CHARACTERIZATION OF THE HOMOGENEOUS POLYNOMIALS $P$ FOR WHICH $(P + Q)(D)$ ADMITS A CONTINUOUS LINEAR RIGHT INVERSE FOR ALL LOWER ORDER PERTURBATIONS $Q$

Rüdiger W. Braun, Reinhold Meise, and B.A. Taylor

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CHARACTERIZATION OF THE HOMOGENEOUS POLYNOMIALS \( P \) FOR WHICH \((P + Q)(D)\) ADMITS A CONTINUOUS LINEAR RIGHT INVERSE FOR ALL LOWER ORDER PERTURBATIONS \( Q \)

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Those homogeneous polynomials \( P \) are characterized for which for arbitrary lower order polynomials \( Q \) the partial differential operator \((P + Q)(D)\) admits a continuous linear right inverse if regarded as an operator from the space of all \( C^\infty \)-functions on \( \mathbb{R}^n \) into itself. It is shown that \( P \) has this property if and only if \( P \) is of principal type and real up to a complex constant and has no elliptic factor.

1. Introduction.

The problem of L. Schwartz to characterize those linear partial differential operators \( P(D) \) with constant coefficients that admit a continuous linear right inverse on \( C^\infty(\Omega) \) or \( \mathcal{D}'(\Omega) \), \( \Omega \) an open set in \( \mathbb{R}^n \), \( n \geq 2 \), was solved in Meise, Taylor, and Vogt [9]. They derived various equivalent conditions for this property. When \( \Omega \) is convex, it is equivalent to a condition PL(\( \Omega, \log \)) of Phragmén-Lindelöf type for plurisubharmonic functions on the algebraic variety

\[
V(P) := \{ z \in \mathbb{C}^n : P(-z) = 0 \}.
\]

Using this characterization they showed in [12], Theorem 4.1, that when \( V(P) \) has PL(\( \Omega, \log \)), then also \( V(P_m) \) has PL(\( \Omega, \log \)), where \( P_m \) denotes the principal part of \( P \), which is a homogeneous polynomial of degree \( m \). In other words, if \( P(D) \) admits a right inverse on \( C^\infty(\Omega) \), so does \( P_m(D) \). The converse implication fails in general, as the example \((\frac{\partial}{\partial x})^2 - (\frac{\partial}{\partial y})^2 + \frac{\partial}{\partial z}\) shows. Since the condition PL(\( \Omega, \log \)) for \( V(P_m) \) is easier to check than for \( V(P) \), one would like to know additional conditions on \( P_m \) which imply that for some or all lower degree perturbations \( Q \) the operator \((P_m + Q)(D)\) admits a right inverse on \( C^\infty(\Omega) \). A first result of this type is Corollary 5.8 of [12] which states the following: If \( P_m \) is homogeneous of degree \( m \), \( \text{grad } P_m(z) \neq 0 \) for all \( z \in \mathbb{C}^n \setminus \{0\} \), and \( V(P_m) \) satisfies PL(\( \mathbb{R}^n, \log \)), then \( V(P_m + Q) \) satisfies PL(\( \mathbb{R}^n, \log \)) for each polynomial \( Q \) of degree less than \( m \).

In the present paper we prove the following extension of this result:
Theorem 1.1. For each polynomial \( P_m \in \mathbb{C}[z_1, \ldots, z_n] \), homogeneous of degree \( m \geq 2 \), the following conditions are equivalent:

1. \((P_m + Q)(D)\) admits a continuous linear right inverse on \( C^\infty(\mathbb{R}^n) \) and/or \( D'(\mathbb{R}^n) \) for each \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) of degree less than \( m \),
2. \( \text{grad} \, P_m(x) \neq 0 \) for each \( x \in \mathbb{R}^n \setminus \{0\} \), \( P_m \) is real up to a complex constant, and each irreducible factor of \( P_m \) has a non-trivial real zero.

In particular, each operator \( P(D) \) of principal type admits a right inverse on \( C^\infty(\mathbb{R}^n) \) and \( D'(\mathbb{R}^n) \) whenever its principal part \( P_m \) is real and no irreducible factor of \( P_m \) is elliptic. Note that these operators \( P(D) \) admit fundamental solutions with large lacunas, as the results of Meise, Taylor, and Vogt \([8, 9]\) imply (see 4.8). Also, Theorem 1.1 proves finally what had been suggested by many examples (see \([12, \text{Example } 4.9, 13, \text{Lemma } 4]\)), namely that the existence of real non-zero singular points in \( V(P_m) \) implies the existence of a perturbation \( Q \) of degree less than \( m \) for which \((P_m + Q)(D)\) does not admit a right inverse on \( C^\infty(\mathbb{R}^n) \).

The proof of Theorem 1.1 in one direction is a modification of the result of Meise, Taylor, and Vogt \([12]\) mentioned above. For the other direction we use the concept of quasihomogeneity of polynomials. We show that this notion together with \([12, \text{Lemma } 4.7]\), provides a systematic method to find necessary conditions for \( V(P_m) \) to satisfy \( \text{PL}(\mathbb{R}^n, \log) \) which can be checked easily and directly on the given polynomial \( P \).

2. Preliminaries.

In this section we introduce some of the definitions that are used in this paper. First we recall the definition of a weight function from \([1]\), then we introduce conditions of Phragmén-Lindelöf type for algebraic varieties according to Meise, Taylor, and Vogt \([9, 11, 12]\) and we explain the significance of these conditions.

Throughout the paper, \(|\cdot|\) will denote the euclidean norm and \( B_\epsilon(z) = \{w \in \mathbb{C}^n : |w - z| < \epsilon\} \) an open ball in that norm. Zero is not a natural number.

**Definition 2.1.** Let \( \omega : [0, \infty[ \to [0, \infty[ \) be continuous and increasing and assume that it has the following properties:

\[
\begin{align*}
(\alpha) \quad & \omega(2t) = O(\omega(t)) \\
(\beta) \quad & \int_1^\infty \frac{\omega(t)}{t^2} \, dt < \infty \\
(\gamma) \quad & \log t = O(\omega(t)), \text{ as } t \text{ tends to infinity} \\
(\delta) \quad & x \mapsto \omega(e^x) \text{ is convex}.
\end{align*}
\]

By abuse of notation, \( \omega : z \mapsto \omega(|z|), z \in \mathbb{C}^n, \) will be called a weight function. Throughout this paper we assume that \( \omega(0) \geq 1 \). It is easy to check that this can be assumed without loss of generality.
Note that each weight function satisfies \( \omega(z) = o(|z|) \). Moreover, each weight function is plurisubharmonic in \( \mathbb{C}^n \) in view of 2.1(δ).

