Pacific Journal of Mathematics

CHARACTERIZATION OF THE HOMOGENEOUS POLYNOMIALS P FOR WHICH $(P+Q)(D)$ ADMITS A CONTINUOUS LINEAR RIGHT INVERSE FOR ALL LOWER ORDER PERTURBATIONS Q

RÜDIGER W. BRAUN, REINHOLD MEISE, AND B.A. TAYLOR

Volume 192 No. 2 February 2000

PACIFIC JOURNAL OF MATHEMATICS
Vol. 192, No. 2, 2000

CHARACTERIZATION OF THE HOMOGENEOUS POLYNOMIALS P FOR WHICH $(P+Q)(D)$ ADMITS A CONTINUOUS LINEAR RIGHT INVERSE FOR ALL LOWER ORDER PERTURBATIONS Q

Rüdiger W. Braun, Reinhold Meise, and B.A. Taylor

Those homogeneous polynomials P are characterized for which for arbitrary lower order polynomials Q the partial differential operator $(P+Q)(D)$ admits a continuous linear right inverse if regarded as an operator from the space of all C^∞-functions on $\mathbb{R}^n$ into itself. It is shown that P has this property if and only if P is of principal type and real up to a complex constant and has no elliptic factor.

1. Introduction.

The problem of L. Schwartz to characterize those linear partial differential operators $P(D)$ with constant coefficients that admit a continuous linear right inverse on $C^\infty(\Omega)$ or $\mathcal{D}'(\Omega)$, Ω an open set in $\mathbb{R}^n$, $n \geq 2$, was solved in Meise, Taylor, and Vogt [**9**]. They derived various equivalent conditions for this property. When Ω is convex, it is equivalent to a condition $\mathrm{PL}(\Omega, \log)$ of Phragmén-Lindelöf type for plurisubharmonic functions on the algebraic variety

$$V(P) := \{z \in \mathbb{C}^n : P(-z) = 0\}.$$

Using this characterization they showed in [**12**], Theorem 4.1, that when $V(P)$ has $\mathrm{PL}(\Omega, \log)$, then also $V(P_m)$ has $\mathrm{PL}(\Omega, \log)$, where P_m denotes the principal part of P, which is a homogeneous polynomial of degree m. In other words, if $P(D)$ admits a right inverse on $C^\infty(\Omega)$, so does $P_m(D)$. The converse implication fails in general, as the example $(\frac{\partial}{\partial x})^2 - (\frac{\partial}{\partial y})^2 + \frac{\partial}{\partial z}$ shows. Since the condition $\mathrm{PL}(\Omega, \log)$ for $V(P_m)$ is easier to check than for $V(P)$, one would like to know additional conditions on P_m which imply that for some or all lower degree perturbations Q the operator $(P_m+Q)(D)$ admits a right inverse on $C^\infty(\Omega)$. A first result of this type is Corollary 5.8 of [**12**] which states the following: If P_m is homogeneous of degree m, $\operatorname{grad} P_m(z) \neq 0$ for all $z \in \mathbb{C}^n \setminus \{0\}$, and $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$, then $V(P_m + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ for each polynomial Q of degree less than m.

In the present paper we prove the following extension of this result:

Theorem 1.1. *For each polynomial $P_m \in \mathbb{C}[z_1, \dots, z_n]$, homogeneous of degree $m \geq 2$, the following conditions are equivalent*:

(1) $(P_m + Q)(D)$ *admits a continuous linear right inverse on* $C^\infty(\mathbb{R}^n)$ *and/or* $\mathcal{D}'(\mathbb{R}^n)$ *for each* $Q \in \mathbb{C}[z_1, \dots, z_n]$ *of degree less than* m,

(2) $\operatorname{grad} P_m(x) \neq 0$ *for each* $x \in \mathbb{R}^n \setminus \{0\}$, P_m *is real up to a complex constant, and each irreducible factor of* P_m *has a non-trivial real zero.*

In particular, each operator $P(D)$ of principal type admits a right inverse on $C^\infty(\mathbb{R}^n)$ and $\mathcal{D}'(\mathbb{R}^n)$ whenever its principal part P_m is real and no irreducible factor of P_m is elliptic. Note that these operators $P(D)$ admit fundamental solutions with large lacunas, as the results of Meise, Taylor, and Vogt [**8**], [**9**] imply (see 4.8). Also, Theorem 1.1 proves finally what had been suggested by many examples (see [**12**], Example 4.9, [**13**], Lemma 4), namely that the existence of real non-zero singular points in $V(P_m)$ implies the existence of a perturbation Q of degree less than m for which $(P_m + Q)(D)$ does not admit a right inverse on $C^\infty(\mathbb{R}^n)$.

The proof of Theorem 1.1 in one direction is a modification of the result of Meise, Taylor, and Vogt [**12**] mentioned above. For the other direction we use the concept of quasihomogeneity of polynomials. We show that this notion together with [**12**], Lemma 4.7, provides a systematic method to find necessary conditions for $V(P)$ to satisfy $\mathrm{PL}(\mathbb{R}^n, \log)$ which can be checked easily and directly on the given polynomial P.

2. Preliminaries.

In this section we introduce some of the definitions that are used in this paper. First we recall the definition of a weight function from [**1**], then we introduce conditions of Phragmén-Lindelöf type for algebraic varieties according to Meise, Taylor, and Vogt [**9**], [**11**], [**12**] and we explain the significance of these conditions.

Throughout the paper, $|\cdot|$ will denote the euclidean norm and $B_\epsilon(z) = \{w \in \mathbb{C}^n : |w - z| < \epsilon\}$ an open ball in that norm. Zero is not a natural number.

Definition 2.1. Let $\omega : [0, \infty[\to]0, \infty[$ be continuous and increasing and assume that it has the following properties:

(α) $\quad \omega(2t) = O(\omega(t))$ $\qquad$ (β) $\quad \displaystyle\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$

(γ) $\quad \log t = O(\omega(t))$, as t tends to infinity $\qquad$ (δ) $\quad x \mapsto \omega(e^x)$ is convex.

By abuse of notation, $\omega : z \mapsto \omega(|z|), z \in \mathbb{C}^n$, will be called a weight function. Throughout this paper we assume that $\omega(0) \geq 1$. It is easy to check that this can be assumed without loss of generality.

Note that each weight function satisfies $\omega(z) = o(|z|)$. Moreover, each weight function is plurisubharmonic in $\mathbb{C}^n$ in view of 2.1(δ).

Definition 2.2. Let V be an algebraic variety of pure dimension k in $\mathbb{C}^n$ and Ω an open subset of V. A function $u : \Omega \to [-\infty, \infty[$ will be called plurisubharmonic if it is locally bounded above, plurisubharmonic in the usual sense on Ω_{reg}, the set of all regular points of V in Ω, and satisfies

$$u(z) = \limsup_{\xi \in \Omega_{\text{reg}}, \xi \to z} u(\xi)$$

at the singular points of V in Ω. By $\mathrm{PSH}(\Omega)$ we denote the set of all plurisubharmonic functions on Ω.

