t^y \partial_0 \right)|_{S_d^+} \). By Lemma 4.5 we conclude that \( D = D_\nu \). \( \square \)

We now summarize the results on derivations of \( S_d(A, T, \varphi, z) \) obtained in this section.

**Theorem 4.9.** Every \( D \in \text{Der}(S_d(A, T, \varphi, z)) \) has the form \( D = \sum_{y \in A} D_y \) for degree \( y \) derivations \( D_y \), such that for each \( u \in S_d(A, T, \varphi, z) \) there only finitely many \( y \in A \) with \( D_y(u) \neq 0 \), where

(a) \( D_y = \text{ad} \left( t^y \partial_0 \right) \) for some \( t^y \partial_0 \in (S_d)_y \) if \( y \in A_d \setminus \{ 0, z \} \);
(b) \( D_y = \text{aad} \left( t^y d_1 \right) \) for some \( a \in F \) if \( y = z \in A_d \);
(c) \( D_y = D_\nu \) for some \( \nu \in \text{hom}(A, F) \) if \( y = 0 \).

As in [3, Proposition 3.3], we also have that the sum \( D = \sum_{y \in A} D_y \) in the above Theorem is finite if \( \text{dim} T < \infty \).

**References**


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