**Definition 2.2.** Let \( V \) be an algebraic variety of pure dimension \( k \) in \( \mathbb{C}^n \) and \( \Omega \) an open subset of \( V \). A function \( u : \Omega \to [-\infty, \infty[ \) will be called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on \( \Omega_{\text{reg}} \), the set of all regular points of \( V \) in \( \Omega \), and satisfies

\[
\limsup_{\xi \in \Omega_{\text{reg}}, \xi \to z} u(\xi) = u(z)
\]

at the singular points of \( V \) in \( \Omega \). By \( \text{PSH}(\Omega) \) we denote the set of all plurisubharmonic functions on \( \Omega \).

**Definition 2.3.** Let \( V \subset \mathbb{C}^n \) be an algebraic variety and let \( \omega \) be a weight function. Then \( V \) satisfies the condition \( \text{PL}(\mathbb{R}^n, \omega) \) if the following holds:

There exists \( A \geq 1 \) such that for each \( \rho > 1 \) there exists \( B > 0 \) such that each \( u \in \text{PSH}(V) \) satisfying (α) and (β) also satisfies (γ), where:

- (α) \( u(z) \leq |\text{Im} z| + O(\omega(z)) \), \( z \in V \),
- (β) \( u(z) \leq \rho |\text{Im} z| \), \( z \in V \),
- (γ) \( u(z) \leq A |\text{Im} z| + B \omega(z) \), \( z \in V \).

**2.4. Phragmén-Lindelöf conditions and continuous linear right inverses.** To explain the significance of the condition \( \text{PL}(\mathbb{R}^n, \omega) \), let \( P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha \) be a complex polynomial of degree \( m > 0 \) and let \( V(P) := \{ z \in \mathbb{C}^n : P(z) = 0 \} \) denote its zero variety. Then \( V(P) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) if and only if the linear partial differential operator

\[
P(D) : \mathcal{E}_\omega(\mathbb{R}^n) \to \mathcal{E}_\omega(\mathbb{R}^n), \quad P(D)f := \sum_{|\alpha| \leq m} a_\alpha i^{-|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}
\]

admits a continuous linear right inverse, where \( \mathcal{E}_\omega(\mathbb{R}^n) \) is the Fréchet space of all \( \omega \)-ultradifferentiable functions of Beurling type (see [1]). This follows from the general characterization in Meise, Taylor, and Vogt [11]. Note that for \( \omega(t) = \log(1+t) \), i.e., \( \mathcal{E}_\omega(\mathbb{R}^n) = C^\infty(\mathbb{R}^n) \), this was obtained earlier in [9] and that Palamodov [15] proved that a differential complex of \( C^\infty \)-functions over \( \mathbb{R}^n \) splits if and only if the associated varieties satisfy \( \text{PL}(\mathbb{R}^n, \log) \).

From Meise, Taylor, and Vogt [12], 4.7, we recall the following lemma which for many examples was the only tool to show that they do not satisfy \( \text{PL}(\mathbb{R}^n, \omega) \) for some weight function \( \omega \).

**Lemma 2.5.** Let \( V \) be an algebraic variety in \( \mathbb{C}^n \) that satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) with constants \( A > 0 \) and \( B_\rho \) for \( \rho > 0 \), according to 2.3. Assume that for some \( M \geq 1 \) and some \( z_0 \in V \) we have \( |\text{Im} z| \leq M|\text{Im} z_0| \) for all \( z \) in the
connected component \( \tilde{V}_{z_0} \) of \( z_0 \) in the set \( V \cap \{ z \in \mathbb{C}^n : |z - z_0| < t|\text{Im} \ z_0| \} \), where \( t \geq 2A + 4 \). Then \( z_0 \) satisfies \( |\text{Im} \ z_0| \leq B(A + 2)M + 1 \omega(z_0) \).

3. Quasihomogeneous Polynomials.

In this section we use the concept of quasihomogeneity together with the lemma of Meise, Taylor, and Vogt \cite{12} stated in 2.5 above to derive conditions on a given polynomial \( P \) which imply that \( V(P) \) fails PL(\( \mathbb{R}^n, \omega \)) for weight functions \( \omega \) which are growing not too fast. These conditions can be checked easily by looking at the powers of the monomials appearing in \( P \).

**Definition 3.1.** For \( d = (d_1, \ldots, d_n) \neq (0, \ldots, 0) \) with \( d_j \in \mathbb{N}_0 \), \( 1 \leq j \leq n \), a polynomial \( P \in \mathbb{C}[z_1, \ldots, z_n] \) is said to be \( d \)-quasihomogeneous of degree \( m \geq 0 \) if

\[
P(z) = \sum_{\langle d, \alpha \rangle = m} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,
\]

where \( \langle d, \alpha \rangle = \sum_{j=1}^n d_j \alpha_j \) and where not all \( a_\alpha \) vanish. The zero polynomial is considered to be \( d \)-quasihomogeneous of degree \(-\infty\).

**Remark.** The concept of quasihomogeneity is widely used in the theory of partial differential operators. We would like to mention, e.g., the theory of semi-elliptic operators (see Hörmander \cite{5}) and the recent books of Gindikin and Volevich \cite{2} and Laurent \cite{6}.

**Lemma 3.2.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be \( d \)-quasihomogeneous of degree \( m > 0 \) and let \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) be a sum of \( d \)-quasihomogeneous polynomials of degrees less than \( m \). Assume further that the following conditions are fulfilled:

1. \( d_1 < d_j \) for \( 2 \leq j \leq n \),
2. there exists \( \zeta = (\zeta_1, \zeta'') \in V(P) \) with \( \zeta_1 \notin \mathbb{R} \), \( \zeta'' \in \mathbb{R}^{n-1} \), and \( \zeta'' \neq 0 \),
3. the polynomial \( \lambda \mapsto P(\lambda, \zeta'') \) does not vanish identically.

If \( V(P + Q) \) satisfies PL(\( \mathbb{R}^n, \omega \)) for some weight function \( \omega \) and \( D = \max\{d_j : \zeta_j \neq 0\} \), then \( \omega \) satisfies \( t^{d_1/D} = O(\omega(t)) \) as \( t \) tends to infinity.