Definition 2.3. Let $V \subset \mathbb{C}^n$ be an algebraic variety and let ω be a weight function. Then V satisfies the condition $\mathrm{PL}(\mathbb{R}^n, \omega)$ if the following holds:

There exists $A \geq 1$ such that for each $\rho > 1$ there exists $B > 0$ such that each $u \in \mathrm{PSH}(V)$ satisfying (α) and (β) also satisfies (γ), where:

(α) $u(z) \leq |\mathrm{Im}\, z| + O(\omega(z)), \quad z \in V,$
(β) $u(z) \leq \rho|\mathrm{Im}\, z|, \quad z \in V,$
(γ) $u(z) \leq A|\mathrm{Im}\, z| + B\omega(z), \quad z \in V.$

2.4. Phragmén-Lindelöf conditions and continuous linear right inverses. To explain the significance of the condition $\mathrm{PL}(\mathbb{R}^n, \omega)$, let $P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha$ be a complex polynomial of degree $m > 0$ and let

$$V(P) := \{z \in \mathbb{C}^n : P(z) = 0\}$$

denote its zero variety. Then $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ if and only if the linear partial differential operator

$$P(D) : \mathcal{E}_{(\omega)}(\mathbb{R}^n) \to \mathcal{E}_{(\omega)}(\mathbb{R}^n), \quad P(D)f := \sum_{|\alpha| \leq m} a_\alpha i^{-|\alpha|} \frac{\partial^{|\alpha|} f}{\partial x^\alpha}$$

admits a continuous linear right inverse, where $\mathcal{E}_{(\omega)}(\mathbb{R}^n)$ is the Fréchet space of all ω-ultradifferentiable functions of Beurling type (see [**1**]). This follows from the general characterization in Meise, Taylor, and Vogt [**11**]. Note that for $\omega(t) = \log(1+t)$, i.e., $\mathcal{E}_{(\omega)}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$, this was obtained earlier in [**9**] and that Palamodov [**15**] proved that a differential complex of C^∞-functions over $\mathbb{R}^n$ splits if and only if the associated varieties satisfy $\mathrm{PL}(\mathbb{R}^n, \log)$.

From Meise, Taylor, and Vogt [**12**], 4.7, we recall the following lemma which for many examples was the only tool to show that they do not satisfy $\mathrm{PL}(\mathbb{R}^n, \omega)$ for some weight function ω.

Lemma 2.5. *Let V be an algebraic variety in $\mathbb{C}^n$ that satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega)$ *with constants $A > 0$ and B_ρ for $\rho > 0$, according to* 2.3. *Assume that for some $M \geq 1$ and some $z_0 \in V$ we have $|\mathrm{Im}\, z| \leq M|\mathrm{Im}\, z_0|$ for all z in the* *connected component* $\widetilde{V}_{z_0}$ *of* z_0 *in the set* $V \cap \{z \in \mathbb{C}^n : |z - z_0| < t|\mathrm{Im}\, z_0|\}$, *where* $t \geq 2A + 4$. *Then* z_0 *satisfies* $|\mathrm{Im}\, z_0| \leq B_{(A+2)M+1}\omega(z_0)$.

3. Quasihomogeneous Polynomials.

In this section we use the concept of quasihomogeneity together with the lemma of Meise, Taylor, and Vogt [**12**] stated in 2.5 above to derive conditions on a given polynomial P which imply that $V(P)$ fails $\mathrm{PL}(\mathbb{R}^n, \omega)$ for weight functions ω which are growing not too fast. These conditions can be checked easily by looking at the powers of the monomials appearing in P.

Definition 3.1. For $d = (d_1, \ldots, d_n) \neq (0, \ldots, 0)$ with $d_j \in \mathbb{N}_0$, $1 \leq j \leq n$, a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ is said to be d-quasihomogeneous of degree $m \geq 0$ if

$$P(z) = \sum_{\langle d, \alpha \rangle = m} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

where $\langle d, \alpha \rangle = \sum_{j=1}^n d_j \alpha_j$ and where not all a_α vanish. The zero polynomial is considered to be d-quasihomogeneous of degree $-\infty$.

Remark. The concept of quasihomogeneity is widely used in the theory of partial differential operators. We would like to mention, e.g., the theory of semi-elliptic operators (see Hörmander [**5**]) and the recent books of Gindikin and Volevich [**2**] and Laurent [**6**].

Lemma 3.2. *Let* $P \in \mathbb{C}[z_1, \ldots, z_n]$ *be* d*-quasihomogeneous of degree* $m > 0$ *and let* $Q \in \mathbb{C}[z_1, \ldots, z_n]$ *be a sum of* d*-quasihomogeneous polynomials of degrees less than* m. *Assume further that the following conditions are fulfilled*:

(1) $d_1 < d_j$ *for* $2 \leq j \leq n$,
(2) *there exists* $\zeta = (\zeta_1, \zeta'') \in V(P)$ *with* $\zeta_1 \notin \mathbb{R}$, $\zeta'' \in \mathbb{R}^{n-1}$, *and* $\zeta'' \neq 0$,
(3) *the polynomial* $\lambda \mapsto P(\lambda, \zeta'')$ *does not vanish identically.*

If $V(P + Q)$ *satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega)$ *for some weight function* ω *and* $D = \max\{d_j : \zeta_j \neq 0\}$, *then* ω *satisfies* $t^{d_1/D} = O(\omega(t))$ *as* t *tends to infinity.*

Proof. By (2) and (3) we can choose

$$0 < \delta \leq \frac{1}{4}|\mathrm{Im}\, \zeta_1|$$

so that ζ_1 is the only zero of $\lambda \mapsto P(\lambda, \zeta_2, \ldots, \zeta_n)$ in the disk $\overline{B_\delta(\zeta_1)}$ and that

$$\eta := \inf_{|\lambda| = \delta} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.$$

By a compactness argument there exists $\varepsilon_0 > 0$ so that whenever $|z_k - \zeta_k| \le \varepsilon_0$ for $2 \le k \le n$ and $|\lambda| = \delta$ we have

$$|P(\zeta_1 + \lambda, z_2, \dots, z_n)| \ge \eta/2. \tag{3.1}$$

Next fix $R \ge 1$ and let

$$s(\lambda) := \frac{1}{R^m}(P+Q)(R^{d_1}(\zeta_1 + \lambda),\ R^{d_2}\zeta_2, \dots, R^{d_n}\zeta_n), \quad \lambda \in \mathbb{C}.$$

By hypothesis, we have $Q = \sum_{k=0}^{m-1} Q_k$ where Q_k is zero or d-quasihomogeneous of degree k. Since P is d-quasihomogeneous of degree m, it follows that

$$s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \dots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(\zeta_1 + \lambda, \zeta_2, \dots, \zeta_n).$$

Hence there exists $R_0 > 1$ such that for $R \ge R_0$

$$|s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \dots, \zeta_n)| \le \eta/4 \quad \text{if } |\lambda| = \delta.$$

Because of this and (3.1), Rouché's theorem implies that for each $R \ge R_0$ there exists $\lambda(R) \in \mathbb{C}$ satisfying $|\lambda(R)| < \delta$ and $s(\lambda(R)) = 0$. Hence

$$z(R) := (R^{d_1}(\zeta_1 + \lambda(R)),\ R^{d_2}\zeta_2, \dots, R^{d_n}\zeta_n)$$

belongs to $V(P+Q)$. By (2) we have

$$|\operatorname{Im} z(R)| = R^{d_1}|\operatorname{Im}(\zeta_1 + \lambda(R))|.$$

Since $|\lambda(R)| < \delta \le \frac{1}{4}|\operatorname{Im}\zeta_1|$, we have

$$\frac{3}{4}R^{d_1}|\operatorname{Im}\zeta_1| \le |\operatorname{Im} z(R)| \le \frac{5}{4}R^{d_1}|\operatorname{Im}\zeta_1|. \tag{3.2}$$

Now assume that $V(P+Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ with constants $A > 0$ and B_ρ for $\rho > 0$, and let $t := 2A + 4$. We claim:

($*$) There exist $R_1 \ge R_0$ and $M > 0$ such that for each $R \ge R_1$ and each z in the connected component $\widetilde{V}_{z(R)}$ containing $z(R)$ of the set

$$V(P+Q) \cap \{z \in \mathbb{C}^n : |z - z(R)| < t|\operatorname{Im} z(R)|\}$$

we have $|\operatorname{Im} z| \ge \frac{1}{M}|\operatorname{Im} z(R)|$.

Assume for a moment that this claim is shown. Then it follows from Lemma 2.5 and (3.2) that for some constant $C > 0$ and all $R \ge R_1$ we have

$$\frac{3}{4}R^{d_1}|\operatorname{Im}\zeta_1| \le |\operatorname{Im} z(R)| \le C\omega(z(R)).$$

It is no restriction to assume $\zeta_n \ne 0$ and $d_n = D$. Then there exists $C_1 > 0$ such that $|z(R)| \le C_1 R^D$ for $R \ge R_1$ and hence

$$R^{d_1} \le C\omega(C_1 R^D).$$

By 2.1(α), this implies $R^{d_1/D} = O(\omega(R))$, as R tends to infinity. Thus the proof of the lemma is complete once we have shown our claim $(*)$. To do so, note that by (1) we can choose $\widetilde{R}_1 \geq R_0$ so large that

$$2t|\mathrm{Im}\,\zeta_1| \leq \varepsilon_0 \widetilde{R}_1^{d_j - d_1} \quad \text{for} \quad 2 \leq j \leq n.$$

Then fix $R \geq \widetilde{R}_1$ and define $\pi_{1,R} : \mathbb{C}^n \to \mathbb{C}$ by $\pi_{1,R}(z) := z_1/R^{d_1}$. Next note that for each $z \in \mathbb{C}^n$ with $|z - z(R)| \leq t|\mathrm{Im}\, z(R)| \leq 2tR^{d_1}|\mathrm{Im}\,\zeta_1|$ its coordinates $z_1, \dots, z_n$ satisfy

$$\begin{aligned} &\left|\frac{z_1}{R^{d_1}} - \zeta_1\right| \leq \left|\frac{z_1}{R^{d_1}} - \pi_{1,R}(z(R))\right| + |\lambda(R)| \leq 3t|\mathrm{Im}\,\zeta_1|, \\ &\left|\frac{z_j}{R^{d_j}} - \zeta_j\right| \leq \frac{2t|\mathrm{Im}\,\zeta_1|}{R^{d_j - d_1}} \leq \varepsilon_0\,, \quad 2 \leq j \leq n. \end{aligned} \tag{3.3}$$

Note further that

$$K := \{w \in \mathbb{C}^n : |w_1 - \zeta_1| \leq 3t|\mathrm{Im}\,\zeta_1|,\ |w_j - \zeta_j| \leq \varepsilon_0,\ 2 \leq j \leq n\}$$

is compact and hence

$$\max_{0 \leq k \leq m-1} \sup_{w \in K} |Q_k(w)| < \infty.$$

Therefore we can choose $R_1 \geq \widetilde{R}_1$ so large that

$$\left|\sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(w)\right| \leq \eta/4 \quad \text{for each} \quad R \geq R_1 \text{ and } w \in K. \tag{3.4}$$

Next fix $R \geq R_1$ and assume that $z \in \mathbb{C}^n$ satisfies the inequalities in (3.3). Then the d-quasihomogeneity properties of P and Q imply

$$\frac{1}{R^m}(P+Q)(z) = P\left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right) + \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k\left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right).$$

By (3.1) and (3.3) this implies

$$\left|\frac{1}{R^m}(P+Q)(z)\right| \geq \eta/4 \quad \text{if} \quad \left|\frac{z_1}{R^{d_1}} - \zeta_1\right| = \delta.$$

This shows that

$$\pi_{1,R}(\widetilde{V}_{z(R)}) \subset \mathbb{C} \setminus \{\lambda \in \mathbb{C} : |\lambda - \zeta_1| = \delta\}.$$

Since $\pi_{1,R}$ is continuous and satisfies $|\pi_{1,R}(z(R)) - \zeta_1| = |\lambda(R)| < \delta$ and since $\widetilde{V}_{z(R)}$ is connected, it follows that

$$\left|\frac{z_1}{R^{d_1}} - \zeta_1\right| < \delta \leq \frac{1}{4}|\mathrm{Im}\,\zeta_1| \quad \text{for each } z \in \widetilde{V}_{z(R)}.$$

Hence we have for each $z \in \widetilde{V}_{z(R)}$

$$|\operatorname{Im} z| \geq |\operatorname{Im} z_1| \geq |\operatorname{Im} R^{d_1}\pi_{1,R}(z(R))| - |\operatorname{Im} R^{d_1}\pi_{1,R}(z(R)) - \operatorname{Im} z_1|$$

$$\geq \frac{3}{4}R^{d_1}|\operatorname{Im}\zeta_1| - \frac{1}{4}R^{d_1}|\operatorname{Im}\zeta_1| = \frac{R^{d_1}}{2}|\operatorname{Im}\zeta_1| \geq \frac{2}{5}|\operatorname{Im} z(R)|.$$

This shows that our claim holds with $M = \frac{5}{2}$. □

Remark. Note that the application of Meise, Taylor, and Vogt [**12**], Lemma 4.7, stated in Lemma 2.5, requires a good understanding of the given variety V in order to find the points $z_0 \in V$ at which one can use this lemma. Lemma 3.2 and also Lemma 3.6 below show that there is a systematic way to find these points in $V(P)$ if P has a non-trivial d-quasihomogeneous principal part with certain other properties. Therefore these lemmas are much easier to use than Lemma 2.5. We demonstrate this in the following examples.

Examples 3.3.

(a) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as

$$P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_2 z_3.$$

If $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega)$ for some weight function ω then $t^{\frac{1}{3}} = O(\omega(t))$ as t tends to infinity. This is an immediate consequence of Lemma 3.2 and the following facts:

(1) P is (1, 2, 3)-quasihomogeneous of degree 5
(2) $(\frac{1}{2}(-1 + i\sqrt{3}), 1, 1) \in V(P)$
(3) $P(\lambda, 1, 1) = \lambda^2 + \lambda + 1$.

(b) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as

$$P(z_1, z_2, z_3) := z_1^2 z_3 + z_1 z_2^2 + z_3^2.$$

If $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega)$ for some weight function ω then $t^{\frac{1}{2}} = O(\omega(t))$ as t tends to infinity. This follows from Lemma 3.2 and the following facts:

(1) P is (2, 3, 4)-quasihomogeneous of degree 8
(2) $(i, 0, 1) \in V(P)$
(3) $P(\lambda, 0, 1) = \lambda^2 + 1$.