**Proof.** By (2) and (3) we can choose

\[
0 < \delta \leq \frac{1}{4}|\text{Im} \ z_1|
\]

so that \( \zeta_1 \) is the only zero of \( \lambda \mapsto P(\lambda, \zeta_2, \ldots, \zeta_n) \) in the disk \( B_\delta(\zeta_1) \) and that

\[
\eta := \inf_{|\lambda| = \delta} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.
\]
By a compactness argument there exists $\varepsilon_0 > 0$ so that whenever $|z_k - \zeta_k| \leq \varepsilon_0$ for $2 \leq k \leq n$ and $|\lambda| = \delta$ we have

\begin{equation}
|P(\zeta_1 + \lambda, z_2, \ldots, z_n)| \geq \eta/2. \tag{3.1}
\end{equation}

Next fix $R \geq 1$ and let

$$s(\lambda) := \frac{1}{R^m} (P + Q)(R^{d_1}(\zeta_1 + \lambda), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n), \quad \lambda \in \mathbb{C}. $$

By hypothesis, we have $Q = \sum_{k=0}^{m-1} Q_k$ where $Q_k$ is zero or $d$-quasihomogeneous of degree $k$. Since $P$ is $d$-quasihomogeneous of degree $m$, it follows that

$$s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n). $$

Hence there exists $R_0 > 1$ such that for $R \geq R_0$

$$|s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| \leq \eta/4 \quad \text{if } |\lambda| = \delta. $$

Because of this and (3.1), Rouché’s theorem implies that for each $R \geq R_0$ there exists $\lambda(R) \in \mathbb{C}$ satisfying $|\lambda(R)| < \delta$ and $s(\lambda(R)) = 0$. Hence

$$z(R) := (R^{d_1}(\zeta_1 + \lambda(R)), R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n)$$

belongs to $V(P + Q)$. By (2) we have

$$|\text{Im } z(R)| = R^{d_1}|\text{Im}(\zeta_1 + \lambda(R))|. $$

Since $|\lambda(R)| < \delta \leq \frac{1}{4}|\text{Im } \zeta_1|$, we have

\begin{equation}
\frac{3}{4} R^{d_1}|\text{Im } \zeta_1| \leq |\text{Im } z(R)| \leq \frac{5}{4} R^{d_1}|\text{Im } \zeta_1|. \tag{3.2}
\end{equation}

Now assume that $V(P + Q)$ satisfies PL$(\mathbb{R}^n, \omega)$ with constants $A > 0$ and $B_\rho$ for $\rho > 0$, and let $t := 2A + 4$. We claim:

(*) There exist $R_1 \geq R_0$ and $M > 0$ such that for each $R \geq R_1$ and each $z$ in the connected component $\bar{V}(z(R))$ containing $z(R)$ of the set

$$V(P + Q) \cap \{ z \in \mathbb{C}^n : |z - z(R)| < t|\text{Im } z(R)| \}$$

we have $|\text{Im } z| \geq \frac{1}{M}|\text{Im } z(R)|$.

Assume for a moment that this claim is shown. Then it follows from Lemma 2.5 and (3.2) that for some constant $C > 0$ and all $R \geq R_1$ we have

$$\frac{3}{4} R^{d_1}|\text{Im } \zeta_1| \leq |\text{Im } z(R)| \leq C\omega(z(R)).$$

It is no restriction to assume $\zeta_n \neq 0$ and $d_n = D$. Then there exists $C_1 > 0$ such that $|z(R)| \leq C_1 R^D$ for $R \geq R_1$ and hence

$$R^{d_1} \leq C\omega(C_1 R^D).$$
By 2.1(α), this implies $R^{d_1/D} = O(\omega(R))$, as $R$ tends to infinity. Thus the proof of the lemma is complete once we have shown our claim (*). To do so, note that by (1) we can choose $\tilde{R}_1 \geq R_0$ so large that

$$2t|\text{Im} \, \zeta_1| \leq \varepsilon_0 \tilde{R}_1^{d_j - d_1} \quad \text{for} \quad 2 \leq j \leq n.$$ 

Then fix $R \geq \tilde{R}_1$ and define $\pi_{1,R} : \mathbb{C}^n \rightarrow \mathbb{C}$ by $\pi_{1,R}(z) := z_1/R^{d_1}$. Next note that for each $z \in \mathbb{C}^n$ with $|z - z(R)| \leq t|\text{Im} \, z(R)| \leq 2tR^{d_1}|\text{Im} \, \zeta_1|$ its coordinates $z_1, \ldots, z_n$ satisfy

$$|z_1/R^{d_1} - \zeta_1| \leq |z_1/R^{d_1} - \pi_{1,R}(z(R))| + |\lambda(R)| \leq 3t|\text{Im} \, \zeta_1|,$$

$$|z_j/R^{d_j} - \zeta_j| \leq 2t|\text{Im} \, \zeta_1|/R^{d_j - d_1} \leq \varepsilon_0, \quad 2 \leq j \leq n.$$ 

Note further that $K := \{w \in \mathbb{C}^n : |w_1 - \zeta_1| \leq 3t|\text{Im} \, \zeta_1|, \ |w_j - \zeta_j| \leq \varepsilon_0, \ 2 \leq j \leq n\}$ is compact and hence

$$\max_{0 \leq k \leq m-1} \sup_{w \in K} |Q_k(w)| < \infty.$$ 

Therefore we can choose $R_1 \geq \tilde{R}_1$ so large that

$$\sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(w) \leq \eta/4 \quad \text{for each} \quad R \geq R_1 \text{ and } w \in K.$$ 

Next fix $R \geq R_1$ and assume that $z \in \mathbb{C}^n$ satisfies the inequalities in (3.3). Then the $d$-quasihomogeneity properties of $P$ and $Q$ imply

$$1/R^m (P + Q)(z) = P\left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right) + \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k \left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right).$$ 

By (3.1) and (3.3) this implies

$$\left|\frac{1}{R^m} (P + Q)(z)\right| \geq \eta/4 \quad \text{if} \quad \left|\frac{z_1}{R^{d_1}} - \zeta_1\right| = \delta.$$ 

This shows that

$$\pi_{1,R}(\tilde{V}_z(R)) \subset \mathbb{C} \setminus \{\lambda \in \mathbb{C} : |\lambda - \zeta_1| = \delta\}.$$ 

Since $\pi_{1,R}$ is continuous and satisfies $|\pi_{1,R}(z(R)) - \zeta_1| = |\lambda(R)| < \delta$ and since $\tilde{V}_z(R)$ is connected, it follows that

$$\left|\frac{z_1}{R^{d_1}} - \zeta_1\right| < \delta \leq \frac{1}{4}|\text{Im} \, \zeta_1| \quad \text{for each} \quad z \in \tilde{V}_z(R).$$
Hence we have for each $z \in \tilde{V}(R)$
\[
|\text{Im } z| \geq |\text{Im } z_1| \geq |\text{Im } R^{d_1} \pi_1, R(z(R))| - |\text{Im } R^{d_1} \pi_1, R(z(R))| - \text{Im } z_1 | \\
\geq \frac{3}{4} R^{d_1} |\text{Im } \zeta_1| - \frac{1}{4} R^{d_1} |\text{Im } \zeta_1| = \frac{R^{d_1}}{2} |\text{Im } \zeta_1| \geq \frac{2}{5} |\text{Im } z(R)| .
\]
This shows that our claim holds with $M = \frac{5}{2}$. \hfill \Box

**Remark.** Note that the application of Meise, Taylor, and Vogt [12], Lemma 4.7, stated in Lemma 2.5, requires a good understanding of the given variety $V$ in order to find the points $z_0 \in V$ at which one can use this lemma. Lemma 3.2 and also Lemma 3.6 below show that there is a systematic way to find these points in $V(P)$ if $P$ has a non-trivial $d$-quasihomogeneous principal part with certain other properties. Therefore these lemmas are much easier to use than Lemma 2.5. We demonstrate this in the following examples.