(c) Let $P \in \mathbb{C}[z_1, z_2, z_3]$ be defined as

$$P(z_1, z_2, z_3) := z_1^2 z_2 - z_3^2.$$

If $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \omega)$ for some weight function ω then $t^{\frac{1}{2}} = O(\omega(t))$ as t tends to infinity. This follows immediately from Lemma 3.2 and the following facts:

(1) P is $(1, 2, 2)$-quasihomogeneous of degree 4
(2) $(i, -1, 1) \in V(P)$

(3) $P(\lambda, 1, -1) = -(\lambda^2 + 1)$.

To indicate that Lemma 3.2 can also be used to disprove conditions of Phragmén-Lindelöf type for homogeneous polynomials which have not been considered so far, we next recall the condition introduced by Hörmander [**3**] to characterize the differential operators $P(D)$ that are surjective on the space $\mathcal{A}(\Omega)$ of all real-analytic functions on a convex open set $\Omega \subset \mathbb{R}^n, n \geq 2$. We restrict our attention here to the case $\Omega = \mathbb{R}^n$.

Definition 3.4. Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree m.

(a) The variety $V(P_m)$ satisfies the condition $\mathrm{HPL}(\mathbb{R}^n)$ if there exists $A \geq 1$ such that each $u \in \mathrm{PSH}(V)$ satisfying (α) and (β) also satisfies (γ), where

(α) $u(z) \leq |z|, \quad z \in V(P_m)$,
(β) $u(z) \leq 0, \quad z \in V(P_m) \cap \mathbb{R}^n$,
(γ) $u(z) \leq A|\mathrm{Im}\, z|, \quad z \in V(P_m)$.

(b) The variety $V(P_m)$ satisfies $\mathrm{HPL}(\mathbb{R}^n, \mathrm{loc})$ at $\xi \in V(P_m) \cap \mathbb{R}^n$ if there exist $A \geq 0$ and $0 < r_2 < r_1$ such that each function u which is plurisubharmonic on $V(P_m) \cap B_{r_1}(\xi)$ and satisfies (α) and (β) also satisfies (γ), where

(α) $0 \leq u \leq 1$ on $V(P_m) \cap B_{r_1}(\xi)$,
(β) $u(z) \leq 0, \quad z \in V(P_m) \cap \mathbb{R}^n \cap B_{r_1}(\xi)$,
(γ) $u(z) \leq A|\mathrm{Im}\, z|, \quad z \in V(P_m) \cap B_{r_2}(\xi)$.

Remark. For $P \in \mathbb{C}[z_1, \ldots, z_n]$ let P_m denote the principal part of P. Hörmander has shown in [**3**] that the operator $P(D)\colon \mathcal{A}(\mathbb{R}^n) \to \mathcal{A}(\mathbb{R}^n)$ is surjective if and only if $V(P_m)$ satisfies $\mathrm{HPL}(\mathbb{R}^n)$. The latter holds if and only if $V(P_m)$ satisfies $\mathrm{HPL}(\mathbb{R}^n, \mathrm{loc})$ at each $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$.

Example 3.5. Let $P \in \mathbb{C}[z_1, \ldots, z_4]$ be defined as

$$P(z_1, \ldots, z_4) := z_1^2 z_4 - z_2^2 z_3.$$

Then $V(P)$ fails $\mathrm{PL}(\mathbb{R}^4, \omega)$ for each weight function ω and $V(P)$ fails $\mathrm{HPL}(\mathbb{R}^4)$. In particular $V(P)$ fails $\mathrm{HPL}(\mathbb{R}^4, \mathrm{loc})$ at some $\xi \in V(P) \cap \mathbb{R}^n$, $|\xi| = 1$.

To show this, note first that P is homogeneous. By Meise, Taylor, and Vogt [**12**], Theorem 4.1 and Corollary 2.9, this implies that $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^4, \log)$ if and only if $V(P)$ satisfies $\mathrm{PL}(\mathbb{R}^4, \omega)$ for each weight function ω. Next note that:

(1) P is (2, 3, 4, 6)-homogeneous of degree 10
(2) $(i, 1, -1, 1) \in V(P)$
(3) $P(\lambda, 1, -1, 1) = \lambda^2 + 1$.

Therefore Lemma 3.2 implies that $V(P)$ fails $\mathrm{PL}(\mathbb{R}^4, t^{1/3})$. Hence it also fails $\mathrm{PL}(\mathbb{R}^4, \log)$. Since P is irreducible and not elliptic, it follows from [**12**], Corollary 3.14, that $V(P)$ does not satisfy $\mathrm{HPL}(\mathbb{R}^4)$. Since $V(P)$ satisfies the dimension condition, $\dim V(P)\cap\mathbb{R}^n = n-1$, Theorem 3.13(4) of [**12**] shows that $V(P)$ fails $\mathrm{HPL}(\mathbb{R}^4, \mathrm{loc})$ at some $\xi \in V(P)\cap\mathbb{R}^n, |\xi| = 1$. Inspection of the proof of Lemma 3.2 shows that $\xi = \lim_{R\to\infty} z(R)/|z(R)|$. So in the present example, $\xi = (0, 0, 0, 1)$.

For our application we also need the following variant of Lemma 3.2, which for $k = 1$ is weaker than that lemma.

Lemma 3.6. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be d-quasihomogeneous of degree m and let $Q \in \mathbb{C}[z_1, \ldots, z_n]$ be the sum of d-quasihomogeneous polynomials of degrees less than m. Assume that for some k, $1 \le k < n$, the following conditions are fulfilled*:

1. *$d_1 = \cdots = d_k < d_j$ for $j > k$,*
2. *there exists $\zeta = (\zeta', \zeta'') \in \mathbb{C}^k \times \mathbb{R}^{n-k}$ satisfying $P(\zeta) = 0$ and $\zeta'' \ne 0$,*
3. *if $P(z', \zeta'') = 0$ then $\operatorname{Im} z' \ne 0$.*

If $V(P + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for some weight function ω and $D = \max\{d_j : \zeta_j \ne 0\}$, then ω satisfies $t^{d_1/D} = O(\omega(t))$ as t tends to infinity.