**Examples 3.3.**

(a) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_2 z_3.
\]
If $V(P)$ satisfies PL($\mathbb{R}^3, \omega$) for some weight function $\omega$ then $t^{\frac{1}{2}} = O(\omega(t))$ as $t$ tends to infinity. This is an immediate consequence of Lemma 3.2 and the following facts:

(1) $P$ is $(1, 2, 3)$-quasihomogeneous of degree 5

(2) $(\frac{1}{2}(-1 + i \sqrt{3}), 1, 1) \in V(P)$

(3) $P(\lambda, 1, 1) = \lambda^2 + \lambda + 1$.

(b) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_2 z_3.
\]
If $V(P)$ satisfies PL($\mathbb{R}^3, \omega$) for some weight function $\omega$ then $t^{\frac{1}{2}} = O(\omega(t))$ as $t$ tends to infinity. This follows from Lemma 3.2 and the following facts:

(1) $P$ is $(2, 3, 4)$-quasihomogeneous of degree 8

(2) $(i, 0, 1) \in V(P)$

(3) $P(\lambda, 0, 1) = \lambda^2 + 1$.

(c) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as
\[
P(z_1, z_2, z_3) := z_1^2 z_2 - z_2^2.
\]
If $V(P)$ satisfies PL($\mathbb{R}^3, \omega$) for some weight function $\omega$ then $t^{\frac{1}{2}} = O(\omega(t))$ as $t$ tends to infinity. This follows immediately from Lemma 3.2 and the following facts:

(1) $P$ is $(1, 2, 2)$-quasihomogeneous of degree 4

(2) $(i, -1, 1) \in V(P)$
Definition 3.4. Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m$.

(a) The variety $V(P_m)$ satisfies the condition HPL($\mathbb{R}^n$) if there exists $A \geq 1$ such that each $u \in \text{PSH}(V)$ satisfying ($\alpha$) and ($\beta$) also satisfies ($\gamma$), where

- ($\alpha$) $u(z) \leq |z|$, $z \in V(P_m)$,
- ($\beta$) $u(z) \leq 0$, $z \in V(P_m) \cap \mathbb{R}^n$,
- ($\gamma$) $u(z) \leq A|\text{Im } z|$, $z \in V(P_m)$.

(b) The variety $V(P_m)$ satisfies HPL($\mathbb{R}^n$, loc) at $\xi \in V(P_m) \cap \mathbb{R}^n$ if there exist $A \geq 0$ and $0 < r_2 < r_1$ such that each function $u$ which is plurisubharmonic on $V(P_m) \cap B_{r_1}(\xi)$ and satisfies ($\alpha$) and ($\beta$) also satisfies ($\gamma$), where

- ($\alpha$) $0 \leq u \leq 1$ on $V(P_m) \cap B_{r_1}(\xi)$,
- ($\beta$) $u(z) \leq 0$, $z \in V(P_m) \cap \mathbb{R}^n \cap B_{r_1}(\xi)$,
- ($\gamma$) $u(z) \leq A|\text{Im } z|$, $z \in V(P_m) \cap B_{r_2}(\xi)$.

Remark. For $P \in \mathbb{C}[z_1, \ldots, z_n]$ let $P_m$ denote the principal part of $P$. Hörmander has shown in [3] that the operator $P(D) : \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective if and only if $V(P_m)$ satisfies HPL($\mathbb{R}^n$). The latter holds if and only if $V(P_m)$ satisfies HPL($\mathbb{R}^n$, loc) at each $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$.

Example 3.5. Let $P \in \mathbb{C}[z_1, \ldots, z_4]$ be defined as

$$P(z_1, \ldots, z_4) := z_1^2 z_4 - z_2^2 z_3.$$ 

Then $V(P)$ fails PL($\mathbb{R}^4$, $\omega$) for each weight function $\omega$ and $V(P)$ fails HPL($\mathbb{R}^4$). In particular $V(P)$ fails HPL($\mathbb{R}^4$, loc) at some $\xi \in V(P) \cap \mathbb{R}^n$, $|\xi| = 1$.

To show this, note first that $P$ is homogeneous. By Meise, Taylor, and Vogt [12], Theorem 4.1 and Corollary 2.9, this implies that $V(P)$ satisfies PL($\mathbb{R}^4$, $\log$) if and only if $V(P)$ satisfies PL($\mathbb{R}^4$, $\omega$) for each weight function $\omega$. Next note that:

1. $P$ is $(2, 3, 4, 6)$-homogeneous of degree 10
2. $(i, 1, -1, 1) \in V(P)$
3. $P(\lambda, 1, -1, 1) = \lambda^2 + 1$.

Therefore Lemma 3.2 implies that $V(P)$ fails PL($\mathbb{R}^4$, $t^{1/3}$). Hence it also fails PL($\mathbb{R}^4$, $\log$). Since $P$ is irreducible and not elliptic, it follows from [12], Corollary 3.14, that $V(P)$ does not satisfy HPL($\mathbb{R}^4$). Since $V(P)$ satisfies the
dimension condition, \( \dim V(P) \cap \mathbb{R}^n = n - 1 \), Theorem 3.13(4) of [12] shows that \( V(P) \) fails HPL(\( \mathbb{R}^d \), loc) at some \( \xi \in V(P) \cap \mathbb{R}^n, |\xi| = 1 \). Inspection of the proof of Lemma 3.2 shows that \( \xi = \lim_{R \to \infty} z(R)/|z(R)| \). So in the present example, \( \xi = (0, 0, 0, 1) \).

For our application we also need the following variant of Lemma 3.2, which for \( k = 1 \) is weaker than that lemma.

**Lemma 3.6.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be \( d \)-quasihomogeneous of degree \( m \) and let \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) be the sum of \( d \)-quasihomogeneous polynomials of degrees less than \( m \). Assume that for some \( k \), \( 1 \leq k < n \), the following conditions are fulfilled:

1. \( d_1 = \cdots = d_k < d_j \) for \( j > k \),
2. there exists \( \zeta = (\zeta', \zeta'') \in \mathbb{C}^k \times \mathbb{R}^{n-k} \) satisfying \( P(\zeta) = 0 \) and \( \zeta'' \neq 0 \),
3. if \( P(z', \zeta'') = 0 \) then \( \Im z' \neq 0 \).

If \( V(P + Q) \) satisfies PL(\( \mathbb{R}^n, \omega \)) for some weight function \( \omega \) and \( D = \max\{d_j : \zeta_j \neq 0\} \), then \( \omega \) satisfies \( t^{d_j/D} = O(\omega(t)) \) as \( t \) tends to infinity.