Proof. From (2) and (3) it follows that the polynomial $z' \mapsto P(z', \zeta'')$ is not constant. Since the hypotheses are invariant under a real linear change of coordinates in the z' variables, we may assume that $z_1 \mapsto P(z_1, \zeta_2, \ldots, \zeta_n)$ is not constant. From this and (2) it follows that we can choose $0 < r < \frac{1}{4}|\operatorname{Im}\zeta'|$ so that

$$\delta := \inf_{|\lambda|=r} |P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n)| > 0.$$

For each $\tau \ge 1$ we get from (3) that P does not vanish on the compact set

$$L(\tau) := \{x \in \mathbb{R}^n : |x_j| \le \tau,\ 1 \le j \le k,\ x_j = \zeta_j,\ k + 1 \le j \le n\}.$$

Hence there exists $\varepsilon = \varepsilon(\tau) > 0$ such that for $B_\varepsilon(0) := \{z \in \mathbb{C}^n : |z| \le \varepsilon\}$ we have

$$\eta(\tau) := \inf\{|P(z)| : z \in L(\tau) + B_{\varepsilon(\tau)}(0)\} > 0.$$

Next note that by hypothesis we have $Q = \sum_{k=0}^{m-1} Q_k$, where Q_k is either zero or d-quasihomogeneous of degree k. Then fix $R \ge 1$ and consider the polynomial

$$s(\lambda) := \frac{1}{R^m}(P + Q)(R^{d_1}(\zeta_1 + \lambda),\ R^{d_2}\zeta_2, \ldots, R^{d_n}\zeta_n).$$

Because of our assumptions on d-quasihomogeneity, we have

$$s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n) = \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k(\zeta_1 + \lambda, \zeta_2, \ldots, \zeta_n).$$

From this and a standard compactness argument it follows that there exists $R_0 \geq 1$ such that for each $R \geq R_0$,

$$\sup_{|\lambda|=r} |s(\lambda) - P(\zeta_1 + \lambda, \zeta_2, \dots, \zeta_n)| \leq \delta/2.$$

Since $P(\zeta) = 0$, our choice of δ shows that we can apply Rouché's theorem to get the existence of a zero $\lambda(R)$ of s satisfying

$$|\lambda(R)| < r \leq \frac{1}{4} |\operatorname{Im} \zeta'|.$$

Now assume that $V(P+Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for some weight function ω with constants $A \geq 1$ and $B_\rho > 0$ for $\rho > 0$. Then let $t := 2A + 4$ and define for $R \geq R_0$

$$z(R) := (R^{d_1}(\zeta_1 + \lambda(R)),\ R^{d_2}\zeta_2, \dots, R^{d_n}\zeta_n).$$

By V_R we denote the set $V(P+Q) \cap B_{t|\operatorname{Im} z(R)|}(z(R))$. We claim that the following holds:

($*$) There exist $R_2 \geq R_0$ and $\sigma > 0$ such that for $R \geq R_2$

$$|\operatorname{Im} z| \geq \sigma |\operatorname{Im} z(R)| \text{ for each } z \in V_R.$$

To prove ($*$) note that the choice of $\lambda(R)$ and $d_1 = \dots = d_k$ imply $z(R) \in V(P+Q)$ and $|\operatorname{Im} z(R)| = R^{d_1} |\operatorname{Im}(\zeta_1 + \lambda(R), \zeta_2, \dots, \zeta_k)|$. By the estimate for $\lambda(R)$ this shows

$$\frac{3}{4} R^{d_1} |\operatorname{Im} \zeta'| \leq |\operatorname{Im} z(R)| \leq \frac{5}{4} R^{d_1} |\operatorname{Im} \zeta'|.$$

For $z \in V_R$ this implies

$$\left| \frac{z_1}{R^{d_1}} - \zeta_1 \right| \leq \frac{t |\operatorname{Im} z(R)|}{R^{d_1}} + |\lambda(R)| \leq 2t |\operatorname{Im} \zeta'|,$$

$$\left| \frac{z_j}{R^{d_j}} - \zeta_j \right| \leq 2t R^{d_1 - d_j} |\operatorname{Im} \zeta'|, \quad 2 \leq j \leq n.$$

Because of this and (1) we can choose $\tau \geq 1$ and $R_1 \geq R_0$ so that

$$\left| \frac{z_j}{R^{d_j}} \right| \leq \tau \text{ for } 1 \leq j \leq k \quad \text{and} \quad \left| \frac{z_j}{R^{d_j}} - \zeta_j \right| \leq \varepsilon(\tau) \text{ for } k+1 \leq j \leq n$$

whenever $R \geq R_1$ and $z \in V_R$. Next note that

$$\begin{aligned} 0 &= (P+Q)(z) \\ &= R^m P\left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right) + R^m \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k \left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right) \end{aligned}$$

implies

$$\left| P\left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right) \right| = \left| \sum_{k=0}^{m-1} \frac{1}{R^{m-k}} Q_k \left(\frac{z_1}{R^{d_1}}, \dots, \frac{z_n}{R^{d_n}}\right) \right| \leq \frac{C}{R}$$

for some constant $C \geq 1$ and all $z \in V_R$, $R \geq R_1$. Hence we can choose $R_2 \geq R_1$, so that $\left|P\left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right)\right| < \eta(\tau)$ whenever $R \geq R_2$, $z \in V_R$. By the definition of $\eta(\tau)$, this implies

$$R^{-d_1}|\operatorname{Im} z'| = \left|\operatorname{Im}\left(\frac{z_1}{R^{d_1}}, \ldots, \frac{z_n}{R^{d_n}}\right)\right| \geq \varepsilon(\tau)$$

and consequently, for $\sigma := \frac{4}{5}\frac{\varepsilon(\tau)}{|\operatorname{Im}\zeta'|}$:

$$|\operatorname{Im} z| \geq |\operatorname{Im} z'| \geq \varepsilon(\tau)R^{d_1} = \sigma\frac{5}{4}R^{d_1}|\operatorname{Im}\zeta'| \geq \sigma|\operatorname{Im} z(R)|,$$

which proves $(*)$.

From $(*)$ and Meise, Taylor, and Vogt [**12**], Lemma 4.7, it follows that there exists $B > 0$ such that

$$|\operatorname{Im} z(R)| \leq B\omega(z(R)) \text{ for } R \geq R_2.$$

It is again no restriction to assume $\zeta_n \neq 0$ and $D = d_n$. From this it follows as in the proof of Lemma 3.2 that $R^{d_1/D} = O(\omega(R))$ as R tends to infinity. □

As an application, we give a short proof of a result of Meise and Taylor [**7**], 2.1.

Corollary 3.7. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be of degree m and assume that its principal part P_m is real. Let $q \in \mathbb{C}[t]$ have degree $k < m$ and non-real leading coefficient. Set $Q(z,t) = P(z) + q(t)$ and let ω be a weight function with $\omega(t) = o(t^{k/m})$. Then $V(Q)$ does not satisfy* $\mathrm{PL}(\mathbb{R}^{n+1}, \omega)$.

Proof. We apply Lemma 3.6 in $n+1$ variables with $d_1 = \cdots = d_n = k < m = d_{n+1}$. Let $b \in \mathbb{C} \setminus \mathbb{R}$ denote the leading coefficient of q. The d-quasihomogeneous principal part of Q is $P_m(z) + bt^k$. Choose $\zeta' \in \mathbb{C}^n$ with $P_m(\zeta') = b$ and set $\zeta'' = -1$. Then (1), (2), and (3) of Lemma 3.6 are obviously satisfied. The claim follows from that lemma. □

4. Main Results.

In this section we use the results of the previous one to characterize the homogeneous polynomials P_m of degree m in n variables ($n \geq 2$) for which $V(P_m + Q)$ satisfies the condition $\mathrm{PL}(\mathbb{R}^n, \log)$ for each perturbation Q of degree less than m. This will also prove Theorem 1.1. For the proof we need the following lemma, which is a variation of Meise, Taylor, and Vogt [**12**], Lemma 5.2.

Lemma 4.1. *For $P \in \mathbb{C}[z_1, \ldots, z_n]$ denote by P_m its principal part and assume that $V(P_m)$ has* $\mathrm{PL}(\mathbb{R}^n, \log)$, *that* $\operatorname{grad} P_m(x) \neq 0$ *for $x \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$, and that for some weight function ω the following condition is fulfilled:*

(C) *For each* $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$, *there exist* $\delta_\xi, C_\xi, R_\xi > 0$ *such that* $\operatorname{dist}(\zeta, V(P_m)) \le C_\xi \omega(\zeta)$ *whenever* $\zeta \in V(P)$ *satisfies* $|\zeta| \ge R_\xi$ *and* $|\frac{\zeta}{|\zeta|} - \xi| < \delta_\xi$.

Then $V(P)$ *satisfies* $\mathrm{PL}(\mathbb{R}^n, \omega)$.

The proof of Lemma 4.1 is quite analogous to that of [**12**], Lemma 5.2. Therefore, we will only sketch its main steps: Since $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^n, \log)$ by hypothesis, it follows from Meise, Taylor, and Vogt [**12**], Theorem 3.13, and [**10**], Theorem 5.1, that $V(P)$ satisfies the condition (RPL) of [**10**], 2.2. Hence there exists $A_0 \ge 1$ such that for each $\rho > 1$, there exists $B_\rho > 0$ such that each $u \in \mathrm{PSH}(V(P))$ satisfying

$$u(z) \le |z| + o(|z|) \quad \text{and} \quad u(z) \le \rho |\operatorname{Im} z|, \quad z \in V(P) \tag{4.1}$$

also satisfies

$$u(z) \le A_0|z| + B_\rho, \quad z \in V(P). \tag{4.2}$$

This a priori estimate and a compactness argument imply that it suffices to prove the desired Phragmén-Lindelöf estimate for each $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$, in the intersection of $V(P)$ with some small cone centered around ξ (for the precise argument we refer to the proof of Meise and Taylor [**7**], 4.5). Using appropriate coordinates in such cones, these estimates are derived from (4.2) similarly as in the proof of [**12**], Lemma 5.2.

To state our main result, we recall the following definition from Hörmander [**5**], 10.4.11.

Definition 4.2. $P \in \mathbb{C}[z_1, \dots, z_n]$ is said to be of principal type if its principal part P_m satisfies

$$\sum_{j=1}^n \left| \frac{\partial P_m}{\partial z_j}(x) \right|^2 \ne 0 \;\; \text{for each} \;\; x \in \mathbb{R}^n \setminus \{0\}.$$

Note that by Euler's rule $\langle x, \operatorname{grad} P_m(x) \rangle = m P_m(x)$, so P is of principal type if and only if

$$\operatorname{grad} P_m(x) \ne 0 \;\; \text{for each} \;\; x \in \mathbb{R}^n \setminus \{0\} \;\; \text{satisfying} \;\; P_m(x) = 0.$$

Theorem 4.3. *Let* $n \ge 2$ *and let* $P_m \in \mathbb{C}[z_1, \dots, z_n]$ *be homogeneous of degree* $m \ge 2$. *Then the following conditions are equivalent*:

(1) *For each* $Q \in \mathbb{C}[z_1, \dots, z_n]$ *with* $\deg Q < m$, *the variety* $V(P_m + Q)$ *satisfies* $\mathrm{PL}(\mathbb{R}^n, \log)$,

(2) P_m *is of principal type,* P_m *is real up to a complex constant, and each irreducible factor* q *of* P_m *has a real zero* $\xi \ne 0$.

Proof. (1) $\Rightarrow$ (2): By hypothesis, $V(P_m)$ has $\mathrm{PL}(\mathbb{R}^n, \log)$. Hence it follows from Meise, Taylor, and Vogt [**12**], Theorem 3.13, that $\dim V(q) \cap \mathbb{R}^n = n-1$

for each irreducible factor q of P_m. Thus, the third condition in (2) is fulfilled.

To prove that P_m is of principal type, note first that by Meise, Taylor, and Vogt [**13**], Lemma 2, there exists $\lambda \in \mathbb{C} \setminus \{0\}$ so that $\lambda P_m \in \mathbb{R}[z_1, \ldots, z_n]$. Hence the second condition of (2) is fulfilled and it is no restriction to assume that P_m has real coefficients. To prove that $\operatorname{grad} P_m$ does not vanish on $V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$ we argue by contradiction and assume that there exists $\theta \in V(P_m) \cap (\mathbb{R}^n \setminus \{0\})$ satisfying $\operatorname{grad} P_m(\theta) = 0$. After a real linear change of variables, we may assume $\theta = e_n = (0, \ldots, 0, 1)$. Then we apply Taylor's formula at θ to get

$$P_m(z', 1) = P_m(\theta + (z', 0)) = \sum_{k=\nu}^{m} q_k(z'), \tag{4.3}$$

where $q_k \in \mathbb{C}[z_1, \ldots, z_{n-1}]$ is zero or homogeneous of degree k and where $q_\nu \neq 0$. Then $2 \leq \nu \leq m$ since P_m and $\operatorname{grad} P_m$ vanish at θ. By the homogeneity of P_m it follows from 4.3 that for $z_n \neq 0$ we have

$$P_m(z', z_n) = z_n^m P_m\left(\frac{z'}{z_n}, 1\right) = z_n^m \sum_{k=\nu}^{m} q_k\left(\frac{z'}{z_n}\right) = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z').$$

By continuity, this holds also when $z_n = 0$. Now let

$$P(z) := P_m(z) + i z_n^{m-1} = \sum_{k=\nu}^{m} z_n^{m-k} q_k(z') + i z_n^{m-1}$$

and $d := (\nu - 1, \ldots, \nu - 1, \nu)$. Then the monomial z_n^{m-1} has d-degree $(m-1)\nu$ and the polynomials $z_n^{m-k} q_k(z')$ have d-degree $\nu m - k$, so they are decreasing in k. Hence the d-quasihomogeneous principal part q of P equals

$$q(z) = z_n^{m-\nu} q_\nu(z') + i z_n^{m-1}.$$

To show that q satisfies the hypotheses of Lemma 3.6, we note that $q_\nu \neq 0$ implies the existence of $\zeta' \in \mathbb{C}^{n-1}$ satisfying $q_\nu(\zeta') = -i$. Then $\zeta := (\zeta', 1)$ satisfies

$$q(\zeta) = q_\nu(\zeta') + i = 0.$$

Hence the conditions (1) and (2) of Lemma 3.6 are fulfilled. To show that also condition 3.6(3) holds, assume that for some $z' \in \mathbb{C}^{n-1}$ we have

$$0 = q(z', 1) = q_\nu(z') + i.$$

Since q_ν has real coefficients, this implies $z' \notin \mathbb{R}^{n-1}$, which proves condition 3.6(3). Hence we can apply Lemma 3.6 to conclude that $V(P)$ does not satisfy $\mathrm{PL}(\mathbb{R}^n, \log)$ in contradiction to the hypothesis (1).