**Proof.** From (2) and (3) it follows that the polynomial \( z' \mapsto P(z', \zeta'') \) is not constant. Since the hypotheses are invariant under a real linear change of coordinates in the \( z' \) variables, we may assume that \( z_1 \mapsto P(z_1, \zeta_2, \ldots, \zeta_n) \) is not constant. From this and (2) it follows that we can choose \( 0 < r < \frac{1}{4} |\Im \zeta'| \) so that

\[
\delta := \inf_{|\lambda|=r} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.
\]

For each \( \tau \geq 1 \) we get from (3) that \( P \) does not vanish on the compact set

\[
L(\tau) := \{x \in \mathbb{R}^n : |x_j| \leq \tau, \ 1 \leq j \leq k, \ x_j = \zeta_j, \ k+1 \leq j \leq n\}.
\]

Hence there exists \( \varepsilon = \varepsilon(\tau) > 0 \) such that for \( B_\varepsilon(0) := \{z \in \mathbb{C}^n : |z| \leq \varepsilon\} \) we have

\[
\eta(\tau) := \inf\{|P(z)| : z \in L(\tau) + B_\varepsilon(0)\} > 0.
\]

Next note that by hypothesis we have \( Q = \sum_{k=0}^{m-1} Q_k \), where \( Q_k \) is either zero or \( d \)-quasihomogeneous of degree \( k \). Then fix \( R \geq 1 \) and consider the polynomial

\[
s(\lambda) := \frac{1}{R^m}(P + Q)(R^{d_1}(\zeta_1 + \lambda), \ R^{d_2}(\zeta_2), \ldots, R^{d_n}(\zeta_n)).
\]

Because of our assumptions on \( d \)-quasihomogeneity, we have

\[
s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n).
\]
From this and a standard compactness argument it follows that there exists \( R_0 \geq 1 \) such that for each \( R \geq R_0 \),
\[
\sup_{|\lambda| = r} |s(\lambda) - P(\xi_1 + \lambda, \xi_2, \ldots, \xi_n)| \leq \delta/2.
\]
Since \( P(\xi) = 0 \), our choice of \( \delta \) shows that we can apply Rouché’s theorem to get the existence of a zero \( \lambda(R) \) of \( s \) satisfying
\[
|\lambda(R)| < r \leq \frac{1}{4} |\text{Im} \xi'|.
\]

Now assume that \( V(P + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \omega) \) for some weight function \( \omega \) with constants \( A \geq 1 \) and \( B_\rho > 0 \) for \( \rho > 0 \). Then let \( t := 2A + 4 \) and define for \( R \geq R_0 \)
\[
z(R) := (R^{d_1}(\xi_1 + \lambda(R)), R^{d_2}\xi_2, \ldots, R^{d_n}\xi_n).
\]
By \( V_R \) we denote the set \( V(P + Q) \cap B_{t|\text{Im}z(R)|}(z(R)) \). We claim that the following holds:

(\ast) There exist \( R_2 \geq R_0 \) and \( \sigma > 0 \) such that for \( R \geq R_2 \)
\[
|\text{Im} z| \geq \sigma |\text{Im} z(R)| \quad \text{for each} \quad z \in V_R.
\]
To prove (\ast) note that the choice of \( \lambda(R) \) and \( d_1 = \cdots = d_k \) imply \( z(R) \in V(P + Q) \) and \( |\text{Im} z(R)| = R^{d_1}|\text{Im}(\xi_1 + \lambda(R), \xi_2, \ldots, \xi_k)| \). By the estimate for \( \lambda(R) \) this shows
\[
\frac{3}{4} R^{d_1}|\text{Im} \xi'| \leq |\text{Im} z(R)| \leq \frac{5}{4} R^{d_1}|\text{Im} \xi'|.
\]
For \( z \in V_R \) this implies
\[
\left| \frac{z_j}{R^{d_j}} - \xi_j \right| \leq \frac{t|\text{Im} z(R)|}{R^{d_j}} + |\lambda(R)| \leq 2t|\text{Im} \xi'|,
\]
\[
\left| \frac{z_j}{R^{d_j}} - \xi_j \right| \leq 2t R^{d_1 - d_j}|\text{Im} \xi'|, \quad 2 \leq j \leq n.
\]
Because of this and (1) we can choose \( \tau \geq 1 \) and \( R_1 \geq R_0 \) so that
\[
\left| \frac{z_j}{R^{d_j}} \right| \leq \tau \quad \text{for} \quad 1 \leq j \leq k \quad \text{and} \quad \left| \frac{z_j}{R^{d_j}} - \xi_j \right| \leq \varepsilon(\tau) \quad \text{for} \quad k + 1 \leq j \leq n
\]
whenever \( R \geq R_1 \) and \( z \in V_R \). Next note that
\[
0 = (P + Q)(z)
\]
\[
= R^m P \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) + R^m \sum_{k=0}^{m-1} \frac{1}{R^{m-k} Q_k} \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right)
\]
implies
\[
\left| P \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) \right| = \left| \sum_{k=0}^{m-1} \frac{1}{R^{m-k} Q_k} \left( \frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}} \right) \right| \leq \frac{C}{R}
\]
for some constant \( C \geq 1 \) and all \( z \in V_R, \ R \geq R_1 \). Hence we can choose \( R_2 \geq R_1 \), so that \( \left| P\left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right)\right| < \eta(\tau) \) whenever \( R \geq R_2, \ z \in V_R \). By the definition of \( \eta(\tau) \), this implies

\[
R^{-d_1} |\text{Im} \ z'| = \left|\text{Im} \left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right)\right| \geq \varepsilon(\tau)
\]

and consequently, for \( \sigma := \frac{4}{5} \frac{\varepsilon(\tau)}{|\text{Im} \ \zeta'|} \):

\[
|\text{Im} \ z| \geq |\text{Im} \ z'| \geq \varepsilon(\tau) R^{d_1} = \sigma \frac{5}{4} R^{d_1} |\text{Im} \ \zeta'| \geq \sigma |\text{Im} \ z(R)|,
\]

which proves (\(*\)).

From (\(*\)) and Meise, Taylor, and Vogt [12], Lemma 4.7, it follows that there exists \( B > 0 \) such that

\[
|\text{Im} \ z(R)| \leq B \omega(z(R)) \quad \text{for} \quad R \geq R_2.
\]

It is again no restriction to assume \( \zeta_n \neq 0 \) and \( D = d_n \). From this it follows as in the proof of Lemma 3.2 that \( R^{d_1/D} = O(\omega(R)) \) as \( R \) tends to infinity.

As an application, we give a short proof of a result of Meise and Taylor [7], 2.1.