(2) $\Rightarrow$ (1): Since P_m is real up to a complex factor, it is no restriction to assume that P_m has real coefficients. By Meise and Taylor [**7**], Lemma 4.6, the hypothesis implies that P_m is a product of distinct, irreducible factors

with real coefficients, each of which is of principal type. This implies that P_m is locally hyperbolic at every real characteristic in the sense of Definition 6.4 of Hörmander [**3**]. Hence it follows from [**3**], Theorem 6.5, that $V(P_m)$ satisfies HPL($\mathbb{R}^n$). By hypothesis, no irreducible component of $V(P_m)$ is elliptic. Hence $V(P_m)$ satisfies PL($\mathbb{R}^n$, log) by Meise, Taylor, and Vogt [**12**], Corollary 3.14. Next fix $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < m$, $Q \neq 0$ and choose $C > 0$ such that $|Q(z)| \leq C(1+|z|^{m-1})$, $z \in \mathbb{C}^n$. By the homogeneity of P_m, the function

$$x \mapsto \sum_{j=1}^{n} \left| \frac{\partial P_m}{\partial x_j}(x) \right|$$

is positively homogeneous of degree $m-1$ and does not vanish for $x \in \mathbb{R}^n \setminus \{0\}$ by hypothesis. This implies the existence of $\delta > 0$ and $A \geq 1$ such that

$$1 + |z|^{m-1} \leq A \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \{z \in \mathbb{C}^n : |\operatorname{Im} z| < \delta |z|\} =: \Gamma.$$

Consequently, there exists A' with

$$|Q(z)| \leq A' \max_{\alpha \neq 0} |P_m^{(\alpha)}(z)|, \quad z \in \Gamma.$$

Now fix $z \in \Gamma \cap (V(P_m + Q) \setminus V(P_m))$ and note that by Hörmander [**4**], Lemma 4.1.1, (which holds also for $\xi \in \mathbb{C}^n$) there exists $D > 0$ such that

$$\operatorname{dist}(\zeta, V(P_m)) \sum_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(\zeta)}{P_m(\zeta)} \right|^{\frac{1}{|\alpha|}} \leq D, \quad \zeta \in \mathbb{C}^n \setminus V(P_m).$$

This and $P_m(z) = -Q(z)$ imply

$$\frac{1}{A'} \leq \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{Q(z)} \right| = \max_{\alpha \neq 0} \left| \frac{P_m^{(\alpha)}(z)}{P_m(z)} \right| \leq \max_{\alpha \neq 0} \left(\frac{D}{\operatorname{dist}(z, V(P_m))} \right)^{|\alpha|}$$

and hence the existence of $E > 0$ such that (by continuity)

$$\operatorname{dist}(z, V(P_m)) \leq E, \quad z \in \Gamma \cap V(P_m + Q).$$

From this we get (1) by Lemma 4.1. □

Remark 4.4. Note that Theorem 4.3 and its Corollary 4.7 below extend Corollary 5.8 of Meise, Taylor, and Vogt [**12**]. Moreover, Theorem 4.3 shows that the characterizing condition is in fact weaker than the sufficient condition given there, since P_m can be of principal type, while $V(P_m)$ has complex singularities. To see this, consider $P_4(x, y, z) := (x^2 + y^2 - z^2)(x^2 + z^2 - y^2/4)$ and note that $\{\lambda \cdot (i\sqrt{3/5}, 2\sqrt{2/5}, 1) : \lambda \in \mathbb{C}\}$ is a singular line for $V(P_4)$.

Remark 4.5. From Meise and Taylor [**7**], 4.8 and 3.4, it follows that each real homogeneous polynomial P_m of principal type for which each irreducible factor has a non-trivial real zero is also stable under certain real perturbations introducing an extra variable. More precisely, the variety

$$\{z \in \mathbb{C}^{n+1} : P_m(z_1, \ldots, z_n) = z_{n+1}\}$$

satisfies $\mathrm{PL}(\mathbb{R}^{n+1}, \log)$, provided that P_m is of real principal type and has no elliptic factor.

In the proof of Theorem 4.3 we used complex polynomials to show that $V(P_m+Q)$ fails $\mathrm{PL}(\mathbb{R}^n, \log)$ if $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ while P_m is not of principal type. In some cases, such as $P_2(x, y, z) = x^2 - y^2$, this is the only possible choice (see Meise, Taylor, and Vogt [**12**], Example 4.9). However, in other cases, real perturbations can also have the same effect, as the following example shows.

Example 4.6. Let $P(x, y, z) := x^2z + yz^2 + yz$. The principal part $P_3(x, y, z) = x^2z + yz^2 = (x^2 + yz)z$ is hyperbolic with respect to $N = (0, -1, 1)$. Hence $V(P_3)$ satisfies $\mathrm{PL}(\mathbb{R}^3, \log)$ by Meise, Taylor, and Vogt [**9**], 3.6, and 4.5 in connection with [**12**], 2.12. Obviously, P_3 is not of principal type. By Example 3.3(a), $V(P)$ does not satisfy $\mathrm{PL}(\mathbb{R}^3, \omega)$ whenever $\omega(t) = o(t^{1/3})$.

Corollary 4.7. *Let $P_m \in \mathbb{C}[z_1, \ldots, z_n]$ be homogeneous of degree $m \geq 2$ and of principal type. Then the following conditions are equivalent*:

1. *$V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$,*
2. *P_m is real up to a complex constant and each irreducible factor q of P_m has a real zero $\xi \neq 0$,*
3. *there exist $k \in \mathbb{N}$, $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q < km$, and a weight function ω so that $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$,*
4. *for each $k \in \mathbb{N}$ and each $Q \in \mathbb{C}[z_1, \ldots, z_n]$ with $\deg Q = l < km$ we have*:
   (a) *if $l \leq k(m-1)$, then $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$,*
   (b) *if $l > k(m-1)$, then $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, 1 + t^\beta)$ for $\beta = 1 + \frac{l}{k} - m$.*

Proof. (1) ⇒ (2): This follows from Meise, Taylor, and Vogt [**13**], Lemma 2, and [**12**], Corollary 3.14.

(2) ⇒ (3): Since P_m is of principal type, (2) implies that condition 4.3(2) is fulfilled. Hence $V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ by Theorem 4.3. Thus (3) holds for $k = 1$ and $Q = 0$.

(3) ⇒ (1): If $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ then $V(P_m^k) = V(P_m)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ by Meise, Taylor, and Vogt [**12**], Theorem 4.1.

(4) ⇒ (3): This holds trivially.

(2) ⇒ (4): Since P_m is of principal type, there exists $\eta > 0$ such that

$$|\mathrm{grad}\, P_m(z)| > 0 \quad \text{for} \quad z \in \mathbb{C}^n, |z| = 1 \quad \text{and} \quad |\mathrm{Im}\, z| \leq \eta.$$

Therefore standard arguments using homogeneity and compactness imply the existence of $\delta > 0$ such that

$$\max_{0<|\alpha|\le k} |(P_m^k)^{(\alpha)}(z)| \ge \delta |z|^{k(m-1)} \quad \text{for } z \in \mathbb{C}^n,\ |\mathrm{Im}\, z| \le \eta |z|,\ |z| \ge 1. \tag{4.4}$$

Now fix $Q \in \mathbb{C}[z_1, \dots, z_n]$ satisfying $\deg Q \le k(m-1)$. Then there exists $D \ge 1$ such that

$$|Q(z)| \le D \max_{|\alpha|>0} \left|(P_m^k)^{(\alpha)}(z)\right| \quad \text{for} \quad z \in \mathbb{C}^n, |z| \ge 1, |\mathrm{Im}\, z| \le \eta |z|. \tag{4.5}$$

Now fix $\xi \in V(P_m) \cap \mathbb{R}^n$, $|\xi| = 1$, and let

$$\Gamma(\xi, \eta, 1) := \left\{ z \in \mathbb{C}^n : \left| \frac{z}{|z|} - \xi \right| \le \eta, |z| \ge 1 \right\}.$$

For $z \in \Gamma(\xi, \eta, 1)$ we have $|\mathrm{Im}\, z| \le \eta |z|$ since ξ is real. Now fix $\zeta \in V(P_m^k + Q) \cap \Gamma(\xi, \eta, 1)$ satisfying $P_m(\zeta) \ne 0$. Then $P_m^k(\zeta) = -Q(\zeta)$ and (4.5) imply the existence of $M \ge 1$ such that

$$\frac{1}{M} \le \sum_{|\alpha|>0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{Q(\zeta)} \right|^{1/|\alpha|} = \sum_{|\alpha|>0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{P_m^k(\zeta)} \right|^{1/|\alpha|}.$$

Since by Hörmander [**4**], Lemma 4.1.1, there exists $C \ge 1$ such that

$$\mathrm{dist}(\zeta, V(P_m^k)) \sum_{|\alpha|>0} \left| \frac{(P_m^k)^{(\alpha)}(\zeta)}{P_m^k(\zeta)} \right|^{1/|\alpha|} \le C$$

we conclude that

$$\mathrm{dist}(\zeta, V(P_m^k)) \le CM \le CM \log(2 + |\zeta|).$$

Since $\xi \in V(P_m) \cap \mathbb{R}^n, |\xi| = 1$, was chosen arbitrarily, it follows from Lemma 4.1 that $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \log)$ in this case.

If $k(m-1) < l = \deg Q < km$ then $0 < \beta := 1 + \frac{l}{k} - m < 1$ and (4.4) implies the existence of $\delta > 0$ such that

$$\begin{aligned} \max_{0<|\alpha|\le k} \left|(P_m^k)^{(\alpha)}(z)\right| |z|^{\beta|\alpha|} &\ge \delta |z|^{k(m-1)-\beta k} \\ &= \delta |z|^l \quad \text{if } |z| \ge 1 \text{ and } |\mathrm{Im}\, z| \le \eta |z|. \end{aligned}$$

Hence there exists $D \ge 1$ such that

$$|Q(z)| \le D \max_{|\alpha|>0} \left|(P_m^k)^{(\alpha)}(z)\right| (1 + |z|^\beta)^{|\alpha|} \quad \text{for} \quad z \in \mathbb{C}^n, |\mathrm{Im}\, z| \le \eta |z|, |z| \ge 1.$$

From this it follows as above that for each $\xi \in V(P_m^k) \cap \mathbb{R}^n$, $|\xi| = 1$, there exists $C_\xi > 0$ such that

$$\mathrm{dist}(\zeta, V(P_m^k)) \le C_\xi (1 + |\zeta|^\beta), \quad \zeta \in \Gamma(\xi, \eta, 1).$$

By Lemma 4.1, this implies that $V(P_m^k + Q)$ satisfies $\mathrm{PL}(\mathbb{R}^n, \omega)$ for $\omega(t) = 1 + t^\beta$, as asserted. $\square$

Theorem 4.3 in connection with Meise, Taylor, and Vogt [**8**], Théorème, also implies the following result on the existence of fundamental solutions with large lacunas.

Corollary 4.8. *Let $P \in \mathbb{C}[z_1, \ldots, z_n]$ be of principal type and of degree $m \geq 2$. If the principal part P_m of P is real up to a complex constant and if each irreducible factor q of P_m has a real zero $\xi \neq 0$ then the following holds:*

For each $r > 0$ there exists $R > 0$ such that for each $\xi \in \mathbb{R}^n$, $|\xi| > R$, there exists a fundamental solution $E_\xi \in \mathcal{D}'(\mathbb{R}^n)$ of $P(D)$ satisfying $\operatorname{Supp} E_\xi \subset \{x \in \mathbb{R}^n : |x - \xi| \geq r\}$.

References

[1] R.W. Braun, R. Meise and B.A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Res. Math., **17** (1990), 207-237.

[2] S.G. Gindikin and L.R. Volevich, *Mixed Problem for Partial Differential Equations with Quasihomogeneous Principal Part*, Amer. Math. Soc., 1996 (translation from the Russian).

[3] L. Hörmander, *On the existence of real analytic solutions of partial differential equations with constant coefficients*, Invent. Math., **21** (1973), 151-183.

[4] ———, *Linear Partial Differential Operators*, Springer 1969.

[5] ———, *The Analysis of Linear Partial Differential Operators*, Vol. II, Springer 1983.

[6] Y. Laurent, *Théorie de la Deuxième Microlocalisation dans le Domaine Complexe*, Birkhäuser, 1985.

[7] R. Meise and B.A. Taylor, *Phragmén-Lindelöf conditions for graph varieties*, Res. Math., to appear.

[8] R. Meise, B.A. Taylor and D. Vogt, *Charactérisation des opérateurs linéaires aux dérivées partielles avec coefficients constants sur $\mathcal{E}(\mathbb{R}^n)$ admettant un inverse à droite qui est linéaire et continu*, C. R. Acad. Sci. Paris, **307** (1988), 239-242.

[9] ———, *Characterization of the linear partial operators with constant coefficients that admit a continuous linear right inverse*, Ann. Inst. Fourier (Grenoble), **40** (1990), 619-655.

[10] ———, *Extremal plurisubharmonic functions of linear growth on algebraic varieties*, Math. Z., **219** (1995), 515-537.

[11] ———, *Continuous linear right inverses for partial differential operators on non-quasianalytic classes and on ultradistributions*, Math. Nachr., **180** (1996), 213-242.

[12] ———, *Phragmén-Lindelöf principles on algebraic varieties*, J. Amer. Math. Soc., **11** (1998), 1-39.

[13] ———, *Continuous linear right inverses for partial differential operators of order* 2 *and fundamental solutions in half spaces*, Manuscr. Math., **90** (1996), 449-464.

[14] R. Nevanlinna, *Eindeutige analytische Funktionen*, Springer, 1974.

[15] V.P. Palamodov, *A criterion for splitness of differential complexes with constant coefficients*, in 'Geometrical and Algebraical Aspects in Several Complex Variables', C.A. Berenstein and D.C. Struppa (Eds.), EditEL, (1991), 265-291.

Received June 4, 1998. The third author gratefully acknowledges support of the A. v. Humboldt Stiftung for visits at Mathematisches Institut der Heinrich-Heine-Universität Düsseldorf, where this research was carried out.

Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstrasse 1
40225 Düsseldorf
Germany
E-mail address: braun@cs.uni-duesseldorf.de

Mathematisches Institut
Heinrich-Heine-Universität
Universitätsstrasse 1
40225 Düsseldorf
Germany
E-mail address: meise@cs.uni-duesseldorf.de

Department of Mathematics
University of Michigan
Ann Arbor, MI 48109
E-mail address: taylor@math.lsa.umich.edu