**Corollary 3.7.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be of degree \( m \) and assume that its principal part \( P_m \) is real. Let \( q \in \mathbb{C}[t] \) have degree \( k < m \) and non-real leading coefficient. Set \( Q(z,t) = P(z) + q(t) \) and let \( \omega \) be a weight function with \( \omega(t) = o(t^{k/m}) \). Then \( V(Q) \) does not satisfy \( PL(R^{n+1}, \omega) \).

**Proof.** We apply Lemma 3.6 in \( n + 1 \) variables with \( d_1 = \cdots = d_n = k < m = d_{n+1} \). Let \( b \in \mathbb{C} \setminus \mathbb{R} \) denote the leading coefficient of \( q \). The \( d \)-quasihomogeneous principal part of \( Q \) is \( P_m(z) + bt^k \). Choose \( \zeta' \in \mathbb{C}^n \) with \( P_m(\zeta') = b \) and set \( \zeta'' = -1 \). Then (1), (2), and (3) of Lemma 3.6 are obviously satisfied. The claim follows from that lemma. \( \square \)

### 4. Main Results.

In this section we use the results of the previous one to characterize the homogeneous polynomials \( P_m \) of degree \( m \) in \( n \) variables \((n \geq 2)\) for which \( V(P_m + Q) \) satisfies the condition \( PL(R^n, \log) \) for each perturbation \( Q \) of degree less than \( m \). This will also prove Theorem 1.1. For the proof we need the following lemma, which is a variation of Meise, Taylor, and Vogt [12], Lemma 5.2.

**Lemma 4.1.** For \( P \in \mathbb{C}[z_1, \ldots, z_n] \) denote by \( P_m \) its principal part and assume that \( V(P_m) \) has \( PL(R^n, \log) \), that \( \text{grad} \ P_m(x) \neq 0 \) for \( x \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\}) \), and that for some weight function \( \omega \) the following condition is fulfilled:
For each $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$, there exist $\delta_\xi, C_\xi, R_\xi > 0$ such that 
\[
\text{dist}(\zeta, V(P_m)) \leq C_\xi \omega(\zeta) \text{ whenever } \zeta \in V(P) \text{ satisfies } |\zeta| \geq R_\xi \text{ and } |rac{\zeta}{|\zeta|} - \xi| < \delta_\xi.
\]
Then $V(P)$ satisfies PL($\mathbb{R}^n, \omega$).

The proof of Lemma 4.1 is quite analogous to that of [12], Lemma 5.2. Therefore, we will only sketch its main steps: Since $V(P_m)$ has PL($\mathbb{R}^n, \log$) by hypothesis, it follows from Meise, Taylor, and Vogt [12], Theorem 3.13, and [10], Theorem 5.1, that $V(P)$ satisfies the condition (RPL) of [10], 2.2.

Hence there exists $A_0 \geq 1$ such that for each $\rho > 1$, there exists $B_\rho > 0$ such that each $u \in \text{PSH}(V(P_m))$ satisfying 
\[
u(z) \leq |z| + o(|z|) \text{ and } u(z) \leq \rho |\text{Im } z|, \quad z \in V(P)
\]
also satisfies 
\[
u(z) \leq A_0 |z| + B_\rho, \quad z \in V(P).
\]
This a priori estimate and a compactness argument imply that it suffices to prove the desired Phragmén-Lindelöf estimate for each $\xi \in V(P_m \cap \mathbb{R}^n, |\xi| = 1$, in the intersection of $V(P)$ with some small cone centered around $\xi$ (for the precise argument we refer to the proof of Meise and Taylor [7], 4.5).

Using appropriate coordinates in such cones, these estimates are derived from (4.2) similarly as in the proof of [12], Lemma 5.2.

To state our main result, we recall the following definition from Hörmander [5], 10.4.11.

**Definition 4.2.** $P \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be of principal type if its principal part $P_m$ satisfies 
\[\sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial z_j} (x) \right|^2 \neq 0 \text{ for each } x \in \mathbb{R}^n \setminus \{0\}.\]

Note that by Euler’s rule $\langle x, \text{grad } P_m(x) \rangle = m P_m(x)$, so $P$ is of principal type if and only if 
\[\text{grad } P_m(x) \neq 0 \text{ for each } x \in \mathbb{R}^n \setminus \{0\} \text{ satisfying } P_m(x) = 0.\]

**Theorem 4.3.** Let $n \geq 2$ and let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$. Then the following conditions are equivalent:

1. For each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < m$, the variety $V(P_m + Q)$ satisfies PL($\mathbb{R}^n, \log$),
2. $P_m$ is of principal type, $P_m$ is real up to a complex constant, and each irreducible factor $q$ of $P_m$ has a real zero $\xi \neq 0$.

**Proof.** (1) $\Rightarrow$ (2): By hypothesis, $V(P_m)$ has PL($\mathbb{R}^n, \log$). Hence it follows from Meise, Taylor, and Vogt [12], Theorem 3.13, that $\dim V(q) \cap \mathbb{R}^n = n-1$
for each irreducible factor $q$ of $P_m$. Thus, the third condition in (2) is fulfilled.

To prove that $P_m$ is of principal type, note first that by Meise, Taylor, and Vogt [13], Lemma 2, there exists $\lambda \in \mathbb{C} \setminus \{0\}$ so that $\lambda P_m \in \mathbb{R}[z_1, \ldots, z_n]$. Hence the second condition of (2) is fulfilled and it is no restriction to assume that $P_m$ has real coefficients. To prove that $P_m$ does not vanish on $V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$ we argue by contradiction and assume that there exists $\theta \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$ satisfying $\text{grad } P_m(\theta) = \mathbf{0}$. After a real linear change of variables, we may assume $\theta = e_n = (0, \ldots, 0, 1)$. Then we apply Taylor’s formula at $\theta$ to get

\[
(4.3) \quad P_m(z', 1) = P_m(\theta + (z', 0)) = \sum_{k=\nu}^{m} q_k(z'),
\]

where $q_k \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ is zero or homogeneous of degree $k$ and where $q_\nu \neq 0$. Then $2 \leq \nu \leq m$ since $P_m$ and $\text{grad } P_m$ vanish at $\theta$. By the homogeneity of $P_m$ it follows from 4.3 that for $z_n \neq 0$ we have

\[
P_m(z', z_n) = z_n^m P_m \left( \frac{z'}{z_n}, 1 \right) = z_n^m \sum_{k=\nu}^{m} q_k \left( \frac{z'}{z_n} \right) = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z').
\]

By continuity, this holds also when $z_n = 0$. Now let

\[
P(z) := P_m(z) + i z_n^{m-1} = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z') + i z_n^{m-1}
\]

and $d := (\nu - 1, \ldots, \nu - 1, \nu)$. Then the monomial $z_n^{m-1}$ has $d$-degree $(m-1)\nu$ and the polynomials $z_n^{m-k} q_k(z')$ have $d$-degree $\nu m - k$, so they are decreasing in $k$. Hence the $d$-quasihomogeneous principal part $q$ of $P$ equals

\[
q(z) = z_n^{m-d} q_\nu(z') + i z_n^{m-1}.
\]

To show that $q$ satisfies the hypotheses of Lemma 3.6, we note that $q_\nu \neq 0$ implies the existence of $\zeta' \in \mathbb{C}^{n-1}$ satisfying $q_\nu(\zeta') = -i$. Then $\zeta := (\zeta', 1)$ satisfies

\[
q(\zeta) = q_\nu(\zeta') + i = 0.
\]

Hence the conditions (1) and (2) of Lemma 3.6 are fulfilled. To show that also condition 3.6(3) holds, assume that for some $z' \in \mathbb{C}^{n-1}$ we have

\[
0 = q(z', 1) = q_\nu(z') + i.
\]

Since $q_\nu$ has real coefficients, this implies $z' \not\in \mathbb{R}^{n-1}$, which proves condition 3.6(3). Hence we can apply Lemma 3.6 to conclude that $V(P)$ does not satisfy $\text{PL}(\mathbb{R}^n, \log)$ in contradiction to the hypothesis (1).

$(2) \Rightarrow (1)$: Since $P_m$ is real up to a complex factor, it is no restriction to assume that $P_m$ has real coefficients. By Meise and Taylor [7], Lemma 4.6, the hypothesis implies that $P_m$ is a product of distinct, irreducible factors.
with real coefficients, each of which is of principal type. This implies that $P_m$ is locally hyperbolic at every real characteristic in the sense of Definition 6.4 of Hörmander [3]. Hence it follows from [3], Theorem 6.5, that $V(P_m)$ satisfies HPL($\mathbb{R}^n$). By hypothesis, no irreducible component of $V(P_m)$ is elliptic. Hence $V(P_m)$ satisfies PL($\mathbb{R}^n$, log) by Meise, Taylor, and Vogt [12], Corollary 3.14. Next fix $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < m$, $Q \neq 0$ and choose $C > 0$ such that $|Q(z)| \leq C(1 + |z|^{m-1})$, $z \in \mathbb{C}^n$. By the homogeneity of $P_m$, the function

$$x \mapsto \sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial x_j}(x) \right|$$

is positively homogeneous of degree $m-1$ and does not vanish for $x \in \mathbb{R}^n \setminus \{0\}$ by hypothesis. This implies the existence of $\delta > 0$ and $A \geq 1$ such that

$$1 + |z|^{m-1} \leq A \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \{z \in \mathbb{C}^n : |\text{Im} z| < \delta|z|\} =: \Gamma.$$

Consequently, there exists $A'$ with

$$|Q(z)| \leq A' \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \Gamma.$$

Now fix $z \in \Gamma \cap (V(P_m + Q) \setminus V(P_m))$ and note that by Hörmander [4], Lemma 4.1.1, (which holds also for $\xi \in \mathbb{C}^n$) there exists $D > 0$ such that

$$\text{dist}(\zeta, V(P_m)) \sum_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(\zeta)}{P_m(\zeta)} \right|^{\frac{1}{\alpha}} \leq D, \quad \zeta \in \mathbb{C}^n \setminus V(P_m).$$

This and $P_m(z) = -Q(z)$ imply

$$\frac{1}{A'} \leq \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{Q(z)} \right| = \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{P_m(z)} \right| \leq \max_{\alpha \neq 0} \left( \frac{D}{\text{dist}(z, V(P_m))} \right)^{|\alpha|}$$

and hence the existence of $E > 0$ such that (by continuity)

$$\text{dist}(z, V(P_m)) \leq E, \quad z \in \Gamma \cap V(P_m + Q).$$

From this we get (1) by Lemma 4.1.

\[\square\]

**Remark 4.4.** Note that Theorem 4.3 and its Corollary 4.7 below extend Corollary 5.8 of Meise, Taylor, and Vogt [12]. Moreover, Theorem 4.3 shows that the characterizing condition is in fact weaker than the sufficient condition given there, since $P_m$ can be of principal type, while $V(P_m)$ has complex singularities. To see this, consider $P_4(x, y, z) := (x^2 + y^2 - z^2)(x^2 + z^2 - y^2/4)$ and note that $\{\lambda \cdot (i\sqrt{3/5}, 2\sqrt{2/5}, 1) : \lambda \in \mathbb{C}\}$ is a singular line for $V(P_4)$.

**Remark 4.5.** From Meise and Taylor [7], 4.8 and 3.4, it follows that each real homogeneous polynomial $P_m$ of principal type for which each irreducible
factor has a non-trivial real zero is also stable under certain real perturbations introducing an extra variable. More precisely, the variety 

\( \{ z \in \mathbb{C}^{n+1} : P_m(z_1, \ldots, z_n) = z_{n+1} \} \)

satisfies PL(\( \mathbb{R}^{n+1}, \log \)), provided that \( P_m \) is of real principal type and has no elliptic factor.

In the proof of Theorem 4.3 we used complex polynomials to show that \( V(P_m + Q) \) fails PL(\( \mathbb{R}^n, \log \)) if \( V(P_m) \) satisfies PL(\( \mathbb{R}^n, \log \)) while \( P_m \) is not of principal type. In some cases, such as \( P_2(x, y, z) = x^2 - y^2 \), this is the only possible choice (see Meise, Taylor, and Vogt [12], Example 4.9). However, in other cases, real perturbations can also have the same effect, as the following example shows.

**Example 4.6.** Let \( P(x, y, z) := x^2 z + y z^2 + y z \). The principal part \( P_3(x, y, z) = x^2 z + y z^2 = (x^2 + y) z \) is hyperbolic with respect to \( N = (0, -1, 1) \). Hence \( V(P_3) \) satisfies PL(\( \mathbb{R}^3, \log \)) by Meise, Taylor, and Vogt [9], 3.6, and 4.5 in connection with [12], 2.12. Obviously, \( P_3 \) is not of principal type. By Example 3.3(a), \( V(P) \) does not satisfy PL(\( \mathbb{R}^3, \omega \)) whenever \( \omega(t) = o(t^{1/3}) \).

**Corollary 4.7.** Let \( P_m \in \mathbb{C}[z_1, \ldots, z_n] \) be homogeneous of degree \( m \geq 2 \) and of principal type. Then the following conditions are equivalent:

1. \( V(P_m) \) satisfies PL(\( \mathbb{R}^n, \log \)),
2. \( P_m \) is real up to a complex constant and each irreducible factor \( q \) of \( P_m \) has a real zero \( \xi \neq 0 \),
3. there exist \( k \in \mathbb{N}, Q \in \mathbb{C}[z_1, \ldots, z_n] \) with \( \deg Q < km \), and a weight function \( \omega \) so that \( V(P_m^k + Q) \) satisfies PL(\( \mathbb{R}^n, \omega \)),
4. for each \( k \in \mathbb{N} \) and each \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) with \( \deg Q = l < km \) we have:
   a. if \( l \leq k(m - 1) \), then \( V(P_m^k + Q) \) satisfies PL(\( \mathbb{R}^n, \log \)),
   b. if \( l > k(m - 1) \), then \( V(P_m^k + Q) \) satisfies PL(\( \mathbb{R}^n, 1 + t^{\beta} \)) for \( \beta = 1 + \frac{l}{k} - m \).

**Proof.** (1) \( \Rightarrow \) (2): This follows from Meise, Taylor, and Vogt [13], Lemma 2, and [12], Corollary 3.14.

(2) \( \Rightarrow \) (3): Since \( P_m \) is of principal type, (2) implies that condition 4.3(2) is fulfilled. Hence \( V(P_m) \) satisfies PL(\( \mathbb{R}^n, \log \)) by Theorem 4.3. Thus (3) holds for \( k = 1 \) and \( Q = 0 \).

(3) \( \Rightarrow \) (1): If \( V(P_m^k + Q) \) satisfies PL(\( \mathbb{R}^n, \omega \)) then \( V(P_m^k) = V(P_m) \) satisfies PL(\( \mathbb{R}^n, \log \)) by Meise, Taylor, and Vogt [12], Theorem 4.1.

(4) \( \Rightarrow \) (3): This holds trivially.

(2) \( \Rightarrow \) (4): Since \( P_m \) is of principal type, there exists \( \eta > 0 \) such that

\[ |\text{grad } P_m(z)| > 0 \text{ for } z \in \mathbb{C}^n, |z| = 1 \text{ and } |\text{Im } z| \leq \eta. \]
Therefore standard arguments using homogeneity and compactness imply the existence of \( \delta > 0 \) such that

\[
(4.4) \quad \max_{0 < |\alpha| \leq k} |(P_m^k(\alpha)(z))| \geq \delta |z|^{k(m-1)} \quad \text{for } z \in \mathbb{C}^n, |\text{Im } z| \leq \eta|z|, |z| \geq 1.
\]

Now fix \( Q \in \mathbb{C}[z_1, \ldots, z_n] \) satisfying \( \deg Q \leq k(m - 1) \). Then there exists \( D \geq 1 \) such that

\[
(4.5) \quad |Q(z)| \leq D \max_{|\alpha| > 0} |(P_m^k(\alpha)(z))| \quad \text{for } z \in \mathbb{C}^n, |z| \geq 1, |\text{Im } z| \leq \eta|z|.
\]

Now fix \( \xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1 \), and let

\[
\Gamma(\xi, \eta, 1) := \left\{ z \in \mathbb{C}^n : \frac{|z|}{|\xi|} - \xi \leq \eta, |z| \geq 1 \right\}.
\]

For \( z \in \Gamma(\xi, \eta, 1) \) we have \(|\text{Im } z| \leq \eta|z| \) since \( \xi \) is real. Now fix \( \xi \in V(P_m^k + Q) \cap \Gamma(\xi, \eta, 1) \) satisfying \( P_m(\xi) \neq 0 \). Then \( P_m(\xi) = -Q(\xi) \) and (4.5) imply the existence of \( M \geq 1 \) such that

\[
\frac{1}{M} \leq \sum_{|\alpha| > 0} \left| \frac{(P_m^k(\alpha)(\xi))}{Q(\xi)} \right|^{1/|\alpha|} = \sum_{|\alpha| > 0} \left| \frac{(P_m^k(\alpha)(\xi))}{P_m^k(\xi)} \right|^{1/|\alpha|}.
\]

Since by Hörmander [4], Lemma 4.1.1, there exists \( C \geq 1 \) such that

\[
\text{dist}(\zeta, V(P_m^k)) \sum_{|\alpha| > 0} \left| \frac{(P_m^k(\alpha)(\xi))}{P_m^k(\xi)} \right|^{1/|\alpha|} \leq C
\]

we conclude that

\[
\text{dist}(\zeta, V(P_m^k)) \leq CM \leq CM \log(2 + |\xi|).
\]

Since \( \xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1 \), was chosen arbitrarily, it follows from Lemma 4.1 that \( V(P_m^k + Q) \) satisfies \( \text{PL}(\mathbb{R}^n, \log) \) in this case.

If \( k(m - 1) < l = \deg Q < km \) then \( 0 < \beta := 1 + \frac{l}{k} - m < 1 \) and (4.4) implies the existence of \( \delta > 0 \) such that

\[
\max_{0 < |\alpha| \leq k} \left| (P_m^k(\alpha)(z)) \right| |z|^{\beta|\alpha|} \geq \delta |z|^{k(m-1)-\beta k}
\]

\[
= \delta |z|^{|l|} \text{ if } |z| \geq 1 \text{ and } |\text{Im } z| \leq \eta|z|.
\]

Hence there exists \( D \geq 1 \) such that

\[
|Q(z)| \leq D \max_{|\alpha| > 0} \left| (P_m^k(\alpha)(z)) \right| (1 + |z|^{\beta|\alpha|}) \quad \text{for } z \in \mathbb{C}^n, |\text{Im } z| \leq \eta|z|, |z| \geq 1.
\]

From this it follows as above that for each \( \xi \in V(P_m^k) \cap \mathbb{R}^n, |\xi| = 1 \), there exists \( C_\xi > 0 \) such that

\[
\text{dist}(\zeta, V(P_m^k)) \leq C_\xi (1 + |\xi|^\beta), \quad \zeta \in \Gamma(\xi, \eta, 1).
\]
By Lemma 4.1, this implies that \( V(P^k_m + Q) \) satisfies PL\((\mathbb{R}^n, \omega)\) for \( \omega(t) = 1 + t^3 \), as asserted.

□

Theorem 4.3 in connection with Meise, Taylor, and Vogt [8], Théorème, also implies the following result on the existence of fundamental solutions with large lacunas.

**Corollary 4.8.** Let \( P \in \mathbb{C}[z_1, \ldots, z_n] \) be of principal type and of degree \( m \geq 2 \). If the principal part \( P_m \) of \( P \) is real up to a complex constant and if each irreducible factor \( q \) of \( P_m \) has a real zero \( \xi \neq 0 \) then the following holds:

For each \( r > 0 \) there exists \( R > 0 \) such that for each \( \xi \in \mathbb{R}^n, |\xi| > R \), there exists a fundamental solution \( E_\xi \in \mathcal{D}'(\mathbb{R}^n) \) of \( P(D) \) satisfying \( \text{Supp} \, E_\xi \subset \{ x \in \mathbb{R}^n : |x - \xi| \geq r \} \).

**References**


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Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstrasse 1
40225 Düsseldorf
Germany
E-mail address: braun@cs.uni-duesseldorf.de

Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstrasse 1
40225 Düsseldorf
Germany
E-mail address: meise@cs.uni-duesseldorf.de

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
E-mail address: taylor@math.lsa.umich.edu